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From repeated games to Brownian games

by

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ABSTRACT. – The subject of this paper is related to the analysis of the convergence rate of the value of the n-times repeated zero-sum game with one sided information and full monitoring. Particularly, the ultimate aim of this work is the proof of the existence of an asymptotic expansion for this value v_n:

$$v_n = v_\infty + \frac{\psi}{\sqrt{n}} + O(\frac{\ell n(n)}{n}).$$

As suggested in the conclusion of [6], the function ψ appearing in this expansion should be regarded as the value of a "continuously repeated" game.

In this paper, we propose and analyze a game of this kind. In this game, the strategies are progressively measurable processes on the filtration generated by a Brownian motion and the payoff function is defined by use of the Itô-integral. Our main result is the proof of the existence of optimal strategies for both players in this game. © Elsevier, Paris

RÉSUMÉ. – L'objet de cet article est lié à l'analyse de la vitesse de convergence de la valeur v_n des jeux à somme nulle n fois répétés et à information incomplète d'un côté. Plus précisément, le but ultime de

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ce travail est la preuve de l'existence d'un développement asymptotique pour v_n:

$$v_n = v_\infty + \frac{\psi}{\sqrt{n}} + O(\frac{\ell n(n)}{n}).$$

Comme le suggérait la conclusion de l'article [6], la fonction ψ qui apparaît dans ce développement devrait être considérée comme la valeur d'un "jeu répété en temps continu".

Dans cet article, nous définissons et analysons un tel jeu. Les stratégies des joueurs y sont des processus progressivement mesurables sur une filtration générée par un mouvement Brownien et la fonction de payement fait intervenir l'intégrale de Itô de ces processus.

1. INTRODUCTION

The notion of Brownian games that will be introduced in this paper is intimately related to the analysis of the asymptotic behavior of the value $v_n(p)$ of the n times repeated zero sum game with one sided information and more specifically with the existence of the following asymptotic expansion for $v_n(p)$:

$$v_n(p) = v_\infty(p) + \frac{\psi(p)}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}). \tag{1}$$

As it will be recalled in the next section, two types of arguments were previously invoked to prove the existence of such an expansion: on one hand, the recurrence formula for v_n yields in the limit a Partial Differential Equation (PDE) for ψ and the main result of [5] states that if a smooth function f with appropriate boundary conditions satisfies to this PDE, then (1) holds with $\psi = f$. This PDE is in general strongly non linear and the existence results of the PDE literature do not apply to this equation. In particular cases however, this PDE can be solved explicitly and its solution is then related to the normal density.

On the other hand, for these particular cases, the expansion (1) is proved in [6] as a consequence of the central limit theorem, giving in this way a rational to the appearance of this normal density.

The approach initiated in this paper is an attempt to prove the existence of solutions the PDE problem in the general non linear case by a probabilistic argument. More specifically, we will introduce in the next section the notion of Brownian games as a kind of limit in continuous time of the n times repeated games.

These Brownian games are stochastic differential games where two completely antagonistic players control the martingale term of a diffusion process. Part of the interest of this paper comes from this particular feature of the Brownian games: "stochastic" differential games are usually considered as "classical" differential games with an additional uncontrolled random perturbation.

The next part of this paper is mainly devoted to the definition of the Brownian games and to the underlying intuition. We also present there the main results concerning these games that will be proved in the remaining parts.

In part 3, we essentially prove that the Brownian games have a value ψ, and both players have optimal strategies. The recursive structure of the Brownian games and the Markovian properties of these optimal strategies are analyzed in part 4.

In the last section of the paper, we prove that the function ψ is a kind of generalized solution of the above PDE: if ψ was a smooth function, it would satisfy a Bellman type equation that coincides with the above mentioned PDE.

The regularity of the function ψ will be proved in a forthcoming paper [7] under a strict ellipticity condition and the validity of (1) will then follow.

2. DEFINITION OF THE BROWNIAN GAMES AND MAIN RESULTS

2.1. On the history of the problem. The repeated zero-sum game with one sided information and full monitoring was first analyzed in the now classical paper by Aumann and Maschler in 1966 (see [1]). They deal there with the following n-times repeated game $\Gamma_n(p)$:

DEFINITION 2.1.1. – *For a finite set $\mathcal{B}$, $\Delta(\mathcal{B})$ will denote in the following the simplex of probability distributions on $\mathcal{B}$.*

Let $\mathcal{K}$ be the set of the K possible states of nature. To each state of nature k corresponds an elementary one shot zero sum game. The action sets for player 1 and player 2 in this game are respectively denoted by $\mathcal{I}$ and $\mathcal{J}$.

They contains respectively I and J elements. A pair (i, j) *of actions in* $\mathcal{I} \times \mathcal{J}$ *yields a payoff* a_{ij}^k *for player 1 in state k.*

For $p \in \Delta(\mathcal{K})$, *the game* $\Gamma_n(p)$ *proceeds as follows: Nature chooses initially and once for all a state k at random with the lottery p. Player 1 is informed by nature about the true state k, while player 2 is not. At each successive stage q of* $\Gamma_n(p)$ $(q = 1, \ldots, n)$, *the players choose, independently of each other, an action* i_q *and* j_q *in their respective action set. They are then informed about the choice of their opponent. At the end of the game player 2 pays the amount* $\frac{1}{n}\sum_{q=1}^{n} a_{i_q j_q}^k$ *to player 1. We assume that both players are aware of the above description of the game. In particular, they know the payoff matrices, as well as the initial probability distribution p.*

Let us remark here that only players' actions are communicated at each stage, but not the corresponding payoffs. As a consequence, player 2 ignores the elementary game he is actually playing up to the end of $\Gamma_n(p)$.

The value $v_n(p)$ of $\Gamma_n(p)$ is known (see e.g. [12] chapter V) to be a concave function of p which decreases, as n increases, to a limit $v_\infty(p) = \text{cav}(u)(p)$, where $u(p)$ denotes the value of the average one shot game $G(p)$ with payoffs $a_{ij}(p) = \sum_{k=1}^{K} a_{ij}^k p_k$. $\text{cav}(u)$ denotes the concavification of this function on the simplex $\Delta(\mathcal{K})$. The proof of this convergence given in [1] is based on a bound on the variation of a bounded martingale and leads to the result $\varepsilon_n(p) := v_n(p) - v_\infty(p) \leq \frac{c}{\sqrt{n}}$ for a positive constant c depending on the payoffs a_{ij}^k.

In [13], $\psi_n(p) := \sqrt{n}\varepsilon_n(p)$ is proved to converge to a limit $\psi(p)$ related to the normal density for a particular two-states-of-nature-game, leading to the asymptotic expansion (1) for v_n.

The reasoning of Mertens and Zamir is generalized in [5] to a broader class of games called $\Delta_{\sigma_0}^0$.

DEFINITION 2.1.2. – *The games in* $\Delta_{\sigma_0}^0$ *are square games (i.e.* $I = J$*) such that the completely mixed (i.e.* $\forall i : \sigma_{0,i} > 0$*) strategy* σ_0 *is the unique optimal strategy for player 1 in* $G(p)$, $\forall p$, *and such that the value* $u(p)$ *of* $G(p)$ *is identically* 0.

The next lemma is proved in [5], Section 4:

LEMMA 2.1.3. – *For games in* $\Delta_{\sigma_0}^0$, *player 2 has a unique optimal strategy* $\rho(p)$ *in* $G(p)$. *This strategy is completely mixed (i.e.* $\rho_j(p) > 0$, $\forall j$*) and the mapping* $\rho : \Delta(\mathcal{K}) \to \Delta(\mathcal{J})$ *belongs to* C^∞, *since,* $\forall j$, $\rho_j(p)$ *may be expressed as a quotient of two polynomials in p.*

Furthermore, the optimal strategies in $G(p)$ are equalizing. With the vector notations $a_{ij} := (a_{ij}^1, \dots, a_{ij}^K)$ and $a_{i\tau} := \sum_{j\in\mathcal{J}} \tau_j a_{ij}$, this means:

$$\forall \tau \in \Delta(\mathcal{J}) : \sum_{i\in\mathcal{I}} \sigma_{0,i} a_{i\tau} = 0, \tag{2}$$

$$\forall p \in \Delta(\mathcal{K}), \forall i \in \mathcal{I} : \langle p | a_{i\rho(p)} \rangle = 0, \tag{3}$$

where $\langle p|a\rangle$ stands for the Euclidean product in $\mathbb{R}^K$: $\langle p|a\rangle := \sum_{k\in\mathcal{K}} p_k a^k$.

The importance of the class $\Delta^0_{\sigma_0}$ is stressed in another paper of Mertens and Zamir [14]: in case $I = J = K = 2$, they prove a convergence faster than $\frac{1}{\sqrt{n}}$, outside $\Delta^0_{\sigma_0}$.

As it follows from lemma 4.3 in [12], chapter V, there is a recurrence relation between the functions $v_{n+1}(\bullet)$ and $v_n(\bullet)$. Since $\psi_n = \sqrt{n} v_n$ for games in $\Delta^0_{\sigma_0}$, this recurrence formula for v_n becomes one for ψ_n that can be written in an abstract way: $\psi_{n+1} = H_n(\psi_n)$, where H_n is a functional operator. So, heuristically, if the ψ_n's were to converge, their limit ψ should be "nearly a fixed point of H_n," as n tends to ∞. By giving the meaning $\|f - H_n(f)\|_\infty = O(n^{-3/2})$ to the expression "f is nearly a fixed point of H_n", the converse of this claim is proved in [5]. More precisely: if f is such a fixed point, bounded on the simplex $\Delta(\mathcal{K})$ and vanishing at its extreme points, then ψ_n converges uniformly to f with a rate of, at least, $n^{-\frac{1}{2}} ln(n)$. When replacing f by its Taylor expansion around p in $H_n(f)(p)$, the "fixed point" property of f becomes a Partial Differential Equation (PDE). Conversely, any sufficiently smooth solution of this PDE, with appropriate boundary conditions will be a "fixed point" for H_n and will therefore be the limit of the ψ_n as proved in [5], theorem 5.4.

This PDE is strongly non linear: it involves the inverse of the Hessian matrix of f. It is however made more appealing by introducing Fenchel duality:

DEFINITION 2.1.4. – *The Fenchel conjugate of a function $f : D \subset \mathbb{R}^K \to \mathbb{R}$ is defined as the function $f^\circ : \mathbb{R}^K \to \mathbb{R}$ that maps x to $f^\circ(x) := \inf_{y\in D}\langle x|y\rangle - f(y)$.*

We are then led to consider a dual game Γ_n^* of Γ_n, whose value v_n^* is the Fenchel conjugate of the concave function v_n.

DEFINITION 2.1.5. – *For x in $\mathbb{R}^K$, $\Gamma_n^*(x)$ is the following game: Player 1 chooses $k \in \mathcal{K}$, without informing player 2. Next, as in Γ_n, the players have to select successively their actions (i_q, j_q). The final payoff player 1 has to* *pay to player 2 is now* $x_k - \frac{1}{n}\sum_{q=1}^{n} a^k_{i_q j_q}$. *(Player 1 is here the minimizer and* x_k *has to be interpreted as a penalty for having chosen the state k.)*

If the factor $\frac{1}{n}$ is replaced with $\frac{1}{\sqrt{n}}$ in the payoff of this dual game, we get an other game whose value ψ^*_n is $(\sqrt{n}v_n)^\circ$. In particular, for games in $\Delta^0_{\sigma_0}$, ψ^*_n is the Fenchel conjugate of ψ_n.

The interest for introducing a dual game comes from the following observation: In the primal game Γ_n, as underlined in [19], player 1 has optimal strategies that are Markovian in the following sense: at each stage, these strategies only depend on the a posteriori probability on $\mathcal{K}$ given the past moves. The process of these a posteriori distributions is thus Markovian, since it only depends on player 1's mixed moves.

In counterpart the dual game Γ^*_n is particularly well suited to analyze the "Markovian" behavior of player 2 as emphasized in [6] section 4.

When restricted to concave upper semi continuous functions, the Fenchel transform is an isometry in the uniform norm. The rate of convergence of v_n may then be analyzed through that of v^*_n and the convergence of ψ_n, through that of ψ^*_n. The recursive formula for ψ_n can be transformed in the following recursive formula for ψ^*_n (see Lemma 5.1 in [5]): $\psi^*_{n+1} = H^*_n(\psi^*_n)$, where the dual recurrence operator H^*_n is defined as

$$H^*_n(f)(x) := \max_{\tau\in\Delta(\mathcal{J})}\min_{i\in\mathcal{I}} \sqrt{\frac{n}{n+1}} f\Big(\sqrt{\frac{n+1}{n}}x - \frac{1}{\sqrt{n}}a_{i\tau}\Big).$$

As in the primal model, proving the convergence of ψ^*_n turns out to be equivalent to finding a function ψ^* satisfying $\|\psi^* - H^*_n(\psi^*)\|_\infty = O(n^{-3/2})$. This finally leads to the following dual PDE for ψ^*:

$$\psi^*(x) = \langle x|\nabla\psi^*(x)\rangle + \sum_{i\in\mathcal{I}} \sigma_{0,i} a^\top_{i\rho(\nabla\psi^*(x))}\psi^{*\prime\prime}(x) a_{i\rho(\nabla\psi^*(x))}, \tag{5}$$

where $a_{i\rho(\nabla\psi^*(x))}$ is considered as a column vector and $\psi^{*\prime\prime}(x)$ stands for the Hessian matrix of ψ^*. This equation is quasi-linear elliptic (see [10], p. 257) and is easier to handle than the primal one. This equation is solved in [5], section 6 for a sub-class R_{σ_0} of $\Delta^0_{\sigma_0}$. Surprisingly, the solution ψ^* then obtained is related to the normal density as happened to be the case in Zamir's game (see [13]).

The appearance of the normal density in a subclass $R^{\mathrm{sym}}_{\sigma_0}$ of R_{σ_0} is explained in [6] as follows:

1) We first prove (see theorem 3.1 in [6]) that, both in $\Gamma^*_n(x)$ and $\Gamma_n(p)$, the knowledge of past actions $(j_1, \ldots, j_{q-1})$ of player 2 is irrelevant for both players when playing stage q.

This allows the players to consider only their "reduced" strategies (see section 3 in [6]). For player 2, a behavioral reduced strategy is an n-uple $(\tau_1(\bullet), \ldots, \tau_n(\bullet))$ where $\tau_q(\bullet)$ is a mapping from $\mathcal{I}^{q-1}$ to $\Delta(\mathcal{J})$. A mixed reduced strategy Σ for player 1 in the normal form game $\Gamma_n^*(x)$ is a probability distribution over $\mathcal{I}^n \times \mathcal{K}$, while in $\Gamma_n(p)$ the marginal distribution of Σ on $\mathcal{K}$ must coincide with p.

Using the following notations

$$S(i_1, \ldots, i_n) := (\Sigma(i_1, \ldots, i_n, 1), \ldots, \Sigma(i_1, \ldots, i_n, K)),$$

the payoff of player 2 in $\Gamma_n^*(x)$ is then written as (see equation (4) in [6]):

$$(5) \qquad \sum_{i_1,\ldots,i_n} \langle S(i_1, \ldots, i_n) | x - \frac{1}{n} \sum_{q=1}^{n} a_{i_q \tau_q(i_1,\ldots,i_{q-1})} \rangle.$$

Similarly, the payoff of player 1 in $\Gamma_n(p)$ is given by:

$$(6) \qquad \sum_{i_1,\ldots,i_n} \langle S(i_1, \ldots, i_n) | \frac{1}{n} \sum_{q=1}^{n} a_{i_q \tau_q(i_1,\ldots,i_{q-1})} \rangle.$$

In $\Delta_{\sigma_0}^0$, we have $\psi_n = \sqrt{n} v_n$, so, relation (5) leads to:

$$(7) \quad \psi_n^*(x) = \max_{\tau_1(\bullet),\ldots,\tau_n(\bullet)} \min_{\pi \in \Delta(\mathcal{I}^n)} E_\pi \left[\gamma^* \left(x - \frac{1}{\sqrt{n}} \sum_{q=1}^{n} a_{i_q \tau_q(i_1,\ldots,i_{q-1})} \right) \right]$$

where, as in [5], $\gamma^*(x)$ denotes $\min\{x_1, \ldots, x_K\}$. In this formula, π is the marginal distribution of Σ on $\mathcal{I}^n$ and the minimization over the conditional probabilities on $\mathcal{K}$ leads to the function γ^*. The min and max operators commute in this formula.

2) For games in $R_{\sigma_0}^{\text{sym}}$, the optimal strategy π^* for player 1 in formula (7) written as a min max is proved (see theorem 5.2. in [6]) to be the i.i.d. σ_0-distribution: $\pi^*(i_1, \ldots, i_n) := \prod_{q=1}^{n} \sigma_{0,i_q}$. This fact together with the particular shape of the vectors a_{ij} in R_{σ_0} leads to:

$$\psi_n^*(x) = E_{\pi^*} [\gamma^*(x - \frac{1}{\sqrt{n}} \sum_{q=1}^{n} A_q)]$$

where the A_q's are i.i.d. random vectors under π^*. The Central Limit Theorem justifies then the appearance of the normal distribution.

2.2. An heuristic approach to the Brownian games. To introduce heuristically the limiting game Γ^*, let us first rewrite formula (7) in a way that emphasizes the role of π^*: σ_0 being completely mixed, any probability distribution π on $\mathcal{I}^n$ has a density y with respect to π^*. So:

(8)
$$\psi_n^*(x) = \max_{\tau_1(\bullet),\dots,\tau_n(\bullet)} \min_{\substack{y\geq 0\\ E_{\pi^*}[y]=1}} E_{\pi^*}\left[y\gamma^*\big(x - \frac{1}{\sqrt{n}}\sum_{q=1}^{n} a_{i_q\tau_q(i_1,\dots,i_{q-1})}\big)\right].$$

The central limit theorem indicates that, asymptotically, the distribution of the sum of n independent F-distributed random variables only depends on the expectation and variance of F.

This suggests that replacing the sum in equation (8) by a sum of random vectors with the same conditional expectation and covariance matrix should not crucially affect the final result.

Since $\tau_q(\bullet)$ is a function of $(i_1,\dots,i_{q-1})$, we get with equation (2):

$$E_{\pi^*}[a_{i_q\tau_q}|i_1,\dots,i_{q-1}] = \sum_{i_q}\sigma_{0,i_q}a_{i_q\tau_q} = 0.$$

Furthermore, the covariance matrix of $a_{i_q\tau_q}$, conditional on $(i_1,\dots,i_{q-1})$, under the probability π^* is $\sum_{i\in\mathcal{I}}\sigma_{0,i}a_{i\tau_q}a_{i\tau_q}^{\top}$ (vectors are considered in this paper as column matrices and $a^{\top}$ denotes the transpose of a).

Let us then replace the random term $a_{i_q\tau_q}$ by $\sum_{i\in\mathcal{I}} z_i^q a_{i\tau_q}\sqrt{\sigma_{0,i}}$, where z^q are independent I-dimensional standard normal random vectors.

Finally, $\frac{z^q}{\sqrt{n}}$ may be seen as the increment $W_{\frac{q}{n}} - W_{\frac{q-1}{n}}$ of the I-dimensional Brownian motion W on a filtration $\mathcal{F}_t$, $t\in[0,1]$. In this way, we get heuristically:

(9)
$$\psi_n^*(x) \approx \max_{\tau_1(\bullet),\dots,\tau_n(\bullet)} \min_{\substack{y\geq 0\\ E[y]=1}} E\left[y\gamma^*\big(x - \sum_{q=1}^{n}\sum_{i\in\mathcal{I}}\sqrt{\sigma_{0,i}}a_{i\tau_q}(W_{i,\frac{q}{n}} - W_{i,\frac{q-1}{n}})\big)\right].$$

In equation (8), τ_q only depends on the past history $(i_1,\dots,i_{q-1})$. This feature will be taken into account in formula (9) by assuming it to be $\mathcal{F}_{\frac{q-1}{n}}$-measurable. Similarly, y, that depends on $(i_1,\dots,i_n)$ in (8), will be $\mathcal{F}_1$-measurable in (9).

Letting now τ denote the $\{\mathcal{F}_t\}$-progressively measurable step process $\tau(\omega,t) := \tau_{q-1}(\omega)$ if $\frac{q-1}{n}\leq t<\frac{q}{n}$, relation (9) becomes:

$$\psi_n^*(x) \approx \sup_{\tau}\inf_{\substack{y\geq 0\\ E[y]=1}} E\left[y\gamma^*\big(x - \int_0^1\sum_{i\in\mathcal{I}}\sqrt{\sigma_{0,i}}a_{i\tau_t}dW_{i,t}\big)\right]. \tag{10}$$

The limiting game $\Gamma^*(x)$ is finally obtained by dispensing with the condition on τ to be a step process and the strategy space for player 2 is chosen to be the set of progressively measurable processes τ valued in $\Delta(\mathcal{J})$.

Before further transformations of formula (10), let us observe here that it formalizes pretty well the optimization problem player 2 is confronted:

The infimum appearing in this formula is the essential infimum of the random variable $\gamma^*\left(x - \int_0^1 \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tau_t} dW_{i,t}\right)$, so player 2 has to control the martingale $x - \int_0^s \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tau_t} dW_{i,t}$ in such a way that it will remain up to time 1 in the orthant $R_a := \{z \in I\!R^K : \gamma^*(z) \geq a\}$ for the highest possible a.

We can still modify formula (10) in order to obtain a payoff function similar to (5), which is linear in both strategies: Since $\gamma^*(x) = \min_{p\in\Delta(\mathcal{K})} \langle p|x\rangle$, we have:

$$\psi_n^*(x) \approx \sup_{\tau} \inf_{\substack{y\geq 0\\ E[y]=1}} \inf_{p\in\mathcal{P}} E\left[\langle yp|x - \int_0^1 \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tau_t} dW_{i,t}\rangle\right],$$

where $\mathcal{P}$ denotes the set of $\mathcal{F}_1$-measurable $\Delta(\mathcal{K})$-valued random vectors. Observing that yp is valued in $I\!R_+^K$ and that $E[yp] \in \Delta(\mathcal{K})$ since $E[y] = 1$, we finally have:

$$\psi_n^*(x) \approx \sup_{\tau} \inf_{\substack{Y\in I\!R_+^K\\ E[Y]\in\Delta(\mathcal{K})}} E\left[\langle Y|x - \int_0^1 \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tau_t} dW_{i,t}\rangle\right]. \tag{11}$$

In analogy with formula (6), we also should have:

$$\psi_n(p) \approx \sup_{\substack{Y\in I\!R_+^K\\ E[Y]=p}} \inf_{\tau} E\left[\langle Y| \int_0^1 \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tau_t} dW_{i,t}\rangle\right]. \tag{12}$$

Let us stress here that the above link between the finitely repeated games and the Brownian games is purely heuristic and the symbol $\approx$ in the above formulas has only a formal meaning: In formula (9), since $W_{i,\frac{q}{n}} - W_{i,\frac{q-1}{n}}$ is normally distributed, the random variable to the right of y in (9) ranges all over $I\!R$. Therefore, if no further condition is imposed on y, the min appearing in this formula equals $-\infty$!

2.3. Definitions and main results. Let T be in $[0,1]$, let $\{\mathcal{F}_t\}_{t\in[T,1]}$ be a filtration on a probability space $(Z, \mathcal{F}, P)$.

DEFINITION 2.3.1. – *For $\alpha \in (1,\infty)$, $\mathcal{M}_2^\alpha(\{\mathcal{F}_t\}_{t\in[T,1]}, I\!R^n)$ will refer here after to the set of $I\!R^n$-valued $\{\mathcal{F}_t\}$-progressively measurable processes c such that $\|c\|_{M_2^\alpha} := E[(\int_T^1 \|c_t\|^2 dt)^{\frac{\alpha}{2}}]^{\frac{1}{\alpha}} < \infty$, where $\|c_t\|$ stands for the Euclidean norm of c_t in $I\!R^n$. If c and c' are such that $\|c - c'\|_{M_2^\alpha} = 0$, we say that c' is a modification of c and we write $c \sim c'$. $M_2^\alpha(\{\mathcal{F}_t\}_{t\in[T,1]}, I\!R^n)$ denotes the quotient of $\mathcal{M}_2^\alpha(\{\mathcal{F}_t\}_{t\in[T,1]}, I\!R^n)$ by the equivalence relation $\sim$.*

These notations are shorten to $\mathcal{M}_2^\alpha$ and M_2^α when the filtration as well as the target space $I\!R^n$ are clearly fixed by the context.

A strategy τ for player 2 in formulas (11) and (12) is valued in $\Delta(\mathcal{J})$ and is therefore in M_2^α, $\forall\alpha > 1$. The Burkholder-Davis-Gundy inequality (see e.g. theorem 4.1 in Chapter IV in [15]) indicates that the Itô-integrals appearing in these formulas belong to $\cap_{\infty>\alpha} L^\alpha$ and the corresponding expectations are thus finite for all Y in $\cup_{\alpha>1} L^\alpha$ as it results from Hölder's inequality.

We are then led to the following definitions of the Brownian games where the origin T of the time interval $[T,1]$ is taken as a parameter to analyze later their recursive structure:

DEFINITION 2.3.2. – *Let T be in $[0,1]$, let $\{\mathcal{F}_t\}_{t\in[T,1]}$ be a filtration on a probability space $(Z,\mathcal{F},P)$, and let W be an I-dimensional $\mathcal{F}_t$-Brownian motion.*

For a process τ in $M_2^\alpha(\{\mathcal{F}_t\}_{t\in[T,1]}, I\!R^J)$, we adopt the following notation:

$$X_{T,s}(\tau) := \int_T^s \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}\, a_{i\tau_t}\, dW_{i,t}. \tag{13}$$

For $p \in \Delta(\mathcal{K})$, the primal Brownian game $\Gamma(p,T)$ is defined by the following elements:

The strategy space for player 1 is the set $\mathcal{Y}_p$ of $I\!R_+^K$-valued random vectors Y in $\cup_{\alpha>1} L^\alpha(\mathcal{F}_1)$ with $E[Y] = p$.

The strategy space $\mathcal{T}$ for player 2 is the set of $\Delta(\mathcal{J})$-valued $\mathcal{F}_t$-progressively measurable processes τ.

The maximizer in the primal game $\Gamma(p,T)$ is player 1 and his payoff is given by

$$g_T(Y,\tau) = E\big[\langle Y | X_{T,1}(\tau)\rangle\big]. \tag{14}$$

DEFINITION 2.3.3. – *For $x \in I\!R^K$ and $\tau \in M_2^\alpha(\{\mathcal{F}_t\}_{t\in[T,1]}, I\!R^J)$, we set*

$$X^*_{x,T,s}(\tau) := x - X_{T,s}(\tau). \tag{15}$$

The dual Brownian game $\Gamma^*(x,T)$ is then defined as follows:

Player 1's strategy space is $\mathcal{Y} := \cup_{p\in\Delta(\mathcal{K})}\mathcal{Y}_p$ (i.e. the set of $\mathbb{R}_+^K$-valued random vectors Y in $\cup_{\alpha>1}L^\alpha(\mathcal{F}_1)$ with $E[Y] \in \Delta(\mathcal{K})$).

Player 2's strategy space is $\mathcal{T}$ as in the primal game.

Player 2 is here the maximizer and his payoff $g^*_{x,T}(Y,\tau)$ is given by

$$g^*_{x,T}(Y,\tau) := E\left[\langle Y|X^*_{x,T,1}(\tau)\rangle\right]. \tag{16}$$

To simplify the presentation of the main results of this paper, let us remind here some fundamental definitions of game theory: We say that a strategy Y of player 1 guarantees a payoff a in $\Gamma(p,T)$ if $\forall\tau \in \mathcal{T}$: $g_T(Y,\tau) \geq a$. The least upper bound of the payoffs player 1 can guarantee is clearly $\sup_{Y\in\mathcal{Y}_p}\inf_{\tau\in\mathcal{T}} g_T(Y,\tau)$. This quantity is referred to as $\sup\inf g_T$. For $\epsilon \geq 0$, an ϵ-optimal strategy of player 1 is a strategy that guarantees $\sup\inf g_T - \epsilon$. It follows from the definition of $\sup\inf g_T$ that there always exist ϵ-optimal strategies for $\epsilon > 0$. The 0-optimal strategies, if any, are called optimal strategies. When there exists such an optimal strategy, the $\sup\inf$ of the game is in fact a $\max\inf$.

Similar definitions hold for player 2 who wants to minimize the payoff. The lowest payoff he ever could guarantee is $\inf\sup g_T := \inf_{\tau\in\mathcal{T}}\sup_{Y\in\mathcal{Y}_p} g_T(Y,\tau)$. The $\inf\sup$ of a game is always greater than the $\sup\inf$. In case both quantities are equal, the game is said to have a value v equal to $v := \inf\sup g_T = \sup\inf g_T$.

These definitions also translate in the framework of the dual Brownian games, but the maximizer is here player 2.

The main results of this paper are presented in the next five theorems. The first one deals with the existence of a value for the Brownian games as well as of optimal strategies for both players.

THEOREM 2.3.4. – *For all $p \in \Delta(\mathcal{K})$ and $x \in \mathbb{R}^K$, $\forall T \in [0,1]$, the games $\Gamma(p,T)$ and $\Gamma^*(x,T)$ have a value that will be respectively denoted $\psi(p,T)$ and $\psi^*(x,T)$. Moreover both players have optimal strategies. In other words:*

$$\psi(p,T) := \max_{Y\in\mathcal{Y}_p}\inf_{\tau\in\mathcal{T}} g_T(Y,\tau) = \min_{\tau\in\mathcal{T}}\sup_{Y\in\mathcal{Y}_p} g_T(Y,\tau) \tag{17}$$

$$\psi^*(x,T) := \min_{Y\in\mathcal{Y}}\sup_{\tau\in\mathcal{T}} g^*_{x,T}(Y,\tau) = \max_{\tau\in\mathcal{T}}\inf_{Y\in\mathcal{Y}} g^*_{x,T}(Y,\tau) \tag{18}$$

These values are independent of the filtration $\{\mathcal{F}_t\}$ and on the $\mathcal{F}_t$-Brownian motion W on which the games are defined.

The two first equalities in the next theorem indicate that primal and dual games are indeed dual models and their values are Fenchel conjugates of each other.

Relations (21) and (22) emphasize the fact that the Brownian game starting at time $T \in [0,1]$ is in fact a rescaling of the Brownian game starting at 0:

THEOREM 2.3.5. – *The function $\psi(\bullet, T)$ (respectively $\psi^*(\bullet, T)$) is concave and continuous on $\Delta(\mathcal{K})$ (resp. on $I\!R^K$). Both functions are linked by the duality relationships:*

$$\forall x \in I\!R^K : \psi^*(x,T) = \min_{p\in\Delta(\mathcal{K})} \langle x|p\rangle - \psi(p,T) \tag{19}$$

$$\forall p \in \Delta(\mathcal{K}) : \psi(p,T) = \inf_{x\in I\!R^K} \langle x|p\rangle - \psi^*(x,T) \tag{20}$$

They fulfill further the following relations, $\forall T \in [0,1), \forall x \in I\!R^K, \forall p \in \Delta(\mathcal{K})$:

$$\psi^*(x,T) = \sqrt{1-T}\psi^*(\frac{x}{\sqrt{1-T}}, 0) \tag{21}$$

$$\psi(p,T) = \sqrt{1-T}\psi(p,0). \tag{22}$$

The next result claims that player 1 has optimal strategies Y in the Brownian games that are uniformly bounded in L^α and completely mixed i.e. $P[Y = 0] = 0$.

THEOREM 2.3.6. – *There exist positive numbers $\alpha > 1$, $\eta > 0$, C, κ depending only on the payoffs such that, for all p in $\Delta(\mathcal{K})$ (respectively $\forall x \in I\!R^K$), player 1 has an optimal strategy Y in $\Gamma(p,T)$ (resp. in $\Gamma^*(x,T)$) fulfilling:*

$$\|Y\|_{L^\alpha} \leq C \tag{23}$$

$$E[\langle u|Y\rangle^{-\eta}] < \kappa, \tag{24}$$

where $u := (1,\ldots,1) \in I\!R^K$. In particular: $P[Y = 0] = 0$.

The next two theorems are devoted to the recursive structure of the dual Brownian games and its consequences: More specifically, a strategy τ of player 2 in $\Gamma^*(x,0)$ can be viewed as a pair $(\tau^<, \tau^>)$ where $\tau^<$ and $\tau^>$

denote the restriction of τ to the respective times intervals $[0,T)$ and $[T,1]$. If X denotes $X^*_{x,0,T}(\tau^<)$, then $X^*_{x,0,1}(\tau) = X^*_{X,T,1}(\tau^>)$ and the payoff $g^*_{x,0}(Y,\tau)$ in $\Gamma^*(x,0)$ can be written as

$$g^*_{x,0}(Y,\tau) = E[\langle Y|X^*_{x,0,1}(\tau)\rangle] = E[y_T E[\langle \frac{Y}{y_T}|X^*_{X,T,1}(\tau^>)\rangle|\mathcal{F}_T]],$$

where y_T is defined as $E[\langle u|Y\rangle|\mathcal{F}_T]$. The conditional expectation in the above equation is in fact the payoff in $\Gamma^*(X^*_{x,0,T}(\tau^<),T)$ resulting from the strategies Y/y_T and $\tau^>$. This suggests that the players will manage to chose for $\tau^>$ and Y/y_T an optimal strategy in $\Gamma^*(X^*_{x,0,T}(\tau^<),T)$.

Two conclusions can be derived from this kind of argument: on one hand, assuming that all the optimal strategies τ^* in $\Gamma^*(x,T)$ start in the same way, i.e. $\tau^*_T = \mathcal{R}(x,T)$ for an appropriate function $\mathcal{R}$, then, if τ is optimal in $\Gamma^*(x,0)$, $\tau^>$ is optimal in $\Gamma^*(X^*_{x,0,T}(\tau^<),T)$ and thus $\tau_T = \tau^>_T$ should be equal to $\mathcal{R}(X^*_{x,0,T}(\tau^<),T)$. This being true for all times T, the process $X^*_{x,0,t}(\tau)$ should satisfy a diffusion equation. This is the content of theorem 2.3.8, where the function $\mathcal{R}$ is specified.

On the other hand, if, after time T, the players play an optimal strategy in $\Gamma^*(X^*_{x,0,T}(\tau^<),T)$, they guarantee a payoff $E[y_T\psi^*(X^*_{x,0,T}(\tau),T)]$ in $\Gamma^*(x,0)$ and we may then conjecture formula (25). Moreover, since optimal strategies in $\Gamma^*(x,0)$ should also be optimal in (25), we conclude with theorem 2.3.6 that player 1 has a completely mixed optimal strategy in the second line of (25), from which (26) can be derived.

THEOREM 2.3.7.

$$(25)\qquad \begin{aligned}\psi^*(x,0) &= \sup_{\tau\in\mathcal{T}}\inf_{Y\in\mathcal{Y}} E[\langle u|Y\rangle\psi^*(X^*_{x,0,T}(\tau),T)]\\ &= \inf_{Y\in\mathcal{Y}}\sup_{\tau\in\mathcal{T}} E[\langle u|Y\rangle\psi^*(X^*_{x,0,T}(\tau),T)]\end{aligned}$$

Furthermore, if τ^* is optimal in $\Gamma^*(x,0)$, then for all $t\in[0,1]$,

$$(26)\qquad \psi^*(X^*_{x,0,t}(\tau^*),t) \overset{a.s.}{=} \psi^*(x,0)$$

Let $\partial\psi^*(x,T)$ denote the super gradient at x of the concave function $\psi^*(\bullet,T)$. The duality relation (19) implies in particular: $\emptyset \neq \partial\psi^*(x,T) \subset \Delta(\mathcal{K})$. We can then introduce the correspondence $\mathcal{R}$ that maps $(x,T)\in \mathbb{R}^K\times[0,1)$ to $\mathcal{R}(x,T) := \{\rho(p)|p\in\partial\psi^*(x,T)\}$, where ρ was introduced in lemma 1.2.3.

THEOREM 2.3.8. – *The correspondence $\mathcal{R}$ is single valued and can therefore viewed as a function. This function maps continuously $\mathbb{R}^K\times[0,1)$ on $\Delta(\mathcal{J})$.*

If τ is optimal for Player 2 in $\Gamma^*(x,0)$, then the process τ and the process r, defined as $r_t := \mathcal{R}(X^*_{x,0,t}(\tau),t)$ are modifications of each others. As a consequence, the process X defined as $X^*_t := X^*_{x,0,t}(\tau)$ is a solution of the stochastic differential equation:

$$dX_t = -\sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}\, a_{i\mathcal{R}(X_t,t)}\, dW_{i,t}.$$

3. EXISTENCE RESULTS FOR UNBOUNDED BROWNIAN GAMES

3.1. The behavioral form of the Brownian games. In this section, we prove that Brownian games are, as stated in theorem 2.3.4, independent of the filtration $\{\mathcal{F}_t\}$ and the Brownian motion on which they are defined. This result will allow us to consider the games in their behavioral form.

Let W be a I-dimensional $\mathcal{F}_t$-Brownian motion on $[T,1]$, let β_j denotes the process $\sum_{\mathcal{I}} \sqrt{\sigma_{0,i}} a_{ij} W_i$ and let $\mathcal{G}_t$ be $\sigma\{\beta_{j,s} | j \in \mathcal{J}; T \le s \le t\}$. We will then refer to the $\mathcal{G}_t$-adapted strategies of player 2 and the $\mathcal{G}_1$-measurable strategies of player 1 as the $\mathcal{G}$-strategies. The set of these strategies will be respectively denoted $\mathcal{GT}$, $\mathcal{GY}_p$ and $\mathcal{GY}$. We have then the theorem:

THEOREM 3.1.1.. – *In the Brownian games $\Gamma(p,T)$ and $\Gamma^*(x,T)$ defined on $\mathcal{F}_t$ and W, there is no loss for the players to restrict themselves to their $\mathcal{G}$-strategies. In other words, both players can guarantee the same payoff in the Brownian games and in the corresponding $\mathcal{G}$-games where they are restricted to their $\mathcal{G}$-strategies. Furthermore, the ϵ-optimal strategies in the $\mathcal{G}$-games are still ϵ-optimal in the corresponding Brownian games.*

Proof. – We will prove the result for $\Gamma(p,T)$, the same argument holds for $\Gamma^*(x,T)$.

Let $\mathcal{X}$ and $\mathcal{GX}$ denote respectively the image of $\mathcal{T}$ and $\mathcal{GT}$ by the mapping $X_{T,1}$ introduced in (13).

Obviously, the operator $E[\bullet|\mathcal{G}_1]$ maps $\mathcal{Y}_p$ onto $\mathcal{GY}_p$.

Similarly, if $\tau \in \mathcal{T}$, $E[X_{T,1}(\tau)|\mathcal{G}_1]$ is the value at time 1 of the $\mathcal{G}_t$-optional projection of the process $X_{T,t}(\tau)$ and can, according to proposition 7 in [7], be written as $X_{T,1}(\tau')$ where τ' is the $\mathcal{G}_t$-predictable projection of τ. τ'_t is thus one version of $E[\tau_t|\mathcal{G}_t]$ and, as such, belongs a.s. to $\Delta(\mathcal{J})$. The process τ' is thus a $\mathcal{G}$-strategy of player 2, and $E[\bullet|\mathcal{G}_1]$ maps therefore $\mathcal{X}$ onto $\mathcal{GX}$.

The result follows then from these onto properties of the mapping $E[\bullet|\mathcal{G}_1]$. Indeed, the payoff guaranteed by a strategy Y of player 1 is then also guaranteed by its $\mathcal{G}$-strategy $E[Y|\mathcal{G}_1]$, since

$$\inf_{X\in\mathcal{X}} E[\langle Y|X\rangle] \leq \inf_{X\in\mathcal{X}} E\big[\big\langle Y\big|E[X|\mathcal{G}_1]\big\rangle\big] = \inf_{X\in\mathcal{X}} E\big[\big\langle E[Y|\mathcal{G}_1]\big|X\big\rangle\big].$$

This indicates that

$$\underline{\psi} := \sup_{Y\in\mathcal{Y}_p}\inf_{X\in\mathcal{X}} E[\langle Y|X\rangle] = \sup_{Y\in\mathcal{G}\mathcal{Y}_p}\inf_{X\in\mathcal{G}\mathcal{X}} E[\langle Y|X\rangle].$$

Moreover, if $Y\in\mathcal{G}\mathcal{Y}_p$ guarantees $\underline{\psi}-\epsilon$ against any $X\in\mathcal{G}\mathcal{X}$, Y still does so against all $X\in\mathcal{X}$, since $E[\langle Y|X\rangle] = E\big[\big\langle Y\big|E[X|\mathcal{G}_1]\big\rangle\big]$.

Similarly, let τ be a strategy of player 2 and let τ' be the $\mathcal{G}_t$-predictable projection of τ, then

$$\begin{aligned}\sup_{Y\in\mathcal{Y}_p} E[\langle Y|X_{T,1}(\tau)\rangle] &\geq \sup_{Y\in\mathcal{Y}_p} E\big[\big\langle E[Y|\mathcal{G}_1]\big|X_{T,1}(\tau)\big\rangle\big]\\ &= \sup_{Y\in\mathcal{Y}_p} E[\langle Y|X_{T,1}(\tau')\rangle].\end{aligned}$$

Therefore,

$$\bar{\psi} := \inf_{X\in\mathcal{X}}\sup_{Y\in\mathcal{Y}_p} E[\langle Y|X\rangle] = \inf_{X\in\mathcal{G}\mathcal{X}}\sup_{Y\in\mathcal{G}\mathcal{Y}_p} E[\langle Y|X\rangle],$$

and if a $\mathcal{G}$-strategy τ of player 2 guarantees him a payoff less than $\bar{\psi}+\epsilon$ against the $\mathcal{G}$-strategies of player 1, it does so against all strategies Y of player 1, since $E[\langle Y|X_{T,1}(\tau)\rangle] = E\big[\big\langle E[Y|\mathcal{G}_1]\big|X_{T,1}(\tau)\big\rangle\big]$. □

It follows from the last theorem that the players may restrict themselves to their $\mathcal{S}_t$-measurable strategies, where $\mathcal{S}_t := \sigma\{W_s : T\leq s\leq t\}$. Since the $\mathcal{S}_t$-Brownian motion W is isomorphic to the Wiener Brownian motion defined on the Wiener probability space, we conclude that:

COROLLARY 3.1.2. – *The payoff a player can guarantee in a Brownian game does not depend on the probability space $(Z,\mathcal{F},P)$, nor on the filtration $\{\mathcal{F}_t\}$, neither on the $\mathcal{F}_t$-Brownian motion on which the game is defined.*

The $\mathcal{G}$-strategies Y of player 1 in $\Gamma(p,T)$ are random vectors in $\cup_{\alpha>1}L^\alpha(\mathcal{G}_1)$ with $E[Y]=p$. Since the Brownian motion has the predictable representation property on the filtration $\{\mathcal{G}_t\}$ (see e.g. theorem 3.5 in chapter V of [15]), the random variable Y may be written as the Itô-integral $Y = p + \int_T^1\sum_{j\in\mathcal{J}} f_{j,t}d\beta_{j,t}$, where f_j is a $(K\times K)$-dimensional process in $\cup_{\alpha>1}M_2^\alpha$. If we define the K-dimensional process b_i as $\sum_{j\in\mathcal{J}} f_j a_{ij}$, it follows from the definition of β_j that $Y = p+\int_T^1\sum_{i\in\mathcal{I}}\sqrt{\sigma_{0,i}}b_{i,t}dW_{i,t}$. It results from lemma 2.1.3 that $\sum_{i\in\mathcal{I}}\sigma_{0,i}b_i = 0$. It is then convenient to introduce the following notations:

DEFINITION 3.1.3. – *From now on, $\mathcal{A}$ will denote the linear space of $b = (b_1, \cdots, b_I)$ in $\mathbb{R}^{(K\times I)}$, such that $\sum_{i\in\mathcal{I}} \sigma_{0,i} b_i = 0$. For $b, b' \in \mathcal{A}$ we set $\langle\langle b|b'\rangle\rangle := \sum_{i\in\mathcal{I}} \sigma_{0,i}\langle b_i|b'_i\rangle$ and for $\tau \in \mathbb{R}^J$, we define $a(\tau)$ as $a(\tau) := (a_{1\tau}, \dots, a_{i\tau})$. Since $\forall i$, $\sigma_{0,i} > 0$, $\langle\langle \bullet|\bullet\rangle\rangle$ is a scalar product on $\mathcal{A}$ and, due to equation (2), $a(\bullet)$ is a linear mapping from $\mathbb{R}^J$ to $\mathcal{A}$.*

With these notations, the payoff $g_T(Y, \tau)$ becomes

$$g_T(Y, \tau) = E[\langle Y|X_{T,1}(\tau)\rangle] = E[\int_T^1 \langle\langle b_t|a(\tau_t)\rangle\rangle dt].$$

We conclude then that the game $\Gamma(p, T)$ is equivalent to its behavioral form:

DEFINITION 3.1.4. – *For $q \in \mathbb{R}^K$ and a $\mathcal{A}$-valued $\mathcal{F}_t$-measurable process b, we set*

$$Y_{T,s}(q, b) := q + \int_T^s \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}\, b_{i,t}\, dW_{i,t} \tag{27}$$

A behavioral strategy for player 1 in $\Gamma(p, T)$ is a $\mathcal{A}$-valued $\mathcal{F}_t$-measurable process b in $\cup_{\alpha>1} M_2^\alpha$ such that $Y_{T,1}(p, b)$ belongs a.s. to $\mathbb{R}_+^K$.

Player 1's payoff $g_T(b, \tau)$ is given by

$$g_T(b, \tau) := E[\langle Y_{T,1}(p, b)|X_{T,1}(\tau)\rangle] = E[\int_T^1 \langle\langle b_t|a(\tau_t)\rangle\rangle dt]. \tag{28}$$

Similarly, a behavioral strategy for player 1 in $\Gamma^*(x, T)$ is a pair (p, b), where $p \in \Delta(\mathcal{K})$ and b is a behavioral strategy in $\Gamma(p, T)$. The behavioral form of player 2's payoff $g^*_{x,T}$ in $\Gamma^*(x, T)$ is: $g^*_{x,T}((p, b), \tau) = \langle p|x\rangle - g_T(b, \tau)$.

3.2. The unbounded Brownian Games. Player 2's strategy space in the Brownian games is closed, bounded, and convex in M_2^α. It is therefore compact in the weak topology of the reflexive space M_2^α. The payoff functions $g_T(Y, \tau)$ and $g^*_{x,T}(Y, \tau)$ are affine and continuous in both strategies. We could then apply Sion's theorem (see [18]) to infer that the Brownian games have a value and player 2 an optimal strategy.

The main difficulty of this approach is nevertheless the proof of the existence of player 1's optimal strategies. This difficulty comes from the asymmetry between both players: player 2's strategy space is bounded, while player 1's is not.

To bypass this difficulty it is technically convenient to first analyze auxiliary games, referred to as the unbounded Brownian games, where both strategy spaces are unbounded, and then to prove the equivalence between bounded and unbounded games.

A strategy for player 2 in the bounded Brownian games takes values in $\Delta(\mathcal{J})$. In the unbounded games, a strategy for player 2 will be a process τ in $M_2^{\beta'}$, for a $\beta' \in (1, \infty)$ to be fixed later, such that $\sum_{j \in \mathcal{J}} \tau_j = 1$.

We have then to reduce player 1 strategy space in $\Gamma(p, T)$ for the payoff $g_T(b, \tau)$ to be defined. To this end, it is convenient to introduce the following definitions:

DEFINITION 3.2.1. – *In the following, $\mathcal{H}$ and $\mathcal{D}$ will denote the set of vectors $h \in I\!R^J$ such that $\sum_{j \in \mathcal{J}} h_j$ is respectively equal to 1 and 0.*

We also set $\mathcal{E}^\perp := a(\mathcal{D})$ and the orthogonal space to $\mathcal{E}^\perp$ in $\mathcal{A}$ with respect to $\langle\langle \bullet | \bullet \rangle\rangle$ will be denoted $\mathcal{E}$. Since $a(\mathcal{H})$ is just a translate of $\mathcal{E}^\perp$, its intersection with $\mathcal{E}$ reduces to a single point e^0. In the following τ^0 will denote a point τ^0 in $\mathcal{H}$ such that $a(\tau^0) = e^0$.

Vectors in $\mathcal{E}$ and $\mathcal{E}$-valued processes are referred to as equalizing. Similarly, vectors in $\mathcal{E}^\perp$ and $\mathcal{E}^\perp$-valued processes are referred to as ortho-equalizing.

The set of equalizing (resp. ortho-equalizing) processes in M_2^α will be denoted $\mathcal{E}^\alpha$ (resp. $\mathcal{E}^{\perp,\alpha}$).

We can then see the strategy τ of player 2 as the sum $\tau^0 + \delta$ of the constant process τ^0 and a $\mathcal{D}$-valued process δ in $M_2^{\beta'}$.

Similarly the strategy b of player 1 can be written in a unique way as the sum $c + z$ of an equalizing process c and an ortho-equalizing process z. We have then

$$g_T(b, \tau) = E[\int_T^1 \langle\langle c_t | a(\tau^0) \rangle\rangle dt] + E[\int_T^1 \langle\langle z_t | a(\tau_t) \rangle\rangle dt], \tag{29}$$

since the expectation $E[\int_T^1 \langle\langle c_t | a(\delta_t) \rangle\rangle dt]$ collapses according to the definition of $\mathcal{E}$.

The constant process τ^0 is in M_2^∞, the first expectation in (29) is defined for all equalizing processes c in M_2^α, for all $\alpha \in (1, \infty)$. The second expectation in turn will be finite for all ortho-equalizing process in $M_2^{\alpha'}$, where $1/\alpha' + 1/\beta' = 1$, since $\tau \in M_2^{\beta'}$.

This leads us to the following definition of unbounded Brownian games in behavioral form:

DEFINITION 3.2.2. – *For α, α' in $(1,\infty)$ the unbounded Brownian game $\hat{\Gamma}_{\alpha,\alpha'}(p,T)$ is defined as follows:*

Player 1's strategy space $\mathcal{Y}_p^{\alpha,\alpha'}$ is the set of processes $b = c+z$ with $c \in \mathcal{E}^\alpha$ and $z \in \mathcal{E}^{\perp,\alpha'}$, such that $Y_{T,1}(p,b)$ is $\mathbb{R}_+^K$-valued.

Player 2's strategy space $\mathcal{T}^{\alpha'}$ is the set of $\mathcal{H}$-valued processes τ in $M_2^{\beta'}$, with β' such that $1/\alpha' + 1/\beta' = 1$. (It does not depend on α)

The payoff function of player 1 is given by (29).

Similarly, the unbounded Brownian game $\hat{\Gamma}^*_{\alpha,\alpha'}(x,T)$ is characterized by the same strategy space $\mathcal{T}^{\alpha'}$ for player 2, the strategy space $\mathcal{Y}^{\alpha,\alpha'}$ for player 1 is the set of pairs (p,b) with $p \in \Delta(\mathcal{K})$ and $b \in \mathcal{Y}_p^{\alpha,\alpha'}$. The payoff function g_T^* is:

$$g_T^*((p,b),\tau) = \langle p|x\rangle - g_T(b,\tau) \tag{30}$$

The main difficulty of this paper will be to find numbers α, α' insuring the existence of a value for the corresponding unbounded Brownian games, as well as the existence of optimal strategies.

3.3. The bounds on player 1's strategies. Since the strategy space of player 2 is an affine space and the payoff functions g_T and g_T^* are respectively linear and affine in τ, we infer that the only strategies b (resp. (p,b)) of player 1 guaranteeing a finite payoff in $\hat{\Gamma}_{\alpha,\alpha'}(p,T)$ (resp. $\hat{\Gamma}^*_{\alpha,\alpha'}(x,T)$) are such that $g_T(b,\tau) = g_T(b,\tau')$ for all strategies τ, τ' of player 2. In turn, this relation indicates that for all $\mathcal{D}$-valued processes δ, $E[\int_T^1 \langle\langle b_t|a(\delta_t)\rangle\rangle dt] = 0$ and b must therefore be $\mathcal{E}$-valued.

The equalizing strategies and the strategies that are nearly equalizing will therefore play a central role in the following argument. We define therefore:

DEFINITION 3.3.1. – *For $R \geq 0$, $V^{\alpha,\alpha'}(R)$ is the set of strategies (p,b) in $\mathcal{Y}^{\alpha,\alpha'}$, with $b = c+z$, $c \in \mathcal{E}^\alpha$, $z \in \mathcal{E}^{\perp,\alpha'}$ such that $\|z\|_{M_2^{\alpha'}} \leq R$.*

The set $V^{\alpha,\alpha'}(0)$ is referred to as the set of equalizing strategies. It does not depend on α' and will be denoted V^α in the following.

For $p \in \Delta(\mathcal{K})$, we also set V_p^α (resp. $V_p^{\alpha,\alpha'}(R)$) for the set of b such that $(p,b) \in V^\alpha$ (resp. $V^{\alpha,\alpha'}(R)$.)

The aim of this section is to prove that $V^{\alpha,\alpha'}(R)$ is bounded in $\Delta(\mathcal{K}) \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'}$ for appropriate choices of α,α'. This will allow us in the next section to apply the separation theorem for convex sets to prove that the unbounded Brownian games have a value. A first result in this direction is the following theorem:

THEOREM 3.3.2. – *There exist positive numbers* $\alpha > 1$, $\eta > 0$, C, C' *depending only on the payoffs* a_{ij}^k *such that* $\forall (p, b) \in \cup_{\alpha>1} V^\alpha$:

$$\|Y_{T,1}(p,b)\|_{L^\alpha} \le C \tag{31}$$

$$E[\langle u|Y_{T,1}(p,b)\rangle^{-\eta}] < C', \tag{32}$$

where $u := (1,\ldots,1) \in I\!R^K$.

The set of $I\!R^K_+$-valued random vectors Y with $E[Y] = p$ is unbounded in $L^\alpha(\mathcal{F}_1)$, $\forall \alpha > 1$ and, to prove this theorem, we have to find a specific property of the equalizing processes to get a bound on $\|Y_{T,1}(p,b)\|_{L^\alpha}$.

If there were in $\mathcal{E}$ a "one dimensional" $c \neq 0$, i.e. a c such that $\forall i : c_i = \lambda_i p$, where $\lambda_i \in I\!R$ and $p \in \Delta(\mathcal{K})$, there would be no hope for such a bound: the process $b_t^n := nc1\!\!1_{\{\theta_n \ge t\}}$ would be an equalizing strategy, where θ_n is the first time the process $Y^n := p + n\sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} c_i W_i$ exits $I\!R^K_+$. The process Y^n being a one dimensional Brownian motion, the set of $Y_{T,1}(p,b^n) = Y^n_{\theta_n}$, $n \in I\!N$, would then be unbounded in any L^α, since the one dimensional Brownian motion killed on one side in not an equi-integrable martingale.

The next lemma indicates that there are no such "one dimensional" c's in $\mathcal{E}$:

LEMMA 3.3.3. – *The following implication holds* $\forall c \in \mathcal{E}$:

$$(\exists p \in \Delta(\mathcal{K}); \exists \lambda_1, \ldots, \lambda_I \in I\!R; \forall i : c_i = \lambda_i p) \Rightarrow c = 0.$$

Proof. – Let indeed $c = (\lambda_1 p, \ldots, \lambda_I p)$ be an equalizing vector. It results then from the definition of $\mathcal{E}$ that for all $\delta \in \mathcal{D}$:

$$\sum_{i\in\mathcal{I}} \sigma_{0,i}\lambda_i \langle p|a_{i\delta}\rangle = 0. \tag{33}$$

Since $\mathcal{E} \subset \mathcal{A}$, we have $0 = \sum_{i\in\mathcal{I}} \sigma_{0,i}\lambda_i$ (see definition 3.1.3), so, for $\epsilon > 0$ small enough, both σ^+ and σ^- are in $\Delta(\mathcal{J})$, where $\sigma_i^+ := \sigma_{0,i}(1 + \epsilon\lambda_i)$ and $\sigma_i^- := \sigma_{0,i}(1 - \epsilon\lambda_i)$. Indeed, σ_0 is completely mixed: $\forall i : \sigma_{0,i} > 0$.

Relation (33) indicates that both strategies σ^+ and σ^- are equalizing in the average game $G(p)$ with payoffs $\langle p|a_{ij}\rangle$. Since $\sum_{i\in\mathcal{I}} \sigma_{0,i}\langle p|a_{ij}\rangle = 0$ (see (2)), one of these two strategies guarantees then a positive payoff and is then optimal in $G(p)$, since $u(p) = 0$. According to definition 2.1.2 of $\Delta^0_{\sigma_0}$, σ_0 is the unique optimal strategy in $G(p)$ and we conclude that

$\sigma_0 = \sigma^+$ or $\sigma_0 = \sigma^-$. It follows then from the definition of σ^+ and σ^- that $\forall i : \lambda_i = 0$ and hence $c = 0$, as announced. $\square$

As a consequence of this lemma, we have:

LEMMA 3.3.4. – *There exists $\kappa > 0$ depending on the payoffs a^k_{ij} such that $\forall p \in \Delta(\mathcal{K})$, $\forall c \in \mathcal{E}$: $\|B - pu^\top B\|^2 \geq \kappa \|u^\top B\|^2$, where $B := [\sqrt{\sigma_{0,1}}c_1, \dots, \sqrt{\sigma_{0,I}}c_I]$.*

Proof. – We will prove an even stronger result: $\exists \kappa'$ such that $\|B - pu^\top B\|^2 \geq \kappa' \|B\|^2$. Since both sides in this relation are, for fixed p, 2-homogeneous in B, we just have to show that $\kappa' := \inf\{\|B - pu^\top B\|^2 : \|B\| = 1; p \in \Delta(\mathcal{K})\}$ is strictly positive.

If, on the contrary, we had $\kappa' = 0$, we would infer by compactness the existence of $c \neq 0$ in $\mathcal{E}$ and $p \in \Delta(\mathcal{K})$ such that $B - pu^\top B = 0$. In turn, this would imply $\forall i : c_i = \langle u | c_i \rangle p$ and a contradiction would follow from lemma 3.3.3, since it would imply $c = 0$. $\square$

The keystone in the proof of theorem 3.3.2 is the following result that is intimately related with the theory of BMO-martingales (see e.g. [11]):

LEMMA 3.3.5. – *Let W be an I-dimensional Brownian motion on a filtration $\mathcal{F}_t$, and let $A \geq 1$ and $\lambda > 0$ be two real numbers.*

If an I-dimensional process v in $\cup_{\alpha > 1} M_2^\alpha$ satisfies, for all $[T, 1]$-valued stopping time θ:

$$E[\exp(\int_\theta^1 \langle v_t | dW_t \rangle + \frac{\lambda - 1}{2} \int_\theta^1 \|v_t\|^2 dt) | \mathcal{F}_\theta] \overset{a.s.}{\leq} A, \tag{34}$$

then the exponential process $y_t := \exp(\int_T^t \langle v_s | dW_s \rangle - \frac{1}{2} \int_T^t \|v_s\|^2 ds)$ fulfills for all stopping times θ:

$$E[(\frac{y_1}{y_\theta})^\alpha | \mathcal{F}_\theta] \overset{a.s.}{\leq} C, \tag{35}$$

where $\alpha := 1 + \frac{\lambda}{1 + 2\sqrt{\lambda}}$ and $C := A^{\frac{1+\sqrt{\lambda}}{1+2\sqrt{\lambda}}}$.

Moreover, there exist $\eta > 0$ and $C' < \infty$ depending only on A and λ such that, if the local martingale y is a martingale, then

$$E[y_1^{-\eta}] \leq C'. \tag{36}$$

Proof. – If we define ξ as $1 + \sqrt{\lambda}$, p as $\frac{1+\sqrt{\lambda}}{1+2\sqrt{\lambda}}$ and p' as $1 - p$, a straightforward computation shows that $(\frac{y_1}{y_\theta})^\alpha = U^p V_1^{p'}$, where $U :=$

$\exp(\int_\theta^1 \langle v_t|dW_t\rangle + \frac{\lambda-1}{2}\int_\theta^1 \|v_t\|^2 dt)$ and $V_s := \exp(\xi \int_T^s 1\!\!1_{\{t\geq\theta\}}\langle v_t|dW_t\rangle - \frac{\xi^2}{2}\int_T^s 1\!\!1_{\{t\geq\theta\}}\|v_t\|^2 dt)$.

From Girsanov's theorem, we know that, a.s., $E[V_1|\mathcal{F}_\theta] \leq V_\theta = 1$, since V is a positive local martingale and is therefore also a supermartingale. Equation (35) follows then readily from Hölder's inequality: $E[(\frac{y_1}{y_\theta})^\alpha|\mathcal{F}_\theta] \leq E[V_1|\mathcal{F}_\theta]^{p'}E[U|\mathcal{F}_\theta]^p \leq A^p$.

Let us next suppose that y is a martingale and let us prove relation (36). Let μ be in $(0,1)$ and let θ_n be the stopping time $\inf\{t : y_t \leq \mu^n\}$, with the convention $\inf(\emptyset) = 1$. Let then π denote the probability $P(\theta_{n+1} < 1|\theta_n < 1)$ and let ϕ be $E[y_{\theta_{n+1}}|\theta_{n+1} = 1; \theta_n < 1]$. Since y is a martingale, we have:

$$\mu^n = E[y_{\theta_n}|\theta_n < 1] = E[y_{\theta_{n+1}}|\theta_n < 1] = \pi\mu^{n+1} + (1-\pi)\phi. \tag{37}$$

Next, relation (35) indicates that:

$$Cy_{\theta_n}^\alpha \geq E[y_1^\alpha|\mathcal{F}_{\theta_n}] \geq E[y_{\theta_{n+1}}^\alpha|\mathcal{F}_{\theta_n}],$$

where the last inequality is a consequence of Jensens' lemma ($\alpha > 1$). The expectation of this relation conditionally to $\{\theta_n < 1\}$ gives:

$$C\mu^{\alpha n} \geq \pi\mu^{\alpha(n+1)} + (1-\pi)E[y_{\theta_{n+1}}^\alpha|\theta_{n+1}=1; \theta_n<1] \geq \pi\mu^{\alpha(n+1)} + (1-\pi)\phi^\alpha.$$

This relation joint with (37) leads then to $C \geq R(\pi)$, where

$$R(\pi) := \pi\mu^\alpha + \frac{(1-\mu\pi)^\alpha}{(1-\pi)^{\alpha-1}}.$$

It is easy to verify that R is increasing from 1 to ∞ on the interval $[0,1)$ and therefore $\pi \leq \rho < 1$, where ρ is the root of the equation $R(\rho) = C$.

Since the previous reasoning holds for all n, we get then

$$P(y_1 < \mu^n) \leq P(\theta_n < 1) = \prod_{j=0}^{n-1} P(\theta_{j+1} < 1|\theta_j < 1) \leq \rho^n$$

Hence, if $\eta > 0$, we have:

$$\begin{aligned} E[y_1^{-\eta}] &= \int_0^\infty P(y_1^{-\eta} > t)dt \\ &\leq 1 + \sum_{n=0}^\infty P(y_1^{-\eta} > \mu^{-\eta n})(\mu^{-\eta(n+1)} - \mu^{-\eta n}) \\ &\leq 1 + \sum_{n=0}^\infty (\rho\mu^{-\eta})^n(\mu^{-\eta} - 1). \end{aligned}$$

Since $\rho < 1$, a strictly positive η may be chosen such that $\rho\mu^{-\eta} < 1$, ensuring the convergence of the last sum and (36) is thus proved. □

We are now ready to prove theorem 3.3.2:

Proof of theorem 3.3.2. – Let (p, b) be in $\cup_{\alpha>1}V^\alpha$. The martingale Y defined as $Y_s := Y_{T,s}(p, b)$ is $I\!R_+^K$-valued since so is Y_1.

Let us first make the additional hypothesis that Y remains, for $\epsilon > 0$, in $C_\epsilon := \{Z \in I\!R_+^K | \epsilon < \langle u|Z\rangle < 1/\epsilon\}$. Let y denote the process $y := \langle u|Y\rangle$ and $\Pi_t := Y_t/y_t$. Let also B_t denotes the $K \times I$-matrix process whose i-th column is $B_{i,t} := \sigma_{0,i}c_{i,t}/y_t$. We have then $dY_t = y_t B_t dW_t$ and $dy_t = y_t u^\top B_t dW_t$. From the last relation, we infer that

$$y_s = \exp(\int_T^s u^\top B_t dW_t - \frac{1}{2}\int_T^s \|u^\top B_t\|^2 dt).$$

Furthermore, since y is uniformly bounded (C_ϵ is bounded), we may apply Girsanov's theorem to the exponential process y: y_1 is a probability density on the probability space $(Z, \mathcal{F}_1, P)$ and, if we endow the space $(Z, \mathcal{F}_1)$ with the probability $\tilde{P} := y_1 \cdot P$, the process $\tilde{W}$ defined by $\tilde{W}_T = 0$, $d\tilde{W}_t = dW_t - B_t^\top u dt$ is a Brownian Motion on $(Z, \{\mathcal{F}_t\}_{t\in[T,1]}, \tilde{P})$. Itô's formula indicates then that $\Pi_s = p + \int_T^s A_t d\tilde{W}_t$, where $A_t := (B_t - \Pi_t u^\top B_t)$. The process Π is therefore a continuous martingale on $(Z, \{\mathcal{F}_t\}_{t\in[T,1]}, \tilde{P})$ that remains in the unit ball $B[0, 1]$ of $I\!R^K$ since Π is valued in the simplex $\Delta(\mathcal{K})$. In particular, Π is an element of $BMO(\tilde{P})$ (see e.g. [11], chapter 2), with $\|\Pi\|_{BMO_2(\tilde{P})} \le 1$. John Nirenberg's inequality (see theorem 2.2 in [11]) implies that for all stopping time θ, for all $\lambda \in [0, 2)$,

$$E_{\tilde{P}}[\exp(\frac{\lambda}{2}\int_\theta^1 \|A_t\|^2 dt)|\mathcal{F}_\theta] \le 2/(2-\lambda).$$

With lemma 3.3.4, we have then

$$E_{\tilde{P}}[\exp(\frac{\lambda\kappa}{2}\int_\theta^1 \|u^\top B_t\|^2 dt)|\mathcal{F}_\theta] \overset{a.s.}{\le} 2/(2-\lambda).$$

Girsanov's theorem indicates that, for any $\mathcal{F}_1$-measurable random variable X, $E_{\tilde{P}}[X|\mathcal{F}_\theta] = E_P[Xy_1/y_\theta|\mathcal{F}_\theta]$. Replacing then the quotient y_1/y_θ by its exponential expression $\exp(\int_\theta^1 u^\top B_t dW_t - \frac{1}{2}\int_\theta^1 \|u^\top B_t\|^2 dt)$, we get finally:

$$E_P[\exp(\int_\theta^1 u^\top B_t dW_t + \frac{\lambda\kappa - 1}{2}\int_\theta^1 \|u^\top B_t\|^2 dt)|\mathcal{F}_\theta] \overset{a.s.}{\le} 2/(2-\lambda).$$

Since for z in $I\!R^K_+$: $||z|| \leq \langle u|z\rangle$, we conclude with (35) in lemma 3.3.5, that

$$E[||Y_{T,1}(p,b)||^{\alpha}] \leq E[y_1^{\alpha}] \leq (2/(2-\lambda))^{\frac{1+\sqrt{\kappa\lambda}}{1+2\sqrt{\kappa\lambda}}}, \tag{38}$$

where $\alpha = 1 + \frac{\kappa\lambda}{1+2\sqrt{\kappa\lambda}}$. This is exactly relation (31). Similarly, relation (32) follows from (36) since y is a martingale.

Observing that the constants α, η, C and C' we obtain in the above argument are independent of ϵ, we dispense now with the hypothesis that Y remains in C_ϵ: Let thus (p,b) be in $\cup_{\alpha>1}V^{\alpha}$ and we set again $Y_t := Y_{T,t}(p,b)$.

For $\epsilon > 0$, let θ_ϵ denote the first exit time of C_ϵ by the process Y:

$$\theta_\epsilon := \inf\{t \leq 1 | Y_t \notin C_\epsilon\},$$

with the convention $\inf(\emptyset) = 1$. Since the process $Y'_t := Y_{t\wedge\theta_\epsilon}$ is C_ϵ-valued, we infer that $E[y^{\alpha}_{\theta_\epsilon}] \leq C$ and $E[y^{-\eta}_{\theta_\epsilon}] \leq C'$. So, to get relations (31) and (32), it is sufficient to prove that a.s. $y_1 = \lim_{\epsilon\to 0} y_{\theta_\epsilon}$.

Since on $\{\theta_\epsilon < 1\}$, $y_{\theta_\epsilon} = \epsilon$ or $y_{\theta_\epsilon} = \epsilon^{-1}$, we conclude that

$$P(\theta_\epsilon < 1) \leq P(y_{\theta_\epsilon} = \epsilon) + P(y_{\theta_\epsilon} = \epsilon^{-1}) \leq \epsilon^{\eta}C' + \epsilon^{\alpha}C.$$

This implies $\lim_{\epsilon\to 0}\theta_\epsilon = 1$ a.s., since θ_ϵ is decreasing in ϵ. Since the sample paths of y are continuous, we infer as announced that a.s. $y_1 = \lim_{\epsilon\to 0} y_{\theta_\epsilon}$.

□

THEOREM 3.3.6. – *There exist $\alpha, \alpha' > 1$, such that, $\forall R < \infty$, the set $V^{\alpha,\alpha'}(R)$ is bounded in $(\Delta(\mathcal{K}) \times \mathcal{E}^{\alpha} \times \mathcal{E}^{\perp,\alpha'})$. In particular, $V_p^{\alpha,\alpha'}(R)$ is bounded in $(\mathcal{E}^{\alpha} \times \mathcal{E}^{\perp,\alpha'})$.*

Moreover, the claims of theorem 3.3.2 hold for this particular α.

Proof. – Let (p,b) be in $V^{\alpha,\alpha'}(R)$, with $b = c + z$, $c \in \mathcal{E}^{\alpha}$ and $z \in \mathcal{E}^{\perp,\alpha'}$. We also set $Z_t := Y_{T,t}(0,z)$, $G_t := Y_{T,t}(0,c)$, $Z^*_s := \sup\{Z_{k,t} : k \in \mathcal{K}; t \in [T,s]\}$ and, for $a \geq 0$, let θ_a be $\inf\{t \in [T,1] : Z^*_t \geq a\}$, with again $\inf(\emptyset) := 1$.

Then, since the process $Y_{T,t}(p,b) = p + G_t + Z_t$ is valued in $I\!R^K_+$, so is the process $Y_t := (p + au + G_{t\wedge\theta_a})/(1+Ka)$, where $u = (1,\ldots,1) \in I\!R^K$. Moreover $p' := (p+au)/(1+Ka) \in \Delta(\mathcal{K})$. The pair $(p', c/(1+Ka))$ belongs thus to V^{α} and theorem 3.3.2 indicates then that $||Y_1||_{L^{\alpha_0}} \leq C$ for an appropriate $\alpha_0 > 1$. As a consequence

$$||G_{\theta_a}||_{L^{\alpha_0}} \leq ||Y_1||_{L^{\alpha_0}}(1+Ka) + ||p+au|| \leq D(1+a),$$

for a constant D depending just on K and C. In turn, this implies:

$$E[\|G_1\|^{\alpha_0}\mathbb{1}_{\{Z_1^*<a\}}] \leq E[\|G_1\|^{\alpha_0}\mathbb{1}_{\{\theta_a=1\}}] \leq E[\|G_{\theta_a}\|^{\alpha_0}] \leq D^{\alpha_0}(1+a)^{\alpha_0}.$$

If we divide this inequality by $(1+a)^\gamma$ with $\gamma > \alpha_0+1$ and if we integrate over $a \in [0,\infty)$, we get with Fubini's theorem:

$$E[\|G_1\|^{\alpha_0}\int_{Z_1^*}^{\infty}(1+a)^{-\gamma}da] \leq D^{\alpha_0}\int_0^{\infty}(1+a)^{\alpha_0-\gamma}da.$$

Hence:

$$E[\frac{\|G_1\|^{\alpha_0}}{(1+Z_1^*)^{\gamma-1}}] \leq \frac{D^{\alpha_0}(\gamma-1)}{\gamma-\alpha_0-1}.$$

If we set next $q := \alpha'/(\gamma+\alpha'-1)$, $q' := 1-q$, $\alpha := q\alpha_0$, we obtain

$$\|G_1\|^\alpha = \left(\frac{\|G_1\|^{\alpha_0}}{(1+Z_1^*)^{\gamma-1}}\right)^q \cdot (1+Z_1^*)^{\alpha' q'}$$

and, according to Hölder's inequality (both q and q' are positive):

$$E[\|G_1\|^\alpha] \leq (D^{\alpha_0}(\gamma-1)/(\gamma-\alpha_0-1))^q E[(1+Z_1^*)^{\alpha'}]^{q'}.$$

It follows from Burkholder-Davis-Gundy's inequality that $\|Z_1^*\|_{L^{\alpha'}}$ is bounded, since $\|z\|_{M_2^{\alpha'}} \leq R$ and the right hand side of the last inequality is therefore bounded.

Taking in the above argument an $\alpha' > \beta_0$, where β_0 denotes the conjugate of α_0 (i.e. $\alpha_0^{-1}+\beta_0^{-1}=1$) and a γ in the interval $(1+\alpha_0, 1+\alpha_0\frac{\alpha'}{\beta_0})$, the resulting α belongs to the interval $(1,\alpha_0)$.

Since $\|p\| \leq 1$, $\|z\|_{M_2^{\alpha'}} \leq R$ and $\|c\|_{M_2^\alpha} \leq \nu\|G_1\|_{L^\alpha}$, where ν is the universal constant of Burkholder-Davis-Gundy's inequality, we conclude as announced that $V^{\alpha,\alpha'}(R)$ is a bounded subset of $\Delta(\mathcal{K}) \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'}$.

Finally, since theorem 3.3.2 holds for $\alpha_0 > \alpha$, it will also hold for α. □

REMARK 3.3.7. – *Theorem 3.3.6 holds for infinitely many pairs (α,α'). We chose once for all one of these pairs (α,α'). From now on we only will be concerned with the corresponding unbounded Brownian games $\hat{\Gamma}_{\alpha,\alpha'}(p,T)$ and $\hat{\Gamma}^*_{\alpha,\alpha'}(x,T)$ that will simply be denoted $\hat{\Gamma}(p,T)$ and $\hat{\Gamma}^*(x,T)$.*

3.4. The value of the unbounded Brownian games

THEOREM 3.4.1. – *$\forall p \in \Delta(\mathcal{K})$, player 1 has an optimal strategy b in $\hat{\Gamma}(p,T)$. This strategy is equalizing and thus satisfies to (32). Similarly,*

$\forall x \in I\!R^K$, player 1 has an optimal strategy (p,b) in $\hat{\Gamma}^*(x,T)$ which is equalizing and fulfills (32).

Proof. – As mentioned in the introduction of the previous section, the only strategies of player 1 guaranteeing a finite payoff in the unbounded Brownian games are the equalizing strategies. There is a lot of such equalizing strategies and we have to prove that one of them maximizes $g_T(b,\tau^0)$ (resp. minimizes $g_T^*((p,b),\tau^0)$).

Since $\mathcal{E}$ is a linear space, the set $\mathcal{E}^\alpha$ is a closed linear subspace of M_2^α. Furthermore, as it follows from Burkholder-Davis-Gundy's inequality, the mapping $Y_{T,1}(p,b)$ defined in (27) is linear and continuous from $(I\!R^K \times M_2^\alpha)$, endowed with the product topology, to $L^\alpha(\mathcal{F}_1)$. We infer then that the set V_p^α (resp. V^α) of the equalizing strategies in $\hat{\Gamma}(p,T)$ (resp. $\hat{\Gamma}^*(x,T)$) is a closed convex subset of M_2^α (resp. $(I\!R^K \times M_2^\alpha)$), since the set of $I\!R_+^K$-valued random vectors is closed and convex in $L^\alpha(\mathcal{F}_1)$. It follows from theorem 3.3.2 that V_p^α (resp. V^α) is also bounded in M_2^α (resp. $(I\!R^K \times M_2^\alpha)$).

The linear functionals $g_T(b,\tau^0)$ and $g_T^*((p,b),\tau^0)$ are continuous on these spaces, as it results from Hölder's inequality and therefore also weakly continuous. Since M_2^α and $(I\!R^K \times M_2^\alpha)$ are reflexive Banach spaces, we conclude then to the existence of optimal strategies for player 1 in $\hat{\Gamma}(p,T)$ and $\hat{\Gamma}^*(x,T)$. Indeed, a bounded closed convex set is compact in the weak topology of a reflexive space (see [9]) and a continuous functional always reaches its maximum on a compact set. □

THEOREM 3.4.2. – *$\forall p \in \Delta(\mathcal{K})$, $\forall x \in I\!R^K$, the games $\hat{\Gamma}(p,T)$ and $\hat{\Gamma}^*(x,T)$ have a value that will be respectively denoted $\hat{\psi}(p,T)$ and $\hat{\psi}^*(x,T)$. In other words:*

$$\hat{\psi}(p,T) := \max_{b\in\mathcal{Y}_p^{\alpha,\alpha'}} \inf_{\tau\in\mathcal{T}^{\alpha'}} g_T(b,\tau) = \inf_{\tau\in\mathcal{T}^{\alpha'}} \sup_{b\in\mathcal{Y}_p^{\alpha,\alpha'}} g_T(b,\tau) \tag{39}$$

$$\begin{aligned} \hat{\psi}^*(x,T) :&= \min_{(p,b)\in\mathcal{Y}^{\alpha,\alpha'}} \sup_{\tau\in\mathcal{T}^{\alpha'}} g^*((p,b),\tau) \\ &= \sup_{\tau\in\mathcal{T}^{\alpha'}} \inf_{(p,b)\in\mathcal{Y}^{\alpha,\alpha'}} g^*((p,b),\tau) \end{aligned} \tag{40}$$

Proof. – Let us first observe that player 1's strategy spaces $\mathcal{Y}_p^{\alpha,\alpha'}$ and $\mathcal{Y}^{\alpha,\alpha'}$ are closed convex subsets of $(\mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ and $(I\!R^K \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ respectively. Indeed, $Y_{T,1}(p,c+z)$ is a continuous linear mapping form $(I\!R^K \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ to $L^{\alpha\wedge\alpha'}$ and the set of $I\!R_+^K$-valued random vectors is closed in $L^{\alpha\wedge\alpha'}$.

Let next $\underline{\psi}$ denote the sup inf of $\hat{\Gamma}(p,T)$ and $\bar{\psi}^*$, the inf sup of $\hat{\Gamma}^*(x,T)$. Let also, for $\epsilon > 0$, $\mathcal{V}_\epsilon$ denote the set c in $\mathcal{E}^\alpha$ such that $g_T(c,\tau^0) = \underline{\psi} + \epsilon$ and $\mathcal{V}_\epsilon^*$ the set of (p,c) in $(I\!R^K \times \mathcal{E}^\alpha)$ such that $g_{x,T}^*((p,b),\tau^0) = \bar{\psi}^* - \epsilon$.

We will first deal with the particular case where one of the sets $\mathcal{V}_\epsilon$ or $\mathcal{V}_\epsilon^*$ is empty: If $\mathcal{V}_\epsilon^*$ is empty, the linear functional $g_{x,T}^*((p,b),\tau^0) = \langle p|x\rangle - g_T(b,\tau^0)$ must be constant on the linear space $I\!R^K \times \mathcal{E}^\alpha$, implying thus $g_T(b,\tau^0) = 0, \forall b \in \mathcal{E}^\alpha$. (the hypothesis $\mathcal{V}_\epsilon = \emptyset$ leads to the same conclusion.)

Since $a(\tau^0)$ belongs to $\mathcal{E}$ (see definition 3.1.3), we have in particular

$$0 = g_T(a(\tau^0),\tau^0) = E[\int_T^1 \langle\!\langle a(\tau^0)|a(\tau^0)\rangle\!\rangle dt],$$

and thus $a(\tau^0) = 0$. The strategy τ^0 guarantees then 0 to player 2 in $\hat{\Gamma}(p,T)$ and $\gamma^*(x)$ in $\hat{\Gamma}^*(x,T)$. Since player 1 can guarantees the same values in these games with the strategies 0 and $(p',0)$ with $p' \in \Delta(\mathcal{K})$ such that $\gamma^*(x) = \langle p'|x\rangle$, we conclude as announced that the games have a value.

So, from the above discussion, we may suppose that for all $\epsilon > 0$, $\mathcal{V}_\epsilon \neq \emptyset \neq \mathcal{V}_\epsilon^*$.

We will now prove that, in $(\mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$, the distance between the convex sets $\mathcal{Y}_p^{\alpha,\alpha'}$ and $\mathcal{V}_\epsilon$ is strictly positive.

If on the contrary, there were a sequence (b_n,d_n) in $(\mathcal{Y}_p^{\alpha,\alpha'} \times \mathcal{V}_\epsilon)$ such that $b_n - d_n$ tends to 0 in $(\mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$, we could then write b_n as the sum $c_n + z_n$, where $c_n \in \mathcal{E}^\alpha$ and $z_n \in \mathcal{E}^{\perp,\alpha'}$ and conclude, since d_n is equalizing, that both $\|c_n - d_n\|_{M_2^\alpha}$ and $\|z_n\|_{M_2^{\alpha'}}$ are tending to 0. So, if R denotes the maximum of the $\|z_n\|_{M_2^{\alpha'}}$ for $n \in I\!N$, we would conclude that b_n belongs to $V_p^{\alpha,\alpha'}(R)$. According to theorem 3.3.6, the sequence b_n would be uniformly bounded in $(\mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ and would therefore have a weak limit point b in $\mathcal{E}^\alpha$ since $\|z_n\|_{M_2^{\alpha'}}$ tends to 0. b would also belong to $\mathcal{Y}_p^{\alpha,\alpha'}$, since $\mathcal{Y}_p^{\alpha,\alpha'}$ is closed and convex and therefore also weakly closed. On the other hand, b would also be a weak limit point of d_n, since $\|c_n - d_n\|_{M_2^\alpha} \to 0$ and therefore $b \in \mathcal{V}_\epsilon$. This is a contradiction: b would then be an equalizing strategy guaranteeing $\underline{\psi} + \epsilon$.

A similar argument indicates that the distance in $(I\!R^K \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ between the convex sets $\mathcal{Y}^{\alpha,\alpha'}$ and $\mathcal{V}_\epsilon^*$ is strictly positive.

Since $(\mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ and $(I\!R^K \times \mathcal{E}^\alpha \times \mathcal{E}^{\perp,\alpha'})$ are Banach spaces, there must then exist two linear functionals H and H^* on these spaces separating respectively $\mathcal{Y}_p^{\alpha,\alpha'}$ from $\mathcal{V}_\epsilon$ and $\mathcal{Y}^{\alpha,\alpha'}$ from $\mathcal{V}_\epsilon^*$:

$$\begin{aligned} &\forall c \in \mathcal{Y}_p^{\alpha,\alpha'}; \forall d \in \mathcal{V}_\epsilon : H(c) < H(d) \\ &\forall (p,c) \in \mathcal{Y}^{\alpha,\alpha'}; \forall (q,d) \in \mathcal{V}_\epsilon^* : H^*(p,c) > H^*(q,d). \end{aligned} \tag{41}$$

Next, the dual spaces of $(\mathcal{E}^{\alpha} \times \mathcal{E}^{\perp,\alpha'})$ and $(I\!R^K \times \mathcal{E}^{\alpha} \times \mathcal{E}^{\perp,\alpha'})$ are respectively $(\mathcal{E}^{\beta} \times \mathcal{E}^{\perp,\beta'})$ and $(I\!R^K \times \mathcal{E}^{\beta} \times \mathcal{E}^{\perp,\beta'})$, where $1/\alpha + 1/\beta = 1 = 1/\alpha' + 1/\beta'$. The linear functionals H and H^* can then be written as:

$$(42)\quad H(b) = E[\int_T^1 \langle\!\langle h_t | b_t \rangle\!\rangle dt] \text{ and } H^*(p,b) = \langle x' | p \rangle - E[\int_T^1 \langle\!\langle h_t^* | b_t \rangle\!\rangle dt],$$

where $h = e + e'$, $h^* = e^* + e^{*\prime}$ with (e, e') and $(e^*, e^{*\prime})$ in $(\mathcal{E}^{\beta} \times \mathcal{E}^{\perp,\beta'})$ and $x' \in I\!R^K$.

Since $\mathcal{V}_\epsilon$ and $\mathcal{V}_\epsilon^*$ are affine spaces of co-dimension 1 in $\mathcal{E}^{\alpha}$ and $I\!R^K \times \mathcal{E}^{\alpha}$ respectively, H and H^* must be constant respectively on $\mathcal{V}_\epsilon$ and on $\mathcal{V}_\epsilon^*$, for relation (41) to hold. Since the functional $g_T(\bullet, \tau^0)$ and $g_{x,T}^*(\bullet, \tau^0)$ have the same property, there must exist two real numbers λ and λ^* such that

(43)
$$\forall c \in \mathcal{E}^{\alpha}, \forall q \in I\!R^K : H(c) = \lambda g_T(c, \tau^0) \text{ and } H^*(q,c) = \lambda^* g_{x,T}^*((q,c), \tau^0).$$

Thus $e = \lambda a(\tau^0)$, $x = \lambda^* x'$ and $e^* = \lambda^* a(\tau^0)$.

From the definition of $\mathcal{V}_\epsilon$, if $c \in \mathcal{Y}_p^{\alpha,\alpha'}$ and $d \in \mathcal{V}_\epsilon$, then $g_T(c, \tau^0) < g_T(d, \tau^0)$. Comparing this with (41), we conclude that $\lambda > 0$. Similarly, $\lambda^* > 0$.

Next, since $\mathcal{E}^{\perp} := a(\mathcal{D})$, the processes e' and $e^{*\prime}$ in $\mathcal{E}^{\perp,\beta'}$ can be written as $e' = \lambda a(\delta)$ and $e^{*\prime} = \lambda^* a(\delta^*)$, for $\mathcal{D}$-valued processes δ and δ^* in $M_2^{\beta'}$.

Relation (42) yields then:

$$H(b)/\lambda = E[\int_T^1 \langle\!\langle b_t | a(\tau_t) \rangle\!\rangle dt] = g_T(b, \tau)$$

and

$$H^*(p,b)/\lambda^* = g_{x,T}^*((p,b), \tau^*),$$

where $\tau := \tau^0 + \delta$ and $\tau^* := \tau^0 + \delta^*$ are strategies of player 2 in the unbounded Brownian game. Relation (41) indicates that, if $d \in \mathcal{V}_\epsilon$, for all strategy b of player 1 in $\hat{\Gamma}(p,T)$: $g_T(b, \tau) < g_T(d, \tau) = \underline{\psi} + \epsilon$. This last inequality indicates that τ is an ϵ-optimal strategy for player 2: up to ϵ, it guarantees to player 2 a lesser payoff than the best one player 1 can guarantee: $\underline{\psi}$. The game $\hat{\Gamma}(p,T)$ has thus a value.

Similarly, we conclude that τ^* guarantees a payoff $\bar{\psi}^* - \epsilon$ to player 2 in $\hat{\Gamma}^*(x,T)$. Since both players can guarantee the same amount, up to an arbitrarily small ϵ, the game has a value. □

3.5. Continuity and duality relations Before dealing with the problem of the existence of optimal strategies for player 2, it is convenient to explore the duality relationships between the primal and the dual games.

LEMMA 3.5.1. – *The function $\hat{\psi}(\bullet, T)$ on $\Delta(\mathcal{K})$ is positive, continuous and concave.*

Proof. – The process $b = 0$ is a strategy for player 1 in $\hat{\Gamma}(p, T)$ since $Y_{T,1}(p, 0) = p \in I\!R^K_+$. Therefore, $\hat{\psi}(p, T) \geq \inf_{\tau \in \mathcal{T}} g_T(0, \tau) = 0$.

Let now F denote the mapping that maps $(p, b) \in V^\alpha$ on $F(p, b) := (p, g_T(b, \tau^0))$. Since V^α is convex closed and bounded in $I\!R^K \times \mathcal{E}^\alpha$ (see theorem 3.3.6) and F is a continuous linear mapping, we conclude that $Q := F(V^\alpha)$ is a closed convex subset of $(\Delta(\mathcal{K}) \times I\!R)$ and so is $Q_- := Q \cup (\Delta(\mathcal{K}) \times I\!R_-)$, since $\forall p \in \Delta(\mathcal{K})$, $(p, 0) \in Q$.

Since $\hat{\psi}(p, T) = \max_{b \in V^\alpha_p} g_T(b, \tau^0)$, the set Q_- turns out to be the hypograph of $\hat{\psi}$ and the result follows since a function with convex closed hypograph is concave and continuous on its domain (see [17], theorems 7.1 and 7.5). □

DEFINITION 3.5.2. – *Hereafter, $\partial\hat{\psi}^*(x, T)$ will denote the super gradient at x of the function $\hat{\psi}^*(\bullet, T)$:*

$$\partial\hat{\psi}^*(x, T) := \{z \in I\!R^K : \forall q \in I\!R^K : \hat{\psi}^*(q, T) \leq \hat{\psi}^*(x, T) + \langle z | q - x\rangle\}.$$

If we define $\hat{\psi}(q, T) := -\infty$ for $q \in I\!R^K - \Delta(\mathcal{K})$, the resulting function $\hat{\psi}(\bullet, T)$ is concave upper semi continuous on $I\!R^K$. The next theorem indicates that $\hat{\psi}^*(\bullet, T)$ and $\hat{\psi}(\bullet, T)$ are Fenchel conjugates of each other:

THEOREM 3.5.3. – *The functions $\hat{\psi}(\bullet, T)$ and $\hat{\psi}^*(\bullet, T)$ are linked by the duality relations:*

$$\forall x \in I\!R^K : \hat{\psi}^*(x, T) = \min_{p \in \Delta(\mathcal{K})} \langle x | p\rangle - \hat{\psi}(p, T) \tag{44}$$

$$\forall p \in \Delta(\mathcal{K}) : \hat{\psi}(p, T) = \inf_{x \in I\!R^K} \langle x | p\rangle - \hat{\psi}^*(x, T) \tag{45}$$

In particular $\hat{\psi}^*(\bullet, T)$ is concave and fulfills:

$$\forall x, x' \in I\!R^K : |\hat{\psi}^*(x, T) - \hat{\psi}^*(x', T)| \leq \|x - x'\|, \tag{46}$$

$$\forall x \in I\!R^K : \partial\hat{\psi}^*(x, T) \subset \Delta(\mathcal{K}). \tag{47}$$

Proof. – We have: $\hat{\psi}^*(x, T) = \inf_{(p,b) \in \mathcal{Y}^{\alpha,\alpha'}} \sup_{\tau \in \mathcal{T}^{\alpha'}} \langle x | p\rangle - g_T(b, \tau)$, as it results from the definition (30). So, by splitting the minimization over

(p, b) in a double minimization, first on b in $\mathcal{Y}_p^{\alpha,\alpha'}$ and then over p, after rearrangement, we get:

$$\begin{aligned}\hat{\psi}^*(x,T) &= \inf_{p\in\Delta(\mathcal{K})} \langle x|p\rangle - \sup_{b\in\mathcal{Y}_p^{\alpha,\alpha'}} \inf_{\tau\in\mathcal{T}^{\alpha'}} g_T(b,\tau) \\ &= \inf_{p\in\Delta(\mathcal{K})} \langle x|p\rangle - \hat{\psi}(p,T).\end{aligned}$$

The continuity of $\hat{\psi}(\bullet, T)$ allows us to replace the inf by min in the last equation and (44) is proved.

As an infimum of affine functionals, the Fenchel conjugate of any function is a concave function and the concavity of $\hat{\psi}^*(\bullet, T)$ follows thus from (44).

Relation (45) is a consequence of (44) joint with the well known relation $(f^\circ)^\circ = f$ for the concave upper semi continuous function f (see [17] theorem 12.2.1.)

For such a function f we also have, as stated in [5], lemma 2.4, $\forall x \in I\!R^K : \partial f^\circ(x) \subset dom(f)$, where $dom(f)$ is the set of points where f is finite. Relation (47) follows then immediately.

Since $\Delta(\mathcal{K})$ is included in the unit ball of $I\!R^K$, we get, with $z \in \partial\hat{\psi}^*(x,T)$: $\forall x' \in I\!R^K : \hat{\psi}^*(x',T) - \hat{\psi}^*(x,T) \leq \langle z|x'-x\rangle \leq \|x'-x\|$. Interchanging x and x' in this inequality, we obtain (46). □

The next corollary indicates that optimal strategies in $\hat{\Gamma}^*(x,T)$ and $\hat{\Gamma}(p,T)$ coincide. This will allow us to infer the existence of player 2's optimal strategies in $\hat{\Gamma}(p,T)$, once proved that player 2 has optimal strategies in $\hat{\Gamma}^*(x,T)$.

COROLLARY 3.5.4. – *If τ is an optimal strategy for player 2 in $\hat{\Gamma}^*(x,T)$, where $x \in \partial\hat{\psi}(p,T)$ for a point $p \in \Delta(\mathcal{K})$, then τ is also optimal in $\hat{\Gamma}(p,T)$.*

Conversely, if b is optimal for player 1 in $\hat{\Gamma}(p,T)$ with $p \in \partial\hat{\psi}^*(x,T)$, then (p,b) is optimal in $\hat{\Gamma}^*(x,T)$.

Proof. – The Fenchel lemma (see [17] theorem 12.2.1) claims that for an upper semi continuous function f : $f(x) + f^\circ(y) = \langle x|y\rangle$ whenever $x \in \partial f^\circ(y)$.

Let now τ be an optimal strategy for player 2 in $\hat{\Gamma}^*(x,T)$ with $x \in \partial\hat{\psi}(p,T)$. We conclude then that $\hat{\psi}^*(x,T) = \langle x|p\rangle - \hat{\psi}(p,T)$. The optimality of τ means that, for all strategy (q,b) of player 1, $g^*_{x,T}((q,b),\tau) = \langle x|q\rangle - g_T(b,\tau) \geq \hat{\psi}^*(x,T)$. Therefore, for all strategy b in $\mathcal{Y}_p^{\alpha,\alpha'}$ $g_T(b,\tau) \leq \langle x|p\rangle - \hat{\psi}^*(x,T) = \hat{\psi}(p,T)$: τ is optimal in $\hat{\Gamma}(p,T)$ and the first claim is proved. A similar argument holds for the converse claim. □

3.6. Existence of optimal strategies for player 2 We have now to prove the existence of optimal strategies for player 2 in the Brownian games. As announced in the last section, we first will deal with the dual games. The central result of this section is theorem 3.6.5. It states that player 2 may restrict himself in $\hat{\Gamma}^*(x,T)$ to a bounded set of strategies. As a corollary, we will then infer the existence of optimal strategies. The proof of this theorem is based on several lemmas.

Let us first explore the notion of guaranteeing for player 2 in the game $\hat{\Gamma}^*(x,T)$. Let τ be a strategy of player 2 guaranteeing a payoff a in $\hat{\Gamma}^*(x,T)$: for all strategy (p,b) of player 1 we have

$$g_T^*((p,b),\tau) = E[\langle Y_{T,1}(p,b)|X_{x,T,1}^*(\tau)\rangle] \geq a. \tag{48}$$

Since the argument used in the proof of theorem 3.1.1 also holds for the unbounded Brownian games, relation (48) means that $E[\langle Y|X_{x,T,1}^*(\tau)\rangle] \geq a$, for all bounded $\mathbb{R}_+^K$-valued $\mathcal{F}_1$-measurable random vector Y in $M_2^{\beta'}$ with $E[Y] \in \Delta(\mathcal{K})$. In turn, this is equivalent to

$$\gamma^*(X_{x,T,1}^*(\tau)) \overset{a.s.}{\geq} a. \tag{49}$$

Lemma 3.6.2 indicates that player 2 can guarantee a finite payoff with uniformly bounded strategies. Its proof relies on the following lemma that is proved in [5], section 7:

Lemma 3.6.1. – *The function $f : x \to f(x) := E[\gamma^*(x+z)]$, where z is a K-dimensional standard normal vector, belongs to $\mathcal{C}^\infty$, is concave, and satisfies to the following properties:*

$$\begin{aligned}
&(A)\exists G \geq 0, \forall x \in \mathbb{R}^K : \gamma^*(x) - G \leq f(x) \leq \gamma^*(x)\\
&(B)\forall x \in \mathbb{R}^K : \nabla f(x) \in \Delta(\mathcal{K})\\
&(C)\exists M \geq 0, \forall x \in \mathbb{R}^K : \|f''(x)\| \leq M,
\end{aligned}$$

where $f''(x)$ denotes the Hessian matrix of f at x.

Lemma 3.6.2. – *There exists a finite payoff D that player 2 can guarantee from any stopping time θ with $\Delta(\mathcal{J})$-valued strategies: for all $[T,1]$-valued stopping time θ, player 2 has a $\Delta(\mathcal{J})$-valued strategy $\tilde{\tau}$ such that*

$$\gamma^*(-\int_\theta^1 \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tilde{\tau}_s} dW_{i,s}) \overset{a.s.}{\geq} -D. \tag{50}$$

Proof. – As stated in lemma 2.1.3, the mapping $\rho(\bullet)$ is $\mathcal{C}^\infty$ on the simplex $\Delta(\mathcal{K})$. Therefore, $\rho(\nabla f(x))$ is well defined, as it follows from property (B) of f and is Lipschitzian on $\mathbb{R}^K$ according to property (C).

Its diffusion term being Lipschitzian in Z, the stochastic differential equation

$$dZ_t = \mathbb{1}_{\{\theta \leq s\}} \sum_{i \in \mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\rho(\nabla f(Z_s))} dW_{i,s}; \ Z_T = 0$$

has a unique solution Z_t according to theorem 2.1, Chapter IX in [15].

Applying Itô's formula, we get:

$$f(Z_1) \overset{a.s.}{=} f(0) - \int_\theta^1 \langle \nabla f(Z_s) | \sum_{i \in \mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\rho(\nabla f(Z_s))} dW_{i,s} \rangle$$
$$+ \frac{1}{2} \int_\theta^1 \sum_{i \in \mathcal{I}} \sigma_{0,i} a_{i\rho(\nabla f(Z_s))}^{\top} f''(Z_s) a_{i\rho(\nabla f(Z_s))} ds.$$

According to relation (2), the first integral collapses.

Moreover, the second integral is bounded, as it follows from property (C) of f. Therefore, using property (A) of f, we infer that $\gamma^*(Z_1)$ is a.s. bounded from below by a constant and the strategy $\tilde{\tau}_t := \rho(\nabla f(Z_t))$ guarantees, as announced, a finite payoff to player 2. □

The following results is a consequence of lemma 2.1.3:

LEMMA 3.6.3. – *There exists a* $\xi > 0$ *such that* $\forall \delta \in \mathcal{D}$, $\forall p \in \Delta(\mathcal{K})$:

$$\sum_{i \in \mathcal{I}} \sigma_{0,i} \langle p | a_{i\delta} \rangle^2 \geq \xi ||\delta||^2$$

Proof. – We just have to show that

$$0 < \xi := \inf\{\sum_{i \in \mathcal{I}} \sigma_{0,i} \langle p | a_{i\delta} \rangle^2 : p \in \Delta(\mathcal{K}), \delta \in \mathcal{D} \text{ and } ||\delta|| = 1\}.$$

If on the contrary, we had $0 = \xi$, we would have $\sum_i \sigma_{0,i} \langle p | a_{i\delta} \rangle^2 = 0$, for some p and $\delta \neq 0$, and hence,

$$\forall i : \langle p | a_{i\delta} \rangle = 0. \tag{51}$$

Since player 2's optimal strategy $\rho(p)$ in $G(p)$ is completely mixed (see lemma 2.1.3), $\tau := \rho(p) + \epsilon\delta$ would then belong to $\Delta(\mathcal{J})$ for an $\epsilon > 0$ and would guarantee a null payoff player 2 in $G(p)$, as indicates (3) joint with (51). τ would therefore be optimal in $G(p)$ and the contradiction would then follow since lemma 2.1.3 would then imply $\tau = \rho(p)$ and hence $\delta = 0$. □

The next lemma is the keystone in the proof of theorem 3.6.5:

LEMMA 3.6.4. – *For all $Q \geq 0$ and $\nu \geq 0$, there exist $\mu > 0$ and $C'' < \infty$ such that if $v \in M_2^\beta(\{\mathcal{F}_t\}, \mathbb{R}_J)$ and h is a progressively measurable one-dimensional process satisfying:*

$$\forall s \in [T,1] : |h_s| \leq \nu \|v_s\|^2, \tag{52}$$

and

$$\forall t \in [T,1] : \|U_t\|_{L^\infty} \leq Q,$$

where U is the semi-martingale $U_t := \int_T^t v_s^\top dW_s + h_s ds$, then

$$E[\mu \exp(\int_T^1 \|v_s\|^2 ds)] \leq C''. \tag{53}$$

Proof. – According to (52), the process $\phi_s := -h_s/\|v_s\|^2$, with the convention $0/0 = 0$, is valued in the interval $[-\nu,\nu]$. Moreover, $dU_s = v_s^\top d\tilde{W}_s$, with $d\tilde{W}_s := dW_s - \phi_s v_s ds$. Let next y be the exponential local martingale $y_t := \exp(\int_T^t \phi_s v_s^\top dW_s - \frac{1}{2}\int_T^t \phi_s^2 \|v_s\|^2 ds)$ and $\theta_n := \inf\{t : y_t \geq n\}$, with the convention $\inf \emptyset := 1$. The process $y'_t := y_{\theta_n \wedge t}$ is then a bounded martingale and y'_1 is a probability density on $(Z, \{\mathcal{F}_t\}, P)$. According to Girsanov's theorem, $\tilde{W}$ is a $\mathcal{F}_t$-Brownian motion up to time θ_n under the probability P', where $dP' := y'_1 dP$. Under P', $U'_t := U_{\theta_n \wedge t}$ is therefore a $[-Q,Q]$-valued martingale. Its $BMO_2(P')$-norm is then less than Q and we conclude with John Nirenberg's inequality (see theorem 2.2 in [11]) that, for all $\lambda < 2/Q^2$, for all stopping time θ:

$$E'[\exp(\frac{\lambda}{2}\int_\theta^1 \|v'_s\|^2 ds)|\mathcal{F}_\theta] \leq \frac{2}{2-\lambda Q^2}, \tag{54}$$

where $v'_t := \mathbb{1}_{t \leq \theta_n} v_t$ and E' denotes the expectation with respect to P'.

Since $|\phi_t| \leq \nu$, we also have

$$E'[\exp(\frac{\lambda}{2\nu^2}\int_\theta^1 \phi_s^2 \|v'_s\|^2 ds)|\mathcal{F}_\theta] \leq \frac{2}{2-\lambda Q^2}.$$

As it follows from Girsanov's theorem, for any random variable X: $E'[X|\mathcal{F}_\theta] = E[X y'_1/y'_\theta|\mathcal{F}_\theta]$. The last relation leads then to:

$$E[\exp(\int_\theta^t \phi_s v_s'^\top dW_s + \frac{\lambda/\nu^2 - 1}{2}\int_\theta^1 \phi_s^2 \|v'_s\|^2 ds)|\mathcal{F}_\theta] \leq \frac{2}{2-\lambda Q^2},$$

and we conclude with (36) in lemma 3.3.5 that there exist $\eta > 0$ and C' depending only on the constant λ/ν^2 and $2/(2-\lambda Q^2)$ such that $E[y_1'^{-\eta}] \leq C'$.

If we set $q := \eta/(\eta + \lambda/2)$, $q' := 1 - q$ and $\mu := q\lambda/2$, we have

$$\exp(\mu \int_T^1 \|v'_s\|^2 ds) = \left(y'_1 \exp(\frac{\lambda}{2} \int_T^1 \|v'_s\|^2 ds)\right)^q (y_1'^{-\eta})^{q'}$$

and we conclude with Hölder inequality and relation (54) that:

$$\begin{aligned} E[\exp(\mu \int_T^{\theta_n} \|v_s\|^2 ds)] &= E[\exp(\mu \int_T^1 \|v'_s\|^2 ds)] \\ &\le (E'[\exp(\frac{\lambda}{2} \int_T^1 \|v'_s\|^2 ds)])^q (E[y_1'^{-\eta}])^{q'} \\ &\le \frac{2^q}{(2 - \lambda Q^2)^q} C'^{q'}. \end{aligned}$$

Since y is a local martingale, the stopping times θ_n increase a. s. to 1 as n tends to ∞ and with Fatou's lemma, we get (53) as wanted. □

We are now able to prove the central result of this section:

THEOREM 3.6.5. – *There exists a constant $N < \infty$ such that, $\forall x \in I\!R^K$, any amount player 2 can guarantee in $\hat{\Gamma}^*(x, T)$ with a strategy τ is also guaranteed by a strategy τ' with*

$$\|\tau'\|_{M_2^{\beta'}} \le N. \tag{55}$$

Proof. – Let τ be a strategy guaranteeing a payoff a to player 2 in $\hat{\Gamma}^*(x, T)$ and let X denote the process $X_t := X_{T,t}(\tau)$. With the usual convention $\inf \emptyset := 1$, we define then the stopping time $\theta := \inf\{t \in [T, 1] : f(x - X_t) \ge a + D\}$, where D is as in lemma 3.6.2 and f as in lemma 3.6.1.

Let also $\tilde{\tau}$ be the $\Delta(\mathcal{J})$-valued strategy corresponding to the stopping time θ in lemma 3.6.2. The new strategy τ' defined by

$$\tau'_t := 1\!\!1_{\{\theta > t\}} \tau_t + 1\!\!1_{\{\theta \le t\}} \tilde{\tau}_t$$

also guarantees a. Indeed, let X' denote the process $X_t := X_{T,t}(\tau')$. According to the definition of τ', we have the relation $X_\theta = X'_\theta$. So, $\gamma^*(X'_1) = \gamma^*(X_1) \ge a$ on $\{\theta = 1\}$. On the other hand, if $\theta < 1$, we have $a + D = f(x - X_\theta) \le \gamma^*(x - X_\theta)$, as it results from property (A) of f. Therefore,

$$\begin{aligned} \gamma^*(x - X'_1) &= \gamma^*(x - X_\theta - \int_\theta^1 \sum_{i \in \mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tilde{\tau}_s} dW_{i,s}) \\ &\ge \gamma^*(x - X_\theta) + \gamma^*(- \int_\theta^1 \sum_{i \in \mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\tilde{\tau}_s} dW_{i,s}) \\ &\ge a, \end{aligned}$$

since $\tilde{\tau}$ fulfills (50).

We will now prove the existence of a bound N on $\|\tau'\|_{M_2^\beta}$ that does not depend on the initial τ nor on x neither on a. Since $\tilde{\tau}$ is $\Delta(\mathcal{J})$-valued, this is equivalent to find a bound N' on $\|\tau''\|_{M_2^{\beta'}}$, where τ'' is the process $\tau''_t := 1\!\!1_{\{\theta>t\}}\tau_t$.

If $f(x) > a+D$, then the stopping time θ is equal to T and hence $\tau'' = 0$. We may therefore suppose $f(x) \leq a+D$. Then, it follows from the definition of θ that the process $X''_t := X_{T,t}(\tau'')$ fulfills: $f(x - X'') \leq a + D$.

On the other hand, since X is a martingale and γ^* a concave function, the guaranteeing relation: a.s. $\gamma^*(x - X_1) \geq a$ implies that the process $\gamma^*(x - X)$ remains above a and so does $\gamma^*(x - X'')$ since $X''_t = X_{\theta\wedge t}$. Therefore, with property (A) of f, we infer that

$$a + D \geq f(x - X'') \geq \gamma^*(x - X'') - G \geq a - G. \tag{56}$$

We apply next Itô's lemma to $f(x - X'')$:

(57)

$$f(x-X''_t) = f(x)-\int_T^t \langle \nabla f(x-X''_s)|dX''_s\rangle+\frac{1}{2}\int_T^t dX''^{\top}_s f''(x-X''_s)dX''_s.$$

It is convenient to introduce here the processes $\tau'''_t := 1\!\!1_{\{t<\theta\}}\rho(\nabla f(x - X''_t))$ and $X'''_t := X_{T,t}(\tau''')$. Since τ''' is uniformly bounded (ρ is $\Delta(\mathcal{J})$-valued), bounding τ'' and bounding $\delta := \tau'' - \tau'''$ are equivalent problems. We express now relations (56) and (57) in terms of δ and $X^\delta_t := X_{T,t}(\delta)$:

Due to equation (3), $\langle \nabla f(x - X''_s)|dX'''_s\rangle = 0$ and thus

$$\langle \nabla f(x - X''_s)|dX''_s\rangle = \langle \nabla f(x - X''_s)|dX^\delta_s\rangle.$$

Moreover, since f is concave, f'' is semi-negative, we have:

$$\begin{aligned} 0 &\geq dX''^{\top}_s f''(x - X''_s)dX''_s \\ &\geq 2\big(dX^{\delta\top}_s f''(x - X''_s)dX^\delta_s + dX'''^{\top}_s f''(x - X''_s)dX'''_s\big). \end{aligned}$$

τ''' being bounded, we conclude that $dX'''^{\top}_s f''(x - X''_s)dX'''_s \geq -Lds$, for a constant $L \geq 0$ depending only on the payoffs of the game and on the constant M in property (C) of f. Therefore $\int_T^t dX'''^{\top}_s f''(x - X''_s)dX'''_s \geq -L(t - T) \geq -L$ since $T \in [0, 1]$. Relations (56) and (57) indicates then that the semi martingale U is valued in the interval $[a - G - f(x), a + D - f(x) + L]$, where

$$U_t := \int_T^t -\langle \nabla f(x-X''_s)|dX^\delta_s\rangle+dX^{\delta\top}_s f''(x-X''_s)dX^\delta_s = \int_T^t v^{\top}_s dW_s+h_s ds,$$

with I-dimensional process v: $v_{is} = -\sqrt{\sigma_{0,i}}\langle\nabla f(x - X''_s)|a_{i\delta_s}\rangle$ and the one dimensional process $h_s := \sum_{i\in\mathcal{I}} \sigma_{0,i} a^{\top}_{i\delta_s} f''(x - X''_s) a_{i\delta_s}$.

Since $a + D \geq f(x) \geq a - G$, as it follows from relation (56) at time T, the interval $[a - G - f(x), a + D - f(x) + L]$ is included in $[-Q, Q]$, where $Q := L + D + G$ and U is thus a $[-Q, Q]$-valued semi-martingale.

Since $\nabla f(x - X''_s)$ is $\Delta(\mathcal{K})$-valued as it follows from property (B) of f and δ is $\mathcal{D}$-valued, we get with lemma 3.6.3: $\|v_s\|^2 \geq \xi\|\delta_s\|^2$. We also have $|h_s| \leq L'\|\delta_s\|^2$ for a constant L' depending only on the payoffs $a^k_{i,j}$ and on the constant M of property (C) of f. Thus $|h_s| \leq \nu\|v_s\|^2$ with $\nu := L'/\xi$.

Lemma 3.6.4 indicates then that $E[\exp(\mu \int_T^1 \|v_s\|^2 ds)] \leq C''$ for $\mu > O$ and C'' depending only on ν and Q and thus $E[\exp(\mu\xi \int_T^1 \|\delta_s\|^2 ds)] \leq C''$. Since for $z \geq 0$, $z^{\frac{\beta'}{2}} \leq (\beta'/(2e\mu\xi))^{\frac{\beta'}{2}} e^{\mu\xi z}$, we conclude as announced that $\|\delta\|^{\beta'}_{M_2^{\beta'}} = E[(\int_T^1 \|\delta_s\|^2 ds)^{\frac{\beta'}{2}}] \leq (\beta'/(2e\mu\xi))^{\frac{\beta'}{2}} C''$. So δ is bounded in $M_2^{\beta'}$ and so is τ' since $\tau'_t = \delta_t + \tau'''_t + \mathbb{1}_{\{t\geq\theta\}}\tilde{\tau}_t$, where the two last terms are $\Delta(\mathcal{J})$-valued. □

COROLLARY 3.6.6. – *$\forall x \in \mathbb{R}^K$, player 2 has an optimal strategy τ in $\hat{\Gamma}^*(x, T)$ satisfying (55).*

For $\epsilon > 0$, the set $\mathcal{T}^{\alpha'}(\epsilon)$ of player 2's ϵ-optimal strategies of in the ball of radius N is a closed convex subset of $M_2^{\beta'}$, as it follows from the linearity and the continuity of the payoff function $g^*_{x,T}((p, b), \bullet)$ in $\hat{\Gamma}^*(x, T)$, for all player 1's strategy (p, b). $\mathcal{T}^{\alpha'}(\epsilon)$ is therefore compact in the weak topology of $M_2^{\beta'}$. Moreover, since ϵ-optimal strategies always exist for $\epsilon > 0$, it follows from theorem 3.6.5 that $\mathcal{T}^{\alpha'}(\epsilon)$ is not empty.

The sets $\mathcal{T}^{\alpha'}(\epsilon)$ for $\epsilon > 0$ decreasing to 0 form then a decreasing system of not empty compact subsets and have therefore a not empty intersection. Any strategy τ in this intersection is optimal and satisfies (55). □

COROLLARY 3.6.7. – *Any optimal strategy τ of player 2 in $\hat{\Gamma}^*(x, T)$ fulfills:*

$$\gamma^*(X^*_{x,T,1}(\tau)) \overset{a.s.}{=} \hat{\psi}^*(x, T) \tag{58}$$

Proof. – An optimal strategy τ of player 2 guarantees by definition $\hat{\psi}^*(x, T)$. It follows then from (49) that a.s. $\gamma^*(X^*_{x,T,1}(\tau)) \geq \hat{\psi}^*(x, T)$.

Let now (p, b) be an optimal strategy of player 1 in $\hat{\Gamma}^*(x, T)$. We have then,

$$\begin{aligned}
\hat{\psi}^*(x, T) &= E[\langle Y_{T,1}(p, b)|X^*_{x,T,1}(\tau)\rangle] \\
&\geq E[\langle Y_{T,1}(p, b)|u\rangle\gamma^*(X^*_{x,T,1}(\tau))] \\
&\geq E[\langle Y_{T,1}(p, b)|u\rangle\hat{\psi}^*(x, T)] \\
&= \hat{\psi}^*(x, T),
\end{aligned}$$

since $E[Y_{T,1}(p,b)] = p \in \Delta(\mathcal{K})$, so that $E[\langle Y_{T,1}(p,b)|u\rangle] = 1$. According to theorem 3.4.1 and relation (32), we have $0 = P(\langle Y_{T,1}(p,b)|u\rangle = 0)$ since (p,b) is optimal and relation (58) follows. □

We will prove in part 4 that player 2 has also optimal strategies in the primal game $\hat{\Gamma}(p,T)$, using the corollary 3.5.4. We postpone this prove up to then since first we want to prove that player 2 has $\Delta(\mathcal{J})$-valued strategies in the dual unbounded Brownian games. This will be a consequence of the recursive structure of these games.

3.7. The time dependence. Our purpose in this section is to show that the Brownian games starting at time $T < 1$ are rescaled Brownian games starting at 0. This will lead to:

LEMMA 3.7.1. – *The next two equations hold:*

$$\forall p \in \Delta(\mathcal{K}), \forall T \in [0,1) : \hat{\psi}(p,T) = \sqrt{1-T}\hat{\psi}(p,0) \tag{59}$$

$$\forall x \in \mathbb{R}^K, \forall T \in [0,1) : \hat{\psi}^*(x,T) = \sqrt{1-T}\hat{\psi}^*(x/\sqrt{1-T}, 0) \tag{60}$$

Proof. – Let W be a Brownian motion on a filtration $\{\mathcal{F}_t\}_{t\in[0,1]}$ and for fixed T in $[0,1)$, let $t(\bullet)$ denotes the mapping from $[T,1]$ to $[0,1]$ defined by $t(t') := (t'-T)/(1-T)$. Then $\mathcal{F}'_{t'} := \mathcal{F}_{t(t')}$ is a filtration indexed by $t' \in [T,1]$ and $W'_{t'} := \sqrt{1-T}W_{t(t')}$ is an $\mathcal{F}'_{t'}$-Brownian motion.

Since the argument used to prove theorem 3.1.1 still holds for the unbounded Brownian games, the values of these games do not depend on the filtration nor on the Brownian motion on which they are defined. To establish relation (59) between $\hat{\psi}(p,0)$ and $\hat{\psi}(p,T)$, we may then consider the game $\hat{\Gamma}(p,0)$ defined on W and the game $\hat{\Gamma}(p,T)$ defined on W'. We denote by $\mathcal{Y}_p^{\alpha,\alpha'}$, $\mathcal{T}^{\alpha'}$ the strategy spaces in $\hat{\Gamma}(p,0)$ and by $\mathcal{Y}_p^{\prime,\alpha,\alpha'}$, $\mathcal{T}^{\prime,\alpha'}$ the strategy spaces in $\hat{\Gamma}(p,T)$.

For $b \in \mathcal{Y}_p^{\alpha,\alpha'}$, let b' denote $b'_{t'} := b_{t(t')}/\sqrt{1-T}$. The relation $Y_{0,1}(p,b) = Y_{T,1}(p,b')$ holds the obviously whenever b is a step process and will then hold for all strategy b by continuity. As a consequence b' belongs to $\mathcal{Y}_p^{\prime,\alpha,\alpha'}$ and the mapping that takes b to b' is one to one between $\mathcal{Y}_p^{\alpha,\alpha'}$ and $\mathcal{Y}_p^{\prime,\alpha,\alpha'}$.

Similarly, if $\tau \in \mathcal{T}^{\alpha'}$, then $\tau' \in \mathcal{T}^{\alpha'}$ with $\tau'_{t'} := \tau_{t(t')}$ and $\sqrt{1-T}X_{0,1}(\tau) = X_{T,1}(\tau')$. Thus

$$\begin{aligned} g_T(b',\tau') &= E[\langle Y_{T,1}(p,b')|X_{T,1}(\tau')\rangle] \\ &= \sqrt{1-T}E[\langle Y_{0,1}(p,b)|X_{0,1}(\tau)\rangle] \\ &= \sqrt{1-T}g_0(b,\tau) \end{aligned}$$

and (59) follows. We get (60) observing that

$$\begin{aligned} g^*_{x,T}((p,b'),\tau') &= \langle x|p\rangle - g_T(b',\tau') \\ &= \sqrt{1-T}(\langle x/\sqrt{1-T}|p\rangle - g_0(b,\tau)) \\ &= \sqrt{1-T}g^*_{x/\sqrt{1-T},0}((p,b),\tau). \quad \square \end{aligned}$$

4. THE MARKOVIAN BEHAVIOR OF PLAYER 2

4.1. The recursive structure of $\hat{\Gamma}^*(x,T)$. Let τ be a strategy of player 2, then $X^*_{x,T,1}(\tau) = X^*_{x,T,t}(\tau) - X_{t,1}(\tau)$. Since player 2 wants to maximize the quantity ess. inf. $\gamma^*(X^*_{x,T,t}(\tau) - X_{t,1}(\tau))$, he should after time t play optimally in $\hat{\Gamma}^*(X^*_{x,T,t}(\tau),t)$. This kind of argument leads to the following theorem:

THEOREM 4.1.1 (The recursive structure of $\hat{\Gamma}^*(x,T)$). –

$$(61)\quad \begin{aligned} \hat{\psi}^*(x,T) &= \sup_{\tau\in\mathcal{T}_T^{\alpha'}} \inf_{(p,b)\in\mathcal{Y}^{\alpha,\alpha'}} E[\langle u|Y_{T,t}(p,b)\rangle\hat{\psi}^*(X^*_{x,T,t}(\tau),t)] \\ &= \inf_{(p,b)\in\mathcal{Y}^{\alpha,\alpha'}} \sup_{\tau\in\mathcal{T}^{\alpha'}} E[\langle u|Y_{T,t}(p,b)\rangle\hat{\psi}^*(X^*_{x,T,t}(\tau),t)]. \end{aligned}$$

Furthermore, any optimal strategy τ of player 2 in $\hat{\Gamma}^*(x,T)$ is optimal in the first line of (61) and any optimal strategy of player 1 in $\hat{\Gamma}^*(x,T)$ is optimal in the second line of formula (61).

Proof. – Let τ be a strategy of player 2. Let next $\{A_n\}_{n\in\mathbb{N}}$ be a countable measurable partition of $\mathbb{R}^K$ with a diameter less than $\epsilon > 0$: $\forall n, \forall x,x' \in A_n : \|x-x'\| \leq \epsilon$. Let also x_n be a sequence of points such that $\forall n : x_n \in A_n$ and let τ_n be an optimal strategy in $\hat{\Gamma}^*(x_n,t)$. It is then easy to see that $\tilde{\tau} := \sum_n \mathbb{1}_{A_n}(X^*_{x,T,t}(\tau))\tau_n$ is a measurable process on the time interval $[t,1]$. Moreover, if the τ_n are chosen uniformly bounded in $M_2^{\beta'}$, as allowed by corollary 3.6.6, the process $\bar{\tau}$ defined as $\bar{\tau}_s := \mathbb{1}_{\{s\leq t\}}\tau_s + \mathbb{1}_{\{s>t\}}\tilde{\tau}_s$ is a strategy for player 2 in $\hat{\Gamma}^*(x,T)$.

If $\tilde{X}$ denotes $\sum_n \mathbb{1}_{A_n}(X^*_{x,T,t}(\tau))x_n$, we clearly have a.s.:$\|X^*_{x,T,t}(\tau) - \tilde{X}\| \leq \epsilon$ and thus $\hat{\psi}^*(\tilde{X},t) \geq \hat{\psi}^*(X^*_{x,T,t}(\tau),t) - \epsilon$, as it results from (46). Moreover, it follows from the definition of $\tilde{\tau}$ and relation (58) that a.s. $\gamma^*(\tilde{X} - X_{t,1}(\tilde{\tau})) = \hat{\psi}^*(\tilde{X},t)$.

If (p,b) is an optimal strategy of player 1 in $\hat{\Gamma}^*(x,T)$, then

$$\begin{aligned}\hat{\psi}^*(x,T) &= E[\langle Y_{T,1}(p,b)|X^*_{x,T,t}(\tau) - X_{t,1}(\tilde{\tau})\rangle] \\ &\geq E[\langle Y_{T,1}(p,b)|u\rangle\gamma^*(X^*_{x,T,t}(\tau) - \tilde{X})] \\ &\quad + E[\langle Y_{T,1}(p,b)|u\rangle\gamma^*(\tilde{X} - X_{t,1}(\tilde{\tau}))] \\ &\geq -\epsilon + E[\langle Y_{T,1}(p,b)|u\rangle\hat{\psi}^*(\tilde{X},t)] \\ &\geq -2\epsilon + E[\langle Y_{T,1}(p,b)|u\rangle\hat{\psi}^*(X^*_{x,T,t}(\tau),t)].\end{aligned}$$

This being true for all $\epsilon > 0$ and all strategies τ of player 2, we conclude that the strategy (p,b) guarantees $\hat{\psi}^*(x,T)$ to player 1 in the second line of (61).

A similar reasoning works in the other direction: let now τ be an optimal strategy of player 2. Let then (p_n,b_n) be an $\mathcal{H}_1$-measurable optimal strategy of player 1 in $\hat{\Gamma}^*(x_n,t)$, where $\mathcal{H}_1$ denotes the σ-algebra generated by $W_s - W_t$, for $s \in [t,1]$. Such a strategy exists as it follows from theorem 3.1.1 in terms of the unbounded Brownian games. Let $\tilde{Y}$ denote $\sum_n \mathbb{1}_{A_n}(X^*_{x,T,t}(\tau))Y(p_n,b_n)$.

Let (p,b) be a strategy of player 1 and let y denote $\langle Y_{T,t}(p,b)|u\rangle$. The random vector $Y := y\tilde{Y}$ can then be expressed as $Y_{T,1}(p,\tilde{b})$ for a strategy $(p,\tilde{b})$ of player 1 in $\hat{\Gamma}^*(x,T)$ and we compute:

$$\begin{aligned}\hat{\psi}^*(x,T) &\leq E[\langle Y|X^*_{x,T,t}(\tau) - X_{t,1}(\tau)\rangle] \\ &\leq E[\langle Y|X^*_{x,T,t}(\tau) - \tilde{X}\rangle] + \\ &\quad + E[y\sum_n \mathbb{1}_{A_n}(X^*_{x,T,t}(\tau))E[\langle Y_n|x_n - X_{t,1}(\tau)\rangle|\mathcal{F}_t]].\end{aligned}$$

Since b_n is $\mathcal{E}$-valued, $\mathcal{H}_1$-measurable and optimal in $\hat{\Gamma}^*(x_n,t)$, we get successively:

$$\begin{aligned}E[\langle Y_n|x_n - X_{t,1}(\tau)\rangle|\mathcal{F}_t] &= \langle p_n|x_n\rangle - E[\int_t^1 \langle\langle b_{n,s}|a(\tau_s)\rangle\rangle ds|\mathcal{F}_t] \\ &= \langle p_n|x_n\rangle - E[\int_t^1 \langle\langle b_{n,s}|a(\tau^0)\rangle\rangle ds|\mathcal{F}_t] \\ &= \langle p_n|x_n\rangle - E[\int_t^1 \langle\langle b_{n,s}|a(\tau^0)\rangle\rangle ds] \\ &= \hat{\psi}^*(x_n,t).\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{\psi}^*(x,T) &\leq E[\langle Y|X^*_{x,T,t}(\tau) - \tilde{X}\rangle] + E[y\hat{\psi}^*(\tilde{X},t)] \\ &\leq 2\epsilon + E[\langle Y_{T,t}(p,b)|u\rangle\hat{\psi}^*(X^*_{x,T,t}(\tau),t)].\end{aligned}$$

This being true for all strategy (p, b) and all $\epsilon > 0$, we conclude that the optimal strategy τ of player 2 in $\hat{\Gamma}^*(x,T)$ guarantees $\hat{\psi}^*(x,T)$ in the first line of (61). The result is thus proved since both players guarantee the same amount in the two lines of (61). □

It results from the first line of (61) that if τ is an optimal strategy for player 2 in $\hat{\Gamma}^*(x,T)$, then a.s.: $\hat{\psi}^*(X^*_{x,T,t}(\tau),t) \geq \hat{\psi}^*(x,T)$.

According to theorem 3.4.1 and relation (32), the optimal strategies (p, b) of player 1 satisfy: $0 = P(\langle Y_{T,1}(p,b)|u\rangle = 0)$. It follows then from the second line of (61) that the inequality may be replaced by an equality in the last formula:

COROLLARY 4.1.2. – *Any optimal strategy τ of player 2 in $\hat{\Gamma}^*(x,T)$ fulfills $\forall t \in [T,1)$:*

$$\hat{\psi}^*(x,T) \overset{a.s.}{=} \hat{\psi}^*(X^*_{x,T,t}(\tau),t) \tag{62}$$

4.2. The stochastic differential equation The payoff functions in $\hat{\Gamma}^*(x,T)$ and $\hat{\Gamma}^*(x+\lambda u,T)$, where $u := (1,\ldots,1) \in I\!R^K$, just differ by the constant amount λ. Both games are thus strategically equivalent and we may therefore restrict our analysis to the case $\hat{\psi}^*(x,T) = 0$.

Let then τ be an optimal strategy of player 2 in $\hat{\Gamma}^*(x,T)$. It follows then jointly from corollary 4.1.2 and relation (60) that: $\hat{\psi}^*(V,0)$ is constantly 0, where V denotes the process $V_t := X^*_{x,T,t}(\tau)/\sqrt{1-t}$.

The function $\hat{\psi}^*(\bullet,0)$ is concave and continuous, as stated in theorem 3.5.3. The correspondence $\partial\hat{\psi}^*(\bullet,0)$ that maps a point $x \in I\!R^K$ to the super gradient $\partial\hat{\psi}^*(x,0)$ of $\hat{\psi}^*(\bullet,0)$ at x has therefore a closed graph.

According to Kuratowski-Ryll-Nardzewski selection theorem (see corollary 1.1 in [16]), there exists a measurable selection $\bar{p}$ in this correspondence, i.e. a measurable mapping $\bar{p}$ from $I\!R^K$ to $\Delta(\mathcal{K})$ such that $\forall x : \bar{p}(x) \in \partial\hat{\psi}^*(x,0)$.

We may then consider the process $\Pi := \bar{p}(V)$ in $M_2^{\beta'}$. Since $\Pi_t \in \partial\hat{\psi}^*(V_t,0)$, we have $\forall t' > t, t' \in [T,1)$:

$$0 = \hat{\psi}^*(V_{t'},0) \leq \hat{\psi}^*(V_t,0) + \langle \Pi_t | V_{t'} - V_t\rangle = \langle \Pi_t | V_{t'} - V_t\rangle. \tag{63}$$

This property implies, as proved below in lemma 4.2.2, that $G_t := \int_T^t \langle \Pi_s | dV_s\rangle$ is an increasing process.

According to Itô's formula, we have:

$$dV_s = \frac{1}{\sqrt{1-s}} dX^*_{x,T,s}(\tau) + \frac{1}{2(1-s)} V_s ds. \tag{64}$$

Therefore,

$$\int_T^{\bullet} \frac{1}{\sqrt{1-s}} \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} \langle \Pi_s | a_{i\tau_s} \rangle dW_{i,s} = \int_T^{\bullet} \frac{1}{2(1-s)} \langle \Pi_s | V_s \rangle ds - G.$$

Since G is increasing and V_s is a.s. finite, we conclude that the process appearing in the right hand side of the last equality is a.s. with finite variation, at least up to any time $t < 1$.

Therefore, the martingale appearing on the left hand side, is a continuous martingale with a.s. finite variation. We conclude then with Proposition (1.2), Chapter IV in [15], that this martingale must be constantly equal to 0. As a consequence, $\forall i \in \mathcal{I}$, the process $\langle \Pi | a_{i\tau} \rangle$ is equal to 0 in $M_2^{\beta'}$. In turn, this indicates that the identity $\tau = \rho(\Pi)$ holds in $M_2^{\beta'}$, as it results from lemma 2.1.3.

From (60), we have $\partial\hat{\psi}^*(x,t) = \partial\hat{\psi}^*(x/\sqrt{1-t}, 0)$ and the mapping $\bar{p}(x,t) := \bar{p}(x/\sqrt{1-t})$ is therefore a measurable selection of the correspondence that maps (x,t) to $\partial\hat{\psi}^*(x,t)$. With these notations, the last relation leads to the following theorem:

THEOREM 4.2.1. – *Let τ be an optimal strategy of player 2 in $\hat{\Gamma}^*(x,T)$ and let X_t denote the associated integral $X_t := X^*_{x,T,t}(\tau)$. Then the process $\rho(\bar{p}(X_t,t))$ is a $\Delta(\mathcal{J})$-valued modification of τ, and the process X is a solution of the stochastic equation: $X_T = x$,*

$$dX_t = -\sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}} a_{i\rho(\bar{p}(X_t,t))} dW_{i,t}.$$

In the above stochastic equation, the diffusion term is apparently depending on the measurable selection $\bar{p}$. In the next section, we will prove that it is in fact independent of $\bar{p}$ and that it is moreover continuous.

To conclude this section, it still remains for us to prove the lemma:

LEMMA 4.2.2. – *The process $\int_T^{\bullet} \langle \Pi | dV \rangle$ is increasing.*

Proof. – Let D_n denote the set of dyadic numbers of order n in $\mathbb{R}$: $D_n := \{m/2^n : m \in \mathbb{Z}\}$. Let also $\phi_n(t)$ denote the highest number in D_n less or equal to t.

In the proof that step processes are dense in M_2^2 made in [8], IX,5, it is proved that, for an appropriate number δ in $[0,1]$, the sequence of step processes $\Pi_t^n := \Pi(T \vee (\delta + \phi_n(t-\delta)))$ converges to Π in M_2^2 as n tends to ∞.

We will prove now the convergence of $\int_T^{\bullet}\langle\Pi^n|dV\rangle$ to $\int_T^{\bullet}\langle\Pi|dV\rangle$. According to (64), this turns out to be equivalent to the convergence, $\forall t < 1$, of the two following integrals to 0:

$$M_n := \int_T^t \frac{1}{\sqrt{1-s}} \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}\langle\Pi_s - \Pi_s^n|a_{i\tau_s}\rangle dW_{i,s}$$

$$N_n := \int_T^t \frac{1}{(1-s)^{3/2}} \langle\Pi_s - \Pi_s^n|X^*_{x,T,s}(\tau)\rangle ds.$$

It results from Hölder's inequality that, a.s.,

$$E[|N_n|] \le \left(E[\int_T^t \frac{\|X^*_{x,T,s}(\tau)\|^{\beta'}}{(1-s)^{3\beta'/2}} ds]\right)^{1/\beta'} \left(E[\int_T^t \|\Pi_s - \Pi_s^n\|^{\alpha'} ds]\right)^{1/\alpha'}.$$

The first factor in the right hand side is bounded since $\tau \in M_2^{\beta'}$ and $t < 1$. Since both Π and Π^n are $\Delta(\mathcal{K})$ valued, the M_2^2 convergence of Π^n to Π implies then the L^1-convergence of N_n to 0.

We also have the $L^{\beta'}$-convergence of M_n to 0, since, $\forall i$:

$$\lim_{n\to\infty} E[\left(\int_T^t \frac{\|a_{i\tau_s}\|^2}{1-s}\|\Pi_s - \Pi_s^n\|^2 ds\right)^{\frac{\beta'}{2}}] = 0,$$

as it follows from Lebesgue's dominated convergence theorem: $\|\Pi - \Pi^n\|$ is bounded and $\tau \in M_2^{\beta'}$.

Thus $\int_T^{\bullet}\langle\Pi^n|dV\rangle$ converges to $\int_T^{\bullet}\langle\Pi|dV\rangle$ in L^1. Next, $\forall t, t' \in \delta + D_m$ with $t' > t$ and $m \le n$, $\int_t^{t'}\langle\Pi^n|dV\rangle$ is just a Riemann sum whose terms are positive as indicates relation (63). Letting then n increase to ∞, we infer that, a.s. $\int_t^{t'}\langle\Pi|dV\rangle \ge 0$, for all $t, t' \in \delta + \cup_m D_m$. Almost every trajectories of the process $\int_T^t\langle\Pi|dV\rangle$ are therefore increasing since they are continuous. $\square$

4.3. The continuity of the optimal strategies. In this section, we will use corollary 4.1.2 joint with the observation that the optimal strategies of player 2 are $\Delta(\mathcal{J})$-valued, to infer that these strategies are nearly constant on small interval of time. As a corollary, this will indicate that player 2's optimal strategies are continuous processes.

Let τ be an optimal strategy of player 2 in $\hat{\Gamma}^*(x, T)$.

Corollary 4.1.2 joint with (60) indicates then that $\forall t \in [T, 1)$:

$$\sqrt{1-t}\hat{\psi}^*(X^*_{x,T,t}(\tau)/\sqrt{1-t}, 0) = \sqrt{1-T}\hat{\psi}^*(x/\sqrt{1-T}, 0).$$

Let now q be any point in $\partial\hat{\psi}^*(x,T)$, which also means that $q \in \partial\hat{\psi}^*(x/\sqrt{1-T},0)$.

According to the definition of $\partial\hat{\psi}^*(x/\sqrt{1-T},0)$, we get:

$$\begin{aligned}\sqrt{\frac{1-T}{1-t}}\hat{\psi}^*(\frac{x}{\sqrt{1-T}},0) &= \hat{\psi}^*(\frac{X^*_{x,T,t}(\tau)}{\sqrt{1-t}},0)\\ &\leq \hat{\psi}^*(\frac{x}{\sqrt{1-T}},0) + \langle q|\frac{X^*_{x,T,t}(\tau)}{\sqrt{1-t}} - \frac{x}{\sqrt{1-T}}\rangle.\end{aligned}$$

After rearranging, this indicates that

$$\begin{aligned}\langle q|X^*_{x,T,t}(\tau) - x\rangle &\geq (\sqrt{1-T} - \sqrt{1-t})(\hat{\psi}^*(\frac{x}{\sqrt{1-T}},0) - \langle q|\frac{x}{\sqrt{1-T}}\rangle)\\ &\geq (\sqrt{1-T} - \sqrt{1-t})\hat{\psi}^*(0,0),\end{aligned}$$

where the last inequality follows from the fact that $q \in \partial\hat{\psi}^*(x/\sqrt{1-T},0)$.

Since the left hand side of the last inequality has a zero expectation, we infer that $\hat{\psi}^*(0,0) \leq 0$ and we conclude that

(65)
$$z_t := \int_T^t \sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}\langle q|a_{i\tau_s}\rangle dW_{i,s} = \langle q|X^*_{x,T,t}(\tau)\rangle \leq \frac{-\hat{\psi}^*(0,0)(t-T)}{\sqrt{1-T}},$$

since, for $t \in [T,1)$, we have: $1 - \sqrt{\frac{1-t}{1-T}} \leq \frac{t-T}{1-T}$.

This last relation joint with the fact that τ is $\Delta(\mathcal{J})$-valued implies the next lemma:

LEMMA 4.3.1. – $\forall t \in [T,1)$:

$$E[z_t^2] \leq \frac{-4M\hat{\psi}^*(0,0)}{\sqrt{2\pi(1-T)}}(t-T)^{3/2}, \tag{66}$$

where M is an upper bound on the payoffs $|a^k_{ij}|$.

Proof. – It will be technically convenient for our purpose to assume that the filtration $\mathcal{F}_t$ and the $\mathcal{F}_t$-Brownian Motion W are defined on the whole time interval $[T,\infty)$. We extend then τ after time 1 in a constant way: let τ' be a non optimal strategy for player 2 in the average game $G(q)$ and we set $\tau_t := \tau'$ for $t > 1$. We have then $dz_t = \sum \sqrt{\sigma_{0,i}}\langle q|a_{i\tau_t}\rangle dW_{i,t}$. So, if v_t denotes the process $\sum \sigma_{0,i}\langle q|a_{i\tau_t}\rangle^2$, and $R_t := \int_T^t v_t ds$, we have $E[z_t^2] = E[R_t]$ according to Itô's formula.

For $s \geq 0$, let us define $\theta_s := \inf\{t \geq T : R_t > s\}$. Then, the stopping time θ_s is always defined and bounded. Indeed, for $s > 1$, we have $v_s = \sum \sigma_{0,i}\langle q|a_{i\tau'}\rangle^2 > 0$.

We may then define: $\beta_s := z_{\theta(s)}$. According to the Dambis-Dubins-Schwarz theorem (see [15], p. 170) β_s is a $\mathcal{F}_{\theta_s}$-Brownian motion, and $z_t = \beta_{R_t}$.

Relation (65) may then be expressed in terms of the process β: if $\theta_s \leq 1$, we have $\beta_s := z_{\theta_s} \leq D(\theta_s - T)$, where $D := -\hat{\psi}^*(0,0)/\sqrt{1-T}$. We get therefore:

$$\sup_{s<t} \beta_s \leq \sup_{s<t} D(\theta_s - T) = D(\theta_t - T), \tag{67}$$

if $\theta_t \leq 1$. For ν in $[0, 1-T)$, let us now compute $P(R_{T+\nu} > t)$. By definition of θ, we have:

$$P(R_{T+\nu} > t) = P(\theta_t \leq T + \nu) = P(D(\theta_t - T) \leq D\nu).$$

According to (67), $P(R_{T+\nu} > t) \leq P(\sup_{s<t} \beta_s \leq D\nu) = P(\sqrt{t}|Z| \leq D\nu)$ where Z is a standard $N(0,1)$-random variable. Indeed $\sup_{s<t} \beta_s$ has the same distribution as $\sqrt{t}|Z|$ (see [15], proposition 3.7 in chapter III.).

Let M be a bound on the payoffs $|a^k_{ij}|$, then $v_t \leq M^2$, since $\tau_s \in \Delta(\mathcal{J})$ and $R_{T+\nu}$ is then bounded by $M^2\nu$. So:

$$\begin{aligned}
E[R_{T+\nu}] &= \int_0^{M^2\nu} P(R_{T+\nu} > t)dt \\
&\leq \int_0^{M^2\nu} P(\frac{D^2\nu^2}{Z^2} \geq t)dt \\
&= E\big[\min\{M^2\nu, \frac{D^2\nu^2}{Z^2}\}\big].
\end{aligned}$$

When computing the last expectation, the density of Z may be bounded by its value at 0: $(2\pi)^{-\frac{1}{2}}$ and we get then $E[R_{T+\nu}] \leq \frac{4MD}{\sqrt{2\pi}}\nu^{\frac{3}{2}}$, which is the announced relation (66). □

According to relation (3) and lemma 3.6.3, we get:

$$\begin{aligned}
v_t &= \sum_i \sigma_{0,i}\langle q|a_{i\tau_t}\rangle^2 \\
&= \sum \sigma_{0,i}\langle q|a_{i\tau_t} - a_{i\rho(q)}\rangle^2 \\
&\geq \xi\|\tau_t - \rho(q)\|^2.
\end{aligned}$$

As a consequence of the last lemma, we have then:

THEOREM 4.2.1. – *There exists a constant R depending only on the payoffs a_{ij}^k of the game such that, $\forall x \in I\!R^K$, if τ is optimal in $\hat{\Gamma}^*(x,T)$ and $q \in \partial\hat{\psi}^*(x,T)$, then:*

$$E[\int_T^t \|\tau_s - \rho(q)\|^2 ds] \leq R\frac{(t-T)^{\frac{3}{2}}}{\sqrt{1-T}}. \tag{68}$$

DEFINITION 4.3.3. – *In the following, we denote by $\mathcal{R}$ the correspondence $\rho \circ \partial\hat{\psi}^*$: $\mathcal{R}(x,t) := \{\rho(q) : q \in \partial\hat{\psi}^*(x,T)\}$.*

COROLLARY 4.3.4. – *For all $x \in I\!R^K$ and all $t < 1$, the set $\mathcal{R}(x,t)$ is a singleton and $\mathcal{R}$ may therefore be considered as a function from $I\!R^K \times [T,1)$ to $\Delta(\mathcal{J})$. This function is continuous.*

Proof. – Let q, q' be two points of $\partial\hat{\psi}(x,T)$ and let τ be an optimal strategy in $\hat{\Gamma}^*(x,T)$. It follows then from the last theorem that:

$$\begin{aligned}\|\rho(q) - \rho(q')\|^2 &= \frac{1}{t-T}E[\int_T^t \|\rho(q) - \rho(q')\|^2 ds]\\ &\leq \frac{2}{t-T}(E[\int_T^t \|\rho(q) - \tau_s\|^2 ds] + E[\int_T^t \|\rho(q') - \tau_s\|^2 ds)\\ &\leq 4R\sqrt{\frac{t-T}{1-T}}.\end{aligned}$$

Since this relation holds $\forall t > T$, we get $\tau(q) = \tau(q')$, as announced.

The correspondence that associates to x the set $\partial\hat{\psi}^*(x,0)$ has a closed graph since $\hat{\psi}^*(\bullet,0)$ is concave and continuous and, according to relation (60), for $a < 1$, the correspondence that associates the set $\partial\hat{\psi}^*(x,T)$ to (x,T) in $(I\!R^K \times [0,a])$ has the same property. Since ρ is continuous, $\mathcal{R}(\bullet,\bullet) = \rho \circ \partial\hat{\psi}^*(\bullet,\bullet)$ has also a closed graph and is therefore continuous. Letting then a increase to 1, we get the continuity of $\mathcal{R}$ on $I\!R^K \times [0,1)$, as announced. □

We can now rephrase theorem 4.2.1 in a way that does not depend on a measurable selection $\bar{p}(x,t)$, since clearly $\mathcal{R}(x,t) = \rho(\bar{p}(x,t))$.

THEOREM 4.3.5. – *Let τ be an optimal strategy of player 2 in $\hat{\Gamma}^*(x,T)$ and let X_t denote the associated integral $X_t := X^*_{x,T,t}(\tau)$. Then the process $\mathcal{R}(X_t,t)$ is a $\Delta(\mathcal{J})$-valued continuous modification of τ and the process X is a solution of the stochastic equation: $X_T = x$,*

$$dX_t = -\sum_{i\in\mathcal{I}} \sqrt{\sigma_{0,i}}a_{i\mathcal{R}(X_t,t)}dW_{i,t}.$$

4.4. The equivalence between the unbounded and the bounded Brownian Games. We first will prove in this section the existence of optimal strategies for player 2 in the unbounded primal game. This will allow us to conclude to the equivalence between unbounded and bounded games and theorems 2.3.4, 2.3.5, 2.3.6, 2.3.7 and 2.3.8 will then follow from the corresponding results in terms of the unbounded games.

THEOREM 4.4.1. – $\forall p \in \Delta(\mathcal{K})$, *player 2 has an optimal* $\Delta(\mathcal{J})$*-valued strategy in* $\hat{\Gamma}(p, T)$.

Proof. – As it results from lemma 3.5.1, $\hat{\psi}(\bullet, T)$ is continuous and concave on $\Delta(\mathcal{K})$. Therefore, for all point p in the relative interior of $\Delta(\mathcal{K})$, $\partial\hat{\psi}(p, T)$ is not empty and the existence of optimal $\Delta(\mathcal{J})$-valued strategies for player 2 follows then from corollary 3.5.4.

For p in the boundary of $\Delta(\mathcal{K})$, we may drop the irrelevant states of nature k for which $p_k = 0$ and consider p as a probability distribution p' on the set $\mathcal{K}' := \{k \in \mathcal{K} : p_k > 0\}$. The game $\hat{\Gamma}(p, T)$ may then be identified with game $\hat{\Gamma}(p', T)$, with $\mathcal{K}'$ as state of nature set. If $\mathcal{K}'$ contains more than one element, p' belongs to the relative interior of $\Delta(\mathcal{K}')$ and the above reasoning leads to the existence of optimal $\Delta(\mathcal{J})$-valued strategies for player 2 in $\hat{\Gamma}(p', T)$ and hence in $\hat{\Gamma}(p, T)$.

So, it just remains for us to deal with the case where $\mathcal{K}'$ reduces to a single element. p is then an extreme point of the simplex $\Delta(\mathcal{K})$ and the strategy 0 guarantees a payoff 0 to player 1, as observed in the proof of theorem 3.5.1. On the other hand, if b is a strategy of player 1 in $\hat{\Gamma}(p, T)$, the $\mathbb{R}_+^K$-valued martingale $Y_{T,s}(p, b)$ must remain a.s. in the half line $\mathbb{R}_+ \cdot p$, since $p = E[Y_{T,1}(p, b)]$ is an extreme point of $\Delta(\mathcal{K})$. As a consequence of relation (3), the constant strategy $\tau_t := \rho(p)$ also guarantees a zero payoff to player 2 in $\hat{\Gamma}(p, T)$, since $X_{t,1}(\tau)$ is then valued in the hyperplane orthogonal to p and hence a.s. $\langle Y_{T,1}(p, b) | X_{T,1}(\tau) \rangle = 0$.

It results then that $\hat{\psi}(p, T) = 0$ and the constant strategy τ is optimal for player 2. The theorem is then proved. □

THEOREM 4.4.2. – *The values of the bounded and unbounded Brownian games are equal and the set of player 2's optimal strategies coincide in both games. Similarly, the sets of* $\mathcal{G}_1$*-measurable optimal strategies of player 1 in both game coincide, where* $\mathcal{G}_1$ *is the* σ*-algebra defined before theorem 3.1.1.*

Proof. – We will prove this result in terms of the primal games, but a similar argument holds in the dual framework.

Since player 1's optimal strategies in strategy $\hat{\Gamma}(p,T)$ are equalizing and are still strategies in $\Gamma(p,T)$, they obviously still guarantee $\hat{\psi}(p,T)$ in $\Gamma(p,T)$.

On the other hand, since the optimal strategies τ of player 2 in $\hat{\Gamma}(p,T)$ are $\Delta(\mathcal{J})$-valued, they are still strategies in $\Gamma(p,T)$. Furthermore, any $L^{\alpha}(\mathcal{G}_1)$ strategy of player 1 in $\Gamma(p,T)$ is the limit in L^{α} of a sequence of strategies of player 1 in $\hat{\Gamma}(p,T)$. Since the payoff function $g_T(\bullet,\tau)$ is continuous on L^{α} when τ is $\Delta(\mathcal{J})$-valued, we conclude that τ still guarantees to player 2 a payoff less than $\hat{\psi}(p,T)$ in $\Gamma(p,T)$.

As a consequence, the game $\Gamma(p,T)$ has a value $\psi(p,T)$ which is exactly $\hat{\psi}(p,T)$ and optimal strategies in $\hat{\Gamma}(p,T)$ are still optimal in $\Gamma(p,T)$.

Since player 1 has less strategies in $\hat{\Gamma}(p,T)$ than in $\Gamma(p,T)$, any optimal strategies of player 2 in $\Gamma(p,T)$ is then also optimal in $\hat{\Gamma}(p,T)$ and we have thus proved that the spaces of optimal strategies for player 2 in $\Gamma(p,T)$ and in $\hat{\Gamma}(p,T)$ coincide.

Let now Y be a $\mathcal{G}_1$-measurable optimal strategy of player 1 in $\Gamma(p,T)$. According to reasoning made to get the behavioral form of $\Gamma(p,T)$ (see definition 3.1.4), Y may be written as $Y_{T,1}(p,b)$ for an $\mathcal{A}$-valued process b in $\cup_{\alpha>1}M_2^{\alpha}$.

Let next S denote the range of the mapping $\rho(\bullet)$ on $\Delta(\mathcal{K})$. Due to the continuity of $\rho(\bullet)$, S is a compact subset of $\Delta(\mathcal{J})$. Moreover, S is included in the relative interior of $\Delta(\mathcal{J})$, since $\rho(p)$ is completely mixed, as stated in lemma 2.1.3. Let then κ denote the distance in between S and the boundary of $\Delta(\mathcal{J})$. Let τ be an optimal strategy of player 2 in $\hat{\Gamma}(p,T)$. According to theorem 4.3.5, τ is S-valued. Therefore, if δ is a process valued in a ball of radius κ of $\mathcal{D}$, $\tau+\delta$ and $\tau-\delta$ are strategies of player 2 in $\Gamma(p,T)$. Since Y and τ are optimal, we infer that

$$\psi(p,T) \leq g_T(Y,\tau\pm\delta) = \psi(p,T) \pm g_T(Y,\delta).$$

As a consequence $g_T(Y,\delta)=0$. This indicates that the process b is $\mathcal{E}$-valued and, since a.s. $Y(p,b)\in I\!R_+^K$, we conclude that $b\in\cup_{\alpha>1}V_p^{\alpha}$. Since theorem 3.3.6 hold for the the particular α chosen in remark 3.3.7, we conclude that b belongs to M_2^{α} and is thus an equalizing strategy of player 1 in $\hat{\Gamma}(p,T)$ that guarantees him a payoff $\psi(p,T)$. b is therefore optimal in $\hat{\Gamma}(p,T)$.

COROLLARY 4.4.3. – *Theorems 2.3.4, 2.3.5, 2.3.6, 2.3.7 and 2.3.8 hold.*

Proof. – Theorem 2.3.4. follows from the last theorem in what concerns the existence of a value and of optimal strategies. That these values are independent on the filtration and on the Brownian motion on which the games are defined is proved in corollary 3.1.2.

Theorem 2.3.5 follows from lemma 3.5.1, theorem 3.5.3 and lemma 3.7.1, since $\psi = \hat{\psi}$ and $\psi^* = \hat{\psi}^*$, as stated theorem 4.4.1.

From theorem 4.4.1, we infer that all $\mathcal{G}_1$-measurable optimal strategies Y in $\Gamma(p,T)$ is optimal in $\hat{\Gamma}(p,T)$ and is thus in V_p^α. Theorem 3.3.2 indicates that the bounds in theorem 2.3.6 hold for all $\mathcal{G}_1$-measurable optimal strategies Y in $\Gamma(p,T)$. This is a slightly stronger result than theorem 2.3.6 that only states the existence of optimal strategies fulfilling these bounds.

Theorem 4.1.1 and corollary 4.1.2 imply theorem 2.3.7 and theorem 2.3.8 is a joint consequence of corollary 4.3.4, theorem 4.3.5 and theorem 4.4.1. □

4.5. Conclusion This first analysis of the Brownian games is still incomplete and the convergence of the finitely repeated games to the Brownian games remains to be proved: the link between both models presented in section 2.2 is purely heuristic.

To conclude this paper, we just want to stress that the function $\psi^*(\bullet,0)$ is a kind of "weak" solution of the PDE (4). Indeed, would this function be $\mathcal{C}^2$, it would be a classical solution of this equation:

Indeed, if τ is an optimal strategy of player 2 in $\Gamma^*(x,0)$, and if X denotes the process $X_t := X^*_{x,0,t}(\tau)$, then we get with Itô's formula joint to relation (26):

$$\begin{aligned}\psi^*(x,0) &= \psi^*(X_T,T)\\ &= \psi^*(x,0) - \int_0^T \sum_{i\in\mathcal{I}} \langle \nabla\psi^*(X_t,t) | a_{i\tau_t^*}\rangle \sqrt{\sigma_{0,i}}dW_{i,t}\\ &\quad + \int_0^T \Big[\frac{\partial\psi^*}{\partial t}(X_t,t) + \frac{1}{2}\sum \sigma_{0,i} a^{\top}_{i\tau_t^*}\psi^{*\prime\prime}(X_t,t)a_{i\tau_t^*}\Big]dt.\end{aligned}$$

According to relations (3) and joint to the definition of $\mathcal{R}$ and theorem 2.3.8, the first integral collapses in the last equation. Therefore, for almost every trajectory of X, for all T,

$$\int_0^T \left[\frac{\partial\psi^*(X_t,t)}{\partial t} + \frac{1}{2}\sum \sigma_{0,i} a^{\top}_{i\tau_t^*}\psi^{*\prime\prime}(X_t,t)a_{i\tau_t^*}\right]dt = 0.$$

Due to the continuity of X_t, this means

$$\frac{\partial\psi^*}{\partial t}(x,0) + \frac{1}{2}\sum \sigma_{0,i} a^{\top}_{i\rho(\nabla\psi^*(x,0))}\psi^{*\prime\prime}(x,0)a_{i\rho(\nabla\psi^*(x,0))} = 0,$$

and with relation (21), we conclude as announced that $\psi^*(\bullet,0)$ solves the PDE (4).

To prove the validity of the expansion (1) for v_n it is then sufficient to establish the regularity of $\psi^*(\bullet, 0)$, as it follows from [5].

This regularity as well as the uniqueness of player 2's optimal strategies is proved in a forthcoming paper [7] under a strict ellipticity condition.

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