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Fluctuation results for Brownian motion in a Poissonian potential

by

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ABSTRACT. – We consider Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. If the Brownian motion starts in 0 and ends in the closed ball with center $y \in \mathbb{R}^d$ and radius 1, then the transverse fluctuation is expected to be of order $|y|^{\xi}$. We prove that $\xi \leq 3/4$ and $\xi \geq 1/(d+1)$, whereas for the lower bound we have to assume that the dimension $d \geq 3$ or that we have a potential with lower bound $\lambda > 0$. As a second result we prove, in dimension $d = 2$, that $\chi \geq 1/8$, where χ is the critical exponent for the fluctutation for certain naturally defined random distance functions. © Elsevier, Paris

RÉSUMÉ. – Nous considérons un mouvement Brownien dans un potentiel Poissonien tronqué atteignant un lieu éloigné. Si le mouvement Brownien démarre en 0 et termine dans la boule de centre $y \in \mathbb{R}^d$ et de rayon 1, alors on attend que la fluctuation transversale soit d'ordre $|y|^{\xi}$. Nous montrons que $\xi \leq 3/4$ et $\xi \geq 1/(d+1)$, où pour la borne inférieure nous devons admettre que la dimension d soit supérieure ou égale à 3 ou que le potentiel ait une borne inférieure $\lambda > 0$. Comme deuxième résultat nous montrons, en dimension $d = 2$ que $\chi \geq 1/8$, où χ est l'exposant critique pour la fluctuation pour certaines fonctions de distance aléatoires définies naturellement. © Elsevier, Paris

0. INTRODUCTION

The theme of random motions in random potentials has attracted much interest recently. In the present work we want to consider Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. Our purpose here is to study some fluctuation properties of certain distance functions.

Description of the model. – Throughout this paper we look at Brownian motion in a truncated Poissonian potential. Let $\mathbb{P}$ stand for the Poisson law with fixed intensity $\nu > 0$ on the space Ω of simple pure point measures ω on $\mathbb{R}^d$. To the points x_i of the Poissonian cloud $\omega = \sum_i \delta_{x_i} \in \Omega$ we want to attach soft obstacles: To model the soft obstacles we take a fixed shape function $W(\cdot) \geq 0$, which is assumed to be bounded, measurable, compactly supported, not a.s. equal to 0 and

$$(0.1) \qquad W(\cdot) \text{ is rotationally invariant.}$$

By $a = a(W) > 0$ we denote the smallest possible $a \in \mathbb{R}^+$ such that $\text{supp}(W) \subset \bar{B}(0,a)$. For $M > 0$ (fixed truncation level), we define the truncated potential as follows:

$$(0.2) \quad V(x,\omega) = \left(\sum_i W(x-x_i)\right) \wedge M = \left(\int_{\mathbb{R}^d} W(x-y)\omega(dy)\right) \wedge M,$$

where $x \in \mathbb{R}^d$ and $\omega = \sum_i \delta_{x_i} \in \Omega$ is a simple pure locally finite point measure on $\mathbb{R}^d$.

In this medium, we look at Brownian motion. For $x \in \mathbb{R}^d$, $d \geq 2$, we denote by P_x the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting from x; $Z_. = Z_.(w)$, $w \in C(\mathbb{R}_+, \mathbb{R}^d)$, stands for the canonical process. For $x, y \in \mathbb{R}^d$, $\lambda \geq 0$, $\omega \in \Omega$, we define the following random variable

$$(0.3) \quad e_\lambda(x,y,\omega) = E_x\left[\exp\left\{-\int_0^{H(y)} (\lambda + V)(Z_s,\omega)ds\right\},\ H(y) < \infty\right],$$

where $H(y) = \inf\{s \geq 0, Z_s \in \bar{B}(y,1)\}$ is the entrance time of $Z_.$ into the closed ball $\bar{B}$ with center y and radius 1. We will call $e_\lambda(x,y,\omega)$ the gauge function for (x,y,ω,λ), it plays the role of the normalizing constant for the path measure of the conditioned process. So the measure of the conditioned process is described by

$$(0.4) \quad \widehat{P}_x^y(dw) = \frac{1}{e_\lambda(x,y,\omega)} \exp\left\{-\int_0^{H(y)} (\lambda + V)(Z_s,\omega)ds\right\} 1_{\{H(y)<\infty\}} P_x(dw).$$

We define

$$a_\lambda(x,y,\omega) = -\inf_{\bar{B}(x,1)} \log e_\lambda(\cdot,y,\omega). \tag{0.5}$$

$a_\lambda(x,y,\omega)$ is a nonnegative random variable that satisfies the triangle inequality. Thus,

$$d_\lambda(x,y,\omega) = \max(a_\lambda(x,y,\omega), a_\lambda(y,x,\omega)) \tag{0.6}$$

is a nonnegative random function that is symmetric and satisfies the triangle inequality [8]. We know that if $d \geq 3$ or $\lambda > 0$ or $\omega \not\equiv 0$ (which is $\mathbb{P}$-a.s. the case) $d_\lambda(\cdot,\cdot,\omega)$ is a distance function on $\mathbb{R}^d$, which induces the usual topology.

From the results in [6], we know that there exist norms $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$ for which

$$\mathbb{P}\text{-a.s.} \quad d_\lambda(0,y,\omega) \sim \alpha_\lambda(y) \text{ as } y \to \infty, \tag{0.7}$$

where in our case of rotationally invariant obstacles the quenched Lyapounov coefficients $\alpha_\lambda(\cdot)$ are proportional to the Euclidean norm on $\mathbb{R}^d$, this will imply that considerations on the Euclidean norm allow us to make statements on the quenched Lyapounov coefficients.

In this work we want to describe, how the Brownian paths are behaving when they feel the presence of the Poissonian distributed soft obstacles. As a first critical exponent, we look at the transverse fluctuation:

If we take a cylinder with axis passing through the origin and through our goal $y \in \mathbb{R}^d$ and with radius $|y|^\gamma$, $\gamma > 3/4$, then we will show in Theorem 1.1 that $\mathbb{P}$-a.s. the $\widehat{P}_0^y$-probablity of the event $A(y,\gamma)$, that the path does not leave this cylinder, tends to 1 as $|y| \to \infty$. On the other hand, if we take $\gamma < 1/(d+1)$, we are able to show (see Theorem 1.3) that for any sequence $(y_n)_n$ of goals tending to infinity, the $\mathbb{E}$-expectation of the random variable $\widehat{P}_0^{y_n}[A(y_n,\gamma)]$ does not tend to 1. These two estimates give us a lower bound and an upper bound on the critical exponent ξ, standing for the transverse fluctuation. Although this subdiffusive lower bound is far from the expected behavior of the paths, the proof is already mathematically involved. In dimension $d = 2$, we expect a superdiffusive behavior of the motion, we guess that the critical exponent ξ should equal $2/3$ (This conjecture is based on the assumption that the behavior of this model should essentially be the same as in the first-passage percolation model (see below). We remark that in a closely related model, we have proved that $\xi \geq 3/5$ if $d = 2$ and $\lambda > 0$ (see Theorem 0.2 of [10])). Whereas in higher dimensions, ξ should be greater or equal to $1/2$ (see Theorem 0.1 of [10] for the related model). But there are no rigorous proofs for these statements in the model considered here. In any case, the bounds which we derive here are a first approach to the expected behavior of the paths.

We also look at a second critical exponent χ, describing the asymptotic behavior of the variance of $-\log e_\lambda(0,y)$ for $|y| \to \infty$. The predicted asymptotic behavior for $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0,y))$ is of the order $|y|^{2\chi}$. We are able to give a nontrivial lower bound on χ (see Theorem 1.2). For general d the following inequality is true:

$$\chi \geq \frac{1-(d-1)\xi}{2}.$$

This (together with Theorem 1.1) gives us a lower bound of $1/8$ in dimension $d = 2$, whereas in dimensions $d \geq 3$ (under the assumption $\xi = 1/2$), we do not get any new interesting features.

Physically, for fixed λ, ω and y, the gauge function $e_\lambda(\cdot, y, \omega)$ can be interpreted as the $\lambda + V(\cdot, \omega)$-equilibrium potential of $\bar{B}(y,1)$ which formally satisfies

$$(0.8)\quad \begin{cases} \frac{1}{2}\Delta u - (\lambda + V)u = 0 & \text{on } \bar{B}(y,1)^c, \\ u = 1 & \text{on } \bar{B}(y,1), \\ u = 0 & \text{at infinity (for typical configurations).} \end{cases}$$

We will see, that the model we study here, has lots of common properties with the models in first-passage percolation (see Kesten [2], [3], Newman-Piza [5], Licea-Newman-Piza [4]). The critical exponents, χ and ξ, for the longitudinal and the transverse fluctuation are expected to depend on d, but nevertheless satisfy the scaling identity $\chi = 2\xi - 1$ for all d. But there is no proof for this scaling identity. In fact, loosely speaking, if χ' denotes the critical exponent for the fluctuation of the random distance function $d_\lambda(0, x, \omega)$ around the Lyapounov coefficient $\alpha_\lambda(x)$ (χ' is an exponent closely related to χ), Theorem 1.1 tells on a heuristic level that $\chi' \geq 2\xi - 1$ (In view of Corollary 3.5 of [8] we see that in any dimension $d \geq 2$, $\chi' \leq 1/2$). Heuristic arguments tell also that $\chi \leq 2\xi - 1$ should be true.

In the next section we give precise statements of all the results and an overview on the results already known. In Sections 2, 3 and 4 all the statements are proved: In Section 2 we will prove the upper bound on ξ, here we use essentially the fact, that we can compare our random distance function $d_\lambda(\cdot, \cdot, \omega)$ to the Euclidean distance. In Section 3 we will prove the lower bound for χ and finally in Section 4 we give the proof of the subdiffusive lower bound for ξ, the main tool for these two bounds will be a martingale method similar to the methods used in the articles of Newman-Piza [5] and Licea-Newman-Piza [4].

1. SETTINGS AND RESULTS

We want to recall that in the whole article we only consider models with rotationally invariant obstacles. It follows that also our quenched Lyapounov coefficients $\alpha_\lambda(\cdot)$ are rotationally invariant. This means that $\alpha_\lambda(\cdot)$ is a norm on $\mathbb{R}^d$, which is proportional to the Euclidean norm.

We take x a non zero vector in $\mathbb{R}^d$. We define the axis L_x to be the line $\{\alpha x \in \mathbb{R}^d \ ; \ \alpha \in \mathbb{R}\}$ through x and the origin. Take $r \geq 0$. We define $\tilde{Z}(x,r) = \{z \in \mathbb{R}^d \ ; \ \mathrm{d}(z, L_x) \leq r\}$ to be the cylinder with axis L_x and radius r, where $\mathrm{d}(\cdot,\cdot)$ is the Euclidean distance. For technical reasons we cut off the ends of the cylinders. For $x \neq 0$ and $1 \geq \gamma > 0$, let $S(x,\gamma)$ be the following slab $S(x,\gamma) = \{z \in \mathbb{R}^d \ ; \ -|x|^\gamma \leq \langle z, \frac{x}{|x|}\rangle \leq |x| + |x|^\gamma\}$, then we define

$$Z(x,\gamma) = \tilde{Z}(x, |x|^\gamma) \cap S(x,\gamma). \tag{1.1}$$

For the boundary of $Z(x,\gamma)$ we use the notation $\partial Z(x,\gamma)$.

We are now able to define the event of our main interest. Let $A(x,\gamma)$ be the event that the path of the Brownian motion starting in 0 with goal $\bar{B}(x,1)$ does **not** leave the cylinder $Z(x,\gamma)$: For $x \neq 0$ and $\gamma > 0$,

$$\begin{aligned} A(x,\gamma) = \big\{ w \in C(\mathbb{R}_+, \mathbb{R}^d);\ w(0) = 0,\ H(x) < \infty \\ \text{and } Z_s(w) \in Z(x,\gamma) \text{ for all } s \leq H(x) \ \big\}. \end{aligned} \tag{1.2}$$

ξ is then the following critical exponent:

$$\xi = \inf\left\{\gamma > 0 \ ; \ \mathbb{P}\text{-a.s.} \lim_{|y|\to\infty} \widehat{P}_0^y[A(y,\gamma)] = 1\right\}. \tag{1.3}$$

We consider for $d \geq 2$ the model described in the introduction, where the obstacles are rotationally invariant and the Poissonian potential is truncated at the level $M > 0$.

THEOREM 1.1.

$$\xi \leq \frac{3}{4}. \tag{1.4}$$

Remark. – The above theorem gives us a superdiffusive upper bound on the transverse fluctuation of our Brownian motion in a truncated Poissonian potential. The proof of the theorem uses essentially the fact, that the obstacles are rotationally invariant and that the Poissonian potential is truncated at a fixed level. In Lemma 2.1 we will show that the random variable $\widehat{P}_0^y[A(y,\gamma)]$ is continuous in y, and therefore we know that $\liminf_{|y|\to\infty} \widehat{P}_0^y[A(y,\gamma)]$ is also a random variable. In the proof of Theorem 1.1, we show the fact that if $\gamma \in (3/4, 1)$, then, $\mathbb{P}$-a.s. for large $|y|$, $\widehat{P}_0^y[A(y,\gamma)] \geq 1 - c|y|^d \exp\{-c'|y|^{2\gamma-1}\}$, for suitable $c > 0$ and $c' > 0$, which of course goes to 1 as $|y|$ tends to infinity.

Once we know this statement, we are able to prove the divergence of the variance of $-\log e_\lambda(0, y, \omega)$ in dimensions $d = 2$. We define the critical exponent for longitudinal fluctuation

$$(1.5) \qquad \chi = \sup\Big\{\kappa \geq 0\ ;\ \exists C > 0 \text{ with } \mathbb{V}\mathrm{ar}(-\log e_\lambda(0,y)) \geq C|y|^{2\kappa} \text{ for all } |y| > 1\Big\}.$$

We have the following theorem:

THEOREM 1.2. – *For* $d \geq 2$,

$$(1.6) \qquad \chi \geq \frac{1-(d-1)\xi}{2}.$$

In view of (1.4) *for* $d = 2$,

$$(1.7) \qquad \chi \geq 1/8.$$

Our next aim is to find a lower bound on the transverse fluctuation. For the start we get a subdiffusive lower bound on ξ_0 defined as

$$(1.8) \qquad \xi_0 = \inf\left\{\gamma \geq 0\ ;\ \limsup_{|y|\to\infty} \mathbb{E}\left[\widehat{P}_0^y[A(y,\gamma)]\right] = 1\right\}.$$

Of course, $\xi_0 \leq \xi$. We want to mention that in the definition of ξ_0 we could restrict ourselves to $y \in \mathbb{R}^d$ of the form $y = (|y|, 0, \ldots, 0)$. Indeed, in the case of rotationally invariant obstacles, the above expectation does not depend on the direction.

THEOREM 1.3. – *Assume* $d \geq 3$ *or* $\lambda > 0$, *then*

$$(1.9) \qquad \xi_0 \geq \frac{1}{d+1}.$$

Remark. – The first statement in Theorem 1.2 and the statement in Theorem 1.3 are also valid in a more general context; in fact in the two proofs we do not use that the shape function is rotationally invariant. We only require this assumption in the proof of Theorem 1.1 and as a consequence for the lower bound on χ in dimension $d = 2$.

We sometimes use the terminology of first-passage percolation because the situation in the case of Brownian motion in a truncated Poissonian potential looks very similar. In first-passage percolation, to prove the statements one has often (unverified) assumptions on the curvature of the asymptotic shape, here we do not have these problems because in the case of rotationally invariant obstacles one knows that the asymptotic shape is a ball with positive radius. This fact indicates an advantage of our model.

To close this chapter we give some further general notations and state some already known results we will need later on. We usually denote positive constants by $c_1, c_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. These constants will only depend on the invariant parameters of our model, namely the dimension d, the intensity ν, the shape function W, the truncation level M and the parameter λ. The constants which are used in the whole article are denoted by γ_i, whereas c_i is only used for local calculations in the proofs.

If U is an open subset of $\mathbb{R}^d$, we introduce the $(\lambda + V)$-Green function relative to U: Take $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ then

$$(1.10) \qquad g_{\lambda,U}(x,y,\omega) = \int_0^\infty e^{-\lambda s} r_U(s,x,y,\omega)\,ds \in (0,\infty],$$

where r_U, for a non void U, is known to be the kernel of the self-adjoint semigroup on $L^2(U, dx)$ generated by $-\frac{1}{2}\Delta + V$ with Dirichlet boundary conditions; for $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ and $t > 0$ is.

$$(1.11) \qquad r_U(t,x,y,\omega) = p(t,x,y) \cdot E^t_{x,y}\left[\exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\}, T_U > t\right],$$

with $p(t,x,y)$ the Brownian transition density, $E^t_{x,y}$ the Brownian bridge in time t from x to y and $T_U = \inf\{s \geq 0; Z_s \notin U\}$ the exit time from U. When $U = \mathbb{R}^d$, we will drop the subscript U from the notation.

Next we recall some properties of $e_\lambda(x,y,\omega)$ and $d_\lambda(x,y,\omega)$. For $e_\lambda(x,y,\omega)$ we have by a tubular estimate for Brownian motion the following nice lower bound (see for instance (1.35) in [8]):

$$(1.12) \qquad e_\lambda(x,y,\omega) \geq \gamma_1 \exp\{-\gamma_2|y-x|\},$$

with $\gamma_1 \in (0,1)$ and $\gamma_2 > 0$.

From [8] Proposition 1.3, we have a shape theorem: on a set of full $\mathbb{P}$-measure we know, for $\lambda \geq 0$, that

$$(1.13) \qquad \lim_{y \to \infty} \frac{1}{|y|} |d_\lambda(0, y, \omega) - \alpha_\lambda(y)| = 0,$$

the convergence also holds in $L^1(\mathbb{P})$, and one can replace $d_\lambda(0, y, \omega)$ by $-\log e_\lambda(0, y, \omega)$, $-\log g_\lambda(0, y, \omega)$ or $a_\lambda(0, y, \omega)$.

Next we introduce a paving of $\mathbb{R}^d$. For $q \in \mathbb{Z}^d$, we consider the cubes of size l and center q

$$(1.14) \qquad C(q) = \left\{ z \in \mathbb{R}^d;\ -\frac{l}{2} \leq z^i - lq^i < \frac{l}{2},\ i = 1, .., d \right\},$$

with $l(d, \nu, a) \in (d(4+8a), \infty)$ fixed, but large enough, see for instance [7] or [8], and for $z \in \mathbb{R}^d$, z^i denotes the i-th coordinate of z for $i = 1, \ldots, d$. We also want to introduce a fixed ordering $q_1, q_2, \ldots$ of all $q \in \mathbb{Z}^d$. So we get also a ordering of our cubes: We define $C_k = C(q_k)$ for all $k \in \mathbb{N}$. For $y \in \mathbb{R}^d$ and $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ with $H(y) < \infty$ we introduce the random lattice animal

$$(1.15) \qquad \mathcal{A}(w) = \{k \in \mathbb{N}\ ; H_k \leq H(y)\},$$

where H_k is the entrance time of $Z.$ into the closed cube $\bar{C}_k$. We know from [8] formula (1.31) that there exists a $\gamma_3(d, \nu, W, M)$ small enough such that for $x \in C(0)$ and $y \in \mathbb{R}^d$:

$$(1.16) \quad E_x\left[\exp\left\{\gamma_3 |\mathcal{A}| - \int_0^{H(y)} V(Z_s, \omega) ds\right\}, H(y) < \infty\right] \leq 2^{N_0(\omega)/2},$$

where $N_0(\omega)$, $\omega \in \Omega$, is a random variable with $\mathbb{E}\left[2^{N_0(\omega)/2}\right] < \infty$, and $|\mathcal{A}|$ denotes the (random) number of cubes visited by the path. With the help of this exponential bound we get very important estimates on the expected value of any power of the number $|\mathcal{A}|$ of visited cubes.

Finally we quote the (for us) important part of Lemma 1.2 of Sznitman [8]. For $|x - y| > 4$ and $\omega \in \Omega$

$$(1.17) \qquad |-\log e_\lambda(x, y, \omega) - d_\lambda(x, y, \omega)| \leq \gamma_4(1 + F_\lambda(x) + F_\lambda(y)),$$

where for $x \in \mathbb{R}^d$ and $\omega \in \Omega$

$$\begin{aligned} F_\lambda(x) &\leq \gamma_5, && \text{if } d \geq 3 \text{ or } \lambda > 0, \\ &\leq \gamma_6\big(1 + \log^+(\log \operatorname{dist}(x, \operatorname{supp}(\omega)))\big), && \text{if } d = 2 \text{ and } \lambda = 0, \end{aligned}$$

provided $\operatorname{supp}(\omega)$ denotes the support of ω.

2. PROOF OF THE SUPERDIFFUSIVE UPPER BOUND ON ξ

First we start with a continuity result. We want to show that the limit in the definition of ξ is a well-defined random variable. Therefore we have to show that $\widehat{P}_0^y[A(y,\gamma)]$ is continuous in y.

LEMMA 2.1. – *For $\lambda \geq 0$ and $\gamma > 0$, the functions $(y,\omega) \to e_\lambda(0,y,\omega)$ and $(y,\omega) \to \widehat{P}_0^y[A(y,\gamma)]$ are measurable in ω and for all $|y| > 1$ continuous in y.*

We give the proof of this lemma in the appendix. Next we state a geometric lemma. If 0 and $y \in \mathbb{R}^d$ are on the axis of the cylinder with radius $|y|^\gamma$ we want to measure the cost of a detour to the boundary of the cylinder, as compared to the "direct way" from 0 to y.

LEMMA 2.2. – *We take $\gamma \in (0,1]$. There exists $\gamma_7 \in (0,\infty)$ such that for all $y \in \mathbb{R}^d$ with $|y| > 1$ and $z \in \partial Z(y,\gamma)$ the following is true:*

$$|0 - z| + |z - y| \geq |0 - y| + \gamma_7 |y|^{2\gamma - 1}. \tag{2.1}$$

Remark. – The lemma will be important for us, because in the case of rotationally invariant obstacles, our quenched Lyapounov coefficients are proportional to the Euclidean norm, so in fact the above lemma is a claim for our quenched Lyapounov coefficients.

Proof. – Take $y \in \mathbb{R}^d$ fixed. Without loss of generality we may assume that y has the same direction as the first unit vector in $\mathbb{R}^d$. By z_1 we denote the first coordinate of the vector $z \in \partial Z(y,\gamma)$.

If $z_1 = -|y|^\gamma$ or if $z_1 = |y| + |y|^\gamma$, then

$$|0 - z| + |z - y| \geq |0 - y| + 2|y|^\gamma \geq |0 - y| + 2|y|^{2\gamma - 1}.$$

If $z_1 \in (-|y|, |y| + |y|^\gamma)$, then we see, that if we embed an ellipsoid into the cylinder with focal points 0 and y and tangent to the cylinder, that $|0 - z| + |z - y|$ is minimal for $z_1 = |y|/2$. Therefore

$$|0 - z| + |z - y| \geq 2\sqrt{\frac{|y|^2}{4} + |y|^{2\gamma}} = 2|y|\sqrt{\frac{1}{4} + |y|^{2\gamma - 2}}.$$

Now there exists a $c_1 > 0$ with

$$\frac{1}{4} + |y|^{2\gamma - 2} \geq \frac{1}{4} + c_1 |y|^{2\gamma - 2} + c_1^2 |y|^{4\gamma - 4} = \left(\frac{1}{2} + c_1 |y|^{2\gamma - 2}\right)^2.$$

So we see that

$$|0-z|+|z-y| \geq 2|y|\sqrt{\left(\frac{1}{2}+c_1|y|^{2\gamma-2}\right)^2} = |y|+2c_1|y|^{2\gamma-1},$$

which completes the proof, if we take $\gamma_7 \in (0,\infty)$ suitable. □

We recall the following result from [8]: Corollary 3.5 tells us that under assumption (0.1) in dimensions $d \geq 2$, $\mathbb{P}$-a.s., for large $|y|$,

$$|d_\lambda(0,y)-\alpha_\lambda(y)| \leq \gamma_8(1+|y|^{1/2}\log^2|y|). \tag{2.2}$$

Our aim is to formulate a similar result for the random distances $d_\lambda(0,z)$ and $d_\lambda(z,y)$ if z is on the boundary of the cylinder $Z(y,\gamma)$.

LEMMA 2.3. – *Assume* (0.1). *When $d \geq 2$, $\gamma \in (0,1]$, then $\mathbb{P}$-a.s., for large $|y|$ and $z \in \partial Z(y,\gamma)$, the following holds*

$$|d_\lambda(z,y)-\alpha_\lambda(y-z)| \leq \gamma_9(1+|y-z|^{1/2}\log^2|y-z|) \tag{2.3}$$

and

$$|d_\lambda(0,z)-\alpha_\lambda(z)| \leq \gamma_{10}(1+|z|^{1/2}\log^2|z|). \tag{2.4}$$

Remark. – If $d \geq 3$ or $\lambda > 0$, one can improve the bounds of the above inequalities. But for our purposes we will not need any better bounds. The important thing in the proof will be that the distance of the two points is growing faster than the big holes in the Poissonian cloud.

Proof. – First we want to prove (2.3) in the case $d \geq 3$ or $\lambda > 0$. We pick a fixed $y \in \mathbb{R}^d$, such that $|y-z| > 4$ for all $z \in \partial Z(y,\gamma)$. Take $z \in \partial Z(y,\gamma)$, then Theorem 2.1 from [8] tells us, that for $0 \leq u \leq c_1|y-z|$,

$$\begin{aligned} \mathbb{P}\Big[|d_\lambda(z,y)-D_\lambda(0,y-z)| > u\sqrt{|y-z|}\Big] \\ = \mathbb{P}\Big[|d_\lambda(0,y-z)-D_\lambda(0,y-z)| > u\sqrt{|y-z|}\Big] \\ \leq c_2\exp\{-c_3u\}, \end{aligned} \tag{2.5}$$

where $D_\lambda(0,x) = \mathbb{E}[d_\lambda(0,x)]$.

The first step is to verify the lemma for a countable set of points because we want to use the Borel-Cantelli lemma. For every $n \in \mathbb{N}$, we take a finite covering of $\partial B(0,n)$ with balls $B(y,1)$ such that $|y| = n$. We denote the set of the centers of these balls by C_n. We may and will choose C_n such that $|C_n| \leq c_4 n^{d-1}$ (By $|A|$ we denote the number of points in A). For $\partial Z(y,\gamma)$ we do exactly the same and we denote the set of centers by $\mathcal{C}_y$. We choose $\mathcal{C}_y$ such that $|\mathcal{C}_y| \leq c_5 |y|^{1+(d-1)\gamma}$. For a fixed $n \in \mathbb{N}$, $y \in C_n$ and $z \in \mathcal{C}_y$ we define the following event

$$A_{n,y,z} = \left\{\omega \in \Omega;\ |d_\lambda(z,y) - \alpha_\lambda(y-z)| \geq c_6(1 + |y-z|^{1/2} \log|y-z|)\right\}.$$

We will choose a suitable but fixed c_6 which is determined in (2.7) below. From Corollary 3.4 of [8] we know that, for $|y| \geq 1$,

$$0 \leq D_\lambda(0,y) - \alpha_\lambda(y) \leq c_7(1 + |y|^{1/2} \log|y|). \tag{2.6}$$

So the triangle inequality and (2.6) imply for $n \in \mathbb{N}$, $y \in C_n$ and $z \in \mathcal{C}_y$

$$A_{n,y,z} \subset \left\{\omega \in \Omega;\ |d_\lambda(z,y) - D_\lambda(0,y-z)| \geq (c_6 - c_7)|y-z|^{1/2} \log|y-z|\right\}.$$

Now we choose c_6. Let c_6 be large enough such that

$$c_3\gamma(c_6 - c_7) - d - (d-1)\gamma \geq 2. \tag{2.7}$$

Then we choose n_0 such that $(c_6 - c_7)\log|y-z| \leq c_1|y-z|$ for all $n \geq n_0$, $y \in C_n$ and $z \in \mathcal{C}_y$. In view of (2.7), we get

$$\begin{aligned} \mathbb{P}[A_{n,y,z}] &\leq c_2 \exp\{-c_3(c_6 - c_7)\log|y-z|\} \\ &\leq c_2|y|^{-c_3\gamma(c_6-c_7)} \\ &\leq c_2 n^{-d-(d-1)\gamma-2} \qquad \text{for all } n \geq n_0. \end{aligned}$$

Therefore, the following sum is finite

$$\sum_{n\geq 1} \sum_{y \in C_n} \sum_{z \in \mathcal{C}_y} \mathbb{P}[A_{n,y,z}] < \infty. \tag{2.8}$$

The proof of (2.3) in the case $d \geq 3$ or $\lambda > 0$ follows with a Borel-Cantelli argument and the observation that $\sup_{|x-y|\leq 1} d_\lambda(x,y)$ is uniformly bounded by (1.12).

The proof of (2.3) in the case $d = 2$ and $\lambda = 0$ is almost the same, the only difference is that one has to use Theorem 2.5 of [8] instead of Theorem 2.1. It is at this point where the upper bound is weakened, i.e., here the power two in the logarithm is coming into the calculations. The proof of (2.4) goes analogously. □

At this stage we can combine Lemma 2.1, Lemma 2.2 and Lemma 2.3 to find the upper bound on ξ. The idea is that with the help of the strong

Markov property and the above lemmas one sees that the detour over the boundary of the cylinder costs too much.

Proof of Theorem 1.1. – Take $\gamma \in (3/4, 1)$ fixed. We want to show that $\mathbb{P}$-a.s., for large $|y|$,

$$\widehat{P}_0^y[A(y,\gamma)] \geq 1 - \gamma_{11}|y|^d \exp\{-\gamma_{12}|y|^{2\gamma-1}\}. \tag{2.9}$$

From Lemma 2.2 we know that for $|y| > 1$ and $z \in \partial Z(y,\gamma)$ the following holds

$$|0-z| + |z-y| \geq |0-y| + \gamma_7|y|^{2\gamma-1}.$$

We multiply the above inequality by $\alpha_\lambda(e_1)$. Thanks to the rotationally invariant obstacles, we get the following inequality for our quenched Lyapounov coefficents $\alpha_\lambda(\cdot)$:

$$\alpha_\lambda(z) + \alpha_\lambda(z-y) \geq \alpha_\lambda(y) + \gamma_7\alpha_\lambda(e_1)|y|^{2\gamma-1}. \tag{2.10}$$

Notice that if our path from 0 to y runs over the boundary of the cylinder $Z(y,\gamma)$, we are allowed to add an extra term of order $2\gamma - 1$ to the right-hand side of the above triangle inequality. In view of Lemma 2.3, (2.2) and (2.10), we get $\mathbb{P}$-a.s., for $|y|$ large enough and any $z \in \partial Z(y,\gamma)$,

$$d_\lambda(0,z) + d_\lambda(z,y) \geq d_\lambda(0,y) + \gamma_7\alpha_\lambda(e_1)|y|^{2\gamma-1} - c_1(1 + |y|^{1/2}\log^2|y|),$$

for a suitable c_1. Here we see the importance of the order of the added term in (2.10). We want to have γ such that the correction term on the right-hand side of the above inequality stays positive. We see that $|y|^{2\gamma-1}$ tends faster to infinity than $|y|^{1/2}\log^2|y|$ whenever $\gamma > 3/4$. Thus, $\mathbb{P}$-a.s., for large $|y|$ and $z \in \partial Z(y,\gamma)$,

$$d_\lambda(0,z) + d_\lambda(z,y) \geq d_\lambda(0,y) + c_2|y|^{2\gamma-1}. \tag{2.11}$$

As in the proof of Lemma 2.3 we take a finite covering of $\partial Z(y,\gamma)$ with balls $B(z,1)$. We denote by $\mathcal{C}_y$ the set of the centers of these balls. We are able to choose $\mathcal{C}_y$ such that $|\mathcal{C}_y| \leq c_3|y|^{1+(d-1)\gamma}$. With the help of the strong Markov property we find

$$\begin{aligned}
\widehat{P}_0^y[A(y,\gamma)^c] &\leq \sum_{z\in\mathcal{C}_y} \widehat{P}_0^y[H(z) \leq H(y)] \\
&= \sum_{z\in\mathcal{C}_y} \frac{1}{e_\lambda(0,y,\omega)} E_0\left[\exp\left\{-\int_0^{H(y)} (\lambda+V)(Z_s)ds\right\}, H(z) \leq H(y) < \infty\right] \\
&\leq \sum_{z\in\mathcal{C}_y} \frac{1}{e_\lambda(0,y,\omega)} e_\lambda(0,z,\omega) \sup_{z'\in B(z,1)} e_\lambda(z',y,\omega).
\end{aligned} \tag{2.12}$$

First we want to treat the case $d \geq 3$ or $\lambda > 0$: We find with the help of (1.17) and (2.11), that $\mathbb{P}$-a.s., for large $|y|$,

$$\begin{aligned}\widehat{P}_0^y[A(y,\gamma)^c] &\leq \sum_{z \in \mathcal{C}_y} \exp\{d_\lambda(0,y,\omega) - d_\lambda(0,z,\omega) - d_\lambda(z,y,\omega) + c_4\} \\ &\leq \sum_{z \in \mathcal{C}_y} c_5 \exp\{-c_2|y|^{2\gamma-1}\} \\ &\leq c_3 c_5 |y|^{1+(d-1)\gamma} \exp\{-c_2|y|^{2\gamma-1}\}.\end{aligned}$$

That proves the theorem in the first case.

Now, the case when $d = 2$ and $\lambda = 0$ is a little bit more difficult, because (1.17) does not have that easy form as in the first case. If the Brownian motion is recurrent, we have to be sure, that there are obstacles in our space to get good results: As in (1.23) of [8] we see that $\mathbb{P}$-a.s., for large $|y|$,

$$\sup\{d(y, \text{supp}(\omega)), d(z, \text{supp}(\omega)), d(0, \text{supp}(\omega))\} \leq c_6 \log^{1/d} |y|.$$

This follows from standard estimates on the Poissonian distributed cloud in $\mathbb{R}^d$. Therefore we have $\mathbb{P}$-a.s., for all large $|y|$, an upper bound on the function F_λ (see (1.17))

$$\sup\{F_\lambda(0), F_\lambda(y), F_\lambda(z)\} \leq c_7 + c_8 \log\log\log |y|.$$

So we find using (2.11) and (2.12) that $\mathbb{P}$-a.s., for large $|y|$,

$$\begin{aligned}\widehat{P}_0^y[A(y,\gamma)^c] &\leq \sum_{z \in \mathcal{C}_y} \exp\{d_\lambda(0,y) - d_\lambda(0,z) - d_\lambda(z,y) \\ &\qquad + c_9 + c_{10} \log\log\log |y|\} \\ &\leq c_{11} |y|^{1+(d-1)\gamma} (\log\log |y|)^{c_{10}} \exp\{-c_2|y|^{2\gamma-1}\}.\end{aligned}$$

This completes the proof of our theorem also in the second case. □

3. PROOF OF THE DIVERGENCE OF THE VARIANCE

We want to derive now the proof of Theorem 1.2. The proof has a very similar structure as an analogous power law result on the divergence of shape fluctuations in first-passage percolation. Our main tool will be the martingale technique used in the spirit of Wehr-Aizenman [9], Aizenman-Wehr [1], Newman-Piza [5] and many other authors.

Proof of Theorem 1.2. – Take U a subset of $\mathbb{R}^d$, we define the following σ-algebra,

$$\mathcal{F}(U) = \sigma\{\omega(A); A \in \mathcal{B}(\mathbb{R}^d) \text{ and } A \subseteq U\}.$$

We introduce then the following filtration $(\mathcal{F}_k)_{k\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \ \mathcal{F}_k = \mathcal{F}\left(\bigcup_{i=1}^{k} C_k\right), \ k \geq 1, \tag{3.1}$$

with C_i, $i \geq 1$, the cubes defined in Chapter 1. For a fixed $y \in \mathbb{R}^d$, introduce the following non-negative martingale

$$M_k = \mathbb{E}[-\log e_\lambda(0,y) \mid \mathcal{F}_k], \ k \geq 0.$$

In view of (1.12) $-\log e_\lambda(0,y)$ is bounded above and M_k converges $\mathbb{P}$-a.s. and in $L^p(\mathbb{P})$, $p \in [1,\infty)$, to $-\log e_\lambda(0,y)$. By standard martingale identities we get

$$\mathbb{V}\text{ar}\,(-\log e_\lambda(0,y)) = \sum_{k\geq 1} \mathbb{V}\text{ar}\,(\Delta M_k), \tag{3.2}$$

where $\Delta M_k = M_k - M_{k-1}$.

We denote by $\mathcal{G}_k = \mathcal{F}(C_k)$. So $\mathcal{F}_k = \mathcal{G}_1 \vee \mathcal{G}_2 \vee \cdots \vee \mathcal{G}_k$, with $\mathcal{G}_1 \vee \mathcal{G}_2$ the smallest σ-algebra containing $\mathcal{G}_1$ and $\mathcal{G}_2$. Because $\mathcal{G}_k \subset \mathcal{F}_k$ and $\mathcal{G}_k \perp \mathcal{F}_{k-1}$, we find

$$\begin{aligned} \mathbb{V}\text{ar}\,(-\log e_\lambda(0,y)) &= \sum_{k\geq 1} \mathbb{V}\text{ar}\,(\Delta M_k) \\ &\geq \sum_{k\geq 1} \mathbb{V}\text{ar}\,(\ \mathbb{E}[\Delta M_k|\mathcal{G}_k]\ ) \\ &= \sum_{k\geq 1} \mathbb{V}\text{ar}\,(\ \mathbb{E}[-\log e_\lambda(0,y)|\mathcal{G}_k]\ ). \end{aligned} \tag{3.3}$$

Our next purpose is to apply similar considerations as Lemma 3 of [5]: If $\omega \in \Omega$ is a cloud configuration, we denote by $\widehat{\omega}_k$ the restriction of ω to C_k^c and by ω_k the restriction of ω to C_k, so we can write $\omega = (\omega_k, \widehat{\omega}_k)$. We consider the following two disjoint events on C_k:

$$D_{0,k} = \{\omega_k;\ \omega_k(\bar{C}_k) = 0\},$$

this is the event that the cube $\bar{C}_k$ receives no point of the cloud. Whereas

$$D_{1,k} = \{\omega_k;\ \omega_k(B(lq_k, 1)) \geq 1\}$$

is the event that we have at least one point of the Poissonian cloud in the center (i.e., in the closed ball with center lq_k and radius 1) of the cube C_k. We then define, for $\delta = 0$ or 1,

$$D_k^\delta = \{\omega \in \Omega \text{ with } \omega_k \in D_{\delta,k}\}.$$

Of course, D_k^0 and D_k^1 are disjoint and $\mathcal{G}_k$ measurable. We denote by $p = \mathbb{P}[D_k^0] > 0$ and by $q = \mathbb{P}[D_k^1] > 0$.

$$\begin{aligned}
&\mathbb{E}[-\log e_\lambda(0, y, \omega)|\mathcal{G}_k] \cdot 1_{D_k^0}(\omega) \\
&= \mathbb{E}[-\log e_\lambda(0, y, \omega) 1_{D_k^0}(\omega)|\mathcal{G}_k] \\
&\leq \mathbb{E}\left[\sup_{\sigma_k^0 \in D_{0,k}} -\log e_\lambda(0, y, (\sigma_k^0, \widehat{\omega}_k)) 1_{D_k^0}(\omega) \middle| \mathcal{G}_k\right] \\
&= \mathbb{E}\left[\sup_{\sigma_k^0 \in D_{0,k}} -\log e_\lambda(0, y, (\sigma_k^0, \widehat{\omega}_k))\right] \cdot 1_{D_k^0}(\omega),
\end{aligned}$$

analogously one gets the following lower bound

$$\begin{aligned}
&\mathbb{E}[-\log e_\lambda(0, y, \omega)|\mathcal{G}_k] \cdot 1_{D_k^1}(\omega) \\
&\geq \mathbb{E}\left[\inf_{\sigma_k^1 \in D_{1,k}} -\log e_\lambda(0, y, (\sigma_k^1, \widehat{\omega}_k))\right] \cdot 1_{D_k^1}(\omega).
\end{aligned}$$

We remark that

$$\sup_{\{\sigma_k^0 \in D_{0,k}\}} -\log e_\lambda(0, y, (\sigma_k^0, \cdot)) \text{ and } \inf_{\{\sigma_k^1 \in D_{1,k}\}} -\log e_\lambda(0, y, (\sigma_k^1, \cdot))$$

are measurable. This can be seen by using an approximation of the cloud configuration by cloud configurations with rational coordinates. We define x_0 and x_1 as follows,

$$\begin{aligned}
x_0 &= \frac{\mathbb{E}\left[\mathbb{E}[-\log e_\lambda(0, y, \omega)|\mathcal{G}_k] \cdot 1_{D_k^0}(\omega)\right]}{\mathbb{P}[D_k^0]} \\
&\leq \mathbb{E}\left[\sup_{\sigma_k^0 \in D_{0,k}} -\log e_\lambda(0, y, (\sigma_k^0, \widehat{\omega}_k))\right],
\end{aligned}$$

and

$$x_1 = \frac{\mathbb{E}\Big[\mathbb{E}[-\log e_\lambda(0,y,\omega)|\mathcal{G}_k]\cdot 1_{D_k^1}(\omega)\Big]}{\mathbb{P}[D_k^1]}$$
$$\geq \mathbb{E}\left[\inf_{\sigma_k^1\in D_{1,k}} -\log e_\lambda(0,y,(\sigma_k^1,\widehat{\omega}_k))\right].$$

To simplify the notation we introduce:

$$E_k(0,y,\widehat{\omega}_k) = \inf_{\sigma_k^1\in D_{1,k}} -\log e_\lambda(0,y,(\sigma_k^1,\widehat{\omega}_k)) - \sup_{\sigma_k^0\in D_{0,k}} -\log e_\lambda(0,y,(\sigma_k^0,\widehat{\omega}_k)).$$

In view of the above bounds on x_0 and x_1, we see that $x_1 - x_0 \geq \mathbb{E}[E_k] \geq 0$ and therefore

$$(3.4) \qquad (x_1 - x_0)^2 \;\geq\; \mathbb{E}[E_k]^2.$$

Using Lemma 3 of [5], we have the following estimate

$$\mathbb{V}\mathrm{ar}\;(\;\mathbb{E}[-\log e_\lambda(0,y)|\mathcal{G}_k]\;) \geq \frac{\mathbb{P}[D_k^0]\mathbb{P}[D_k^1]}{\mathbb{P}[D_k^0]+\mathbb{P}[D_k^1]}\;(x_1-x_0)^2.$$

Therefore, together with (3.3) and (3.4), this is yielding a lower bound for the variance of $-\log e_\lambda(0,y)$

$$(3.5) \qquad \mathbb{V}\mathrm{ar}\;(-\log e_\lambda(0,y)) \geq \frac{pq}{p+q}\sum_{k\geq 1}\mathbb{E}[E_k]^2.$$

Take $\gamma > \xi$ and define $\mathcal{E}_y = \{k \in \mathbb{N} \text{ with } C_k \cap Z(y,\gamma) \neq \emptyset\}$, to be all the cubes, that intersect the cylinder $Z(y,\gamma)$ with radius $|y|^\gamma$. We know that $|\mathcal{E}_y| \leq c_1|y|^{1+(d-1)\gamma}$ for a suitable $c_1 \in (0,\infty)$. With the help of Cauchy-Schwarz inequality, we get

$$\mathbb{V}\mathrm{ar}\;(-\log e_\lambda(0,y)) \geq \frac{pq}{p+q}\sum_{k\in\mathcal{E}_y}\mathbb{E}[E_k]^2$$
$$\geq \frac{pq}{p+q}\frac{1}{|\mathcal{E}_y|}\left(\sum_{k\in\mathcal{E}_y}\mathbb{E}[E_k]\right)^2$$
$$\geq c_2|y|^{-1-(d-1)\gamma}\left(\sum_{k\in\mathcal{E}_y}\mathbb{E}[E_k]\right)^2.$$

If we are able to prove that there exists a $c_3 \in (0, \infty)$ with

$$\liminf_{|y|\to\infty} \frac{1}{|y|} \sum_{k \in \mathcal{E}_y} \mathbb{E}[E_k] \geq c_3, \tag{3.6}$$

then $\chi \geq (1-(d-1)\gamma)/2$ for all $\gamma > \xi$, so the claim of Theorem 1.2 follows for $d \geq 2$, whereas for the bound $1/8$ for $d = 2$ one simply inserts the bound for ξ given in Theorem 1.1. It remains to prove (3.6).

We will therefore prove two technical lemmas, the proof of the two lemmas will essentially be the same as the proof of formulas (2.10), (2.11) and (2.13) of [8]: For $k \geq 1$, take $\sigma_k^\delta \in D_{\delta,k}$, with $\delta = 0$ or 1, $\omega \in \Omega$ and $x \in \mathbb{R}^d$. We define the potential

(3.7)

$$V_k^\delta(x, \widehat{\omega}_k) = M \wedge \left(\int_{C_k} W(x-y)\sigma_k^\delta(dy) + \int_{C_k^c} W(x-y)\widehat{\omega}_k(dy) \right).$$

If $\tilde{C}_k$ is the closed a-neighborhood of C_k, then notice that

$$V_k^1 = V_k^0 \text{ on } \tilde{C}_k^c \quad \text{and} \quad V_k^1 \geq V_k^0 \text{ on } \tilde{C}_k, \tag{3.8}$$

and there exists a $c_4 > 0$ and a domain $G \subset C_k$ (depending on $\sigma_k^1 \in D_{1,k}$) such that G has positive Lebesgue measure and the difference $V_k^1 - V_k^0 > c_4$ on G. Define $\tilde{H}_k = H_{\tilde{C}_k}$ to be the entrance time of $Z_.$ into $\tilde{C}_k$, and $H_k = H_{C_k}$ to be the entrance time of $Z_.$ into the closed cube $\bar{C}_k$. So we denote the path measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ generated by V_k^δ with start in $x \in \mathbb{R}^d$ and goal in $y \in \mathbb{R}^d$ as follows

$$\begin{aligned} (3.9)\quad d\widehat{P}_x^{k,\delta} &= d\widehat{P}_{x,y}^{k,\delta} \\ &= \frac{1}{e_\lambda(x, y, \sigma_k^\delta, \widehat{\omega}_k)} \exp\left\{ -\int_0^{H(y)} (\lambda + V_k^\delta)(Z_s)ds \right\} 1_{\{H(y)<\infty\}} dP_x, \end{aligned}$$

to avoid overloaded notations we will drop the brackets for the cloud configuration in the gauge function.

LEMMA 3.1. – *With the above notations the following statements are true:*

$$\frac{e_\lambda(0, y, \sigma_k^1, \widehat{\omega}_k)}{e_\lambda(0, y, \sigma_k^0, \widehat{\omega}_k)} = 1 + \widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \frac{e_\lambda(Z_{\tilde{H}_k}, y, \sigma_k^1, \widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k}, y, \sigma_k^0, \widehat{\omega}_k)} - 1 \right],$$

and

$$\frac{e_\lambda(0, y, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(0, y, \sigma_k^1, \widehat{\omega}_k)} = 1 + \widehat{E}_0^{k,1}\left[\tilde{H}_k \leq H(y), \frac{e_\lambda(Z_{\tilde{H}_k}, y, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k}, y, \sigma_k^1, \widehat{\omega}_k)} - 1 \right].$$

Proof of the lemma. – With the help of the strong Markov property follows

$$\frac{e_\lambda(0,y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(0,y,\sigma_k^0,\widehat{\omega}_k)} = \widehat{P}_0^{k,0}[\tilde{H}_k > H(y)] + \widehat{E}_0^{k,0}\left[\tilde{H}_k \le H(y), \frac{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)}\right] = 1 + \widehat{E}_0^{k,0}\left[\tilde{H}_k \le H(y), \frac{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)} - 1\right].$$

Whereas for the second claim of the lemma one has to exchange the role of σ_k^0 and σ_k^1. This finishes the proof of Lemma 3.1. □

The second lemma tells us, how to handle the fraction on the right-hand side of the equation in the above lemma. For $k \ge 1$, take $\sigma_k^\delta \in D_{\delta,k}$, with $\delta = 0$ or 1, $\omega \in \Omega$ and $y \in \mathbb{R}^d$, we denote by $g_{\lambda,k}^{y,\delta}(\cdot,\cdot)$ the $(\lambda + V_k^\delta)$-Green function on $U = B(y,1)^c$, that is, for $x, z \in \mathbb{R}^d$,

$$g_{\lambda,k}^{y,\delta}(x,z) = \int_0^\infty e^{-\lambda s} p(s,x,z) E_{x,z}^s\left[\exp\left\{-\int_0^s V_k^\delta(Z_s)ds\right\}, s < H(y)\right] ds,$$

see also formula (1.10).

LEMMA 3.2. – *With the above notations the following two statements are true:*

$$e_\lambda(x,y,\sigma_k^0,\widehat{\omega}_k) - e_\lambda(x,y,\sigma_k^1,\widehat{\omega}_k) = \int_{\tilde{C}_k} g_{\lambda,k}^{y,1}(x,z)(V_k^1 - V_k^0)(z) e_\lambda(z,y,\sigma_k^0,\widehat{\omega}_k)dz,$$

and

$$e_\lambda(x,y,\sigma_k^0,\widehat{\omega}_k) - e_\lambda(x,y,\sigma_k^1,\widehat{\omega}_k) = \int_{\tilde{C}_k} g_{\lambda,k}^{y,0}(x,z)(V_k^1 - V_k^0)(z) e_\lambda(z,y,\sigma_k^1,\widehat{\omega}_k)dz.$$

Proof of the lemma. – By a classical differentiation and integration argument one has for $w \in C(\mathbb{R}_+, \mathbb{R}^d)$, with $H(y) < \infty$,

$$\exp\left\{-\int_0^{H(y)} (V_k^0 - V_k^1)(Z_s)ds\right\} = 1 + \int_0^{H(y)} (V_k^1 - V_k^0)(Z_s)\exp\left\{-\int_s^{H(y)} (V_k^0 - V_k^1)(Z_u)du\right\}ds.$$

To prove the first claim, we multiply both sides by

$$\exp\left\{-\int_0^{H(y)} (\lambda + V_k^1)(Z_s)ds\right\}$$

and take the integration with respect to $1_{\{H(y)<\infty\}}P_x$. Whereas for the second claim one has to exchange the role of σ_k^0 and σ_k^1. This finishes the proof of Lemma 3.2. □

Now we are able to prove (3.6). Take $\sigma_k^0 \in D_{0,k}$. (In fact $D_{0,k}$ contains only one element.) Then

$$E_k(0,y,\widehat{\omega}_k) = \inf_{\sigma_k^1 \in D_{1,k}} -\log\left(\frac{e_\lambda(0,y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(0,y,\sigma_k^0,\widehat{\omega}_k)}\right) \geq 0. \tag{3.10}$$

We want to find a good lower bound on the term on the right-hand side of the above equation. We take a fixed $\sigma_k^1 \in D_{1,k}$. Then by Lemma 3.1 and Lemma 3.2 we find

$$\begin{aligned}
&-\log\left(\frac{e_\lambda(0,y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(0,y,\sigma_k^0,\widehat{\omega}_k)}\right)\\
&= -\log\left(1+\widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \frac{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)}-1\right]\right)\\
&= -\log\Bigg(1-\widehat{E}_0^{k,0}\Bigg[\tilde{H}_k \leq H(y),\\
&\int_{\tilde{C}_k} g_{\lambda,k}^{y,1}(Z_{\tilde{H}_k},z)(V_k^1-V_k^0)(z)\frac{e_\lambda(z,y,\sigma_k^0,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)}dz\Bigg]\Bigg).
\end{aligned}$$

Now, we have to distinguish whether $\tilde{C}_k$ is a neighboring box of our goal or not: Choose R minimal, such that $\tilde{C}_k \subset B(lq_k,R)$. We say $\tilde{C}_k$ is a neighboring box of $y \in \mathbb{R}^d$ if y is contained in the closure of $B(lq_k,R+2)$. Define

$$N_y = \left\{k \in \mathbb{N};\ \tilde{C}_k \text{ is a neighboring box of } y\right\}.$$

Of course the number of points contained in N_y is bounded by a constant only depending on a, l and d.

First case, $k \in N_y$.

$$-\log\left(\frac{e_\lambda(0,y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(0,y,\sigma_k^0,\widehat{\omega}_k)}\right) \geq 0.$$

Second case, $k \notin N_y$. From Harnack's inequality (see for instance [6] after (1.28)), we get, for $k \geq 1$,

$$\frac{\inf_{z\in\tilde{C}_k} e_\lambda(z,y)}{\sup_{z\in\tilde{C}_k} e_\lambda(z,y)} \geq c_5(d,\lambda,M,a). \tag{3.11}$$

Thus, because $(V_k^1 - V_k^0)(z) \geq 0$ we see that

$$\widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \int_{\tilde{C}_k} g_{\lambda,k}^{y,1}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z)\frac{e_\lambda(z,y,\sigma_k^0,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)}dz\right]$$
$$\geq c_5\widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \int_{\tilde{C}_k} g_{\lambda,k}^{y,1}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z)dz\right]$$
$$\geq c_5\widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \int_{C_k} g_{\lambda,k}^{y,1}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z)dz\right].$$

On $k \notin N_y$, there exists a $c_6 > 0$, independent of k, such that for all $x \in \partial\tilde{C}_k$ and all $\sigma_k^1 \in D_{1,k}$,

$$\int_{C_k} g_{\lambda,k}^{y,1}(x,z)(V_k^1 - V_k^0)(z)dz$$
$$\geq \int_{C_k} g_{\lambda+M,\bar{B}(y,1)^c}(x,z,\omega=0)(V_k^1 - V_k^0)(z)dz$$
$$\geq \int_{C_k} g_{\lambda+M,\bar{B}(lq_k,R+1)}(x,z,\omega=0)(V_k^1 - V_k^0)(z)dz \geq c_6.$$

Therefore, we find

$$\widehat{E}_0^{k,0}\left[\tilde{H}_k \leq H(y), \int_{\tilde{C}_k} g_{\lambda,k}^{y,1}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z)\frac{e_\lambda(z,y,\sigma_k^0,\widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k},y,\sigma_k^0,\widehat{\omega}_k)}dz\right]$$
$$\geq c_7\widehat{P}_0^{k,0}\left[\tilde{H}_k \leq H(y)\right].$$

If we insert this result into formula (3.10), we get

$$E_k(0,y,\widehat{\omega}_k) = \inf_{\sigma_k^1\in D_{1,k}} -\log\left(\frac{e_\lambda(0,y,\sigma_k^1,\widehat{\omega}_k)}{e_\lambda(0,y,\sigma_k^0,\widehat{\omega}_k)}\right)$$
$$\geq \inf_{\sigma_k^1\in D_{1,k}} -\log\left(1 - c_7\widehat{P}_0^{k,0}\left[\tilde{H}_k \leq H(y)\right]\right)$$
$$\geq c_7\widehat{P}_0^{k,0}\left[\tilde{H}_k \leq H(y)\right] \geq c_7\widehat{P}_0^{y}\left[\tilde{H}_k \leq H(y)\right]$$
$$\geq c_7\widehat{P}_0^{y}[H_k \leq H(y)],$$

where we have used Lemma 3.3 which follows below. Thus, we find the following lower bound

$$E_k(0,y,\widehat{\omega}_k) \geq \begin{cases} 0 & \text{if } k \in N_y \\ c_7\widehat{P}_0^{y}[H_k \leq H(y)] & \text{if } k \notin N_y \end{cases}. \tag{3.12}$$

If $A(y,\gamma)$ again denotes the event that the path of the Brownian motion starting in 0 with goal $B(y,1)$ does not leave the cylinder $Z(y,\gamma)$, we know, because we have taken $\gamma > \xi$, that $\mathbb{P}$-a.s. $\liminf_{|y|\to\infty} \widehat{P}_0[A(y,\gamma)] = 1$. Therefore,

$$\begin{aligned}\sum_{k\in\mathcal{E}_y} \mathbb{E}[E_k] &\geq \sum_{k\in\mathcal{E}_y\setminus N_y} \mathbb{E}[E_k] \\ &\geq c_7\mathbb{E}\left[\sum_{k\in\mathcal{E}_y\setminus N_y} \widehat{P}_0^y[H_k \leq H(y)]\right] \\ &\geq c_7\mathbb{E}\left[\widehat{E}_0^y\left[\sum_{k\in\mathcal{E}_y\setminus N_y} 1_{\{H_k\leq H(y)\}}\cdot 1_{A(y,\gamma)}\right]\right] \\ &\geq c_7\mathbb{E}\left[\widehat{P}_0^y[A(y,\gamma)]\frac{|y|}{2\sqrt{d}l}\right] \quad \text{for all large } |y|.\end{aligned}$$

So the claim (3.6) follows by the lemma of Fatou, this completes the proof of our theorem. □

LEMMA 3.3. – *Take $y \in \mathbb{R}^d$, $k \geq 1$, $\omega \in \Omega$, $\sigma_k^0 \in D_{0,k}$ and $\sigma_k^1 \in D_{1,k}$. Then we have*

$$\widehat{P}_0^{k,0}\left[\tilde{H}_k \leq H(y)\right] \geq \widehat{P}_0^{k,1}\left[\tilde{H}_k \leq H(y)\right] \tag{3.13}$$

and

$$\widehat{P}_0^{k,0}\left[\tilde{H}_k \leq H(y)\right] \geq \widehat{P}_0^{y}\left[\tilde{H}_k \leq H(y)\right]. \tag{3.14}$$

Proof. – We know by (3.8) that $V_k^0(\widehat{\omega}_k) \leq V_k^1(\widehat{\omega}_k)$ and also $V_k^0(\widehat{\omega}_k) \leq V(\omega)$. Therefore it suffices to prove the first claim, the second one then follows analogously. Let us define for $\delta = 0, 1$, $w \in C(\mathbb{R}_+, \mathbb{R}^d)$,

$$f(\widehat{\omega}_k, \sigma_k^\delta, w) = \exp\left\{-\int_0^{H(y)} (\lambda + V_k^\delta)(Z_s,\omega)ds\right\}\cdot 1_{\{H(y)<\infty\}},$$

and observe that

$$f(\widehat{\omega}_k, \sigma_k^1, w)\cdot 1_{\{\tilde{H}_k > H(y)\}} = f(\widehat{\omega}_k, \sigma_k^0, w)\cdot 1_{\{\tilde{H}_k > H(y)\}} \tag{3.15}$$

and

$$f(\widehat{\omega}_k, \sigma_k^1, w)\cdot 1_{\{\tilde{H}_k \leq H(y)\}} \leq f(\widehat{\omega}_k, \sigma_k^0, w)\cdot 1_{\{\tilde{H}_k \leq H(y)\}}. \tag{3.16}$$

Therefore by (3.15) and (3.16) we get

$$\begin{aligned}
&\widehat{P}_0^{k,1}\Big[\tilde{H}_k \le H(y)\Big]\\
&= \frac{E_0\Big[f(\widehat{\omega}_k, \sigma_k^1, w)\cdot 1_{\{\tilde{H}_k\le H(y)\}}\Big]}{E_0\Big[f(\widehat{\omega}_k, \sigma_k^1, w)\cdot \Big(1_{\{\tilde{H}_k\le H(y)\}} + 1_{\{\tilde{H}_k> H(y)\}}\Big)\Big]}\\
&\le \frac{E_0\Big[f(\widehat{\omega}_k, \sigma_k^0, w)\cdot 1_{\{\tilde{H}_k\le H(y)\}}\Big]}{E_0\Big[f(\widehat{\omega}_k, \sigma_k^0, w)\cdot 1_{\{\tilde{H}_k\le H(y)\}}\Big] + E_0\Big[f(\widehat{\omega}_k, \sigma_k^1, w)\cdot 1_{\{\tilde{H}_k> H(y)\}}\Big]}\\
&= \widehat{P}_0^{k,0}\Big[\tilde{H}_k \le H(y)\Big].
\end{aligned}$$

This completes the proof of the lemma. □

4. PROOF OF THE SUBDIFFUSIVE LOWER BOUND OF ξ_0

The strategy of the proof of Theorem 1.3 is an extention of the arguments used in the proof of Theorem 1.2. For a similar result in first-passage percolation we want to refer to Section 3 of [4]. The idea of the proof is to compare $-\log e_\lambda(\cdot,\cdot)$ for the two different starting points 0 and m and the two different endpoints y and $y+m$. One easily gets an upper bound on the variance of the difference of these two "passage times": If the distance between 0 and m is of order $|y|^\gamma$ then $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0,y)+\log e_\lambda(m,y+m)) = O(|y|^{2\gamma})$. On the other hand we will get, by the methods presented in the preceding chapter, a lower bound of the form $c|y|^{1-(d-1)\gamma}$. So if we choose γ, the power of the cylinder radius, smaller than $1/(d+1)$ we get a contradiction, to the two bounds mentioned above, this leads us to the claim of Theorem 1.3.

Proof of Theorem 1.3. – Assume that for a fixed $\gamma \in [0,1]$, there exists a sequence $(y_n)_n \subset \mathbb{R}^d$ with $|y_n| \to \infty$ such that

$$\mathbb{E}\Big[\widehat{P}_0^{y_n}[A(y_n,\gamma)]\Big] \to 1 \quad \text{as } n \to \infty. \tag{4.1}$$

In fact, by rotation invariance of the obstacles, we can and will choose the sequence $(y_n)_n$ such that $y_n = (|y_n|, 0, \ldots, 0)$ for all n. We want to show that under these assumptions $\gamma \ge 1/(d+1)$.

Define, for $\omega \in \Omega$, $n \ge 1$, the difference of an "upper" and a "lower" passage time as follows

$$\delta \log e_n(\omega) = -\log e_\lambda(0, y_n, \omega) - (-\log e_\lambda(m_n, y_n + m_n, \omega)),$$

where we choose $m_n \in \mathbb{R}^d$ as follows: $m_n = (0, |m_n|, 0, \ldots, 0)$ points into the direction of the second coordinate axis, and $|m_n|$ is minimal such that the two cylinders $\tilde{Z}(y_n, |y_n|^\gamma + \sqrt{d}l)$ and $\tilde{Z}(y_n, |y_n|^\gamma + \sqrt{d}l) + m_n$ are disjoint. We remark that $|m_n| \leq c_1|y_n|^\gamma$ for all large n because of our choice of the sequence $(y_n)_n$. Using the strong Markov property, we see that

$$e_\lambda(0, y_n) \geq e_\lambda(0, m_n) \inf_{z \in \bar{B}(m_n, 1)} e_\lambda(z, y_n + m_n) \inf_{z \in \bar{B}(y_n + m_n, 1)} e_\lambda(z, y_n).$$

By Harnack's inequality (see (3.11)), one gets

$$-\log\left(\frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)}\right) \leq c_2 - \log e_\lambda(0, m_n) - \log e_\lambda(y_n + m_n, y_n),$$

analogously, by exchanging the role of the passage times,

$$-\log\left(\frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)}\right) \geq -c_2 + \log e_\lambda(m_n, 0) - \log e_\lambda(y_n, y_n + m_n).$$

Therefore the following estimate on $|\delta \log e_n|$ holds,

$$\left|-\log\left(\frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)}\right)\right| \leq c_2 - \log e_\lambda(0, m_n) - \log e_\lambda(y_n + m_n, y_n)$$
$$- \log e_\lambda(m_n, 0) - \log e_\lambda(y_n, y_n + m_n).$$

In view of (1.12), we find a suitable constant $c_3 \in (0, \infty)$ such that for all large n

$$\text{(4.2)} \qquad \mathbb{V}\text{ar}(\delta \log e_n) \leq c_3|m_n|^2 \leq c_1^2 c_3 |y_n|^{2\gamma}.$$

Hence, we have found the desired upper bound on the variance of the difference of these two passage times. During the rest of the proof we try to show that one gets the following lower bound on the same variance:

$$\text{(4.3)} \qquad \mathbb{V}\text{ar}(\delta \log e_n) \geq c_4 |y_n|^{1-(d-1)\gamma}.$$

If we have verified those two bounds, we see that $2\gamma \geq 1 - (d-1)\gamma$ from which our claim, $\gamma \geq 1/(d+1)$, follows. It remains to find the lower bound (4.3) on the variance of the difference of the two passage times.

With the same notations as in the previous chapters, we get by martingale identities as before

$$\text{(4.4)} \qquad \mathbb{V}\text{ar}(\delta \log e_n) \geq \sum_{k \geq 1} \mathbb{V}\text{ar}(\ \mathbb{E}[\delta \log e_n | \mathcal{G}_k]\ ).$$

So we are again interested in a lower bound for $\mathbb{V}\text{ar}(\ \mathbb{E}[\delta \log e_n | \mathcal{G}_k]\ )$. We introduce the same events $D_{0,k}$, $D_{1,k}$, D_k^0 and D_k^1 on our cubes C_k

as in Chapter 3. We also keep the notation for the decomposition of $\omega = (\omega_k, \widehat{\omega}_k) \in \Omega$ on the cube C_k and on its complement. In view of Lemma 3 in [5], we find

$$\mathbb{V}\mathrm{ar}(\ \mathbb{E}[\delta \log e_n | \mathcal{G}_k]\ ) \geq \frac{pq}{p+q}(x_1 - x_0)^2,$$

where $p = \mathbb{P}[D_k^0] > 0$, $q = \mathbb{P}[D_k^1] > 0$ and

$$x_\delta = \mathbb{E}\left[\frac{\mathbb{E}[\delta \log e_n | \mathcal{G}_k] 1_{D_k^\delta}]}{\mathbb{P}[D_k^\delta]}\right], \quad \text{for } \delta = 0 \text{ or } 1.$$

We have to find a "good" lower bound on $x_1 - x_0$. For $\omega \in \Omega$, we define the random variable

$$G_k(\widehat{\omega}_k) = \inf_{\sigma_k^1 \in D_{1,k}} \delta \log e_n(\sigma_k^1, \widehat{\omega}_k) - \sup_{\sigma_k^0 \in D_{0,k}} \delta \log e_n(\sigma_k^0, \widehat{\omega}_k).$$

As in the preceding chapter, wee see that $x_1 - x_0 \geq \mathbb{E}[G_k]$, thus

$$|x_1 - x_0| \geq (\mathbb{E}[G_k])_+.$$

Further, we define $\mathcal{E}_n = \{k \in \mathbb{N},\ C_k \cap Z(y_n, \gamma) \neq \emptyset\}$ to be the cubes that intersect our cylinder $Z(y_n, \gamma)$. Therefore, we find with the Cauchy-Schwarz inequality

$$\begin{aligned}
(4.5) \qquad \mathbb{V}\mathrm{ar}(\delta \log e_n) &\geq \frac{pq}{p+q} \sum_{k \geq 1} (\mathbb{E}[G_k]_+)^2 \\
&\geq \frac{pq}{p+q} \sum_{k \in \mathcal{E}_n} (\mathbb{E}[G_k]_+)^2 \\
&\geq \frac{pq}{p+q} \frac{1}{|\mathcal{E}_n|} \left(\sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k]\right)_+^2 \\
&\geq c_5 |y_n|^{1-(d-1)\gamma} \left(\frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k]\right)_+^2.
\end{aligned}$$

To prove (4.3) it remains to show that

$$(4.6) \qquad \liminf_{n \to \infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k] > 0.$$

We denote the only element in $D_{0,k}$ by σ_k^0. For $\omega \in \Omega$ is

$$G_k(\widehat{\omega}_k) \geq \inf_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(0, y_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(0, y_n, \sigma_k^1, \widehat{\omega}_k)} + \inf_{\sigma_k^1 \in D_{1,k}} - \log \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)}.$$

Thus, it suffices to prove

$$\liminf_{n\to\infty} \frac{1}{|y_n|} \sum_{k\in\mathcal{E}_n} \mathbb{E}\left[\inf_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(0, y_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(0, y_n, \sigma_k^1, \widehat{\omega}_k)}\right] > 0, \tag{4.7}$$

and

$$\limsup_{n\to\infty} \frac{1}{|y_n|} \sum_{k\in\mathcal{E}_n} \mathbb{E}\left[\sup_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)}\right] = 0. \tag{4.8}$$

The proof of claim (4.7) follows exactly in the same way as the proof of (3.6). So we want to show (4.8): By Lemma 3.1 and Lemma 3.2 follows, for $\sigma_k^1 \in D_{1,k}$,

$$\log\left(\frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)}\right) = \log\Bigg(1 + \widehat{E}_{m_n}^{k,1}\Bigg[\tilde{H}_k \leq H(y_n + m_n), \int_{\tilde{C}_k} g_{\lambda,k}^{y_n+m_n,0}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z) \frac{e_\lambda(z, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)}{e_\lambda(Z_{\tilde{H}_k}, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)} dz\Bigg]\Bigg);$$

for all $k \in \mathcal{E}_n$, $\tilde{C}_k$ is not a neighboring box of our goal $y_n + m_n$, so we can use Harnack's inequality. Therefore the last member of the above equality is smaller than

$$\leq \log\left(1 + c_6 M \widehat{E}_{m_n}^{k,1}\left[\tilde{H}_k \leq H(y_n + m_n), \int_{\tilde{C}_k} g_{\lambda,k}^{y_n+m_n,0}(Z_{\tilde{H}_k}, z) dz\right]\right).$$

In the case $d \geq 3$ or $\lambda > 0$, $g_{\lambda,k}^{y_n+m_n,0}(\cdot,\cdot)$ is smaller than $g_\lambda^{y_n+m_n}(\cdot,\cdot,\omega = 0)$ the λ-Green function for Brownian motion, so the last expression is smaller than

$$\leq \log\left(1 + c_7 \widehat{P}_{m_n}^{k,1}\left[\tilde{H}_k \leq H(y_n + m_n)\right]\right) \leq c_7 \widehat{P}_{m_n}^{k,1}\left[\tilde{H}_k \leq H(y_n + m_n)\right].$$

For $k \in \mathcal{E}_n$, we get with the above considerations, using the independence of the Poissonian process and Lemma 3.3

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \widehat{\omega}_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \widehat{\omega}_k)}\right] \\
&\leq c_7 \mathbb{E}\left[\sup_{\sigma_k^1 \in D_{1,k}} \widehat{P}_{m_n}^{k,1}\left[\tilde{H}_k \leq H(y_n + m_n)\right]\right] \\
&= \frac{c_7}{p}\mathbb{E}\left[\sup_{\sigma_k^1 \in D_{1,k}} \widehat{P}_{m_n}^{k,1}\left[\tilde{H}_k \leq H(y_n + m_n)\right] 1_{D_k^0}\right] \\
&\leq \frac{c_7}{p}\mathbb{E}\left[\widehat{P}_{m_n}^{k,0}\left[\tilde{H}_k \leq H(y_n + m_n)\right] 1_{D_k^0}\right] \\
&\leq \frac{c_7}{p}\mathbb{E}\left[\widehat{P}_{m_n}^{y_n+m_n}\left[\tilde{H}_k \leq H(y_n + m_n)\right]\right].
\end{aligned}$$

Therefore to prove claim (4.8) it suffices to show that

$$\lim_{n\to\infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}\left[\widehat{P}_{m_n}^{y_n+m_n}\left[\tilde{H}_k \leq H(y_n + m_n)\right]\right] = 0. \tag{4.9}$$

We consider now the random lattice animal $\mathcal{A} = \{k \in \mathbb{N} \ ; \ H_k \leq H(y_n + m_n)\}$. By the Cauchy-Schwarz inequality and the fact that the distance between the two disjoint cylinders $Z(y_n, \gamma)$ and $Z(y_n, \gamma) + m_n$ is $2\sqrt{d}l$, we find

$$\begin{aligned}
&\frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}\left[\widehat{P}_{m_n}^{y_n+m_n}\left[\tilde{H}_k \leq H(y_n + m_n)\right]\right] \\
&= \mathbb{E}\left[\frac{1}{|y_n|}\widehat{E}_{m_n}^{y_n+m_n}\left[\#\{k \in \mathcal{E}_n \ ; \ \tilde{H}_k \leq H(y_n + m_n)\}\right]\right] \\
&\leq 3^d \mathbb{E}\left[\frac{1}{|y_n|}\widehat{E}_{m_n}^{y_n+m_n}\left[|\mathcal{A}| \cdot 1_{A_{m_n}(y_n,\gamma)^c}\right]\right] \\
&\leq 3^d \mathbb{E}\left[\widehat{E}_{m_n}^{y_n+m_n}\left[\frac{|\mathcal{A}|^2}{|y_n|^2}\right]\right]^{1/2} \mathbb{E}\left[\widehat{P}_{m_n}^{y_n+m_n}\left[A_{m_n}(y_n, \gamma)^c\right]\right]^{1/2},
\end{aligned} \tag{4.10}$$

where $A_{m_n}(y_n, \gamma)$ denotes the event that the paths of Brownian motion starting in m_n with goal $B(y_n + m_n, 1)$ do not leave the cylinder $Z(y_n, \gamma) + m_n$. By translation invariance is $\mathbb{E}\left[\widehat{P}_{m_n}^{y_n+m_n}[A_{m_n}(y_n, \gamma)^c]\right] = \mathbb{E}\left[\widehat{P}_0^{y_n}[A(y_n, \gamma)^c]\right]$, which tends to 0 as n goes to infinity. Therefore it remains to show that the first term on the right-hand side of inequality (4.10) stays bounded for all large n. By Jensen's inequality, Cauchy-Schwarz inequality and with the estimates (1.12) and (1.16) (using translation invariance), we see that for a suitable constant c_8:

$$\begin{aligned}
&\mathbb{E}\left[\widehat{E}_{m_n}^{y_n+m_n}\left[\frac{|\mathcal{A}|^2}{|y_n|^2}\right]\right] \\
&\leq c_8 \mathbb{E}\left[\widehat{E}_{m_n}^{y_n+m_n}\left[\exp\left\{\frac{\gamma_3}{2}\frac{|\mathcal{A}|}{|y_n|}\right\}\right]\right] \\
&\leq c_8 \mathbb{E}\left[\widehat{E}_{m_n}^{y_n+m_n}\left[\exp\left\{\frac{\gamma_3}{2}|\mathcal{A}|\right\}\right]\right]^{1/|y_n|} \\
&\leq c_8 \Bigg(\mathbb{E}\left[\frac{1}{e_\lambda(m_n, y_n+m_n)^2}\right]^{1/2} \cdot \\
&\qquad \mathbb{E}\left[E_{m_n}^{y_n+m_n}\left[\exp\left\{\gamma_3|\mathcal{A}| - \int_0^{H(y_n+m_n)} (\lambda+V)(Z_s)ds\right\},\right.\right. \\
&\qquad \left.\left. H(y_n+m_n)<\infty\right]\right]^{1/2}\Bigg)^{1/|y_n|} \\
&\leq c_8 \left(\frac{1}{\gamma_1^2}\exp\{\gamma_2|y_n|\}\right)^{1/(2|y_n|)} \mathbb{E}\left[2^{N_0(\omega)/2}\right]^{1/(2|y_n|)}.
\end{aligned}$$

The above expression is bounded, this completes the proof. □

APPENDIX A. MEASURABILITY

In this appendix we prove Lemma 2.1:

LEMMA 2.1. – *For* $\lambda \geq 0$ *and* $\gamma > 0$, *the functions* $(y, \omega) \to e_\lambda(0, y, \omega)$ *and* $(y, \omega) \to \widehat{P}_0^y[A(y, \gamma)]$ *are measurable in* ω *and for all* $|y| > 1$ *continuous in* y.

Proof. – For the measurability see Lemma 1.1 of [6]. Let us prove the continuity of $e_\lambda(0, y, \omega)$ in y. Define for $y, y' \in \mathbb{R}^d$, $w \in C(\mathbb{R}_+, \mathbb{R}^d)$

$$\begin{aligned}
\text{(A.1)}\quad & f(y, y', w) \\
&= \exp\left\{-\int_0^{H(y)} (\lambda+V)(Z_s,\omega)ds\right\} \cdot 1_{\{H(y)<\infty\}}(w) \\
&\quad - \exp\left\{-\int_0^{H(y')} (\lambda+V)(Z_s,\omega)ds\right\} \cdot 1_{\{H(y')<\infty\}}(w).
\end{aligned}$$

First we choose $w \in \{H(y') \leq H(y)\}$:

$$\text{(A.2)} \quad \begin{aligned} &\left|f(y,y',w) \cdot 1_{\{H(y')\leq H(y)\}}\right| \\ &\leq (1-\exp\{-(\lambda+M)(H(y)-H(y'))\}) \cdot 1_{\{H(y)<\infty, H(y')\leq H(y)\}} \\ &\quad + 1_{\{H(y)=\infty, H(y')<\infty\}}. \end{aligned}$$

Therefore, if we take the expectation with respect to the Brownian path measure P_0, we get the following upper estimate

$$\text{(A.3)} \quad \begin{aligned} &|E_0[f(y,y',w) \cdot 1_{\{H(y')\leq H(y)\}}]| \\ &\leq E_0[(1-\exp\{-(\lambda+M)(H(y)-H(y'))\}) \\ &\qquad \cdot 1_{\{H(y)<\infty, H(y')\leq H(y)\}}] \\ &\quad + P_0[H(y)=\infty, H(y')<\infty]. \end{aligned}$$

The second term on the right-hand side of (A.3) is zero for $d=2$ and it tends to zero as $|y'-y| \to 0$ in dimensions $d \geq 3$. So we want to focus on the first term on the right-hand side of (A.3).

For all $\varepsilon > 0$ there exists a small $b = b(\varepsilon) > 0$ such that $\exp\{-(\lambda+M)b\} \geq 1-\varepsilon/8$. Thus, by the strong Markov property, there exists a $\delta_1 > 0$ such that

$$\text{(A.4)} \quad \begin{aligned} &E_0[(1-\exp\{-(\lambda+M)(H(y)-H(y'))\}) \cdot 1_{\{H(y)<\infty, H(y')\leq H(y)\}}] \\ &\leq 1-\exp\{-(\lambda+M)b\} \\ &\quad + P_0[H(y)<\infty, H(y')\leq H(y), H(y)-H(y')>b] \\ &\leq \varepsilon/8 + \sup_{x\in\partial B(y',1)} P_x[b<H(y)<\infty] < \varepsilon/4, \\ &\quad \text{for all } y' \text{ with } |y'-y| \leq \delta_1. \end{aligned}$$

Of course, we have symmetry in y and y', so the same estimates hold in the case $w \in \{H(y) < H(y')\}$. The continuity of $e_\lambda(0,\cdot,\omega)$ now easily follows.

To see the continuity of $\widehat{P}_0^y[A(y,\gamma)]$ in y, define for $y \in \mathbb{R}^d$, $w \in C(\mathbb{R}_+, \mathbb{R}^d)$,

$$\text{(A.5)} \quad g(w,y) = \exp\left\{-\int_0^{H(y)} (\lambda+V)(Z_s,\omega)ds\right\} \cdot 1_{\{H(y)<\infty\}}(w),$$

and

$$h(w, y) = 1_{A(y,\gamma)} \cdot 1_{\{H(y)<\infty\}}(w). \tag{A.6}$$

Using the continuity of $e_\lambda(0, \cdot, \omega)$ and the tubular estimate (1.12), we see that it suffices to show that $E_0[g(\omega, \cdot)h(\omega, \cdot)]$ is continuous. For $y, y' \in \mathbb{R}^d$, $\omega \in \Omega$, we have, using the triangle inequality,

$$\begin{aligned} &|E_0[g(\omega, y)h(\omega, y)] - E_0[g(\omega, y')h(\omega, y')]| \\ &\leq \left|E_0[(g(w, y) - g(w, y'))h(w, y)]\right| \\ &+ \left|E_0[g(w, y')(h(w, y) - h(w, y'))]\right|. \end{aligned} \tag{A.7}$$

The first term on the right-hand side of (A.7) tends to zero as $|y - y'| \to 0$, using the same observations as done in the proof of the continuity of $e_\lambda(0, y, \omega)$. So we want to focus on the second term on the right-hand side of (A.7). First we choose $w \in \{H(y') < H(y)\}$, then we have

$$\begin{aligned} &|E_0[g(w, y')(h(w, y) - h(w, y')) \cdot 1_{\{H(y')<H(y)\}}]| \\ &\leq E_0[|h(w, y) - h(w, y')| \cdot 1_{\{H(y')<H(y)<\infty\}}] \\ &\quad + P_0[H(y') < \infty,\ H(y) = \infty]. \end{aligned} \tag{A.8}$$

For the second term on the right-hand side of (A.8) we use the same remark as after (A.3), whereas for the first term we see, using the strong Markov property, that it tends to zero as $|y - y'| \to 0$. This finishes the proof of the lemma. □

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