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Rates of convergence to the local time of a diffusion

by

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ABSTRACT. – In this paper we consider the approximation of the local time L_t of a 1-dimensional diffusion process X at some level x , say $x = 0$ by normalized sums, say U_t^n , of functions of the values $X_{\frac{i}{n}}$ for $i \leq nt$ as $n \rightarrow \infty$. Our main aim is to prove an associated functional central limit theorem giving a mixed normal limiting value to the sequence of processes $n^\alpha(U_t^n - L_t)$, for a suitable value of α . © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous considérons l'approximation du temps local L_t d'un processus de diffusion uni-dimensionnel au niveau $x = 0$ par des sommes normalisées U_t^n de fonctions des valeurs $X_{\frac{i}{n}}$ pour $i \leq nt$ quand $n \rightarrow \infty$. Notre objectif principal est de montrer un théorème central limite fonctionnel, donnant la convergence de la suite $n^\alpha(U_t^n - L_t)$ vers une limite qui est un mélange de processus gaussien, pour une valeur convenable de α . © Elsevier, Paris

1. INTRODUCTION AND MAIN RESULTS

1-1) It is well known that one can approximate the local time of a Brownian motion, and more generally of continuous semimartingales, in many ways by some sorts of discretizations: one may either discretize “in space”, that is use the random times at which the process hits a grid of mesh $1/n$, say (this includes counting the number of upcrossings from 0 to $1/n$), or one

may discretize “in time”, that is use the values of the process at times i/n , and in both cases let n go to ∞ .

When the basic process is the Brownian motion, space-discretization approximations together with their rates are known. Rates for time-discretization seem to be unknown except in some special cases (see below), and a fortiori no rates are known when the basic process is a diffusion process: finding these rates for time-discretizations, in the form of a central limit theorem, is the main aim of this paper.

Let us introduce our basic assumptions. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ on which is defined a continuous adapted 1-dimensional process X of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma(X_s) dW_s, \quad (1.1)$$

where W is a standard Brownian motion. The assumptions on b, σ are such:

HYPOTHESIS A. — σ is a continuously differentiable positive (non vanishing) function on $\mathbb{R}$, such that the equation $dY_t = \sigma(Y_t) dW_t$ with $Y_0 = X_0$ has a (necessarily unique and strong) non-exploding solution. Further, the process b is such that the laws of X and of Y are locally equivalent. $\square$

Next, we denote by L the local time of the process X at level 0, given by

$$L_t = |X_t| - |X_0| - \int_0^t \text{sign}(X_s) dX_s. \quad (1.2)$$

Next we describe the processes which approximate L . The simplest way is to count how many times $X_{i/n}$ is close enough to 0, that is to consider the processes

$$t \rightsquigarrow \sum_{i=1}^{[nt]} 1_{\{|X_{(i-1)/n}| \leq 1/u_n\}} \quad (1.3)$$

for a sequence u_n of positive numbers going to infinity (here, $[y]$ denotes the integer part of $y \geq 0$; we take $(i-1)/n$ instead of i/n for coherence with further notation). These processes will converge in probability, after normalization, to L . A bit more generally we can consider the processes

$$V(u_n, g)_t^n = \sum_{i=1}^{[nt]} g\left(u_n X_{\frac{i-1}{n}}\right) \quad (1.4)$$

for a function g which “goes to 0 fast enough” at infinity: clearly (1.3) is $V(u_n, g)_t^n$ for $g(x) = 1_{[-1,1]}$. Even more general, and of interest for statistical applications, are the processes

$$U(u_n, h)_t^n = \sum_{i=1}^{[nt]} h\left(u_n X_{\frac{i-1}{n}}, \sqrt{n}\left(X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right)\right). \quad (1.5)$$

It is known that under suitable assumptions on g and assumptions slightly stronger than (A) on the coefficients b, σ , the processes $\frac{1}{\sqrt{n}}U(\sqrt{n}, g)_t^n$ converge in probability to cL for some constant c depending on g and on the function σ . Further when $b = 0$ and $\sigma = 1$ (i.e. X is a standard Brownian motion), the normalized differences $n^{1/4}(\frac{1}{\sqrt{n}}U(\sqrt{n}, g)_t^n - cL_t)$ converge in law to $c'W'_{L_t}$ where W' is another Wiener process, independent of X : for example, Azaïs [3] has shown that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} 1_{\{X_{(i-1)/n} X_{i/n} < 0\}} \quad (1.6)$$

converges in probability (and also in $\mathbb{L}^2$) to $\frac{1}{\sigma(0)}\sqrt{\frac{2}{\pi}}L_t$. (1.6) counts the number of “crossings” of the level 0 for the discrete-time process, and it is equal to $\frac{1}{\sqrt{n}}U(\sqrt{n}, h)_t^n$ for $h(x, y) = 1_{\{x(x+y) < 0\}}$. The associated central limit theorem mentioned above has been shown by Borodin ([4],[5]) in a more general context, where $X_{i/n}$ is replaced by $\frac{1}{\sqrt{n}}(Z_1 + \dots + Z_i)$ and (Z_i) are centered i.i.d. variables (and in [5] many other related results are exhibited).

Here we will first show that under quite general conditions, the processes $\frac{u_n}{n}U(u_n, h)_t^n$ converge in probability to the process cL , where c is a number defined below (depending only on h and $\sigma(0)$), as soon as $u_n/n \rightarrow 0$ and $u_n \rightarrow \infty$. Some related results have been proved by Florens-Zmirou [6].

Next, for the rates of convergence, we restrict to the case where

$$u_n = n^\alpha \quad \text{for some } \alpha \in (0, 1). \quad (1.7)$$

Then we prove that if $\delta = ((1 - \alpha) \wedge \alpha)/2$, the processes $n^\delta(\frac{1}{n^{1-\alpha}}U(n^\alpha, h)_t^n - cL)$ converge in law to a non-trivial limit, provided $\alpha > 1/3$, and we do not know what happens when $\alpha \leq 1/3$. When $\alpha = 1/2$ this essentially reproves the results of Borodin (relative to the Brownian motion), with a different method allowing processes of the form (1.1). Observe that the rate of convergence n^δ is biggest when $\alpha = 1/2$, in which case it is $n^{1/4}$: this is a bit surprising at first glance, but may be

interpreted as such: if $\alpha > 1/2$, then $U(n^\alpha, h)^n$ is of “order of magnitude” $n^{1-\alpha}$, that is $n^{1-\alpha}$ is the “number” of nonnegligible terms in (1.5), and it is only natural that the normalizing factor in the associated limit theorem be $n^{(1-\alpha)/2}$. When $\alpha < 1/2$, we still have $n^{1-\alpha}$ nonnegligible terms, but most of them concern values of $X_{(i-1)/n}$ which are too far away from 0 to give an appropriate information about the local time at level 0 (see Remark 2) after Theorem 1-2 for more explanations on this phenomenon).

1-2) Now we proceed to state the main results. Let us begin with some notation. If g is a Borel function on $\mathbb{R}$ and $\gamma \geq 0$ we set

$$\beta_\gamma(g) = \int |x|^\gamma |g(x)| dx \quad \lambda(g) = \int g(x) dx. \quad (1.8)$$

Let ρ denote the density of the standard normal law $\mathcal{N}(0, 1)$. For small enough Borel functions h on $\mathbb{R}^2$, set

$$H_h(x) = \int h(x, y) \rho(y) dy. \quad (1.9)$$

We will assume that h satisfies the following with some $\gamma \geq 0$:

HYPOTHESIS B- γ . — *the function h on $\mathbb{R}^2$ is Borel and satisfies $h(x, v) \leq \hat{h}(x)e^{a|v|}$, where $a \in \mathbb{R}$ and $\hat{h}$ is bounded with $\beta_\gamma(\hat{h}) < \infty$. $\square$*

Finally, associate with any $u > 0$ and any function h on $\mathbb{R}^2$ the function

$$h_u(x, y) = h(ux, uy), \quad (1.10)$$

and consider the following condition on the sequence (u_n) :

$$\frac{u_n}{n} \rightarrow 0, \quad u_n \rightarrow \infty. \quad (1.11)$$

Recall also that a sequence $(Z^n)_{n \geq 1}$ of processes is said to *converge locally uniformly in time, in probability* to a limiting process Z if for any $t \in \mathbb{R}_+$ the sequence $\sup_{s \leq t} |Z_s^n - Z_s|$ goes to 0 in probability.

THEOREM 1.1. — *Assume (A) and (1.11), and let h satisfy (B-0). The processes $\frac{u_n}{n} U(u_n, h)^n$ converge locally uniformly in time, in probability, to $\frac{1}{\sigma(0)} \lambda(H_{h_{\sigma(0)}}) L$.*

For example $h(x, y) = 1_{\{x(x+y) < 0\}}$ satisfies (B- γ) for all γ with $a = 1$ and $\hat{h}(x) = e^{-|x|}$, and $\lambda(H_{h_{\sigma(0)}}) = \sqrt{\frac{2}{\pi}}$. So we recover Azaïs' result in a slightly more general situation.

Remark. – When u_n/n does not go to 0, the theorem cannot be valid in general, since the first summand in $V(u_n, g)_t^n$ is $g(u_n x)$ if $X_0 = x$, and this quantity does not go to 0 in general.

When u_n does not go to infinity, the result is also wrong: for example if $u_n = u$ is a constant, by Riemann approximation $\frac{1}{n}V(u, g)_t^n$ converges to $\int_0^t g(X_s)ds$ as soon as g is continuous. $\square$

1-3) Let us turn now to the rates of convergence. For this we need first to recall some facts about *stable convergence*. Let Y_n be a sequence of random variables with values in a Polish space E , all defined on the same probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub- σ -field of $\mathcal{F}$. We say that Y_n *converges $\mathcal{G}$ -stably in law* to Y , if Y is an E -valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the original space and if

$$\lim_n E(Uf(Y_n)) = \tilde{E}(Uf(Y)) \quad (1.12)$$

for every bounded continuous $f : E \rightarrow \mathbb{R}$ and all bounded $\mathcal{G}$ -measurable random variables U (and then (1.12) holds for all integrable U). This convergence was introduced by Renyi [10] and studied by Aldous and Eagleson [1], see also [8]. It is obviously stronger than the convergence in law, and below it will be applied to càdlàg processes Y^n with E being the space of càdlàg functions endowed with the Skorokhod topology.

For the *conditional Gaussian martingales*, we refer to [9]: denote by $(\mathcal{G}_t)$ the filtration generated by the process X , and $\mathcal{G} = \bigvee \mathcal{G}_t$. A (possibly multidimensional) process Y defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{G}}, (\tilde{\mathcal{G}}_t), \tilde{P})$ of the original filtered space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ is a $\mathcal{G}$ -conditional Gaussian continuous martingale with bracket C if the process C is adapted to the filtration $(\mathcal{G}_t)$ and if Y is a continuous (necessarily Gaussian) martingale with bracket C , for a regular version of the conditional probability knowing $\mathcal{G}$.

Next, we need a whole set of new notation. If f and g are functions on $\mathbb{R}^2$ and $\mathbb{R}$ respectively, we put

$$\bar{H}_{f,g}(x) = \int f(x, y)g(x + y)\rho(y)dy. \quad (1.13)$$

Next, set

$$\hat{g}(x) = \int \rho(y)(|x + y| - |x|)dy,$$

$$\text{which has: } \beta_\alpha(\hat{g}) < \infty \quad \forall \alpha > 0, \quad \lambda(\hat{g}) = 1 \quad (1.14)$$

by a simple calculation.

Next, $(P_t)_{t \geq 0}$ denotes the Brownian semi-group, given by $P_t k(x) = \int k(x + y\sqrt{t})\rho(y)dy$. The following two estimates, where k denotes a Lebesgue-integrable function, are well known (and for the convenience of the reader they will be reproved, along with other related estimates, in Lemma 3-1 below):

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t}| \leq \begin{cases} K\lambda(|k|)/\sqrt{t} \\ Kt^{-3/2}(\beta_1(k) + \beta_2(k)|x|), \end{cases} \quad (1.15)$$

where K denotes a universal constant. Hence if $\beta_1(k) < \infty$ and $\beta_2(k) < \infty$ and $\int k(x)dx = 0$, the series

$$F(k)(x) = \sum_{j \in \mathbb{N}} P_j k(x) \quad (1.16)$$

is absolutely convergent and $|F(k)(x)| \leq K(\beta_1(k) + \beta_2(k)|x|)$. If f has (B-2) the function $k = H_f - \lambda(H_f)\hat{g}$ satisfies the above-mentioned conditions, hence $F(k)$ exists; further, if $\bar{\rho}$ denotes the law of the pair (B_1, ℓ_1) on $\mathbb{R} \times \mathbb{R}_+$, where B is a standard Brownian motion starting at 0, with its local time ℓ at level 0, the following expression is well defined:

$$\begin{aligned} \delta(f) &= \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{1}{r} - 1} dr \int \bar{\rho}(dx, dy) y F(H_f - \lambda(H_f)\hat{g})(x\sqrt{1-r}) \\ &\quad + \int_0^1 \sqrt{\frac{1}{r} - 1} dr \int \rho(z) |z| dz \int \bar{\rho}(dx, dy) y f(z\sqrt{r}, x\sqrt{1-r} - z\sqrt{r}). \end{aligned} \quad (1.17)$$

When f and f' have (B-2), we set

$$\begin{aligned} \eta(f, f') &= \lambda\left(H_{ff'} + \bar{H}_{f, F(H_{f'} - \lambda(H_{f'})\hat{g})} + \bar{H}_{f', F(H_f - \lambda(H_f)\hat{g})}\right) \\ &\quad + \frac{8}{3\sqrt{2\pi}} \lambda(H_f)\lambda(H_{f'}) - \lambda(H_f)\delta(f') - \lambda(H_{f'})\delta(f). \end{aligned} \quad (1.18)$$

Using (1.15) again, if $\beta_2(k) < \infty$ and $\lambda(|k|) < \infty$, the integral

$$G(k)(x) = \int_0^\infty \left(P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t} \right) dt \quad (1.19)$$

absolutely converges and $|G(k)(x)| \leq K_k(1 + |x|)$. So if f and f' have (B-2), we can set

$$\begin{aligned} \eta'(f, f') &= \lambda(H_f G(H_{f'}) + H_{f'} G(H_f)) \\ &\quad - \lambda(H_f)G(H_{f'})(0) - \lambda(H_{f'})G(H_f)(0). \end{aligned} \quad (1.20)$$

Clearly $\eta(\cdot, \cdot)$ and $\eta'(\cdot, \cdot)$ are bilinear. That $\eta(f, f) \geq 0$ and $\eta'(f, f) \geq 0$ is not obvious from (1.18) and (1.20), but it follows from the fact that these quantities are limits of nonnegative numbers.

Below we state the results for a d -dimensional function $h = (h^i)_{1 \leq i \leq d}$ on $\mathbb{R}^2$, so the processes $U(u_n, h)^n$ are d -dimensional, as well as H_h and $\lambda(H_h)$.

THEOREM 1.2. – Assume (A), let $h = (h^i)_{1 \leq i \leq d}$ be a d -dimensional function satisfying (B-r) for some $r > 3$, and set $\delta = ((1 - \alpha) \wedge \alpha)/2$. Under either one of the following:

- (i) the function σ is a constant (so X is a Brownian motion plus a random drift),
- (ii) the function h is differentiable in the first variable, with a partial derivative satisfying (B-1),

the processes $n^\delta (\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - \lambda(H_{h_{\sigma(0)}})_{\frac{1}{\sigma(0)}} L)$ converge $\mathcal{G}$ -stably in law to a process $Y = (Y^i)_{1 \leq i \leq d}$, defined on an extension of the space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$, and which is a $\mathcal{G}$ -conditional Gaussian continuous martingale with brackets $\langle Y^i, Y^j \rangle$ as given below, in the following cases:

- a) If $\alpha = 1/2$ (hence $\delta = 1/4$), with

$$\langle Y^i, Y^j \rangle = \frac{1}{\sigma(0)} \eta(h_{\sigma(0)}^i, h_{\sigma(0)}^j) L. \quad (1.21)$$

- b) If $1/2 < \alpha < 1$ (hence $\delta = (1 - \alpha)/2$), with

$$\langle Y^i, Y^j \rangle = \frac{1}{\sigma(0)} \lambda(H_{h_{\sigma(0)}^i} h_{\sigma(0)}^j) L. \quad (1.22)$$

- c) If $1/3 < \alpha < 1/2$ (hence $\delta = \alpha/2$), with

$$\langle Y^i, Y^j \rangle = \frac{1}{\sigma(0)} \eta'(h_{\sigma(0)}^i, h_{\sigma(0)}^j) L. \quad (1.23)$$

Remark 1. – There is another, equivalent, way to characterize the limit Y above, when $\langle Y^i, Y^j \rangle = a_{ij} L$. Namely one can construct on an extension of the space a Wiener process $W = (W^i)_{1 \leq i \leq d}$ having $E(W_t^i W_t^j) = a_{ij} t$ and independent of X , and we set $Y_t = W_{L_t}$. This formulation is closer to the formulation of Borodin [4] when $\alpha = 1/2$. In this case, the expression (1.18) which is used to compute a_{ij} seems quite different from the corresponding expression in [4], but of course the two agree.

Remark 2. – Suppose that we are in the Brownian case, i.e. $b = 0$ and $\sigma = 1$, and also that $h(x, y) = g(x)$. Denote by L^x the local time at level x , and set

$$A_t^n = \frac{u_n}{n} V(u_n, g)_t^n - \int g(x) L_{[nt]/n}^{x/u_n} dx. \quad (1.24)$$

We have

$$\begin{aligned} \frac{u_n}{n} V(u_n, g)_t^n - \lambda(g) L_t &= A_t^n + \int g(x) (L_{[nt]/n}^{x/u_n} - L_{[nt]/n}) dx \\ &+ \lambda(g) (L_{[nt]/n} - L_t). \end{aligned} \quad (1.25)$$

If g is twice differentiable, with g, g', g'' satisfying (B-1), one can prove that when $u_n = n^\alpha$ and $\alpha < 1/2$, then $n^{\alpha/2} A_t^n$ goes to 0 in probability, while because L is Hölder in time with any index smaller than $1/2$ we also have $n^{\alpha/2} (L_{[nt]/n} - L_t) \rightarrow 0$: then the leading term in Statement (c) of Theorem 1-2 is the second term in the right side of (1.25). When $\alpha > 1/2$ on the contrary, this second term is of order $n^{-\alpha/2}$ (because of the Hölder properties of L_t^x in x) and the leading term in Statement (b) is A_t^n . When $\alpha = 1/2$, both the first and the second terms in (1.25) have an influence on the limit in Statement (a). In a sense, it is more natural to look at the processes (1.24) rather than $\frac{u_n}{n} V(u_n, g)^n - \lambda(g) L$. $\square$

When $f(x, y) = g(x)$ the expression for $\delta(f)$ becomes (see the end of Section 6):

$$\begin{aligned} \delta(f) &= \int g(x) \hat{g}(x) dx \\ &+ \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{1}{r} - 1} \, dr \int \bar{\rho}(dx, dy) y (F(g - \lambda(H_g) \hat{g})(x\sqrt{1-r})). \end{aligned} \quad (1.26)$$

If $H_f = \lambda(H_f) \hat{g}$ and $H_{f'} = \lambda(H_{f'}) \hat{g}$, all the terms in (1.18) in which $F(\cdot)$ shows up disappear. In particular if $f(x, y) = \hat{g}(x)$, hence $\lambda(H_f) = 1$, (1.18) becomes

$$\eta(\hat{g}, \hat{g}) = \frac{8}{3\sqrt{2\pi}} - \lambda(\hat{g}^2) = \frac{8}{3\sqrt{\pi}} (\sqrt{2} - 1). \quad (1.27)$$

The remainder of the paper is organised as follows: in Section 2 we show how to reduce the proof to the case where X is a standard Brownian motion. Section 3 is devoted to some preliminary estimates on the semi-group and on the local time of the Brownian motion. In Section 4 we prove Theorem 1-1. For Theorem 1-2, in Section 5 the problem is reduced to a central limit theorem for a suitable martingale (pretty much as one classically proves a central limit theorem for mixing sequences by reduction to a similar result for martingales), and we consider separately the cases $\alpha = 1/2$, $\alpha > 1/2$ and $\alpha < 1/2$ in Sections 6, 7 and 8.

2. REDUCTION TO THE BROWNIAN CASE

In this section, and throughout a number of steps, we show how our results for the process X having (1.1) can be reduced to the case where X is the standard Brownian motion.

2-1) We will use several times the following method for constructing the limiting processes Y and Z with a bracket of the form cL in Theorem 1-2 (c is a nonnegative symmetric $d \times d$ matrix): according to [9], the process Y itself may be taken as the canonical process on the canonical space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t))$ of all continuous $\mathbb{R}^d$ -valued functions on $\mathbb{R}_+$; then $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t))$ is the product of $(\Omega, \mathcal{G}, (\mathcal{G}_t))$ by $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t))$; finally, the measure on the extension is $\tilde{P}(d\omega, d\hat{\omega}) = P(d\omega)Q(\omega, d\hat{\omega})$, where $Q(d\omega, \cdot)$ is the unique measure under which Y is a Gaussian martingale with deterministic bracket $cL(\omega)$ and $Y_0 = 0$ (observe that Q is a transition probability from $(\Omega, \mathcal{G})$ into $(\hat{\Omega}, \hat{\mathcal{F}})$, and also from $(\Omega, \mathcal{G}_t)$ into $(\hat{\Omega}, \hat{\mathcal{F}}_t)$ for all t , because L is $(\mathcal{G}_t)$ -adapted by (1.2)).

2-2) Here we prove that one may assume $\mathcal{F}_t = \mathcal{G}_t$ (recall that $(\mathcal{G}_t)$ is the filtration generated by X).

First, if X satisfies (1.1) and (A) relative to $(\mathcal{F}_t)$, it also satisfies (1.1) and (A) relative to $(\mathcal{G}_t)$, with the same function σ and a drift process b' and a Brownian motion W' adapted to $(\mathcal{G}_t)$: indeed, X is a continuous semimartingale w.r.t. $(\mathcal{F}_t)$, hence w.r.t. $(\mathcal{G}_t)$ as well, with the $(\mathcal{G}_t)$ -canonical decomposition $X = X_0 + B_t + M_t$ (where M is the martingale part), and the quadratic variation of M is $\int_0^\cdot \sigma(X_s)^2 ds$. Thus $W'_t = \int_0^t (1/\sigma(X_s)) dX_s$ is a $(\mathcal{G}_t)$ -Brownian motion. Further, that $B_t = \int_0^t b'_s ds$ for some b'_s follows for example from Girsanov's Theorem and Assumption (A), and clearly the pair (b', σ) satisfies (A) as well.

Next, the local time L as defined by (1.2) is the same for $(\mathcal{F}_t)$ and for $(\mathcal{G}_t)$. The processes $U(u_n, h)^n$ do not depend on the filtration, and Step 2-1 above yields that the limits in Theorem 1-2 depend only on L and on the functions h and σ . Therefore one may always replace $\mathcal{F}_t$ by $\mathcal{G}_t$, or in other words we can and will assume that $\mathcal{F}_t = \mathcal{G}_t$.

2-3) Here we show that we can replace the original space by the “canonical” space.

More precisely let $(\Omega', \mathcal{F}', (\mathcal{F}'_t))$ be the canonical filtered space of all real-valued continuous functions on $\mathbb{R}_+$, endowed with the canonical process X' . Define $\varphi : \Omega \rightarrow \Omega'$ by $X = X' \circ \varphi$, so that the law of X under P is $P' = P \circ \varphi^{-1}$. Then standard arguments on changes of space yield that if X satisfies (1.1) and (A) with (b, σ) the process X' (under P') satisfies

(1.1) and (A) with a b' having $b = b' \circ \varphi$ and the same σ , and if L' is the local time of X' under P' , then $L' \circ \varphi$ is a version of L . Further the process $U'(u_n, h)^n$ associated with X' by (1.5) has $U'(u_n, h)^n = U'(u_n, h)^n \circ \varphi$.

Finally if we define the extension of $(\Omega', \mathcal{F}', (\mathcal{F}_t), P')$ as in Step 2-1 with L' , we observe that $Q(\omega, \cdot) = Q'(\varphi(\omega), \cdot)$. Therefore, with obvious notation, we have $\tilde{E}'(U'f(Y)) = \tilde{E}(U' \circ \varphi f(Y))$, while any random variable on $(\Omega, \mathcal{G})$ is a.s. of the form $U' \circ \varphi$: all these facts imply that if our theorems hold for X' under P' , they also hold for X .

Henceforth, we may assume in the sequel that $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ is the canonical space, endowed with the canonical process X .

2-4) Here prove the following property: if our results hold for a given pair (σ, b) satisfying (A), they hold for any other such pair (σ, b') with the same function σ .

Indeed, denote by P' the law of the solution of (1.1) with b' . The two measures P and P' are equivalent on each σ -field $\mathcal{F}_t$ (recall that by Step 2-3 we are on the canonical space). By (1.2), the process L is also a version of the local time of X under P' . Further if a sequence A_n of $\mathcal{F}_t$ -measurable variables converge in P_x -measure to a limit A (necessarily $\mathcal{F}_t$ -measurable as well), we also have $A_n \rightarrow A$ in P'_x -measure. So our claim is true for Theorem 1-1.

Now we define the extension for P' as in Step 2-1, except that the measure is now $\tilde{P}'(d\omega, d\hat{\omega}) = P'(d\omega)Q(\omega, d\hat{\omega})$. If $E(Uf(Y^n)) \rightarrow \tilde{E}(Uf(Y))$ for all integrable variable U and bounded continuous function f on the space of càdlàg functions, we need to prove the same thing for P' when U is in addition bounded. Since Skorokhod convergence is “local” in time, it is enough to prove it when U is $\mathcal{F}_t$ -measurable and f depends only on the function up to time t , for any finite t . But if Z_t denotes the density of P' w.r.t. P on the σ -field $\mathcal{F}_t$, and since $Q(\cdot, A)$ is $\mathcal{F}_t$ -measurable when $A \in \hat{\mathcal{F}}_t$, we have $E'(Uf(Y^n)) = E(Z_t Uf(Y^n))$, and $\tilde{E}'(Uf(Y)) = \tilde{E}(Z_t Uf(Y))$. Since UZ_t is P_x -integrable when U is bounded, we deduce the result: hence our claim is true also for Theorem 1-2.

2-5) Suppose that our results hold for (1.1), when σ , $1/\sigma$ and σ' are bounded. We prove here, via a well-known localization procedure, that they also hold without the boundedness of σ , $1/\sigma$ and σ' .

In view of 2-4), it is enough to prove this result when $b = 0$. For each $p \geq 1$ choose a continuously differentiable function σ_p which is bounded, as well as $1/\sigma_p$ and σ'_p , and such that $\sigma_p(x) = \sigma(x)$ whenever $|x| \leq p$. Observe that since $b = 0$, Equation (1.1) may be “inverted” to give $W_t = \int_0^t (1/\sigma(X_s)) dX_s$. Now let $X(p)$ be the (strong) solution to (1.1),

w.r.t. the same W . If $T_p = \inf(t : |X_t| \geq p)$, we clearly have $X_t = X(p)_t$ a.s. for all $t \leq T_p$. Hence all the processes showing up in our results coincide a.s. for X and $X(p)$ on the interval $[0, T_p]$. Since $T_p \uparrow \infty$, the claim is thus obvious.

Hence all what precedes shows that it is enough to prove the results when $\mathcal{F}_t = \mathcal{G}_t$ and σ , $1/\sigma$ and σ' are bounded, and X is the solution to the equation

$$dX_t = \frac{1}{2}(\sigma\sigma')(X_t)dt + \sigma(X_t)dW_t. \quad (2.1)$$

2-6) In our last step, we consider Equation (2.1) with σ , σ' and $1/\sigma$ bounded, and we show how to reduce the two theorems to the case where X is a standard Brownian motion starting at a fixed point x .

Consider an arbitrary twice continuously differentiable function $T : \mathbb{R} \rightarrow \mathbb{R}$ such that for some $\varepsilon > 0$:

$$T(0) = 0, \quad T'(0) = 1, \quad \varepsilon \leq T'(x) \leq \frac{1}{\varepsilon}, \quad |T''(x)| \leq \frac{1}{\varepsilon}. \quad (2.2)$$

With any function h on $\mathbb{R}^2$, and with the function T having (2.2) and the sequence (u_n) of positive numbers, we associate the following functions:

$$h_n(x, y) = h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right)\right). \quad (2.3)$$

Next, if $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ is the canonical space endowed with the canonical process X , we denote by P_x the unique measure under which X is a standard Brownian motion starting at x . The following auxiliary result will be proved in Section 4:

PROPOSITION 2.1. — *In the above setting, let h satisfy (B-0) and assume (I.11). Let T satisfy (2.2) and associate h_n with T , u_n and h by (2.3). Then*

- The processes $\frac{u_n}{n}U(u_n, h_n)^n$ tend to $\lambda(H_h)L$ locally uniformly in time in P_x -probability.*
- If h has (B-1), if h is differentiable in the first variable with a partial derivative satisfying (B-1), and if $n/u_n^3 \rightarrow 0$, the processes $\sqrt{\frac{u_n}{n}}U(u_n, h_n - h)^n$ tend to 0 locally uniformly in time in P_x -probability.*

Suppose also that *Theorem 1-2* holds in the above canonical setting, for each measure P_x . We will presently see how to deduce the results in the general case.

First there is a version of the local time L which works under each P_x . Let μ be any probability measure on $\mathbb{R}$, and set $P = \int \mu(dx)P_x$ (so under P , X is a standard Brownian motion starting at X_0 , and the law of X_0 is μ). Since L is the same under each P_x , it is also a version of the local time under P , and it is obvious that Proposition 2-1 hold also for the measure P . Since in Step 2-1 the transition measure Q depends only on ω through $L(\omega)$, if $\tilde{P}_x$ is the extension of P_x , then with the extension of P given by $\tilde{P} = \int \mu(dx)\tilde{P}_x$ it is also obvious that Theorem 1-2 holds for P .

By Step 2-3, Theorem 1-2 and Proposition 2-1 hold also when X is a standard Brownian motion on an arbitrary space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with an arbitrary (possibly random) initial value, provided $(\mathcal{F}_t)$ is the filtration generated by X .

Let us now consider the case of Equation (2.1) with σ , σ' and $1/\sigma$ bounded. Set

$$S(x) = \int_0^x \frac{1}{\sigma(y)} dy. \quad (2.4)$$

This function is of class C^2 , so by Ito's formula, we immediately deduce from (2.1) that the process $X'_t = S(X_t)$ is a standard Brownian motion, starting at the random point $X'_0 = S(X_0)$. From the above, and since $(\mathcal{F}_t)$ is also the filtration generated by X' because S is invertible, Theorem 1-2 and Proposition 2-1 hold for X' . Moreover, we have the

LEMMA 2.2. – *The process $L' = L/\sigma(0)$ is a version of the local time of X' at level 0.*

Proof. – Let X^+ and X^- (resp. X'^+ and X'^-) be the positive and negative parts of X and X' , and let L' be the local time of X' . Not only do we have (1.2), but also

$$X_t^+ = X_0^+ + \frac{1}{2}L_t + \int_0^t 1_{\{X_s > 0\}} dX_s, \quad X_t^- = X_0^- + \frac{1}{2}L_t - \int_0^t 1_{\{X_s < 0\}} dX_s.$$

Now, $\text{sign}(S(x)) = \text{sign}(x)$, hence $X'^+ = S(X^+)$ and $X'^- = -S(-X^-)$, so the property $S'(0) = 1/\sigma(0)$, the fact that L charges only the set where

$X_s = 0$ and Ito's formula yield

$$\begin{aligned} X_t^+ &= X_0^+ + \frac{1}{2\sigma(0)}L_t + \int_0^t S'(X_s^+)1_{\{X_s>0\}}dX_s \\ &\quad + \frac{1}{2}\int_0^t S''(X_s^+)1_{\{X_s>0\}}\sigma(X_s)^2ds, \\ X_t^- &= X_0^- + \frac{1}{2\sigma(0)}L_t \\ &\quad - \int_0^t S'(-X_s^-)1_{\{X_s<0\}}dX_s - \frac{1}{2}\int_0^t S''(-X_s^-)1_{\{X_s<0\}}\sigma(X_s)^2ds. \end{aligned}$$

Adding these two equations yields

$$\begin{aligned} |X_t'| &= |X_0'| + \frac{1}{\sigma(0)}L_t + \int_0^t S'(X_s)\text{sign}(X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t S''(X_s)\text{sign}(X_s)\sigma(X_s)^2ds. \end{aligned}$$

On the other hand, (1.2) applied to X' , together with the fact that $dX_t' = S'(X_t)dX_t + \frac{1}{2}S''(X_t)\sigma(X_t)^2dt$ (by Ito's formula again) and that $\text{sign}(S(x)) = \text{sign}(x)$ again yields

$$|X_t'| = |X_0'| + L_t' + \int_0^t S'(X_s)\text{sign}(X_s)dX_s + \frac{1}{2}\int_0^t S''(X_s)\text{sign}(X_s)\sigma(X_s)^2ds.$$

Comparing the last two equalities gives $L' = L/\sigma(0)$. $\square$

The function S is invertible, and if S^{-1} denotes its inverse we set

$$T(x) = \frac{1}{\sigma(0)}S^{-1}(x). \quad (2.5)$$

This function is twice differentiable and satisfies (2.2). Let h be an $\mathbb{R}^d$ -valued function, with which we associate $h' = h_{\sigma(0)}$ by (1.10), while h'_n is associated with h' (and T as above and a given sequence (u_n)) by (2.3). Denote by $U'(u_n, h)^n$ the process defined by (1.5) with X' in place of X . Since $X = \sigma(0)T(X')$ a simple calculation shows:

$$U(u_n, h)^n = U'(u_n, h'_n)^n. \quad (2.6)$$

If h satisfies (B-r), so does h' . Thus (2.6) and Lemma 2-2 yield that Theorem 1-1 is exactly Proposition 2-1(a) for X' . It is also clear that (2.6),

Lemma 2-2 and Proposition 2-1(b) for X' and Theorem 1-2 for X' yield Theorem 1-2 for X , since if $u_n = n^\alpha$ with $\alpha > 1/3$ implies $n/u_n^3 \rightarrow 0$ and since $\sqrt{\frac{u_n}{n}} \geq n^{\delta+\alpha-1}$ (when σ is a constant, we have $T(x) = x$ and thus Proposition 2-1(b) is trivial without any regularity assumption on h because $h_n = h$).

To summarize, at this point we are left to prove Theorem 1-2 and Proposition 2-1, on the canonical space, with the canonical process X and the Wiener measures P_x .

3. SOME ESTIMATES

We give here some estimates on the semigroup (P_t) of the standard Brownian motion. These are more or less known, but simple to prove. Below, K denotes a constant which may change from line to line; when it depends on an additional parameter u , we write it K_u .

LEMMA 3.1. – *If $t > s > 0$ and $\gamma \geq 0$ we have*

$$|P_t k(x)| \leq K \frac{\lambda(|k|)}{\sqrt{t}}. \quad (3.1)$$

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t}| \leq \frac{K_\gamma}{t} \left(\frac{\beta_1(k)}{1 + |x/\sqrt{t}|^\gamma} + \frac{\beta_{1+\gamma}(k)}{1 + |x|^\gamma} \right), \quad (3.2)$$

$$|P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t}| \leq \frac{K}{t^{3/2}} (\beta_2(k) + \beta_1(k)|x|), \quad (3.3)$$

$$|P_t k(x) - P_t k(y)| \leq K \frac{|x - y|}{t} \lambda(|k|), \quad (3.4)$$

$$|P_t k(x) - P_s k(x)| \leq K \frac{t - s}{s^{3/2}} \lambda(|k|). \quad (3.5)$$

Proof. – The density of the law $\mathcal{N}(0, t)$ is $\rho_t(u) = \frac{1}{\sqrt{t}} \rho(\frac{u}{\sqrt{t}})$. Since $P_t k(x) = \int \rho_t(y) k(x + y) dy$ and ρ is bounded, we have (3.1). Next,

$$i \in \mathbb{N} \Rightarrow |u/\sqrt{t}|^i \rho_t(u) \leq K_i \rho_{2t}(u) \leq K'_i / \sqrt{t}, \quad (3.6)$$

$$\frac{\partial \rho_t(u)}{\partial u} = -\frac{u}{t} \rho_t(u), \quad \frac{\partial \rho_t(u)}{\partial t} = \frac{1}{2t} \left(\frac{u^2}{t} - 1 \right) \rho_t(u). \quad (3.7)$$

Taylor's formula yields, with $\rho'_t(u) = \partial \rho_t(u)/\partial u$:

$$\begin{aligned} P_t k(x) - \frac{\lambda(k)}{\sqrt{2\pi t}} e^{-x^2/2t} &= \int k(u)(\rho_t(x-u) - \rho_t(x)) du \\ &= - \int_0^1 d\theta \int k(u) u \rho'_t(x - \theta u) du. \end{aligned} \quad (3.8)$$

If $|x - \theta u| \geq |x|/2$ (3.6) and (3.7) yield $|\rho'_t(x - \theta u)| \leq \frac{K}{t} e^{-x^2/16t}$, while otherwise $|\theta u| > |x|/2$, hence $|u| > |x|/2$ and $|\rho'_t(x - \theta u)| \leq \frac{K}{t} \leq \frac{K_\gamma}{t} \frac{1+|u|^\gamma}{1+|x|^\gamma}$. Hence for all $x, u \in \mathbb{R}$ and $\theta \in [0, 1]$ we get

$$|\rho'_t(x - \theta u)| \leq \frac{K_\gamma}{t} \left(e^{-x^2/16t} + \frac{1 + |u|^\gamma}{1 + |x|^\gamma} \right).$$

Since furthermore $e^{-x^2/16t} \leq K_\gamma \frac{1}{1+|x/\sqrt{t}|^\gamma}$, (3.2) readily follows from (3.8).

Next, (3.6) and (3.7) yield $|\rho_t(x-u) - \rho_t(x)| \leq \frac{K}{t^{3/2}} |u|(|u| + |x|)$, hence (3.3) follows from (3.8).

Next, we have

$$P_t k(x) - P_t k(y) = \int k(u)(\rho_t(x-u) - \rho_t(y-u)) du.$$

Again (3.6) and (3.7) yield $|\rho_t(x-u) - \rho_t(y-u)| \leq K|x-y|/t$, hence (3.4).

Finally,

$$P_t k(x) - P_s k(x) = \int k(u)(\rho_t(x-u) - \rho_s(x-u)) du.$$

A further application of (3.6) and (3.7) yields $|\rho_t(x-u) - \rho_s(x-u)| \leq K(t-s)/s^{3/2}$, hence (3.5). $\square$

LEMMA 3.2. — If $|k(x)| \leq \frac{1}{1+|x/\delta|^\gamma}$ for some $\delta \geq 1$ and $\gamma > 0$, we have for all t :

$$|P_t k(x)| \leq K_\gamma \frac{1 + t^{\gamma/2}}{1 + |x/\delta|^\gamma}. \quad (3.9)$$

Proof. — Since $|P_t k(x)| \leq \int \frac{1}{1+|(x+y\sqrt{t})/\delta|^\gamma} \rho(y) dy$ and $\frac{1}{1+|(x+y\sqrt{t})/\delta|^\gamma} \leq \frac{K_\gamma(1+|y\sqrt{t}|^\gamma)}{1+|x/\delta|^\gamma}$ by an easy calculation, the result is obvious. $\square$

LEMMA 3.3. – If $\gamma(k, x)_t^n = E_x(\sum_{i=2}^{[nt]} k(\sqrt{n}X_{\frac{i-1}{n}}))$, we have

$$|\gamma(k, x)_t^n| \leq K\lambda(|k|)\sqrt{nt}. \quad (3.10)$$

If furthermore $\lambda(k) = 0$, then

$$\left. \begin{aligned} |\gamma(k, x)_t^n| &\leq K(\beta_2(k) + \beta_1(k)|x|\sqrt{n}) \\ |\gamma(k, x)_t^n| &\leq K\beta_1(k)(1 + \log^+(nt)). \end{aligned} \right\} \quad (3.11)$$

Proof. – We have $E_x(k(\sqrt{n}X_{\frac{i-1}{n}})) = P_{i-1}k(x\sqrt{n})$, hence $\gamma(k, x)_t^n \leq \sum_{i=1}^{[nt]-1} |P_i k(x\sqrt{n})|$. The estimates (3.1), (3.2), (3.3), $\sum_{i=1}^{[nt]} \frac{1}{\sqrt{i}} \leq 2\sqrt{nt}$, $\sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty$ and $\sum_{i=1}^{[nt]} \frac{1}{i} \leq 1 + \log^+(nt)$ give the results. $\square$

We end this section with some simple calculations on the local time L . Put

$$G(f, q, x) = E_x(L_1^q f(X_1)), \quad q \in \mathbb{N}, \quad (3.12)$$

where f satisfies $|f(x)| \leq Ke^{a|x|}$ and $q > 0$. First, according to Revuz and Yor [11], if $L(a)$ denotes the local time of X at level a , under P_0 the processes $(X_t, L(a)_t)$ and $(\frac{1}{c}X_{tc^2}, \frac{1}{c}L(ac)_{tc^2})$ have the same law. Hence

$$\begin{aligned} n^{q/2} E_{x/\sqrt{n}}(L_{1/\sqrt{n}}^q f(\sqrt{n}X_{1/n})) \\ = n^{q/2} E_0(L(-x/\sqrt{n})_{1/\sqrt{n}}^q f(x + \sqrt{n}X_{1/n})) \\ = E_0(L(-x)_1^q f(x + X_1)) = G(f, q, x). \end{aligned} \quad (3.13)$$

Similarly, for $t > 0$:

$$E_0(L_t^q f(X_t)) = t^{q/2} E_0(L_1^q f(X_1/\sqrt{t})). \quad (3.14)$$

Moreover, one knows (see e.g. [11]) that L_1 under P_0 has the same law than $|X_1|$, hence

$$E_0(L_1) = \sqrt{\frac{2}{\pi}}, \quad E_0(L_1^2) = 1. \quad (3.15)$$

The hitting time T_x of $\{x\}$ for X , under P_0 , has the density $r \rightsquigarrow \frac{|x|}{\sqrt{2\pi}r^{3/2}} e^{-x^2/2r} 1_{\mathbb{R}_+}(r)$. Then (3.13) and the Markov property yield

$$G(f, q, x) = \begin{cases} E_0(L_1^q f(X_1)) & \text{if } x = 0 \\ \int_0^1 \frac{|x|}{\sqrt{2\pi}r^{3/2}} e^{-\frac{x^2}{2r}} (1-r)^{q/2} & \text{if } x \neq 0. \\ \quad \times E_0(L_1^q f(X_1\sqrt{1-r})) dr & \end{cases} \quad (3.16)$$

4. THEOREM 1-1

4-1) Recall once more that we are on the canonical space, with the canonical process X and the Wiener measure P_x . We first prove a version of Theorem 1-1 in the case $u_n = \sqrt{n}$, and for the processes (1.4). In this setting, this version is indeed more general than Theorem 1-1, and has interest on its own.

THEOREM 4.1. – *a) If g_n is a sequence of functions on $\mathbb{R}$ satisfying for all $x \in \mathbb{R}$, as $n \rightarrow \infty$:*

$$\lambda(|g_n|) \rightarrow 0, \quad \frac{g_n(x\sqrt{n})}{\sqrt{n}} \rightarrow 0, \quad (4.1)$$

then the processes $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)^n$ converge locally uniformly in time, in $\mathbb{L}^1(P_x)$, to 0.

b) Let g_n be a sequence of functions on $\mathbb{R}$ satisfying for all $x \in \mathbb{R}$, as $n \rightarrow \infty$:

$$\frac{g_n(x\sqrt{n})^2}{n} + \frac{\lambda(g_n^2)}{\sqrt{n}} + \frac{\beta_1(g_n)|g_n(x\sqrt{n})|\log n}{n} + \frac{\beta_1(g_n)\lambda(|g_n|)\log n}{\sqrt{n}} \rightarrow 0. \quad (4.2)$$

If $\lambda(g_n) \rightarrow \lambda$ we have for all t :

$$\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n \xrightarrow{P_x} \lambda L_t. \quad (4.3)$$

If furthermore $\sup_n \lambda(|g_n|) < \infty$, the processes $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)^n$ converge locally uniformly in time, in P_x -probability, to λL .

Let us begin with a lemma.

LEMMA 4.2. – *If the functions g_n satisfy (4.2) and $\lambda(g_n) = 0$, we have*

$$E_x(|\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n|^2) \rightarrow 0. \quad (4.4)$$

Proof. – With $\delta(g)_t^n = \sup_{y \in \mathbb{R}, s \in [0, t]} |\gamma(g, y)_s^n|$, we can write (use the Markov property):

$$\begin{aligned} & E_x(|V(\sqrt{n}, g)_t^n|^2) \\ &= \sum_{i=1}^{[nt]} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})^2) + 2 \sum_{i \leq i < j \leq [nt]} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})g(\sqrt{n}X_{\frac{j-1}{n}})) \\ &= g^2(\sqrt{n}x) + \gamma(g^2, x)_t^n + 2 \sum_{i=1}^{[nt]-1} E_x(g(\sqrt{n}X_{\frac{i-1}{n}})\gamma(g, X_{\frac{i-1}{n}})_{t-(i-1)/n}^n) \\ &\leq g^2(\sqrt{n}x) + \gamma(g^2, x)_t^n + 2\delta(g)_t^n(|g(\sqrt{n}x)| + \gamma(|g|, x)_{t-1/n}^n). \end{aligned}$$

If $\lambda(g) = 0$, (3.11) yields $\delta(g)_t^n \leq K_t \beta_1(g) \log n$ for $n \geq 2$, hence by (3.10):

$$\begin{aligned} E_x(|V(\sqrt{n}, g)_t^n|^2) \\ \leq K_t(g^2(\sqrt{n}x) + \lambda(g^2)\sqrt{n} + \beta_1(g)(\log n)(|g(\sqrt{n}x)| + \lambda(|g|)\sqrt{n})), \end{aligned}$$

and (4.4) follows. $\square$

Proof of Theorem 4.1. — We have $E_x(\sup_{s \leq t} |V(\sqrt{n}, g_n)_s^n|) \leq |g_n(x\sqrt{n})| + \gamma(|g_n|, x)_t$, so (a) obviously follows from (3.10).

Let us prove (b). The function $\hat{g}$ of (1.14) is $\hat{g}(x) = E_x(|X_1| - |X_0|)$, and by definition of the local time,

$$\begin{aligned} E_x(|X_{\frac{i}{n}}| - |X_{\frac{i-1}{n}}| | \mathcal{F}_{\frac{i-1}{n}}) &= E_x(L_{\frac{i}{n}} - L_{\frac{i-1}{n}} | \mathcal{F}_{\frac{i-1}{n}}) \\ &= \frac{1}{\sqrt{n}} \hat{g}(\sqrt{n} X_{\frac{i-1}{n}}), \end{aligned} \quad (4.5)$$

so by Lemma (2.14) of [7] we have the following convergence for all x :

$$\frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})_t^n \xrightarrow{P_x} L_t. \quad (4.6)$$

Now let g_n be a sequence satisfying (4.2) and $\lambda(g_n) \rightarrow \lambda$. We set $g'_n = g_n - \lambda(g_n)\hat{g}$. (1.14) implies that the sequence g'_n satisfies (4.2) as well and $\lambda(g'_n) = 0$, hence Lemma 4-2 yields $\frac{1}{\sqrt{n}} V(\sqrt{n}, g'_n)_t^n \xrightarrow{P_x} 0$. Since $V(\sqrt{n}, g_n)^n = V(\sqrt{n}, g'_n)^n + \lambda(g_n)V(\sqrt{n}, \hat{g})^n$, (4.3) follows from (4.6).

If finally $\sup_n \lambda(|g_n|) < \infty$, up to taking a subsequence we may assume that $\lambda(|g_n|) \rightarrow \lambda'$. Then $\lambda(g_n^+) \rightarrow b_+ := \frac{\lambda' + \lambda}{2}$ and $\lambda(g_n^-) \rightarrow b_- := \frac{\lambda' - \lambda}{2}$. The processes $\frac{1}{\sqrt{n}} V(\sqrt{n}, g_n^+)^n$ and $\frac{1}{\sqrt{n}} V(\sqrt{n}, g_n^-)^n$ converge in P_x -probability to $b_+ L$ and $b_- L$ for all t , and since they are non-decreasing and with a continuous limit this convergence is locally uniform in time. By difference we deduce the second claim in (b). $\square$

4-2) Proof of Theorem 1-1 (under P_x). For further convenience, we first introduce some new notation and some simple properties. For any function h on $\mathbb{R}^2$ having (B-0), the process

$$M(h)^n = U(u_n, h)^n - V(u_n, H_h)^n \quad (4.7)$$

is a P_x -martingale w.r.t. the filtration $\mathcal{F}_t^n = \mathcal{F}_{[nt]/n}$, with predictable bracket given by

$$\langle M(h)^n, M(h)^n \rangle_t = V(u_n, H_{h^2} - (H_h)^2)^n \leq V(u_n, H_{h^2})^n. \quad (4.8)$$

Observe also that for any function g on $\mathbb{R}$ we have

$$V(u_n, g)_t^n = V(\sqrt{n}, g_n)_t^n \quad \text{with} \quad g_n(x) = g\left(\frac{u_n x}{\sqrt{n}}\right). \quad (4.9)$$

In (4.9) above, we have $\lambda(|g_n|) = \frac{\sqrt{n}}{u_n} \lambda(|g|)$. Hence (3.10) and (4.9) yield the following useful estimate:

$$\begin{aligned} E_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} V(u_n, g)_t^n \right| \right) &\leq E_x \left(\frac{u_n}{n} V(u_n, |g|)_t^n \right) \\ &\leq K \left(\frac{u_n}{n} |g(u_n x)| + \sqrt{t} \lambda(|g|) \right). \end{aligned} \quad (4.10)$$

Now we prove three lemmas, which are indeed too strong for proving Theorem 1-1 but will be useful for Proposition 2-1.

LEMMA 4.3. – *Let h_n be a sequence of functions satisfying $|h_n(x, y)| \leq \hat{h}_n(x) e^{a|y|}$ for some $a \in \mathbb{R}_+$, and such that*

$$\sup_n \lambda(\hat{h}_n) < \infty, \quad \sup_{n,x} \hat{h}_n(x) < \infty. \quad (4.11)$$

Then under (1.11) the processes $\frac{u_n}{n} U(u_n, h_n - H_{h_n})^n$ converge locally uniformly in time, in P_x -probability, to 0.

Proof. – Observe that $H_{h_n} \leq K \hat{h}_n$, so a combination of (4.7) and (4.8) yields that the martingale $M(h_n)^n$ has a bracket smaller than $KV(u_n, \hat{h}_n)^n$ (use the second part of (4.11)). Therefore $E(\langle \frac{u_n}{n} M(h_n)^n, \frac{u_n}{n} M(h_n)^n \rangle_t) \rightarrow 0$ by (4.10) and (4.11) and (1.11), and the result follows from Doob's inequality. $\square$

LEMMA 4.4. – *Let g_n be a sequence of functions satisfying*

$$\sup_n \lambda(|g_n|) < \infty, \quad \sup_{n,x} |g_n(x)| < \infty, \quad (4.12)$$

$$\lim_q \limsup_n \int_{|x|>q} |g_n(x)| dx = 0. \quad (4.13)$$

If further $\lambda(g_n) \rightarrow \alpha$ and u_n satisfies (1.11) and $\frac{\log n}{u_n} \rightarrow 0$, the processes $\frac{u_n}{n} V(u_n, g_n)^n$ converge locally uniformly in time, in P_x -probability, to αL .

Proof. – a) Assume first that $\sup_n \beta_1(g_n) < \infty$. In view of (4.9), and if $k_n(x) = \frac{u_n}{\sqrt{n}} g_n(\frac{u_n x}{\sqrt{n}})$, we need to prove the convergence, locally uniformly in time in P_x -probability, of the sequence $\frac{1}{\sqrt{n}} V(\sqrt{n}, k_n)^n$ to αL .

We readily check that $|k_n| \leq Ku_n/\sqrt{n}$ and $\lambda(|k_n|) \leq K$ and $\beta_1(k_n) \leq K\sqrt{n}/u_n$ and $\lambda(k_n^2) \leq Ku_n/\sqrt{n}$. Hence by (1.11) and $\frac{\log n}{u_n} \rightarrow 0$ the sequence (k_n) satisfies (4.2), while $\lambda(k_n) = \lambda(g_n) \rightarrow \alpha$ by (a): the last statement in Theorem 4-1 gives the result.

b) Let us now consider the general case. Up to taking a subsequence, we can assume that $\int_{\{|x| \leq q\}} g_n(x) ds \rightarrow \alpha_q$ for any $q \in \mathbb{N}$, and (4.13) readily gives that $\alpha_q \rightarrow \alpha$ as $q \rightarrow \infty$.

Set $k_q^n(x) = g_n(x)1_{\{|x| \leq q\}}$. We have $\beta_1(k_q^n) \leq q\lambda(|g_n|) \leq Kq$ by (4.12), so (a) yields that $\frac{u_n}{n}V(u_n, k_q^n)^n$ converges locally uniformly in time in P_x -probability to $\alpha_q L$. Since $\alpha_q \rightarrow \alpha$ it thus remain to show that

$$\limsup_n E_x \left(\sup_{s \leq t} \frac{u_n}{n} |V(u_n, g_n - k_q^n)_s^n| \right) \rightarrow 0 \quad (4.14)$$

as $q \rightarrow \infty$. By (4.10), the expectation above is smaller than $K(\frac{u_n}{n} + \sqrt{t}\lambda(|g_n - k_q^n|))$. Thus the left side of (4.14) is smaller than $K_t \limsup_n \lambda(|g_n - k_q^n|)$ by (1.11), and this quantity goes to 0 as $q \rightarrow \infty$ by (4.13): hence (4.14) holds. $\square$

LEMMA 4.5. – *Let g be a bounded and integrable function of $\mathbb{R}$, and let T be a function satisfying (2.2), and set $g_n(x) = g(u_n T(x/u_n))$. If u_n satisfies (1.11) and $u_n^2/n \rightarrow 0$ the processes $\frac{u_n}{n}V(u_n, g_n)^n$ converge locally uniformly in time in P_x -probability to $\lambda(g)L$.*

Proof. – By the same argument as in the proof of Theorem 4-1, it suffices to prove the result when $g \geq 0$, in which case it is enough to have the convergence for each time t .

We can write

$$\frac{u_n}{n}V(u_n, g_n)_t^n = \int g_n(x) L_{[nt]/n}^{x/u_n} dx - B_t^n - \alpha_t^n, \quad (4.15)$$

where (by the occupation time formula):

$$\alpha_t^n = \frac{u_n}{n} \int_0^{nt-[nt]} (g_n(u_n X_{\frac{[nt]}{n} + \frac{s}{n}}) - g_n(u_n X_{\frac{[nt]}{n}})) ds,$$

$$B_t^n = \sum_{i=1}^{[nt]} \beta_i^n, \quad \beta_i^n = \frac{u_n}{n} \int_0^1 (g_n(u_n X_{\frac{i-1}{n} + \frac{s}{n}}) - g_n(u_n X_{\frac{i-1}{n}})) ds.$$

Since g is bounded, $\alpha_t^n \rightarrow 0$ follows from (1.11). Next, the variables $L_{[nt]/n}^{x/u_n} - L_t$ tend to 0 and remain smaller (for all x and n) than a fixed

finite variable U . Therefore

$$\left| \int g_n(x) (L_{[nt]/n}^{x/u_n} - L_t) dx \right| \leq K \int_{\{|x| \leq q\}} |L_{[nt]/n}^{x/u_n} - L_t| dx + U \int_{\{|x| > q\}} \left| g \left(u_n T \left(\frac{x}{u_n} \right) \right) \right| dx.$$

The first term in the right side above goes to 0 as $n \rightarrow \infty$ for all q , and the second term is smaller than $U \int_{\{|x| > q/\varepsilon\}} |g(x)| dx$, which goes to 0 as $q \rightarrow \infty$: thus

$$\int g_n(x) L_{[nt]/n}^{x/u_n} dx - \lambda(g_n) L_t \rightarrow 0.$$

Furthermore $\lambda(g_n) = \int g(x) \bar{T}'(\frac{x}{u_n}) dx$, where $\bar{T}$ is the inverse function of T . Since $\bar{T}'(x/u_n)$ tends to 1 and remains bounded by (2.2), it follows that $\lambda(g_n) \rightarrow \lambda(g)$. Therefore the first term of the right side of (4.15) goes (for all ω) to $\lambda(g) L_t$.

It remains to prove that $B_t^n \xrightarrow{P_x} 0$. Let $C_t^n = \sum_{i=1}^{[nt]} \gamma_i^n$, where $\gamma_i^n = E_x(\beta_i^n | \mathcal{F}_{\frac{i-1}{n}})$. We have

$$E_x(|B_t^n - C_t^n|^2) = E_x \left(\sum_{i=1}^{[nt]} (\beta_i^n - \gamma_i^n)^2 \right) \leq 2E_x \left(\sum_{i=1}^{[nt]} |\beta_i^n|^2 \right), \quad (4.16)$$

and $|\beta_i^n| \leq K u_n^2/n^2$. The above sum is thus smaller than $K t u_n^2/n$, which goes to 0 by hypothesis: so it remains to prove that

$$E_x \left(\sum_{i=1}^{[nt]} |\gamma_i^n| \right) \rightarrow 0.$$

Set $v_n = u_n^2/n$. A simple calculation shows that $\gamma_i^n = \frac{u_n}{n} h_n(u_n X_{\frac{i-1}{n}})$, where $h_n = \int_0^1 (P_{sv_n} g_n - g_n) ds$. Since $|g_n| \leq K$, by (4.10) yields that the left side of (4.16) is smaller than $K(\frac{u_n}{n} + \sqrt{t} \lambda(|f_n|))$, and we are left to prove that $\lambda(|f_n|) \rightarrow 0$.

We can find for each integer p a function k_p which is Lipschitz with compact support and such that $\lambda(|g - k_p|) \leq 1/p$. The function $k_p^n(x) = k_p(u_n T(x/u_n))$ has $\lambda(|g_n - k_p^n|) \leq 1/p\varepsilon$. Since the action of the kernel P_t is a convolution, we also have $\lambda(|P_{sv_n} g_n - P_{sv_n} k_p^n|) \leq 1/p\varepsilon$. Therefore

$$\lambda(|f_n|) \leq \frac{2}{p\varepsilon} + \int_0^1 \lambda(|P_{sv_n} k_p^n - k_p^n|) ds.$$

So it remains to prove that for each p , the last term above goes to 0 as $n \rightarrow \infty$. By Lebesgue Theorem, it is even enough to prove that for each s , then $\lambda(|P_{sv_n} k_p^n - k_p^n|) \rightarrow 0$. But if Y denotes an $\mathcal{N}(0, 1)$ random variable, this quantity is less than

$$E\left(\int \left| k_p\left(u_n T\left(\frac{x}{u_n}\right)\right) - k_p\left(u_n T\left(\frac{x + Y\sqrt{sv_n}}{u_n}\right)\right) \right| dx\right).$$

Since $v_n \rightarrow 0$ and since k_p is Lipschitz with compact support, and in view of (2.2), this last expression clearly goes to 0 as $n \rightarrow \infty$, and we are finished. $\square$

Theorem 1-1 obviously follows from these results: assume (1.11) and let h satisfy (B-0). When $(\log n)/u_n \rightarrow 0$, we can apply Lemma 4-3 with $h_n = h$ and Lemma 4-4 with $g_n = H_h$ and (4.7). Otherwise we apply again Lemma 4-3 with $h_n = h$ and Lemma 4-5 with $g = H_h$ and $T(x) = x$ and (4.7) again.

4-3) Proof of Proposition 2-1. This proof will go through three steps.

Step 1. – First, we prove the claim (a). We assume that h satisfies (B-0) with a and $\hat{h}$. Define h_n by (2.3), and set $k_n = H_{h_n}$. In view of (2.2), it is clear that $|h_n(x, y)| \leq \hat{h}_n(x)e^{a'|y|}$, where $a' = a/\varepsilon$ and $\hat{h}_n(x) = \frac{1}{\varepsilon}\hat{h}(u_n T(x/u_n))$.

Obviously (2.2) yields that the sequence $\hat{h}_n$ satisfies (4.11), hence Lemma 4-3 yields that $\frac{u_n}{n}U(u_n, h_n - k_n)^n$ goes to 0 locally uniformly in time in P_x -probability. So it remains to prove that $\frac{u_n}{n}V(u_n, k_n)^n$ tends to $\lambda(H_h)L$ locally uniformly in time in P_x -probability. For this, by using exactly the same argument than in the end of the proof of Lemma 4-4, we can and will assume that $\beta_1(\hat{h}) < \infty$.

Observe that

$$k_n(x) = \int h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right)\right) \rho(y) dy.$$

Set $\alpha_n(x, y) = \sqrt{n}\left(\bar{T}\left(T\left(\frac{x}{u_n}\right) + \frac{y}{\sqrt{n}}\right) - \frac{x}{u_n}\right)$. A change of variable yields

$$k_n(x) = \int h\left(u_n T\left(\frac{x}{u_n}\right), y\right) \delta_n(x, y) dy,$$

where

$$\delta_n(x, y) = \rho(\alpha_n(x, y)) \bar{T}'\left(T\left(\frac{x}{u_n}\right) + \frac{y}{\sqrt{n}}\right).$$

Now, (2.2) implies that $|\alpha_n(x, y) - y| \leq K(\frac{y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$ and $|\bar{T}'(\frac{y}{\sqrt{n}} + T(\frac{x}{u_n})) - 1| \leq K(\frac{|y|}{\sqrt{n}} + |T(\frac{x}{u_n})|)$ and $\varepsilon \leq \frac{\alpha_n(x, y)}{y} \leq \frac{1}{\varepsilon}$. Thus $\rho(\alpha_n(x, y)) \leq K\rho(y\varepsilon)$, while $|\rho'(y)| \leq K\rho(y/2)$. It follows by Taylor's formula that $|\rho(\alpha_n(x, y)) - \rho(y)| \leq K\rho(y\varepsilon/2)(\frac{y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$, and thus $|\delta_n(x, y) - \rho(y)| \leq K\rho(y\varepsilon/2)(\frac{1+y^2}{\sqrt{n}} + |T(\frac{x}{u_n})|)$. Therefore if $f_n(x) = H_h(u_n T(x/u_n))$ we have

$$|k_n(x) - f_n(x)| \leq K\hat{h}\left(u_n T\left(\frac{x}{u_n}\right)\right)\left(\frac{1}{\sqrt{n}} + \left|T\left(\frac{x}{u_n}\right)\right|\right). \quad (4.17)$$

Combining (4.10) and the facts that $\hat{h}$ is bounded and that $\beta_1(\hat{h}) < \infty$, we deduce that $E_x(\frac{u_n}{n}V(u_n, |k_n - f_n|_t^n) \leq Kt(\frac{1}{u_n} + \frac{1}{\sqrt{n}})$, and thus $\frac{u_n}{n}V(u_n, k_n - f_n)^n$ tends to 0 in P_x -probability locally uniformly in time: therefore it remains to prove that $\frac{u_n}{n}V(u_n, f_n)^n$ tends to $\lambda(H_h)L$ in P_x -probability locally uniformly in time.

Suppose first that $(\log n)/u_n \rightarrow 0$. The sequence f_n satisfies (4.12) and (4.13) (for the later, observe that $|f_n| \leq \hat{h}_n$ and that $\int_{|x|>q} \hat{h}_n(x)dx \leq \frac{1}{\varepsilon} \int_{|x|>q\varepsilon} \hat{h}(x)dx$). On the other hand, a change of variable yields

$$\lambda(f_n) = \int H_h(x) \bar{T}'\left(\frac{x}{u_n}\right) dx,$$

which clearly goes to $\lambda(H_h)$ (since $\bar{T}'$ is bounded and goes to 0 at 0): the result then follows from Lemma 4-4.

Next, suppose that $u_n^2/n \rightarrow 0$: the result readily follows from Lemma 4-5.

Step 2. – From now on we assume that h satisfies (B-1). In this step we prove that with notation (4.7), the process $M^n = \sqrt{\frac{u_n}{n}}M(h_n - h)^n$ tends to 0 locally uniformly in time in P_x probability. In view of (4.8) and of Doob's inequality, it is enough to prove that $E_x(\frac{u_n}{n}V(u_n, H_{(h_n-h)^2})_t^n) \rightarrow 0$.

Set $g_n(x) = H_{(h_n-h)^2}$. As seen at the beginning of Step 1, $|h_n(x, y)| \leq \hat{h}_n(x)e^{a'|y|}$, where $a' = a/\varepsilon$ and $\hat{h}_n(x) = \frac{1}{\varepsilon}\hat{h}(u_n T(x/u_n))$: it readily follows that $g_n(x)$ is bounded uniformly in x, n . Thus in view of (1.11) and (4.10), it remains to show that $\lambda(g_n) \rightarrow 0$. We have

$$\begin{aligned} \lambda(g_n) = \int & \left(h\left(u_n T\left(\frac{x}{u_n}\right), \sqrt{n}\left(T\left(\frac{x}{u_n} + \frac{y}{\sqrt{n}}\right) - T\left(\frac{x}{u_n}\right)\right) \right) \right. \\ & \left. - h(x, y) \right)^2 \rho(y) dx dy. \end{aligned} \quad (4.18)$$

Denote below by $\|g\|$ the quantity $(\int g(x, y)^2 \rho(y) dx dy)^{1/2}$. For any $q \in \mathbb{N}$ one may find a continuous function f_q on $\mathbb{R}^2$ with compact support, such that $\|h - g_q\| \leq 1/q$. With the notation of Step 1, a change of variables yields

$$\begin{aligned} & \int \left((h - g_q)^2 \left(u_n T \left(\frac{x}{u_n} \right), \right. \right. \\ & \quad \left. \left. \sqrt{n} \left(T \left(\frac{x}{u_n} + \frac{y}{\sqrt{n}} \right) - T \left(\frac{x}{u_n} \right) \right) \right) - h(x, y) \right) \rho(y) dx dy \\ &= \int (h - g_q)^2(x, y) \bar{T}' \left(\frac{x}{u_n} \right) \rho \left(\alpha_n \left(\bar{T} \left(\frac{x}{u_n} \right), y \right) \right) \bar{T}' \left(\frac{x}{u_n} + \frac{y}{\sqrt{n}} \right) dx dy \\ &\leq K \|h - g_q\|^2 \end{aligned}$$

for a constant K depending only on ε in (2.2). Now, if α_q^n denotes the right side of (4.18) with g_q instead of h , it then follows that $\sqrt{\lambda(g_n)} \leq \frac{1+\sqrt{K}}{p} + \sqrt{\alpha_q^n}$. Furthermore, since g_q is continuous with compact support, it is immediate (by (2.2) again) that $\alpha_q^n \rightarrow 0$ as $n \rightarrow \infty$, for all q : thus $\lambda(g_n) \rightarrow 0$ readily follows, and Step 2 is complete.

Step 3. – In view of Step 2 and of (4.7), in order to obtain Proposition 2-1(b) it remains to prove that $\sqrt{\frac{u_n}{n}} V(u_n, H_{h_n-h})^n$ tends to 0 locally uniformly in time in P_x -probability when $n/u_n^3 \rightarrow 0$ and when h is differentiable in the first variable with a partial derivative satisfying (B-1).

We use again the notation of Step 1. In particular $H_{h_n-h} = k_n - H_h$. Using (4.17), we obtain $\lambda(|k_n - f_n|) \leq K(\frac{1}{\sqrt{n}} + \frac{1}{u_n})$, while by definitions of k_n and f_n and the boundedness of $\hat{h}$ we have $|k_n - f_n| \leq K$. Then (4.10) implies that

$$E_x \left(\sup_{s \leq t} \left| \sqrt{\frac{u_n}{n}} V(u_n, k_n - f_n)_s^n \right| \right) \leq K \left(\sqrt{\frac{u_n}{n}} + \frac{1}{\sqrt{u_n}} + \frac{\sqrt{n}}{u_n^{3/2}} \right).$$

Since we have (1.11) and $n/u_n^3 \rightarrow 0$, the above goes to 0, and we are left to prove the convergence of $\sqrt{\frac{u_n}{n}} V(u_n, f_n - H_h)^n = \frac{1}{\sqrt{n}} V(\sqrt{n}, g_n)^n$ to 0, where $g_n(x) = \sqrt{u_n}(f_n - H_h)(xu_n/\sqrt{n})$ (use (4.9)).

We assume that $h' = \partial h / \partial x$ exists and has (B-1). We have

$$g_n(x) = \sqrt{u_n} \left(T \left(\frac{x}{\sqrt{n}} \right) - \frac{x}{\sqrt{n}} \right) \int H_{h'} \left(u_n \left(\frac{x}{\sqrt{n}} + v \left(T \left(\frac{x}{\sqrt{n}} \right) - \frac{x}{\sqrt{n}} \right) \right) \right) dv$$

because $H_{h'}$ is the derivative of H_h . Since $|T(\frac{x}{\sqrt{n}}) - \frac{x}{\sqrt{n}}| \leq K|x|/\sqrt{n}$ by (2.2), it follows that

$$|g_n(x)| \leq \sqrt{u_n} \frac{|x|}{\sqrt{n}} \int |H_{h'}| \left(u_n \left(\frac{x}{\sqrt{n}} + v \left(T \left(\frac{x}{\sqrt{n}} \right) - \frac{x}{\sqrt{n}} \right) \right) \right) dv.$$

Since h' , hence $H_{h'}$, satisfy (B-2), we easily deduce that $\lambda(|g_n|) \leq K\sqrt{n}/u_n^{3/2}$, while on the other hand $|g_n| \leq K\sqrt{u_n}$ by construction of g_n : hence the sequence g_n satisfies (4.1), and the result follows from Theorem 4-1.

4-4) Finally we give a result which is simple, but will be used later.

THEOREM 4.6. — Assume (1.11) and (A), and let $h = (h^i)_{1 \leq i \leq d}$ be a d -dimensional function satisfying (B-0). Then the processes $Y^n = \sqrt{\frac{u_n}{n}}(U(u_n, h)^n - V(u_n, H_h)^n)$ converge stably in law to a process Y , defined on an extension of the space, and which is an $\mathcal{F}$ -conditional Gaussian continuous martingale with brackets

$$\langle Y^i, Y^j \rangle = \lambda(H_{h^i h^j} - H_{h^i} H_{h^j})L.$$

Proof. — We have that $Y^n = \sqrt{u_n/n}M(h)^n$ (notation (4.7)) is a martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t \geq 0}$. Further, under P_x any martingale (w.r.t. $(\mathcal{F}_t)$) orthogonal to X is constant. Hence by Theorem 3-2 of [9], the result will follow from the next three properties, where for any process Z we put $\Delta_i^n Z = Z_{i/n} - Z_{(i-1)/n}$:

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} \lambda(H_{h^i h^j} - H_{h^i} H_{h^j})L_t, \quad (4.19)$$

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nj} \Delta_i^n X | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} 0, \quad (4.20)$$

$$\sum_{i=1}^{[nt]} E_x(|\Delta_i^n Y^{nj}|^2 1_{\{|\Delta_i^n Y^{nj}| > \varepsilon\}} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} 0 \quad \forall \varepsilon > 0. \quad (4.21)$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel.

First, by (4.8) the left side of (4.19) is $\frac{u_n}{n} V(u_n, H_{h^2} - (H_h)^2)_t^n$, hence (4.19) obtains by Theorem 1-1.

Second, by a simple computation the left side of (4.20) is $\frac{\sqrt{u_n}}{n} V(u_n, g)_t^n$, where $g(x) = \int h(x, y) y \rho(y) dy$, hence (4.20) obtains by Theorem 1-1 and $u_n \rightarrow \infty$.

Third, another computation yields that $|\Delta_i^n Y^n| \leq K \hat{h}(u_n X_{(i-1)/n}) \sqrt{u_n/n} e^{a|\sqrt{n}\Delta_i^n X|}$ (where a and $\bar{h}$ are as in (B-0)). Hence $|\Delta_i^n Y^n| > \varepsilon \Rightarrow |\sqrt{n}\Delta_i^n X| > K_\varepsilon \log(n/u_n)$ for some

$K_\varepsilon > 0$. Therefore

$$E_x(|\Delta_i^n Y^{nj}|^2 1_{\{|\Delta_i^n Y^{nj}| > \varepsilon\}} | \mathcal{F}_{\frac{i-1}{n}}) \leq \frac{u_n \gamma_n}{n} \hat{h}(u_n X_{\frac{i-1}{n}}),$$

where $\gamma_n = K e^{-K_\varepsilon^2 (\log(n/u_n))/8}$

(the constant K depends on a and on $\bar{h}$). Therefore the left side of (4.21) is smaller than $(\gamma_n u_n/n) V(u_n, \hat{h})_t^n$: we conclude by Theorem 1-1 once more and by the property $\gamma_n \rightarrow 0$.

5. A BASIC MARTINGALE

It remains to prove Theorem 1-2, for the Brownian measures P_x on the canonical space; the d -dimensional function h satisfies (B-r) for some $r > 3$, and $u_n = n^\alpha$ with $1/3 < \alpha < 1$, and $\delta = ((1-\alpha) \wedge \alpha)/2$. For proving the convergence of the sequence $n^\delta (\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - \lambda(H_h)L)$ we will use the same method as for Theorem 4-6. However the above processes are not in general martingales, and our first task is to write them as sums of a sequence of martingales to which the previous method applies, and a sequence of processes which go to 0. In this section, we perform this decomposition.

The processes of interest may be decomposed in a sum of four terms:

$$\frac{1}{n^{1-\alpha}} U(n^\alpha, h)^n - \lambda(H_h)L = M^n + N^n + \theta R^n + \theta Q^n, \quad (5.1)$$

where $M^n = \frac{1}{n^{1-\alpha}} M(h)^n$ (see (4.7)), and $\theta = \lambda(H_h)$, and (recall (1.14) for $\hat{g}$):

$$\left. \begin{aligned} N^n &= \frac{1}{n^{1-\alpha}} V(n^\alpha, H_h)^n - \theta \frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})^n, \\ R_t^n &= \frac{1}{\sqrt{n}} V(\sqrt{n}, \hat{g})_t^n - L_{[nt]/n}, \quad Q_t^n = L_{[nt]/n} - L_t. \end{aligned} \right\} \quad (5.2)$$

As seen before, the process M^n is a martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t \geq 0}$, as well as R^n by (4.5). It is not true for N^n and Q^n , but in this section we prove that $n^\delta Q^n$ is “negligible”, while $n^\delta N^n$ is a martingale plus a “negligible” term.

The term Q^n is easy to deal with: the local time L has Hölder paths with index ε for any $\varepsilon < 1/2$ (this is classical result, following for example

from Kolmogorov's criterion combined with (3.14)). Hence, since $\delta < 1/2$, we have for all $\omega \in \Omega$:

$$\sup_{s \leq t} |n^\delta Q_s^n(\omega)| \rightarrow 0. \quad (5.3)$$

The case of N^n is more complicated. By (4.9), we have $n^\delta N^n = \frac{1}{\sqrt{n}} V(\sqrt{n}, k_n)^n$, where

$$k_n(x) = n^\delta \left(n^{\alpha-1/2} H_h(n^{\alpha-1/2} x) - \theta \hat{g}(x) \right). \quad (5.4)$$

In the rest of the paper, the constants K, K_γ may depend on the function h , via $\hat{h}$ and a . Observe that (B-r) and (1.14) yield

$$|k_n(x)| \leq K n^\delta, \quad \beta_\gamma(k_n) \leq K n^\delta (1 + n^{(\gamma-1)(1/2-\alpha)})$$

for $\gamma \in [0, r]$, $\lambda(k_n) = 0$. (5.5)

We also consider $\beta \in (0, 1)$, to be chosen later, and set $w_n = [n^\beta]$ and

$$F_n = \sum_{j=0}^{w_n} P_j k_n, \quad \bar{F}_n = \sum_{j=1}^{w_n} P_j k_n, \quad \hat{F}_n = \sum_{j=1}^{w_n+1} P_j k_n, \quad \check{F}_n = \sum_{j=0}^{w_n+1} P_j k_n. \quad (5.6)$$

Due to (5.5) and Lemma 3-1 we have for $\gamma \in [0, r-1]$:

$$|\hat{F}_n(x)| + |\bar{F}_n(x)| \leq \begin{cases} K(1 + n^{1/2-\alpha}) n^\delta \log n \\ K_\gamma n^\delta (\log n) \left(\frac{1 + n^{1/2-\alpha}}{1 + |x n^{-\beta/2}|^\gamma} + \frac{1 + n^{(\gamma+1)(1/2-\alpha)}}{1 + |x|^\gamma} \right) \\ K(1 + |x|) n^\delta (1 + n^{1-2\alpha}), \end{cases} \quad (5.7)$$

$$|F_n(x)| + |\check{F}_n(x)| + |\hat{F}_n(x)| + |\check{F}_n(x)| \leq K n^\delta (\log n + n^{1/2-\alpha} \log n + n^{\alpha-1/2}), \quad (5.8)$$

$$|P_{w_n+1} k_n(x)| \leq K_\gamma n^{\delta-\beta} \left(\frac{1 + n^{1/2-\alpha}}{1 + |x n^{-\beta/2}|^\gamma} + \frac{1 + n^{(\gamma+1)(1/2-\alpha)}}{1 + |x|^\gamma} \right). \quad (5.9)$$

On the other hand, put

$$\left. \begin{aligned} \zeta_i^n &= \sum_{j=0}^{w_n} \left(E_x(k_n(\sqrt{n} X_{\frac{i+j}{n}} | \mathcal{F}_{\frac{i}{n}})) - E_x(k_n(\sqrt{n} X_{\frac{i+j}{n}} | \mathcal{F}_{\frac{i-1}{n}})) \right), \\ W_t^n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \zeta_i^n, \end{aligned} \right\} \quad (5.10)$$

$$\left. \begin{aligned} A_t^n &= \frac{1}{\sqrt{n}} (F_n(\sqrt{n}X_0) - F_n(\sqrt{n}X_{[nt]/n})), \\ B^n &= \frac{1}{\sqrt{n}} V(\sqrt{n}, P_{w_n+1}k_n)^n. \end{aligned} \right\} \quad (5.11)$$

Observe that

$$\begin{aligned} \zeta_i^n &= k_n(\sqrt{n}X_{\frac{i}{n}}) + \bar{F}_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}}) \\ &= F_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}}), \end{aligned} \quad (5.12)$$

so a simple computation yields

$$n^\delta N^n = W^n + A^n + B^n. \quad (5.13)$$

First (5.8) and (5.11) yield that $\sup_t |A_t^n|$ is smaller, up to a multiplicative constant, than $(n^{\delta-1/2} + n^{\delta-\alpha} + n^{\delta+\alpha-1}) \log n$. Now, the definition of δ implies that all the powers of n in the previous bound are negative, and thus

$$\sup_{t,\omega} |A_t^n(\omega)| \rightarrow 0. \quad (5.14)$$

LEMMA 5.1. — *a) If $\beta \in (1 - \alpha, 1)$, then B^n goes to 0 in P_x -probability, uniformly over all finite intervals.*

b) If $\beta \in ((1 - 2\alpha)^+, 1)$, then $E_x(|B_t^n|^2) \rightarrow 0$.

Proof. — Set $g_n = P_{w_n+1}k_n$. Observe that $\lambda(P_t k) = 0$ as soon as $\lambda(k) = 0$, so here (5.5) yields $\lambda(g_n) = 0$. Hence (a) (resp. (b)) follows from Theorem 4-1(a) (resp. Lemma 4-2) if we prove that the sequence g_n satisfies (4.1) when $\beta > 1 - \alpha$ (resp. (4.2) when $\beta > (1 - 2\alpha)^+$).

When $\alpha \geq 1/2$, (5.9) yields

$$\begin{aligned} |g_n| &\leq Kn^{(1-\alpha)/2-\beta}, & \lambda(|g_n|) &\leq Kn^{(1-\alpha-\beta)/2}, \\ \beta_1(g_n) &\leq Kn^{(1-\alpha)/2}, & \lambda(|g_n|^2) &\leq Kn^{1-\alpha-3\beta/2}, \end{aligned}$$

and all the claims are obvious.

When $\alpha < 1/2$, (5.9) yields for all $\varepsilon \in (0, r - 3]$:

$$\begin{aligned} |g_n| &\leq Kn^{1/2-\beta-\alpha/2}, & \lambda(|g_n|) &\leq K_\varepsilon(n^{(1-\alpha-\beta)/2} + n^{\varepsilon-\beta+1-3\alpha/2}), \\ \beta_1(g_n) &\leq K_\varepsilon(n^{(1-\alpha)/2} + n^{\varepsilon+3/2-\beta-5\alpha/2}), \\ \lambda(|g_n|^2) &\leq K_\varepsilon(n^{1-3\beta/2-\alpha} + n^{\varepsilon+3/2-2\beta-2\alpha}). \end{aligned}$$

Then again, all the claims are easy to check, since $\alpha > 1/3$. $\square$

LEMMA 5.2. — If $\beta \in ((1 - 2\alpha)^+, 1)$, then $\sup_{s \leq t} |n^\delta N_s^n - W_s^n| \xrightarrow{P_x} 0$ for all t .

Proof. — Let β be as above, and choose β' in $(1 - \alpha, 1)$. With β' we associate the processes W'^n , A'^n , and B'^n . Let $C^n := A^n + B^n = n^\delta N^n - W^n$ and $C'^n = A'^n + B'^n$.

By the previous lemma and (5.14), C'^n tends to 0 in probability, locally uniformly in time, and also C_t^n and $C_t'^n$ tend to 0 in $\mathbb{L}^2(P_x)$. Then the processes $Z^n = C^n - C'^n = W'^n - W^n$ are martingales and satisfy $E_x(|Z_t^n|^2) \rightarrow 0$ for all t .

Following Aldous [2], we deduce that in fact Z^n tends in probability to 0, locally uniformly in time, hence the same holds for C^n . $\square$

6. THEOREM 1-2, THE CASE $\alpha = 1/2$

Here we prove Theorem 1-2 when $\alpha = 1/2$, hence $\delta = 1/4$. Let us take $\beta = 1/4$ in the definition (5.10) of W^n . In view of (5.1), (5.3) and Lemma 5-2, we are left to prove that the sequence

$$Y^n = n^{1/4} M^n + W^n + n^{1/4} \theta R^n \quad (6.1)$$

stably converges to the limit Y , as described in Theorem 1-2. Note that Y^n is a locally square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t \geq 0}$, so exactly as for Theorem 4-6 the result will follow from (4.21), (4.22) and the following (which replaces (4.19)):

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} \eta(h^l, h^j) L_t, \quad (6.2)$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. Combining (6.1), (5.2), (5.10), (5.12), (4.7), plus the facts that $k_n = n^{1/4}(H_h - \theta \hat{g})$ and that $\check{F}_n = \hat{F}_n + k_n$, we obtain

$$\begin{aligned} \Delta_i^n Y^n &= \frac{1}{n^{1/4}} (h(\sqrt{n} X_{\frac{i-1}{n}}, \sqrt{n} \Delta_i^n X) - \theta \sqrt{n} \Delta_i^n L) \\ &\quad + \frac{1}{\sqrt{n}} (F_n(\sqrt{n} X_{\frac{i}{n}}) - \check{F}_n(\sqrt{n} X_{\frac{i-1}{n}})). \end{aligned} \quad (6.3)$$

Proof of 4.21. — By (6.3), (5.8) and (B-r), $|\Delta_i^n Y^n|^6 \leq K n^{-3/2} (e^{6a|\sqrt{n} \Delta_i^n X|} + (\log n)^6 + n^3 |\Delta_i^n L|^6)$. Since $E_x(n^3 |\Delta_i^n L|^6 | \mathcal{F}_{\frac{i-1}{n}}) =$

$G(1, 6, \sqrt{n}X_{\frac{i-1}{n}})$ by (3.13), while (3.16) yields that $G(1, 6, \cdot)$ is bounded, we get $E_x(|\Delta_i^n Y^n|^6 | \mathcal{F}_{\frac{i-1}{n}}) \leq K(\log n)^6/n^{3/2}$. This yields $E_x(|\Delta_i^n Y^n|^2 1_{|\Delta_i^n Y^n| > \varepsilon} | \mathcal{F}_{\frac{i-1}{n}}) \leq K\varepsilon^{-4}(\log n)^6/n^{3/2}$, and thus (4.21) holds.

Proof of 4.20. – In view of (6.3) and (3.13), the left side of (4.20) is $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n}, g'_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n}, g''_n)_t^n$, where

$$g_n(x) = \frac{1}{n^{1/4}} \int y \left(h(x, y) + \frac{1}{n^{1/4}} k_n(x + y) \right) \rho(y) dy,$$

$$g'_n(x) = \frac{1}{\sqrt{n}} \int y \bar{F}_n(x + y) \rho(y) dy,$$

$$g''_n(x) = \frac{\theta}{n^{1/4}} (xG(1, 1, x) - G(f, 1, x)), \quad \text{with } f(x) = x.$$

First, by Cauchy-Schwarz inequality $|g_n(x)| \leq \frac{1}{n^{1/4}} \int |yh(x, y)| \rho(y) dy + \frac{1}{\sqrt{n}} \sqrt{P_1 k_n^2(x)}$. Then (5.5) and (3.2) give $P_1 k_n^2(x) \leq K\sqrt{n}/(1 + |x|^{r-1})$, hence

$$|g_n(x)| \leq \frac{K}{n^{1/4}} \left(\bar{h}(x) + \frac{1}{1 + |x|^{(r-1)/2}} \right).$$

It follows that $|g_n|$ and $\lambda(|g_n|)$ are smaller than $Kn^{-1/4}$, so the sequence g_n satisfies (4.1), and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n \rightarrow^{P_x} 0$.

Next, $|g'_n(x)| \leq \frac{1}{\sqrt{n}} (P_1 \bar{F}_n^2(x))^{1/2}$. We have $\bar{F}_n(x)^2 \leq Kn^{2\delta}(\log n)^2/(1 + |xn^{-1/8}|^{2r-2})$ by (5.7). Then (3.9) yields $(P_1 \bar{F}_n^2(x))^{1/2} \leq Kn^\delta(\log n)/(1 + |xn^{-1/8}|^{r-1})$ and thus

$$|g'_n(x)| \leq K \frac{\log n}{n^{1/4}(1 + |xn^{-1/8}|^{r-1})}.$$

Hence $|g_n|$ and $\lambda(|g_n|)$ are smaller than $K(\log n)/n^{1/8}$, so the sequence g_n satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g'_n)_t^n \rightarrow^{P_x} 0$ by Theorem 4-1.

Finally, consider g''_n . Since $E_0(L_1 X_1) = 0$ for symmetry reasons, one has $G(f, 1, x) = 0$ by (3.16), so (3.15) and (3.16) give

$$|g''_n(x)| = \begin{cases} 0 & \text{if } x = 0 \\ \frac{|\theta|}{\pi n^{1/4}} \int_0^1 \frac{x^2}{r^{3/2}} e^{-x^2/2r} \sqrt{1-r} dr & \text{if } x \neq 0. \end{cases}$$

By the change of variable $r = x^2/s$ we obtain

$$|g''_n(x)| \leq Kn^{-1/4} |x| \int_{x^2}^\infty \frac{1}{\sqrt{s}} e^{-s/2} ds \leq Kn^{-1/4} e^{-x^2/4}.$$

Then the sequence g_n'' satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n'')^n \xrightarrow{P_x} 0$ by Theorem 4-1 again.

Proof of 6.2. – In this proof, we set $k = H_h - \theta\hat{g} = k_n/n^{1/4}$ and $F'_n = F_n/n^{1/4}$ and $\check{F}'_n = \check{F}_n/n^{1/4}$. Observe that $\lambda(k) = 0$, so $F := F(k)$ may be defined by (1.16), and by (5.7) we see that both $F'_n(x)$ and $\check{F}'_n(x)$ converge to $F(x)$ and stay smaller than $K(1 + |x|)$.

By (6.3) and a simple computation, and if $f_{n,x}(y) = h(x, y - x) + F'_n(y)$, the left side of (6.2) is $\frac{1}{\sqrt{n}}(V(\sqrt{n}, g_n)_t^n + \theta^2 V(\sqrt{n}, g')_t^n - 2\theta V(\sqrt{n}, g_n'')_t^n)$, where (recall (1.13), (3.12) and (3.13)):

$$g_n(x) = H_{h^2}(x) + P_1 F_n'^2(x) - \check{F}_n'(x)^2 + 2\bar{H}_{h, F_n'}(x),$$

$$g'(x) = G(1, 2, x), \quad g_n''(x) = G(f_{n,x}, 1, x).$$

First we observe that by (3.15) and (3.16),

$$g'(x) = \begin{cases} 1 & \text{if } x = 0 \\ \int_0^1 \frac{|x|}{\sqrt{2\pi}r^{3/2}} e^{-\frac{x^2}{2r}} (1-r) dr & \text{if } x \neq 0. \end{cases}$$

Hence $\lambda(g') = \sqrt{\frac{2}{\pi}} \int_0^1 (\frac{1}{\sqrt{r}} - \sqrt{r}) dr = \frac{8}{3\sqrt{2\pi}}$, and we obtain that $\frac{1}{\sqrt{n}}V(\sqrt{n}, g')_t^n$ converges in P_x -probability to $\frac{8}{3\sqrt{3\pi}}L_t$ by Theorem 1-1. So in order to prove (6.2), and in view of Theorem 4-1, it suffices to show that the two sequences g_n and g_n'' satisfy (4.2) and

$$\lambda(g_n) \rightarrow \lambda(H_{h^2} + 2\bar{H}_{h, F}), \quad \lambda(g_n'') \rightarrow \delta(h). \quad (6.4)$$

The sequence g_n : First we have

$$\begin{aligned} \lambda(g_n) &= \lambda(H_{h^2} + F_n'^2 - \check{F}_n'^2 + 2\bar{H}_{h, F_n'}) \\ &= \lambda(H_{h^2} - (P_{w_{n+1}}k)(2F_n' + P_{w_{n+1}}k) + 2\bar{H}_{h, F_n'}). \end{aligned}$$

Since $|P_{w_{n+1}}k(x)| \leq K/n^{1/4}(1 + |x/n^{1/8}|^{s-1})$ by (5.9) and $|F_n'| \leq K \log n$ by (5.8), we see that $\lambda((P_{w_{n+1}}k)(2F_n' + P_{w_{n+1}}k)) \rightarrow 0$. Since $F_n'(x) \rightarrow F(x)$ and $|F_n'(x)| \leq K(1 + |x|)$, we see that $\bar{H}_{h, F_n'}(x) \rightarrow \bar{H}_{h, F}(x)$ and that $|\bar{H}_{h, F_n'}(x)| \leq K\bar{h}(x)(1 + |x|)$; so $\lambda(\bar{H}_{h, F_n'}) \rightarrow \lambda(\bar{H}_{h, F})$, hence the first property (6.4).

If $\bar{F}_n' = \bar{F}_n/n^{1/4}$, we have $P_1 F_n'^2 \leq 2P_1 k^2 + 2P_1 \bar{F}_n'$ and $\bar{H}_{h, F_n'} = \bar{H}_{h, k} + \bar{H}_{h, \bar{F}_n'}$. (B-r) implies $\beta_s(k^2) < \infty$ for all $s \in [0, r]$, so (3.2) yields $P_1 k^2(x) \leq K(e^{-x^2/2} + \frac{1}{1+|x|^r}) \leq \frac{K}{1+|x|^r}$, while clearly $|\bar{H}_{h, k}| \leq \bar{h}\sqrt{P_1 k^2} \leq K\bar{h}$ by Cauchy-Schwarz inequality. On the other hand we have seen in the

proof of (4.20) that $P_1 \bar{F}_n^2(x) \leq K\sqrt{n}(\log n)^2/(1 + |xn^{-1/8}|^{2r-2})$, and by (5.7) the same majoration holds for $\bar{F}_n^2(x)$, while $|\bar{H}_{h,F'_n}| \leq K\bar{h}\sqrt{P_1 F_n'^2}$ by Cauchy-Schwarz inequality again. Putting all these together yields

$$|g_n(x)| \leq K \left(\bar{h} + \frac{1}{1 + |x|^r} + \frac{(\log n)^2}{1 + |x/n^{1/8}|^{2r-2}} \right).$$

Then $|g_n| \leq K(\log n)^2$ and $\lambda(|g_n|) \leq K(\log n)^2 n^{1/8}$ and $\lambda(g_n^2) \leq K(\log n)^4 n^{1/4}$ and $\beta_1(g_n) \leq K(\log n)^2 n^{1/4}$. It readily follows that the sequence g_n satisfies (4.2).

The sequence g_n'' : (3.16) yields for $x \neq 0$:

$$g_n''(x) = \begin{cases} E_0(L_1(h(0, X_1) + F'_n(X_1))) & \text{if } x = 0 \\ \int_0^1 \frac{|x|}{\sqrt{2\pi r^{3/2}}} e^{-\frac{x^2}{2r}} \sqrt{1-r} & \text{if } x \neq 0 \\ \quad \times E_0(L_1(h(x, X_1\sqrt{1-r} - x)) + F'_n(X_1\sqrt{1-r})) dr & \end{cases} \quad (6.5)$$

The change of variable $x = y\sqrt{r}$ gives

$$\lambda(g_n'') = \int_0^1 \sqrt{\frac{1}{r} - 1} dr \int |y| E_0(L_1(h(y\sqrt{r}, X_1\sqrt{1-r} - y\sqrt{r}) + F'_n(X_1\sqrt{1-r}))) \rho(y) dy.$$

Then $F'_n(x) \rightarrow F(x)$ and $|F'_n(x)| \leq K(1 + |x|)$ readily give the last property (6.4) (recall (1.17), and observe that $\bar{\rho}$ is the law of (X_1, L_1) under P_0). Further, (H-r) implies that the expectation in the expression (6.5) for $x \neq 0$ is smaller than $Ke^{a|x|}$. Hence

$$|g_n''(x)| \leq K|x|e^{a|x|} \int_{x^2}^{\infty} \frac{1}{\sqrt{y}} e^{-y/2} dy \leq Ke^{a|x|-x^2/2}.$$

It is then obvious that the sequence g_n'' satisfies (4.2).

Proof of 1.26. – Finally, let us verify that (1.26) holds when $h(x, y) = g(x)$. First we have $H_h = g$. Second,

$$\int_0^1 \left(\frac{1}{\sqrt{r}} - \sqrt{r} \right) dr \int |y| g(y\sqrt{r}) E_0(L_1) \rho(y) dy = \int g(x) G(1, 1, x) dx$$

by (3.16) and the change of variable $x = y\sqrt{r}$, while $G(1, 1, x) = \hat{g}(x)$.

7. THEOREM 1-2, THE CASE $\alpha > 1/2$

Here we prove Theorem 1-2 when $\alpha \in (1/2, 1)$, hence $\delta = (1 - \alpha)/2$. We will choose β such that $0 < \beta < (1 - \alpha) \wedge (2\alpha - 1)$: this choice is possible, and yields $\beta < 1/3$ and $\beta < \alpha$.

First, observe that $n^{1/4}(\frac{1}{\sqrt{n}}V(\sqrt{n}, \hat{g})^n - L) = n^{1/4}(R^n + Q^n)$ (this is (5.1) when $h(x, y) = \hat{g}(x)$), and by Theorem 1-2 applied with $\alpha = 1/2$ and $h(x, y) = \hat{g}(x)$ these processes converge stably in law. In view of (5.3), it follows that $n^{1/4}R^n$ stably converge in law as well. Coming back to the case $\alpha > 1/2$, hence $\delta < 1/4$, we deduce that $n^\delta R^n$ converges in law to 0. This and (5.3) and Lemma 5-2 imply that for Theorem 1-2 with $\alpha > 1/2$ we are left to prove that the sequence

$$Y^n = n^\delta M^n + W^n \quad (7.1)$$

stably converges to the limit as described in the theorem. Again Y^n is a locally square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{[nt]/n})_{t \geq 0}$, so exactly as for Theorem 4-6 the result will follow from the properties (4.20), (4.21) and

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{ni} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} \lambda(H_{h^t h^j}) L_t. \quad (7.2)$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. By (4.7), (5.10), (5.12) and (7.1), we have

$$\begin{aligned} \Delta_i^n Y^n &= \frac{1}{n^\delta} (h(n^\alpha X_{\frac{i-1}{n}}, \sqrt{n} \Delta_i^n X) - H_h(n^\alpha X_{\frac{i-1}{n}})) \\ &\quad + \frac{1}{\sqrt{n}} (F_n(\sqrt{n} X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n} X_{\frac{i-1}{n}})). \end{aligned} \quad (7.3)$$

Proof of 4.21. – (B-r), (5.8) and (7.3) yield $|\Delta_i^n Y^n| \leq K(n^{-\delta} e^{a|\sqrt{n} \Delta_i^n X|} + n^{\delta-1/2} \log n)$, hence $E(|\Delta_i^n Y^n|^q | \mathcal{F}_{\frac{i-1}{n}}) \leq K_q(n^{-q\delta} + n^{q(\delta-1/2)} \log n)$. This quantity is smaller than $1/n^2$ if q is large enough, hence (4.21) follows from Tchebicheff's inequality.

Proof of 4.20. – By (7.3), the left side of (4.20) is $\frac{1}{\sqrt{n}} V(\sqrt{n}, g_n)_t^n$, where

$$g_n(x) = \int y \left(\frac{1}{n^{(1-\alpha)/2}} h(xn^{\alpha-1/2}, y) + \frac{1}{\sqrt{n}} F_n(x+y) \right) \rho(y) dy.$$

In view of Theorem 4-1(a), it suffices to prove that the sequence g_n satisfies (4.1). Since $\alpha > 1/2$, it follows from (5.7) and (3.9) that

$P_1 \bar{F}_n^2(x) \leq K n^{1-\alpha} (\log n)^2 / (1 + |x n^{-\beta/2}|^{2r-2})$. On the other hand, (5.5) and (1.9) yield $P_1 k_n^2(x) \leq K n^{1-\alpha} (e^{-x^2/2} + 1/(1 + |x|^{r-1}))$ hence

$$P_1 F_n^2(x) \leq K n^{1-\alpha} (\log n)^2 \left(\frac{1}{1 + |x n^{-\beta/2}|^{2r-2}} + \frac{1}{1 + |x|^{r-1}} \right). \quad (7.4)$$

By Cauchy-Schwarz inequality $|g_n(x)| \leq \frac{1}{n^{(1-\alpha)/2}} \bar{h}(x n^{\alpha-1/2}) + \frac{1}{\sqrt{n}} \sqrt{P_1 F_n^2(x)}$. Thus (7.4) yields

$$|g_n(x)| \leq K \left(\frac{1}{n^{(1-\alpha)/2}} \bar{h}(x n^{\alpha-1/2}) + \frac{\log n}{n^{\alpha/2} (1 + |x/n^{\beta/2}|^{r-1})} + \frac{1}{n^{\alpha/2} (1 + |x|^{r/2-1/2})} \right).$$

Hence $|g_n| \leq K n^{-(1-\alpha)/2}$ and $\lambda(|g_n|) \leq (\log n)/n^{(\alpha-\beta)/2}$: since $\beta < \alpha$, we obtain the desired result.

Proof of 7.2. — In view of (7.3), the left side of (7.2) is $n^{\alpha-1} V(n^\alpha, H_{h^2} - (H_h)^2)_t^n + \frac{1}{\sqrt{n}} V(\sqrt{n}, g_n)_t^n + \frac{1}{\sqrt{n}} V(\sqrt{n}, \ell_n)_t^n$, where

$$g_n = \frac{1}{\sqrt{n}} (P_1 F_n^2 - \hat{F}_n^2),$$

$$\ell_n(x) = \frac{2}{n^\delta} \int (h(n^{\alpha-1/2}x, y) - H_h(n^{\alpha-1/2}x)) F_n(x+y) \rho(y) dy.$$

First, Theorem 1-1 gives that $n^{\alpha-1} V(n^\alpha, H_{h^2} - (H_h)^2)_t^n \rightarrow \lambda(H_{h^2} - (H_h)^2) L_t$ in P_x -probability. Thus it is enough to prove that

$$\frac{1}{\sqrt{n}} V(\sqrt{n}, g_n)_t^n \xrightarrow{P_x} \lambda(H_h^2) L_t, \quad \frac{1}{\sqrt{n}} V(\sqrt{n}, \ell_n)_t^n \xrightarrow{P_x} 0. \quad (7.5)$$

Let us study first g_n : we will prove that this sequence satisfies (4.2) and $\lambda(g_n) \rightarrow \lambda(H_h^2)$, thus obtaining the first condition in (7.5) by Theorem 4-1. By (5.7) and (7.4) we have

$$|g_n(x)| \leq K_\gamma n^{1/2-\alpha} (\log n)^2 \left(\frac{1}{1 + |x n^{-\beta/2}|^{2r-2}} + \frac{1}{1 + |x|^{r-1}} \right).$$

Hence $|g_n| \leq K (\log n)^2 / n^{\alpha-1/2}$, and $\lambda(|g_n|) \leq K (\log n)^2 / n^{\alpha-1/2-\beta/2}$, and $\beta_1(g_n) \leq K (\log n)^2 / n^{\alpha-1/2-\beta}$ and $\lambda(|g_n|^2) \leq K (\log n)^4 / n^{2\alpha-1-\beta/2}$: in view of $\beta < \alpha$ and $\beta < 2\alpha - 1$, we readily observe that the sequence g_n satisfies (4.2).

Next, we easily obtain $\lambda(g_n) = \lambda(\frac{1}{\sqrt{n}}(F_n^2 - \hat{F}_n^2))$. But we may write $\frac{1}{\sqrt{n}}(F_n^2 - \hat{F}_n^2) = v_n + y_n + z_n$, where

$$v_n(x) = n^{\alpha-1/2} H_h(n^{\alpha-1/2} x)^2,$$

$$y_n(x) = n^{1/2-\alpha} \theta^2 \hat{g}(x)^2 - 2n^{1/2-\alpha} \hat{g}(x) n^{\alpha-1/2} H_h(n^{\alpha-1/2} x),$$

$$z_n = \frac{1}{\sqrt{n}}(k_n(\hat{F}_n + \bar{F}_n) - (P_{w_n+1} k_n)(F_n + \hat{F}_n)).$$

Then $|y_n(x)| \leq K n^{1/2-\alpha} (\hat{g}(x) + n^{\alpha-1/2} \bar{h}(n^{\alpha-1/2} x))$, and it follows that $\lambda(y_n) \rightarrow 0$. Moreover, (5.5), (5.7), (5.8) and (5.9) give $|z_n(x)| \leq K n^{1/2-\alpha} (\log n) / (1 + |x n^{-\beta/2}|^{r-1})$, hence $\lambda(|z_n|) \leq K n^{-(\alpha-1/2-\beta/2)} \log n \rightarrow 0$. Finally, we trivially have $\lambda(v_n) = \lambda(H_h^2)$: hence the first part of (7.5) holds.

Let us turn now to the sequence ℓ_n . We have $|\ell_n(x)| \leq K n^{-\delta} \bar{h}(n^{\alpha-1/2} x) \sqrt{P_1 F_n^2(x)}$. By (7.4) we obtain that $|\ell_n(x)| \leq K (\log n) \bar{h}(n^{\alpha-1/2} x)$: thus $|\ell_n| \leq K \log n$ and $\lambda(|\ell_n|) \leq K n^{1/2-\alpha} \log n \rightarrow 0$, and Theorem 4-1(a) yields the second condition in (7.5).

8. THEOREM 1-2, THE CASE $\alpha < 1/2$

Here we prove Theorem 1-2 when $\alpha \in (1/3, 1/2)$, hence $\delta = \alpha/2$. We will choose β such that

$$1-2\alpha < \beta < \frac{1}{3} \quad (\Rightarrow \quad \beta < 1-\alpha, \quad \beta < 2\alpha, \quad 3\beta < 1+2\alpha). \quad (8.1)$$

This choice is possible, since $1/3 < \alpha < 1/2$.

As seen in Section 7, $n^{1/4} R^n$ stably converge in law. Since $\delta < 1/4$, we deduce that $n^\delta R^n$ converges in law to 0. On the other hand, $n^{(1-\alpha)/2} M^n$ converges in law by Theorem 4-6 and since $\delta < (1-\alpha)/2$ it follows that the sequence $n^\delta M^n$ tends in law to 0. These two facts, plus (5.3) and Lemma 5-2, imply that for Theorem 1-2 with $\alpha < 1/2$ we are left to prove that the sequence of martingales $Y^n = W^n$ stably converges to the limit as described in the theorem. Hence once again the result will follow from the properties (4.20), (4.21) and

$$\sum_{i=1}^{[nt]} E_x(\Delta_i^n Y^{nl} \Delta_i^n Y^{nj} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{P_x} \eta'(h^l, h^j) L_t. \quad (8.2)$$

By polarization, it is enough to prove these when h is 1-dimensional, which we assume in the sequel. By (5.10) and (5.12), we have

$$\Delta_i^n Y^n = \frac{1}{\sqrt{n}} (F_n(\sqrt{n}X_{\frac{i}{n}}) - \hat{F}_n(\sqrt{n}X_{\frac{i-1}{n}})). \quad (8.3)$$

Proof of 4.21. – (5.8) yields $|\Delta_i^n Y^n| \leq K(\log n)/n^{\alpha/2}$, and thus (4.21) obviously holds.

Proof of 4.20. – In view of (8.3), the left side of (4.20) is $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n + \frac{1}{\sqrt{n}}V(\sqrt{n}, g'_n)_t^n$, where

$$g_n(x) = \frac{1}{\sqrt{n}} \int y k_n(x+y) \rho(y) dy, \quad g'_n(x) = \frac{1}{\sqrt{n}} \int y \bar{F}_n(x+y) \rho(y) dy.$$

First, (1.9) and (5.5) yield $P_1 k_n^2(x) \leq K_\gamma n^\alpha (e^{-x^2/2} + \frac{n^{\gamma(1/2-\alpha)}}{1+|x|^\gamma})$ for $\gamma \in [1, r-1]$, hence

$$|g_n(x)| \leq K_\gamma \frac{n^{\alpha/2+\gamma(1/2-\alpha)-1/2}}{1+|x|^\gamma} \quad \text{for } \gamma \in [1/2, (r-1)/2].$$

Then clearly $|g_n| \leq K n^{-1/4}$ and $\lambda(|g_n|) \leq K_\varepsilon n^{\varepsilon-\alpha/2}$ for $\varepsilon > 0$ small enough: the sequence g_n satisfies (4.1) and $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n \xrightarrow{P_x} 0$ by Theorem 4-1.

Next, observe that $\lambda(k_n) = 0$, hence $\lambda(F_n) = 0$ and $\lambda(g'_n) = 0$. So in view of Theorem 4-1, it suffices to prove that the sequence g'_n satisfies (4.2). It follows from (5.7) and (3.9) that for $\gamma \in [0, r-1]$:

$$\sqrt{P_1 \bar{F}_n^2(x)} \leq K_\gamma n^\delta (\log n) \left(\frac{n^{1/2-\alpha}}{1+|x n^{-\beta/2}|^\gamma} + \frac{n^{(\gamma+1)(1/2-\alpha)}}{1+|x|^\gamma} \right).$$

By Cauchy-Schwarz inequality $|g'_n(x)| \leq \frac{1}{\sqrt{n}}(P_1 \bar{F}_n^2(x))^{1/2}$. Thus for $\varepsilon > 0$ small enough,

$$|g'_n(x)| \leq K \frac{\log n}{n^{\alpha/2}}, \quad \lambda(|g'_n|) \leq K_\varepsilon (\log n) (n^{\beta/2-\alpha/2} + n^{3\alpha/2-1/2+\varepsilon}),$$

$$\beta_1(g'_n) \leq K_\varepsilon (\log n) (n^{\beta-\alpha/2} + n^{5\alpha/2-1+\varepsilon}),$$

$$\lambda(|g'_n|^2) \leq K_\varepsilon (\log n)^2 (n^{\beta/2+1/2-\alpha} + n^{5\alpha/2-1+\varepsilon}).$$

In view of (8.1) it is then obvious that (4.2) is met by g'_n .

Proof of 8.2. – By (8.3), the left side of (8.2) is $\frac{1}{\sqrt{n}}V(\sqrt{n}, g_n)_t^n$, where $g_n = \frac{1}{\sqrt{n}}(P_1 F_n^2 - \hat{F}_n^2)$. In fact $\hat{F}_n = F_n - k_n + P_{w_n+1}k_n$, so we can write $g_n = y_n + z_n + z'_n$, with

$$y_n = \frac{2}{\sqrt{n}}k_n F_n, \quad z_n = -\frac{1}{\sqrt{n}}(P_{w_n+1}k_n)(\bar{F}_n + \check{F}_n),$$

$$z'_n = \frac{1}{\sqrt{n}}(P_1 F_n^2 - F_n^2).$$

Then (8.3) will follow from the next three properties:

$$\begin{aligned} \frac{1}{\sqrt{n}}V(\sqrt{n}, y_n)_t^n &\rightarrow \eta'(h, h)L_t, & \frac{1}{\sqrt{n}}V(\sqrt{n}, z_n)_t^n &\rightarrow 0, \\ \frac{1}{\sqrt{n}}V(\sqrt{n}, z'_n)_t^n &\rightarrow 0. \end{aligned} \quad (8.4)$$

2) By (5.8) we have $|z_n| \leq K n^{-\alpha/2}(\log n)|P_{w_n+1}k_n|$, while (5.9) gives $|P_{w_n+1}k_n| \leq K n^{1/2-\beta-\alpha/2}$ and $\lambda(|P_{w_n+1}k_n|) \leq K n^{(1-\beta-\alpha)/2}$ because $\beta > 1 - 2\alpha$. It readily follows that the sequence z_n satisfies (4.1), and the second property in (8.4) is satisfied.

3) In order to prove the last property in (8.4) it is enough by Theorem 4-1 to show that the sequence z'_n satisfies (4.2), since obviously $\lambda(z'_n) = 0$. By (3.4) and (5.5) we have $|P_j k_n(x+y) - P_j k_n(x)| \leq K|y|n^{\alpha/2}/j$. Then $|F_n(x+y) - F_n(x)| \leq K n^{\alpha/2}(1+|y|)\log n$, and

$$\begin{aligned} |z'_n(x)| &\leq K n^{\alpha/2-1/2}(\log n) \int \rho(y)dy(1+|y|)(|F_n(x+y)| + |F_n(x)|) \\ &\leq K n^{\alpha/2-1/2}(\log n) \left(|k_n(x)| + \int \rho(y)(1+|y|)|k_n(x+y)|dy \right) \\ &\quad + n^{\alpha-1/2}(\log n)^2 \left(\frac{n^{1/2-\alpha}}{1+|x|n^{-\beta/2}|^\gamma} + \frac{n^{(\gamma+1)(1/2-\alpha)}}{1+|x|^\gamma} \right), \end{aligned}$$

where the last equality follows from (5.7) and (3.9). It follows clearly (using again $\beta > 1 - 2\alpha$, and (5.5)) that

$$\begin{aligned} |z'_n| &\leq K(\log n)^2, & \lambda(|z'_n|) &\leq K(\log n)^2 n^{\beta/2}, \\ \beta_1(z'_n) &\leq K(\log n)^2 n^\beta, & \lambda(z'^2_n) &\leq K(\log n)^4 n^{\beta/2}. \end{aligned}$$

Hence the sequence z'_n satisfies (4.2) as soon as $\beta < 1/3$, which is met by (8.1): so we have the last property in (8.4).

4) In view of Theorem 4-1, the first property in (8.4) will hold if we prove that the sequence y_n satisfies (4.2) and $\lambda(y_n) \rightarrow \eta'(h, h)$. By (5.8), we have $|y_n| \leq n^{-\alpha/2}(\log n)|k_n|$, therefore (5.5) yields $|y_n| \leq K \log n$, and $\lambda(|y_n|) \leq K \log n$, and $\beta_1(y_n) \leq Kn^{1/2} - \alpha \log n$ and $\lambda(y_n^2) \leq K(\log n)^2$, which clearly imply that (4.2) is satisfied.

5) To simplify the notation we write $v_n = n^{\alpha-1/2}$. Set $\ell_n(x) = H_h(x) - \theta \hat{g}(x/v_n)/v_n$, so $k_n(x) = v_n n^\delta \ell_n(v_n x)$. A simple calculation shows $P_j k_n(x) = v_n n^\delta P_{jv_n^2} \ell_n(v_n x)$, and thus

$$F_n(x) = v_n n^\delta \sum_{j=0}^{w_n} P_{jv_n^2} \ell_n(v_n x). \quad (8.5)$$

Now, $\beta_i(|\ell_n|) \leq K$ for $i = 0, 1, 2$ (use the fact that $v_n \leq 1$). So we deduce from (3.1), from (3.3) and $\lambda(\ell_n) = 0$, and from (3.5), that

$$\begin{aligned} \left| \frac{1}{v_n} \int_0^{v_n^2} (P_s \ell_n) ds \right| &\leq K, \\ \left| \frac{1}{v_n} \int_{(w_n+1)v_n^2}^\infty (P_s \ell_n) ds \right| &\leq K \frac{1+|x|}{v_n^2 \sqrt{w_n}}, \\ \left| v_n P_{jv_n^2} \ell_n - \frac{1}{v_n} \int_{jv_n^2}^{(j+1)v_n^2} (P_s \ell_n) ds \right| &\leq \frac{K}{j^{3/2}}. \end{aligned}$$

Putting all these together with (8.5) yields

$$\left| F_n(x) - \frac{n^\delta}{v_n} \int_0^\infty P_s \ell_n(v_n x) ds \right| \leq Kn^\delta \left(1 + \frac{1}{v_n^2 \sqrt{w_n}} + \frac{|x|}{v_n \sqrt{w_n}} \right).$$

Hence $y'_n(x) = 2n^{-\delta} k_n(x) \int_0^\infty P_s \ell_n(v_n x) ds$ has $|y_n(x) - y'_n(x)| \leq K|k_n(x)|(n^{(\alpha-1)/2} + n^{1-\beta-3\alpha/2} + |x|n^{-(\alpha+\beta)/2})$. Since $\lambda(|k_n|) \leq Kn^\delta$ and $\beta_1(k_n) \leq Kn^{\delta-\alpha+1/2}$, we obtain $\lambda(|y_n - y'_n|) \leq K(n^{\alpha-1/2} + n^{(1-\beta-2\alpha)/2}) \rightarrow 0$ because $\beta + 2\alpha > 1$ and $\alpha < 1/2$. Further if $y''_n = 2\ell_n \int_0^\infty P_s \ell_n ds$ we have $\lambda(y'_n) = \lambda(y''_n)$. Hence it remains to prove that $\lambda(y''_n) \rightarrow \eta'(h, h)$.

6) Now we study $P_s \ell_n = P_s H_h - \theta P_s \hat{g}_n$, where $\hat{g}_n(x) = \hat{g}(x/v_n)/v_n$. In fact we compare $\int_0^\infty P_s \ell_n(x) ds$ with $G(x)$, where $G = G(H_h)$.

If we set $\gamma_s(x) = P_s H_h(x) - \frac{\theta}{\sqrt{2\pi s}} e^{-x^2/2s}$ we have $G(x) = \int_0^\infty \gamma_s(x) ds$. Set also $\gamma_s^n(x) = \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} - P_s \hat{g}_n(x)$, so that

$$\int_0^\infty P_s \ell_n(x) ds - G(x) = \theta \int_0^\infty \gamma_s^n(x) ds. \quad (8.6)$$

Now, $\beta_i(\hat{g}_n) = v_n^i \beta_i(\hat{g})$, so (3.1) and (3.3) and $v_n \leq 1$ yield

$$|\gamma_s(x)| + |\gamma_s^n(x)| \leq \frac{K}{\sqrt{s}}, \quad (8.7)$$

$$|\gamma_s(x)| \leq K \frac{1+|x|}{s^{3/2}}, \quad |\gamma_s^n(x)| \leq K v_n \frac{1+|x|}{s^{3/2}}. \quad (8.8)$$

Dividing the integral in (8.6) in two pieces, from 0 to ε and from ε to ∞ , and using (8.7) for the first piece and (8.8) for the second piece, we get

$$\left| \int_0^\infty P_s \ell_n(x) ds - G(x) \right| \leq K \left(\sqrt{\varepsilon} + \frac{v_n}{\sqrt{\varepsilon}} (1 + |x|) \right).$$

If $f_n = 2\ell_n G$, and taking $\varepsilon = v_n$ above, we thus get $|y_n''(x) - f_n(x)| \leq K |\ell_n(x)| \sqrt{v_n} (1 + |x|)$. Thus $\lambda(|y_n'' - f_n|) \leq K \sqrt{v_n} \rightarrow 0$, and it remains to prove that $\lambda(f_n) \rightarrow \eta'(h, h)$.

7) Apply (3.4) and the majorations $|\frac{1}{\sqrt{2\pi s}}(e^{-x^2/2s} - 1)| \leq K|x|/s$ (easily deduced from (3.6) and (3.7)) to obtain $|\gamma_s(x) - \gamma_0(x)| \leq K|x|/s$. Thus if we cut the integral $G(v_n x) - G(0) = \int_0^\infty (\gamma_s(v_n x) - \gamma_s(0)) ds$ in three pieces, from 0 to ε (use (8.7)), from ε to A (use what precedes) and from A to ∞ (use (8.8)), we get if $0 < \varepsilon < 1 < A < \infty$:

$$|G(v_n x) - G(0)| \leq K \left(\sqrt{\varepsilon} + v_n |x| \left(\log \frac{A}{\varepsilon} \right) + \frac{1 + v_n |x|}{\sqrt{A}} \right).$$

Taking $\varepsilon = v_n$ and $A = 1/v_n^2$ gives

$$|G(v_n x) - G(0)| \leq K \sqrt{v_n} (1 + |x|). \quad (8.9)$$

Now we can write

$$\begin{aligned} \lambda(f_n) &= 2 \int H_h(x) G(x) dx - 2\theta \int \hat{g} \left(\frac{x}{v_n} \right) G(x) \frac{1}{v_n} dx \\ &= 2\lambda(H_h G) - 2\theta \int \hat{g}(x) G(v_n x) dx. \end{aligned}$$

Recalling that $\eta'(h, h) = 2\lambda(H_h G) - 2\theta G(0)$ and that $\lambda(\hat{g}) = 1$, it follows that

$$\begin{aligned} |\lambda(f_n) - \eta'(h, h)| &\leq K \int \hat{g}(x) |G(v_n x) - G(0)| dx \\ &\leq K \sqrt{v_n} (1 + \beta_1(\hat{g})) \rightarrow 0 \end{aligned}$$

by (8.9), and we are finished.

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