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An invariance principle for Markov processes and Brownian particles with singular interaction

by

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Dedicated to professor Shinzo Watanabe on his 60th birthday

ABSTRACT. – We prove an invariance principle for functionals of Markov processes. As an application we prove an invariance principle for tagged particles of Brownian particles with non-symmetric interactions. © Elsevier, Paris

Key words: An invariance principle for Markov processes

RÉSUMÉ. – Nous prouvons un principe d'invariance pour des fonctionnelles additives de processus de Markov. En application, nous démontrons un principe d'invariance pour une particule marquée dans un système de particules browniennes avec interactions non-symétriques. © Elsevier, Paris

0. INTRODUCTION

An invariance principle for Markov processes is a theorem to claim a diffusive scaling limit of a functional $X = X_t$ of Markov processes $(Y, \{\mathbb{P}_\theta\})$ converges to (a constant multiplication of) Brownian motion;

$$\lim_{\epsilon \to 0} \epsilon X_{t/\epsilon^2} = \sigma B_t. \tag{0.1}$$

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This has been studied by [8], [1] for the reversible case, and by [21], [29] for the non reversible case under the strong sector condition.

Some of main assumptions in these works are the following; (1) X is an additive functional of Y, and (2) the mean forward velocity of X exists. Here the mean forward velocity φ of X means, roughly speaking, the following quantity;

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_\theta[X_t] = \varphi(\theta) \quad \text{or} \quad \lim_{\lambda\to\infty}\lambda^2\mathbb{E}_\theta[\int_0^\infty e^{-\lambda t}X_t dt] = \varphi(\theta).$$

Here $\mathbb{E}_\theta$ is the expectation with respect to $\mathbb{P}_\theta$. In continuous models, it is often difficult to check these assumptions; even in the simple case such as homogenization of reflecting diffusions in random domains we need a long argument to prove (2) (see [21, Sect. 3]). One of our purpose is to improve such a situation by assuming the condition—(A.4) and (A.5) defined later—on associated Dirichlet forms; under this condition, (1) is unnecessary and (2) is automatically satisfied. This condition is not restrictive. It is still a very mild assumption satisfied by many models (see Sect. 1).

Our second purpose is to obtain an explicit expression of the limit matrix, that is, the average of $\frac{1}{2}\sigma\sigma^*$. By product we give a universal inequality of limit matrices, which says limit matrices are always greater than or equal to that of the symmetrized process, or in other words, if we add a skew symmetric part to a reversible Dirichlet form, then the limit matrix always increases.

Our work is motivated by the following problem: Consider infinitely many hard core Brownian balls $(X^i)_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ and tag one particle, say X^{i_0}. Then the problem is to prove (0.1) for X^{i_0}. For this the previous results are not sufficient. Indeed, X^{i_0} has very singular drifts caused by collisions to other infinitely many Brownian balls. So its mean forward velocity has an extreme roughness. To prove the existence of such a singular mean forward velocity in infinitely dimensional situation, we develop a new technique in a general context.

Let Θ be a Hausdorff topological space and $\mathfrak{B}(\Theta)$ denote its Borel σ-algebra. We assume $\mathfrak{B}(\Theta) = \sigma[C(\Theta)]$, where $C(\Theta)$ is the set of all continuous functions. Let μ be a probability measure on $\mathfrak{B}(\Theta)$ and $\mathcal{D}_Y$ a dense subspace in $L^2(\Theta,\mu)$. Let $\mathcal{E}_Y : \mathcal{D}_Y \times \mathcal{D}_Y \to \mathbb{R}$ be a bilinear form. We assume:

(A.1) $(\mathcal{E}_Y, \mathcal{D}_Y)$ is a quasi-regular Dirichlet form on $L^2(\Theta,\mu)$.

(A.2) $1 \in \mathcal{D}_Y$ and $\mathcal{E}_Y(1,1) = 0$.

(A.3) $(\mathcal{E}_Y, \mathcal{D}_Y)$ satisfies the strong sector condition;

$$\mathcal{E}_Y(g_1,g_2) \le C_1\mathcal{E}_Y(g_1,g_1)^{1/2}\mathcal{E}_Y(g_2,g_2)^{1/2} \quad \text{for all } g_1, g_2 \in \mathcal{D}_Y.$$

By (A.1) and Ma-Röckner's result [12, Ch. IV Theorem 3.5] there exists a special standard process $\mathbb{H}_Y = (\Omega, \mathfrak{G}, \mathfrak{G}_t, Y_t, \{\mathbb{P}_\theta\}_{\theta\in\Theta})$ associated with $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu)$.

When Θ is a Polish or, more generally, metrizable Lusin space, then $\mathbb{H}_Y$ becomes a Hunt process. By (A.2) and (A.3) $\mathbb{H}_Y$ is conservative with invariant probability measure μ. This statement is clear if Θ is a metrizable Lusin space. Otherwise, we can prove this by using the transfer method in [12], [3].

In previous works X has been an additive functional of Y. One of our key idea is to extend Y to be a Markov process $(\tilde{X}, Y)$ on $\mathbb{R}^d \times \Theta$ in such a way that $X_t = \tilde{X}_t - \tilde{X}_0$. For this we assume $(\mathcal{E}_Y, \mathcal{D}_Y)$ has the following structure.

(A.4) There exists a positive closed form $(\mathcal{E}_Y^1, \mathcal{D}_Y)$ on $L^2(\Theta, \mu)$ satisfying:

(0.2) $\mathcal{E}_Y^2 := \mathcal{E}_Y - \mathcal{E}_Y^1$ with $\mathcal{D}_Y$ is a positive closed form on $L^2(\Theta, \mu)$,

$$\mathcal{E}_Y^1(g, g) = \int_\Theta \sum_{i,j=1}^d a_{ij} D_i g D_j g d\nu. \tag{0.3}$$

Here ν is a probability measure on $(\Theta, \mathcal{B}(\Theta))$, $D_i : \mathcal{D}_Y \to L^2(\Theta, \mu)$ $(1 \le i \le d)$ are linear operators satisfying $D_i 1 = 0$ and $a_{ij} : \Theta \to \mathbb{R}$ are uniformly elliptic, bounded measurable functions; there exist positive constants C_2 and C_3 such that $C_2|\xi|^2 \le \sum_{i,j=1}^d a_{ij}\xi_i\xi_j$, and $|\sum_{i,j=1}^d a_{ij}\xi_i\eta_j| \le C_3|\xi||\eta|$ for all $\xi = (\xi_i), \eta = (\eta_i) \in \mathbb{R}^d$.

Let $\partial = (\partial_i)_{1\le i\le d}$, where $\partial_i = \frac{\partial}{\partial x_i}$. We regard D_i and ∂_i as operators on $C^\infty(\mathbb{R}^d)\otimes\mathcal{D}_Y$ by $D_i h = \sum_k f_k \otimes D_i g_k$ and $\partial_i h = \sum_k (\partial_i f_k)\otimes g_k$, where $h = \sum_k f_k \otimes g_k := \sum_k f_k(x) g_k(\theta)$ and $C^\infty(\mathbb{R}^d)\otimes\mathcal{D}_Y$ is the algebraic tensor product of $C^\infty(\mathbb{R}^d)$ and $\mathcal{D}_Y$. Let $\mathcal{E}_{XY}^1$ and $\mathcal{E}_{XY}^2$ be bilinear forms on $C_0^\infty(\mathbb{R}^d)\otimes\mathcal{D}_Y$ given by

$$\mathcal{E}_{XY}^1(h_1, h_2) = \int_{\mathbb{R}^d\times\Theta} \sum_{i,j=1}^d a_{ij}(D_i - \partial_i)h_1(D_j - \partial_j)h_2 dx d\nu \tag{0.4}$$

$$\mathcal{E}_{XY}^2(h_1, h_2) = \sum_{k,l} \Big(\int_{\mathbb{R}^d} f_{1k} f_{2l} dx\Big) \cdot \mathcal{E}_Y^2(g_{1k}, g_{2l}). \tag{0.5}$$

Here $h_i = \sum_m f_{im} \otimes g_{im}$. Let $\mathcal{E}_{XY}$ be the bilinear form on $C_0^\infty(\mathbb{R}^d)\otimes\mathcal{D}_Y$ given by

$$\mathcal{E}_{XY} = \mathcal{E}_{XY}^1 + \mathcal{E}_{XY}^2.$$

We assume:

(A.5) $(\mathcal{E}_{XY}, C_0^\infty(\mathbb{R}^d)\otimes\mathcal{D}_Y)$ is closable on $L^2(\mathbb{R}^d\times\Theta, dx\times\mu)$ and its closure $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ is a quasi-regular Dirichlet form on $L^2(\mathbb{R}^d\times\Theta, dx\times\mu)$.

(A.6) $(\mathcal{E}^1_{XY}, \mathcal{D}_{XY})$ is strong local (see [3]).

By (A.5) there exists a special standard process $\mathbb{H}_{XY} = (\Omega, \mathfrak{H}, \mathfrak{H}_t, (X_t, Y_t), \{\mathbb{P}_{x\theta}\})$ associated with $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d\times\Theta, dx\times\mu)$. Here $\mathfrak{H}_t$ is the natural filtration. We prove in Lemma 2.3 that $\mathbb{H}_{XY}$ is conservative with invariant measure $dx\times\mu$. We also prove in Lemma 5.3 that X_t is a continuous process by (A.6); we note here (X_t, Y_t) is not necessarily continuous.

We recall systems of Markovian measures $\{\mathbb{P}_\theta\}$ and $\{\mathbb{P}_{x\theta}\}$ in $\mathbb{H}_Y$ and $\mathbb{H}_{XY}$ are unique up to *quasi everywhere*. We will prove in Lemma 2.3 that there exist versions of $\{\mathbb{P}_\theta\}$ and $\{\mathbb{P}_{x\theta}\}$ satisfying the following

$$\mathbb{P}_{x\theta}\circ(X_t - X_0, Y_t)^{-1} \text{ is independent of } x\in\mathbb{R}^d,$$
$$\mathbb{P}_{x\theta}\circ Y^{-1} = \mathbb{P}_\theta \quad \text{for all } x\in\mathbb{R}^d.$$

So we will take these versions. We are interested in asymptotic behavior of X. Let

$$(0.6)\qquad P^\epsilon_\theta = \mathbb{P}_{x\theta}\circ(X^\epsilon_\cdot - X^\epsilon_0)^{-1}, \quad \text{where } X^\epsilon_t = \epsilon X_{t/\epsilon^2}.$$

By the reason above P^ϵ_θ is independent of x.

THEOREM 1. – *Assume that* (A.1)–(A.6) *hold. Then there exist* $\widehat{a}_{ij}:\Theta\to\mathbb{R}$ *such that*

$$(0.7)\qquad \lim_{\epsilon\to 0} P^\epsilon_\theta = \widehat{P}_\theta \quad \textit{in f.d.d. in } \mu\textit{-measure}.$$

Here $\widehat{P}_\theta$ *is the distribution of d-dimensional continuous martingale* $\widehat{X}_t$ *such that*

$$< \widehat{X}^i, \widehat{X}^j >_t = 2\widehat{a}_{ij}(\theta)t, \quad \widehat{X}_0 = 0.$$

In addition, when the matrix $a = (a_{ij})$ *and the bilinear form* $\mathcal{E}^2_Y$ *are symmetric, the convergence in* (0.7) *is strengthened to be weakly in* $C([0,\infty);\mathbb{R}^d)$ *in* μ*-measure.*

Remark 0.1. – (1) If $(Y, \mathbb{P}_\mu)$ is ergodic under the time shift, then $\widehat{a}_{ij}(\theta)$ are constant.
(2) By definition $P^\epsilon_\theta = \mathbb{P}_{x\theta}\circ(X^\epsilon_\cdot - \epsilon x)^{-1}$. So (0.7) implies

$$\lim_{\epsilon\to 0}\mathbb{P}_{x\theta}\circ(X^\epsilon_\cdot)^{-1} = \widehat{P}_\theta \quad \text{in f.d.d. in } \mu\text{-measure for all } x\in\mathbb{R}^d.$$

We next proceed with the identification of the limit matrix. Let

$$\alpha_{ij} = \int_{\Theta} \widehat{a}_{ij}(\theta) d\mu. \tag{0.8}$$

We call $\alpha = (\alpha_{ij})$ the limit matrix. It is determined by $\mathcal{E}_Y$, a_{ij} and D_i. We write $\alpha = \alpha[\mathcal{E}_Y]$ when we emphasize the dependence on $\mathcal{E}_Y$. Let

$$\widetilde{\mathcal{D}} = \underbrace{L^2(\Theta,\mu) \times \cdots \times L^2(\Theta,\mu)}_{d \text{ times}} \times (\mathcal{D}_Y/\mathcal{E}_Y^2)$$

Here $\mathcal{D}_Y/\mathcal{E}_Y^2$ is the quotient space of $\mathcal{D}_Y$ with the equivalence relation $\sim$ such that $f \sim g$ if and only if $\mathcal{E}_Y^2(f-g, f-g) = 0$. Let $\widetilde{\mathcal{E}}_Y : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \mathbb{R}$ such that

$$\widetilde{\mathcal{E}}_Y(\mathfrak{f}, \mathfrak{g}) = \int_{\Theta} \sum_{i,j=1}^{d} a_{ij} f_i g_j d\nu + \mathcal{E}_Y^2(f, g).$$

Here $\mathfrak{f} = (f_1, \dots, f_d, \widetilde{f}) \in \widetilde{\mathcal{D}}$, and $\widetilde{f} \in \mathcal{D}_Y/\mathcal{E}_Y^2$ is the element whose representative is $f \in \mathcal{D}_Y$. We set $\mathfrak{g}$ similarly. Let $\widetilde{\mathcal{E}}_Y^{\mathrm{sym}}$ be the inner product on $\widetilde{\mathcal{D}}$ given by

$$\widetilde{\mathcal{E}}_Y^{\mathrm{sym}}(\mathfrak{f}, \mathfrak{g}) = \{\widetilde{\mathcal{E}}_Y(\mathfrak{f}, \mathfrak{g}) + \widetilde{\mathcal{E}}_Y(\mathfrak{g}, \mathfrak{f})\}/2.$$

Then $\widetilde{\mathcal{D}}$ is a Hilbert space with inner product $\widetilde{\mathcal{E}}_Y^{\mathrm{sym}}$. Let $\widetilde{\mathcal{D}}_{00}$ be a subset of $\widetilde{\mathcal{D}}$ given by

$$\widetilde{\mathcal{D}}_{00} = \{\mathfrak{g} \in \widetilde{\mathcal{D}}; \mathfrak{g} = (D_1 g, \dots, D_d g, \widetilde{g}), g \in \mathcal{D}_Y\}.$$

Let $\widetilde{\mathcal{D}}_0$ denote the closure of $\widetilde{\mathcal{D}}_{00}$ with respect to $\widetilde{\mathcal{E}}_Y^{\mathrm{sym}}$. Let $\mathbf{e}^i \in \widetilde{\mathcal{D}}$ $(i = 1, \cdots, d)$ such that $\mathbf{e}^i = (\delta_{ij})_{1 \le j \le \widetilde{d+1}}$, where δ_{ij} is the Kronecker delta. We consider the equation on $\psi^i \in \widetilde{\mathcal{D}}_0$ for each $1 \le i \le d$ given by:

$$\widetilde{\mathcal{E}}_Y(\psi^i, \mathfrak{g}) = \widetilde{\mathcal{E}}_Y(\mathbf{e}^i, \mathfrak{g}) \quad \text{for all } \mathfrak{g} \in \widetilde{\mathcal{D}}_0. \tag{0.9}$$

It is easy to see that (0.9) has a unique solution ψ^i (see [21, Lemma 2.1]).

We now state the expression of the limit matrix.

THEOREM 2. – *Under the same assumptions in Theorem 1 the following hold:*

$$\alpha_{ij} = \widetilde{\mathcal{E}}_Y^{\mathrm{sym}}(\mathbf{e}^i - \psi^i, \mathbf{e}^j - \psi^j) \tag{0.10}$$

Remark 0.2. – (1) If $\mathbf{e}^i \in \widetilde{\mathcal{D}}_0$, then $\psi^i = \mathbf{e}^i$. By (0.10) $\alpha_{ii} > 0$ if and only if $\mathbf{e}^i \notin \widetilde{\mathcal{D}}_0$.

(2) Positivity of limit matrix α depends on individual structures of each models. We will prove it in [20] for tagged particles of infinitely many hard core Brownian balls (see Remark 1.3 below). We refer to [1], [13] for the case of soft core, and [8], [26] in case of exclusion processes on $\mathbb{Z}^d$. See [17], [27] for reflecting barrier Brownian motions in random domains.

(3) α is called effective constants in homogenization problem, and self-diffusion constants for tagged particles of infinitely many particle systems.

(4) Suppose X is the additive functional of $\mathbb{H}_Y$ and Dirichlet forms are symmetric. If the mean forward velocity φ is in $L^2(\Theta, \mu)$ and in the domain of the generator of $\mathbb{H}_Y$, then α is given by the following (see [1], [26]);

$$\alpha_{ij} = \mathbb{E}_\mu[(X_1^i - X_0^i)(X_1^j - X_0^j)] - 2\mathbb{E}_\mu[\int_0^\infty \varphi^i(X_0)\varphi^j(X_t)dt].$$

The second term is called the integral of velocity autocorrelation function. This expression does not make sense when $\varphi \notin L^2(\Theta, \mu)$. One of advantages of (0.10) is that (0.10) holds even if the mean forward velocity φ is a distribution, which is the case of hard core Brownian motions.

As an application of Theorem 2, we obtain a universal inequality on limit matrices: Let $\mathcal{E}^{\text{sym}}(\mathfrak{f}, \mathfrak{g}) = \{\mathcal{E}(\mathfrak{f}, \mathfrak{g}) + \mathcal{E}(\mathfrak{g}, \mathfrak{f})\}/2$ and $\mathcal{E}^{\text{sym},1}(\mathfrak{f}, \mathfrak{g}) = \{\mathcal{E}^1(\mathfrak{f}, \mathfrak{g}) + \mathcal{E}^1(\mathfrak{g}, \mathfrak{f})\}/2$. Then $(\mathcal{E}^{\text{sym}}, \mathcal{E}^{\text{sym},1}, \partial)$ also satisfies the assumptions (A.1)–(A.6).

THEOREM 3. – *Under the same assumptions in Theorem 1 the following hold:*

(1) *Let* $\alpha[\mathcal{E}_Y^{\text{sym}}]$ *and* $\alpha[\mathcal{E}_Y]$ *be the limit matrices associated with* $\mathcal{E}_Y^{\text{sym}}$ *and* $\mathcal{E}_Y$ *respectively. Then*

$$\alpha[\mathcal{E}_Y^{\text{sym}}] \leq \alpha[\mathcal{E}_Y]. \tag{0.11}$$

Here the inequality means $\alpha[\mathcal{E}_Y] - \alpha[\mathcal{E}_Y^{\text{sym}}]$ *is a positive definite matrix.*

(2) *We have a variational formula of the limit matrix for the symmetric case;*

$$\sum_{i,j=1}^{d} \xi_i\xi_j\alpha_{ij}[\mathcal{E}_Y^{\text{sym}}] = \inf\{\widetilde{\mathcal{E}}_Y^{\text{sym}}(\xi - \mathfrak{g}, \xi - \mathfrak{g}); \mathfrak{g} \in \widetilde{\mathcal{D}}_0\}. \tag{0.12}$$

Here $\xi = (\xi_i) \in \mathbb{R}^d$.

Remark 0.3. – When φ is in $L^2(\Theta, \mu)$, inequality (0.11) was known for specific models (see [29], [13]). Our contribution here is to prove inequality (0.11) with a great generality.

We now explain some idea of the proof. In previous works [8], [1], [21], [29], X was assumed to be an additive functional of $\mathbb{H}_Y$ and have a mean forward velocity. It is difficult to prove the existence of the mean forward velocity for general additive functionals even if $\mathbb{H}_Y$ is symmetric. However, if X is an additive functional of the form

$$X_t = A_t^{[f]} := \hat{f}(Y_t) - \hat{f}(Y_0) \tag{0.13}$$

for some $f \in \mathcal{D}_Y$ ($\hat{f}$ is the quasi continuous modification of f, see [3], [12]), then the existence of the mean forward velocity φ is trivial from the well known relation (see, e.g., [12, Theorem I 2.13 (iii)]);

$$\lim_{\lambda \to \infty} \lambda(\lambda R_{Y,\lambda} f - f, v)_{L^2(\Theta,\mu)} = -\mathcal{E}_Y(f, v) \quad \text{for all } v \in \mathcal{D}_Y$$

and φ is identified, where $R_{Y,\lambda}$ is the λ-resolvent of $\mathbb{H}_Y$. (When the limit $\lim_{\epsilon \to 0} \epsilon X_{t/\epsilon^2}$ is non-degenerate, this is something hardly expected because $\mathbb{H}_Y$ has invariant probability measure and f is in $L^2(\Theta, \mu)$). Taking this into account, we consider the new Markov process $\mathbb{H}_{XY}$ on the extended space $\mathbb{R}^d \times \Theta$. (For this we assumed (A.4) and (A.5)). Then $X_t - X_0$ of $\mathbb{H}_{XY}$ is an additive functional of the form $A^{[x]}$, like as (0.13), with coordinate function x. We next introduce a weighted non-symmetric form and a weighted L^2 space, associated with $\mathbb{H}_{XY}$, in such a way that x is in its domain. Then, as we see in Lemma 3.1, we obtain the existence of the mean forward velocity of X.

At a first glance one may think our formulation is complicated; however, it nicely fit concrete problems. In order to convince readers to this point, we give applications in the next section. The proof of main theorems will be started from Section 2.

The organization of this paper is as follows: In Sect. 1 we apply main theorems to central limit theorems for tagged particles of interacting Brownian motions with skew symmetric drifts. We also refer to the homogenization of non-symmetric reflecting diffusion processes in $\mathbb{R}^d$. In Sect. 2 we introduce a weighted form and prove $\mathbb{H}_{XY}$ is conservative. In Sect. 3 we prove the existence of mean forward velocity. In Sect. 4 we complete the proof of Theorems 1–3. In Sect. 5 we collect some results from Dirichlet form theory. These results are used in preceding sections.

1. Applications

In this section we give applications.

1. Tagged particles of interacting Brownian motions

We first give a rough sketch of the problem. Let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a measurable function such that $\Phi(x) = \Phi(-x)$. We consider a diffusion on $(\mathbb{R}^d)^{\mathbb{N}}$ formally given by the following SDE;

$$dX_t^i = dB_t^i - \sum_{j=1, j\neq i}^{\infty} \frac{1}{2} \nabla\Phi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of the natural numbers and B_t^i $(i \in \mathbb{N})$ are independent Brownian motion on $\mathbb{R}^d$. We tag one particle, X^{i_0} say. The problem is to prove $\lim_{\epsilon \to 0} \epsilon X_{t/\epsilon^2}^{i_0} = \sigma B_t$.

When $\Phi \in C_0^3(\mathbb{R}^d)$, SDE (1.1) was solved by Lang [9], [10]. (See also [25], [4], [28]). We however use here the Dirichlet form construction due to [18] because it does not need the smoothness of Φ and gives associated Dirichlet forms explicitly.

We now proceed to the precise formulation of the problem: Let Θ be the set of all locally finite configurations in $\mathbb{R}^d$, where a locally finite configuration means a Radon measure θ of the form $\theta = \sum_n \delta_{x_n}$. Here $\{x_n\}$ is a finite or infinite sequence in $\mathbb{R}^d$ with no cluster points and δ_a is the delta measure at a. By convention we regard zero measure as a configuration. We equip Θ with the vague topology. Θ is a Polish space with this topology. (cf. [23]. The Θ-valued process associated with (1.1) is

$$\mathbb{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}. \tag{1.2}$$

In order to construct dynamics (1.2) we introduce a bilinear form: Let $\Theta^i = \{\theta \in \Theta; \theta(\mathbb{R}^d) = i\}$ for $i \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Let $\mathbb{R}^{(i)} = \mathbb{R}^{di}$ for $i \in \mathbb{N}$, and $\mathbb{R}^{(\infty)} = \{(x_i)_{i \in \mathbb{N}};\ (x_i)_{i\in\mathbb{N}}$ have no cluster points in $\mathbb{R}^d\}$. A map $\mathbf{x}^i : \Theta^i \to \mathbb{R}^{(i)}$ $(i \in \mathbb{N} \cup \{\infty\})$ is called a $\mathbb{R}^{(i)}$-coordinate of θ if $\theta = \sum_{j=1}^{i} \delta_{x^j(\theta)}$ for all $\theta \in \Theta^i$, where $\mathbf{x}^i(\theta) = (x^1(\theta), \ldots, x^i(\theta))$. Let for $i \in \mathbb{N} \cup \{\infty\}$ and $f, g \in C^\infty(\mathbb{R}^{(i)})$

$$D^i[f, g](\mathbf{x}) = \frac{1}{2} \sum_{j=1}^{i} \nabla_j f(\mathbf{x}) \cdot \nabla_j g(\mathbf{x}). \tag{1.3}$$

Here $\nabla_j = (\frac{\partial}{\partial x_{jk}})_{1 \le k \le d}$, $\mathbf{x} = (x^j) \in \mathbb{R}^{(i)}$ and $\cdot$ means the inner product on $\mathbb{R}^d$. Let $\mathcal{D}_\infty$ be the set of all local, smooth functions on Θ (see [18, (1.2)] for the precise definition). For $\mathfrak{f}, \mathfrak{g} \in \mathcal{D}_\infty$ we set $\mathbf{D}[\mathfrak{f}, \mathfrak{g}] : \Theta \to \mathbb{R}$ by

$$\mathbf{D}[\mathfrak{f}, \mathfrak{g}](\theta) = \begin{cases} D^i[f^i, g^i](\mathbf{x}^i(\theta)) & \text{for } \theta \in \Theta^i,\ i \in \mathbb{N} \cup \{\infty\} \\ 0 & \text{for } \ \theta \in \Theta^0. \end{cases} \tag{1.4}$$

Here $\mathbf{x}^i(\theta)$ is a $\mathbb{R}^{(i)}$-coordinate, and f^i is the permutation invariant function on $\mathbb{R}^{(i)}$ such that $\mathfrak{f}(\theta) = f^i(\mathbf{x}^i(\theta))$ for all $\theta \in \Theta^i$. We set g^i similarly. Note that such f^i and g^i are unique for each $i \in \mathbb{N} \cup \{\infty\}$ and $\mathbf{D}$ is well defined. For a probability measure μ on $(\Theta, \mathcal{B}(\Theta))$ we set

$$\mathcal{E}^\mu(\mathfrak{f}, \mathfrak{g}) = \int_\Theta \mathbf{D}[\mathfrak{f}, \mathfrak{g}](\theta) d\mu. \tag{1.5}$$

Our dynamics are diffusion $\{\mathbb{P}_\theta^\mu\}$ associated with $(\mathcal{E}^\mu, \mathcal{D}_\infty)$ on $L^2(\Theta, \mu)$.

Remark 1.1. – (1) If we take μ to be the Poisson random measure whose intensity measure is Lebesgue measure, then $\{\mathbb{P}_\theta^\mu\}$ is given by Θ-valued Brownian motion $\mathbb{B}_t$; that is, $\mathbb{B}_t = \sum_{i=1}^\infty \delta_{B_t^i}$. Here B^i are independent copies of Brownian motion.

(2) Typical examples of μ are grand canonical Gibbs measures with potential Φ. See [18] for the definition; there they are called Gibbs measures.

(3) If we take μ to be a grand canonical Gibbs measure with hard core potential given in Example 1.1, then $\{\mathbb{P}_\theta^\mu\}$ describe the motion of infinitely many hard core Brownian balls.

Let $Q_r = \{x \in \mathbb{R}^d; |x| \le r\}$ and Q_r^i be the i times product of Q_r. We denote by σ_r^i the density functions of μ on Q_r^i (see [18] for the definition). We also denote by $\{\rho^i\}_{i=0,1,\dots}$ the infinite volume correlation functions of μ if exist. (see [24]). Let $\tau_a : \Theta \to \Theta$ denote the translation given by $\tau_a\theta = \sum_n \delta_{x_n + a}$ for $\theta = \sum_n \delta_{x_n}$. We assume:

(M.1) $(\mathcal{E}^\mu, \mathcal{D}_\infty)$ is closable on $L^2(\Theta, \mu)$.
(M.2) $\sigma_r^i \in L^\infty(Q_r^i, dx)$ for all $1 \le i, r < \infty$, $\rho^1 < \infty$ exists.
(M.3) $\mu \circ \tau_a^{-1} = \mu$ for all $a \in \mathbb{R}^d$.
(M.4) $\mathrm{Cap}(\mathcal{N}) = 0$, where $\mathcal{N} = \{\theta; \theta(\{x\}) \ge 2$ for some $x \in \mathbb{R}^d\}$.

By (M.1) we denote by $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ the closure of $(\mathcal{E}^\mu, \mathcal{D}_\infty)$ on $L^2(\Theta, \mu)$. We note by (M.3) $\rho^1(x) = $ constant.

LEMMA 1.1. – *Assume* (M.1)–(M.4). *Then we have the following:*

(1) $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ *is a quasi-regular Dirichlet form on* $L^2(\Theta, \mu)$ *and there exists a diffusion* $\{\mathbb{P}_\theta^\mu\}$ *associated with* $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ *on* $L^2(\Theta, \mu)$.

(2) $\mathbb{P}_\theta^\mu(\mathbb{X}_t \in \cdot) = \mathbb{P}_{\tau_a\theta}^\mu(\tau_a\mathbb{X}_t \in \cdot)$ for all $a \in \mathbb{R}^d$.

(3) $\mathbb{P}_\theta^\mu(\mathbb{X}_t \in \mathcal{N}$ for some $t) = 0$ *q.e.* θ.

Proof. – Since ρ^1 is constant, we have $\sum_{i=1}^{\infty} i\mu(\Theta_r^i) = \int_{Q_r} \rho^1 dx < \infty$ for all $r < \infty$, where $\Theta_r^i = \{\theta \in \Theta; \theta(Q_r) = i\}$. Hence (1) follows from [18, Theorem 1]. (2) follows from (M.3). (3) follows from (M.4). □

Let μ_x denote the conditional probability given by $\mu_x = \mu(\cdot|\theta(\{x\}) = 1)$. By (M.3) we can choose the version μ_x in such a way that $\mu_x = \mu_{\mathbf{0}} \circ \tau_x^{-1}$ for all $x \in \mathbb{R}^d$. Here the subscript $\mathbf{0}$ of $\mu_{\mathbf{0}}$ denotes the origin in $\mathbb{R}^d$. We set

$$\Theta_x = \{\theta; \theta(\{x\}) = 1, \mathbb{P}_\theta^\mu(\mathbb{X}_t \in \mathcal{N} \text{ for some } t) = 0\}.$$

We can write $\mathbb{X} \in C([0,\infty) \to \Theta)$ as $\mathbb{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$, where $X_t^i \in C(I_i \to \mathbb{R}^d)$ and I_i is an interval in $[0,\infty)$ of the form $[0,b)$ or (a,b). (Although we can prove $I_i = [0,\infty)$ for all i from (M.3), we consider here a general case). Then I_i are unique up to numbering. If $\mathbb{X}_0 = \theta \in \Theta_x$, then there exists an $i(x,\theta)$ such that $X_0^{i(x,\theta)} = x$ and such $\mathbb{R}^d$-valued path $X^{i(x,\theta)} = X_t^{i(x,\theta)}$ is unique. For each $\theta \in \Theta_x$ we call $(X^{i(x,\theta)}, \mathbb{P}_\theta^\mu)$ a tagged particle starting from x. We want to prove the convergence of $\lim_{\epsilon\to\infty} \epsilon X_{\cdot/\epsilon^2}^{i(x,\theta)}$. So for $\theta \in \Theta_x$ we set $P_\theta^{\mu,\epsilon} = \mathbb{P}_\theta^\mu \circ (\epsilon X_{\cdot/\epsilon^2}^{i(x,\theta)})^{-1}$.

THEOREM1.2. – *Assume that μ satisfies* (M.1)–(M.4). *Then for each $x \in \mathbb{R}^d$ we obtain*

(1.6) $$\lim_{\epsilon\to 0} P_\theta^{\mu,\epsilon} = \widehat{P}_\theta \quad \textit{weakly in } C([0,\infty) \to \mathbb{R}^d) \textit{ in } \mu_x\textit{-measure,}$$

where $\widehat{P}_\theta$ is the distribution of d-dimensional continuous martingale $\widehat{X}_t$ such that

$$< \widehat{X}^i, \widehat{X}^j >_t = 2\widehat{a}_{ij}(\tau_x\theta)t, \quad \widehat{X}_0 = 0.$$

Here $\widehat{a}_{ij} : \Theta_0 \to \mathbb{R}$.

We will reduce Theorem 1.2 to Theorems 1 and 2. As previous works [6], [1] we consider the dynamics so called *environments seen from the tagged particle*. For this we introduce new Dirichlet forms: Let $D : \mathcal{D}_\infty \to (\mathcal{D}_\infty)^d$ such that

$$D\mathfrak{f}(\theta) = \begin{cases} \frac{1}{\sqrt{2}} \sum_{j=1}^{i} \nabla_j f^i(\mathbf{x}^i(\theta)) & \text{for } \theta \in \Theta^i \quad (i \in \mathbb{N} \cup \{\infty\}) \\ 0 & \text{for } \theta \in \Theta^0. \end{cases}$$

Here ∇_j, $\mathbf{x}^i(\theta)$ and $f^i : \mathbb{R}^{(i)} \to \mathbb{R}$ are same as in (1.5). We note $\sqrt{2}D$ is the generator of a family of unitary operators $\{U_a\}_{a\in\mathbb{R}^d}$ on $L^2(\Theta,\mu)$ given by $U_a\mathfrak{f}(\theta) = \mathfrak{f}(\tau_a\theta)$. Let $\partial = (\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i})_{1\le i\le d}$ and $\mu_0 = \mu_{\mathbf{0}}(\cdot - \delta_{\mathbf{0}})$. We set

$$\mathcal{E}_Y(\mathfrak{g}_1, \mathfrak{g}_2) = \int_\Theta \{D\mathfrak{g}_1 \cdot D\mathfrak{g}_2 + \mathbf{D}[\mathfrak{g}_1, \mathfrak{g}_2]\} d\mu_0$$

$$\mathcal{E}_{XY}(\mathfrak{h}_1, \mathfrak{h}_2) = \int_{\mathbb{R}^d \times \Theta} \{(D-\partial)\mathfrak{h}_1 \cdot (D-\partial)\mathfrak{h}_2 + \mathbf{D}[\mathfrak{h}_1, \mathfrak{h}_2]\} dx d\mu_0.$$

Here $\cdot$ is the inner product in $\mathbb{R}^d$ as before, and for $\mathfrak{h}_i = \sum_m \mathfrak{f}_{im} \otimes \mathfrak{g}_{im}$ we set

$$\mathbf{D}[\mathfrak{h}_1, \mathfrak{h}_2] = \sum_{k,l} \mathfrak{f}_{1k} \mathfrak{f}_{2l} \mathbf{D}[\mathfrak{g}_{1k}, \mathfrak{g}_{2l}].$$

LEMMA 1.3. – (1) $(\mathcal{E}_Y, \mathcal{D}_\infty)$ *is closable on* $L^2(\Theta, \mu_0)$, *and its closure* $(\mathcal{E}_Y, \mathcal{D}_Y)$ *is a quasi regular Dirichlet form on* $L^2(\Theta, \mu_0)$.

(2) $(\mathcal{E}_{XY}, C_0^\infty \otimes \mathcal{D}_\infty)$ *is closable on* $L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0)$, *and its closure* $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ *is a quasi regular Dirichlet form on* $L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0)$.

(3) *Let* $(\{\mathbb{P}_{x\theta}\}_{(x,\theta) \in \mathbb{R}^d \times \Theta}, (X_t, Y_t))$ *denote the diffusion associated with* $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ *on* $L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0)$. *Then* $\mathbb{P}_\theta^\mu \circ (X_t^{i(x,\theta)})^{-1} = \mathbb{P}_{x\theta} \circ X_t^{-1}$.

Remark 1.2. – A diffusion associated with $\mathcal{E}_Y$ describes the motion of environments seen from the tagged particle, and $\mathcal{E}_{XY}$ corresponds to the motion of the coupling of the tagged particle and environments seen from the tagged particle. The former diffusion is represented by

$$\mathbb{Y}_t := \sum_{j \neq i(x,\theta)} \delta_{X_t^j - X_t^{i(x,\theta)}} \quad \text{under } \mathbb{P}_\theta^\mu.$$

Here $X_t^{i(x,\theta)}$ is the (position of) tagged particle. This representation makes sense until $t < \sigma := \sup\{t; X_t^{i(x,\theta)} \in \mathbb{R}^d\}$. Here, in other words, σ is the right end point of the random interval $[0, \sigma)$, where the tagged particle $X_t^{i(x,\theta)}$ is defined. We see eventually $\sigma = \infty$ a.s. by Lemma 2.3.

Proof. – For the sake of brevity we only sketch the proof; the details will appear in [19]. The closability of $\mathcal{E}_{XY}$ can be proved in precisely the same fashion as [18, Theorems 1 and 4].

In order to prove the quasi regularity of $\mathcal{E}_{XY}$, we first construct the associated diffusion $\{\mathbb{P}_{x\theta}\}$. For $\theta \in \Theta_x$ we consider a family of finite measures $\{\mathbb{P}_{x\theta}\}$ given by

$$\mathbb{P}_{x\theta}(*) = \mathbb{P}_\theta^\mu((X^{i(x,\theta)}, \mathbb{Y}) \in * \cap \{\sigma < \infty\}).$$

Then it is clear that $\{\mathbb{P}_{x\theta}\}$ is a diffusion with state space $\mathbb{R}^d \times \Theta$.

Let T_t denote the semigroup associated with $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0)$. Then it is not difficult to see

$$(1.7) \quad T_t h(x, \theta) = \mathbb{E}_{x\theta}[h(X_t, Y_t); t < \sigma] \quad \text{for } h \in L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0).$$

Here $\mathbb{E}_{x\theta}$ is the expectation with respect to $\mathbb{P}_{x\theta}$ and $\sigma = \sup\{t; |X_t| < \infty\}$. We can prove (1.7) by using an approximating sequence of finite dynamics.

We thus see $\{\mathbb{P}_{x\theta}\}$ is the diffusion associated with $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d \times \Theta, dx \times \mu_0)$.

Now by Ma-Röckner's result [12, Ch. IV Theorem 5.1], the quasi regularity follows form the existence of the associated diffusion $\{\mathbb{P}_{x\theta}\}$. We thus proved (2). (3) is clear from the construction of $\{\mathbb{P}_{x\theta}\}$.

Closability of $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu_0)$ follows from that of $(\mathcal{E}_{XY}, C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y)$. Let $\mathbb{P}_\theta = \mathbb{P}_{x\theta} \circ \mathbb{Y}^{-1}$. Then it is clear that $\{\mathbb{P}_\theta\}$ is a diffusion associated with $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu_0)$. Hence by Ma-Röckner's result we obtain the quasi-regularity of $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu_0)$, which yields (1). □

Proof of Theorem. 1.2. – By Lemma 1.3 (1) and (2), we see $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu_0)$ and ∂ satisfy the assumptions in Theorem 1. (μ in Theorem 1 corresponds to μ_0 here). By Lemma 1.3 (3)

$$\lim_{\epsilon \to 0} P_\theta^{\mu, \epsilon} = \lim_{\epsilon \to 0} P_{x\theta} \circ (\epsilon X_{\cdot/\epsilon^2})^{-1} = \lim_{\epsilon \to 0} P_{x\theta} \circ (\epsilon X_{\cdot/\epsilon^2} - \epsilon X_0)^{-1}.$$

Hence Theorem 1.2 follows from Theorem 1. □

We now give a class of measures μ satisfying the assumptions in Theorem 1.2. As we stated before, this class consist of grand canonical Gibbs measures μ with pair potential $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. For the existence of Gibbs measures and dynamics $\{\mathbb{P}_\theta\}$ we assume Φ satisfies the following:

(Φ.1) Φ is super stable in the sense of Ruelle [24].

(Φ.2) Φ is regular in the sense of Ruelle [24]; there exists a positive decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}$ and a constant R_1 such that $\int_0^\infty \phi(t) t^{d-1} dt < \infty$, $\Phi(x) \geq -\phi(|x|)$ for all x, and $\Phi(x) \leq \phi(|x|)$ for $|x| \geq R_1$.

(Φ.3) Φ is upper semicontinuous.

By (Φ.3) $\Gamma = \{x \in \mathbb{R}^d; \Phi(x) = \infty\}$ is a closed set. We call Γ the core of particles. Let $\mathcal{G}(\Phi)_z$ denote the set of all grand canonical Gibbs measures with activity z and potential Φ satisfying (M.1)–(M.4).

LEMMA 1.4. – *Assume that Φ satisfies (Φ.1)–(Φ.3) and $d \geq 2$. Then translation invariant, grand canonical Gibbs measures obtained by Ruelle* [24] *are elements of $\cup_{z>0} \mathcal{G}(\Phi)_z$. In particular, $\mathcal{G}(\Phi)_z \neq \emptyset$ for all $z > 0$.*

Proof. – By results in [24] assumptions (Φ.1) and (Φ.2) imply, for each activity $z > 0$, the existence of grand canonical Gibbs measures satisfying (M.2), (M.3) and having infinite volume correlation functions ρ^i such that

(1.8) $$\rho^i(x^1, \ldots, x^i) \leq C_4^i \quad (C_4 \text{ is a constant}).$$

Moreover (M.1) follows from [18, Theorem 4]. (M.4) follows from (1.8) with $i = 2$ and $d \geq 2$. □

We proceed with the main theorem in this subsection:

THEOREM 1.5. – *Assume that* Φ *satisfies* (Φ.1)–(Φ.3) *and* $d \geq 2$. *Then for each* $z > 0$ *for each* $\mathcal{G}(\Phi)_z \neq \emptyset$, *and for each* $\mu \in \cup_{z>0}\mathcal{G}(\Phi)_z$ *we obtain* (1.6).

Proof. – Theorem 1.5 follows from Theorem 1.2 and Lemma 1.4 immediately.

Remark 1.3. – (1) In [20] we will prove the limit matrix is strictly positive if $d \geq 2$ and Φ satisfies the following:

(Φ.4) Γ is convex. (Φ.5) Γ has positive Lebesgue measure.

We conjecture that (Φ.5) is unnecessary. On the other hand, (Φ.4) seems essential; we conjecture that there exists a potential Φ whose limit matrix is degenerate when (Φ.4) is not satisfied.

(2) When $d = 1$ and $\Phi \in C_0^3(\mathbb{R})$, then (M.4) is not satisfied in general. So it may happen that $\mathcal{G}(\Phi)_z = \emptyset$.

(3) When $\Phi \in C_0^3(\mathbb{R}^d)$ and $\#\mathcal{G}_z(\Phi) = 1$, Guo [6] proved the convergence in f.d.d.. In [1] DeMasi et al proved the convergence weak in $C([0,\infty) \to \mathbb{R}^d)$, not only the f.d.d. convergence, when $\Phi \in C_0^3(\mathbb{R}^d)$ and $\sum_i \nabla\Phi(x^i) \in L^2(\Theta, \mu_0)$. They used the Kipnis-Varadhan argument. Compared with these results, our results require no such restrictions on Φ and, in addition, the convergence is weak in $C([0,\infty) \to \mathbb{R}^d)$. We will study non symmetric interacting Brownian motions in the next subsection, which are also excluded in [6], [1].

We give three examples of Φ which satisfy assumptions (Φ.1)–(Φ.3). No example below are covered by [6], [1].

Example 1.1. – (hard core Brownian balls). Let Φ_{hard} be a hard core potential such that

$$\Phi_{\text{hard}}(x) = \infty \text{ for } |x| < R, \quad \Phi_{\text{hard}}(x) = 0 \text{ for } |x| \geq R.$$

Here $R > 0$ is a constant. Let μ_{hard} be a grand canonical Gibbs measure with potential Φ_{hard} and activity $z > 0$. Then the associated diffusion $\{\mathbb{P}_\theta^\mu\}$ for μ_{hard} describe the motion of infinitely many hard core Brownian balls with diameter R. In [20] we will prove, if $d \geq 2$, then the limit matrix is strictly positive for all $z > 0$.

Example 1.2. – (Lennard-Jones 6-12 potentials) Let $d = 3$ and

$$\Phi_{6,12}(x) = 2\gamma\{|x|^{-12} - |x|^{-6}\} \qquad (\gamma > 0 \text{ is a constant}).$$

In this case the corresponding SDE is

$$dX_t^i = dB_t^i + \sum_{j=1, j\neq i}^{\infty} \gamma(X_t^i - X_t^j)\{12|X_t^i - X_t^j|^{-14} - 6|X_t^i - X_t^j|^{-8}\}dt \quad (i \in \mathbb{N}).$$

Example 1.3. – (Lennard-Jones type potentials). Let $a > d$. Set $\Phi_a(x) = 2|x|^{-a}$. In this case the corresponding SDE is

$$dX_t^i = dB_t^i + \sum_{j=1, j\neq i}^{\infty} a(X_t^i - X_t^j)|X_t^i - X_t^j|^{-a-2}dt \quad (i \in \mathbb{N}).$$

1.2. Interacting Brownian motions with skew symmetric drifts

We consider non-symmetric Dirichlet forms obtained by adding skew symmetric forms to Dirichlet forms in Sect. 2.1. Let $\Psi = (\Psi_{mn})_{m,n=1,\ldots,d} : \mathbb{R}^d \to \mathbb{R}^{d^2}$ be a measurable function satisfying the following:

$(\Psi.1)$ $\qquad \Psi(x) = \Psi(-x) \quad \text{for all } x \in \mathbb{R}^d$

$(\Psi.2)$ $\qquad \Psi_{mn} = -\Psi_{nm} \quad$ (skew symmetry).

For $i \in \mathbb{N} \cup \{\infty\}$ and $f, g \in C^\infty(\mathbb{R}^{(i)})$ we set

$$D^{\Psi,i}[f,g](\mathbf{x}) = \sum_{j=1}^{i} \sum_{k=1,k\neq j}^{i} \nabla_j g(\mathbf{x}) \cdot \Psi(x^j - x^k)\nabla_j f(\mathbf{x}).$$

Here ∇_j, $\mathbf{x} = (x^j) \in \mathbb{R}^{(i)}$ and $\cdot$ are same as in (1.3). We remark, if f and g are permutation invariant, then $D^{\Psi,i}[f,g]$ is also permutation invariant. So for $\mathfrak{f}$ and $\mathfrak{g} \in \mathcal{D}_\infty$ we define $\mathbf{D}^\Psi[\mathfrak{f},\mathfrak{g}] : \Theta \to \mathbb{R}$ by

$$\mathbf{D}^\Psi[\mathfrak{f},\mathfrak{g}](\theta) = \begin{cases} D^{\Psi,i}[f^i, g^i](\mathbf{x}^i(\theta)) & \text{for } \theta \in \Theta^i,\ i \in \mathbb{N} \cup \{\infty\} \\ 0 & \text{for } \theta \in \Theta^0. \end{cases}$$

Here $\mathbf{x}^i(\theta)$, f^i and g^i are same as in (1.4). Let

$$\mathcal{E}^\Psi(\mathfrak{f},\mathfrak{g}) = \int_\Theta \mathbf{D}^\Psi[\mathfrak{f},\mathfrak{g}]d\mu, \quad \mathcal{E}^{\mu,\Psi} = \mathcal{E}^\mu + \mathcal{E}^\Psi.$$

Note that $\mathcal{E}^\Psi(\mathfrak{f},\mathfrak{f}) = 0$ by $(\Psi.2)$. So we have $\mathcal{E}^{\mu,\Psi}(\mathfrak{f},\mathfrak{f}) = \mathcal{E}^\mu(\mathfrak{f},\mathfrak{f})$. We assume

$(\Psi.3)$ $\qquad \mathcal{E}^{\mu,\Psi}(\mathfrak{f},\mathfrak{g}) \leq C_5 \mathcal{E}^{\mu,\Psi}(\mathfrak{f},\mathfrak{f})^{1/2}\mathcal{E}^{\mu,\Psi}(\mathfrak{g},\mathfrak{g})^{1/2}$

Remark 1.4. – If Γ has positive Lebesgue measure and $\sup\{|\Psi|; x \notin \Gamma\} < \infty$, and Ψ is finite range, then (Ψ.3) is satisfied. We also remark, for soft core interaction, (Ψ.3) should not be true; so it is still an open problem.

By (Ψ.3) and Lemma 1.3 it is clear that $(\mathcal{E}^{\mu,\Psi}, \mathcal{D}^{\mu})$ is a quasi-regular Dirichlet form on $L^2(\Theta, \mu)$. So let $\{\mathbb{P}_\theta^{\mu,\Psi}\}$ denote the associated diffusion. Since $\mathcal{N}$ defined in (M.4) is also capacity zero for $(\mathcal{E}^{\mu,\Psi}, \mathcal{D}^{\mu}, L^2(\Theta, \mu))$, we define the tagged particle starting from x similarly as before and denote it by $X^{i(x,\theta)}$. By Theorem 1 and 3 we obtain

THEOREM 1.6. – *Assume* (Φ.1)–(Φ.3), $\mu \in \mathcal{G}_z(\Phi)$ *and* (Ψ.1)–(Ψ.3). *Then for* $\theta \in \Theta_x$ *we set* $P_\theta^{\mu,\Psi,\epsilon} = \mathbb{P}_\theta^{\mu,\Psi} \circ (\epsilon X^{i(x,\theta)}_{\cdot/\epsilon^2})^{-1}$. *Then for each* $x \in \mathbb{R}^d$

$$\lim_{\epsilon \to 0} P_\theta^{\mu,\Psi,\epsilon} = \widehat{P}_\theta^{\mu,\Psi} \qquad \textit{in f.d.d. in } \mu_x\textit{-measure.}$$

Here $\widehat{P}_\theta^{\mu,\Psi}$ *is the distribution of the d-dimensional continuous martingale* X *such that*

$$< \widehat{X}^i, \widehat{X}^j >_t = 2\widehat{a_{ij}^{\mu,\Psi}}(\tau_x\theta)t, \quad \widehat{X}_0 = 0.$$

Moreover its limit matrix $\alpha[\mathcal{E}^{\mu,\Psi}] = \int \widehat{a_{ij}^{\mu,\Psi}}(\theta)d\mu_0$ *satisfies the following inequality;*

$$\alpha[\mathcal{E}^{\mu}] \le \alpha[\mathcal{E}^{\mu,\Psi}].$$

Example 1.4. – (Hard core vortexes) Let μ_{hard} and R be as in Example 1.1. Let $d = 2$ and

$$G_{R_1,R_2}(x) = \begin{cases} -(2\pi)^{-1}\log|x| & \text{if } R_1 < |x| < R_2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $0 \le R \le R_1 < R_2 \le \infty$ are constants. Let $\Psi_{12}(x) = -\Psi_{21}(x) = G_{R_1,R_2}(x)$ and $\Psi_{11}(x) = \Psi_{22}(x) = 0$. If we assume $R, R_1 > 0$ and $R_2 < \infty$, then the assumptions in Theorem 1.6 are fulfilled. In case of $R = R_1 = 0$ and $R_2 = \infty$, we do not know whether (Ψ.3) is satisfied or not. In this case the associated SDE becomes

$$(1.9) \qquad \begin{cases} dX_t^i = dB_t^{i,1} - \sum_{j=1, j\ne i}^{\infty} \frac{Y_t^i - Y_t^j}{2\pi|Z_t^i - Z_t^j|^2} dt \\ dY_t^i = dB_t^{i,2} + \sum_{j=1, j\ne i}^{\infty} \frac{X_t^i - X_t^j}{2\pi|Z_t^i - Z_t^j|^2} dt \end{cases} \qquad (i \in \mathbb{N}),$$

where $Z_t^i = (X_t^i, Y_t^i)$ are $\mathbb{R}^2$-valued processes, and $\{(B_t^{i,1}, B_t^{i,2})\}_{i\in\mathbb{N}}$ are independent copies of two dimensional Brownian motion. This describes the motion of vortexes with the same vorticity in viscous planer fluid. This model has particular interests and, in case of finite number vortexes, has been studied from various motivations (c.f. [2], [11], [15], [16], [5]). When the number of vortexes is finite, (1.9) was solved in [2], [14], [15]. On the other hand if the number of vortexes is infinite, (1.9) has not been solved yet. This problem was proposed in [4].

Remark 1.5. – Let Φ satisfy (Φ.1)–(Φ.3) and (Φ.5) and let $\mu \in \mathcal{G}_z(\Phi)$. Let G be a bounded measurable function with compact support on $\mathbb{R}^2$. Then by replacing μ_{hard} and $G_{R_1.R_2}$ in Example 1.4 by μ and G we obtain Ψ satisfying the assumptions in Theorem 1.6. This argument can be generalized to any *even* dimensional $\mathbb{R}^d$.

1.3. Reflecting diffusions in random domains

In [21] we studied homogenization of reflecting diffusion in random domains. By applying Theorems 1–3 to this problem, we obtain better results than ones in [21, Sect. 3]. Indeed, assumptions (3.3) and (3.4) of [21, Theorem 3.1] become unnecessary.

2. WEIGHTED DIRICHLET FORMS

In this section we introduce a weighted form $(\mathcal{E}^\rho, \mathcal{D}^\rho)$ on a weighted L^2-space $L^2(\rho)$ such that $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ is associated with $\mathbb{H}_{XY}$ and that $\mathcal{E}^\rho_\alpha = \mathcal{E}^\rho + \alpha(*, *)_{L^2(\rho)}$ is a positive form for large α. Let

$$\rho(x) = (1 + |x|^{2(d+4)})^{-1/2}/C,$$

where $C > 0$ is a constant such that $\int_{\mathbb{R}^d} \rho(x)^2 dx = 1$, and

$$L^2(\rho) = L^2(\mathbb{R}^d \otimes \Theta, \rho^2 dx \times \mu).$$

We set

$$\mathcal{E}^\rho(h_1, h_2) = \mathcal{E}_{XY}(h_1, \rho^2 h_2) \quad \text{for } h_1, h_2 \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_{\mathcal{Y}}. \tag{2.1}$$

We easily see

$$\mathcal{E}^2_{XY}(h_1, \rho^2 h_2) = \mathcal{E}^2_{XY}(\rho h_1, \rho h_2) \quad (\text{ by } (0.5)), \tag{2.2}$$

$$C_7 := \sup_{x \in \mathbb{R}^d} \{ \sum_{i=1}^d (|\partial_i \rho| / \rho)^2 \} < \infty. \tag{2.3}$$

LEMMA 2.1. – (1) *Let* $\mathcal{E}^\rho_\lambda = \mathcal{E}^\rho + \lambda(\cdot, \cdot)_{L^2(\rho)}$. *Then there exists* $\lambda_0 > 0$ *such that* $(\mathcal{E}^\rho_\lambda, C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y)$ *are closable on* $L^2(\rho)$ *for all* $\lambda > \lambda_0$.

Moreover, its closure $(\mathcal{E}^\rho_\lambda, \mathcal{D}^\rho)$ *on* $L^2(\rho)$ *for* $\lambda > \lambda_0$ *is a quasi regular Dirichlet form.*

(2) $x^i \otimes 1, (x^i)^2 \otimes 1 \in \mathcal{D}^\rho$ $(i = 1, \ldots, d)$.

Proof. – We prepare a subsidiary form $\mathcal{E}^*$ given by $\mathcal{E}^*(h_1, h_2) = \mathcal{E}(\rho h_1, \rho h_2)$. It is clear that $(\mathcal{E}^*, C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y)$ is closable on $L^2(\rho)$, and its closure $(\mathcal{E}^*, \mathcal{D}^*)$ on $L^2(\rho)$ is a quasi regular Dirichlet form. A straightforward calculation shows

$$(2.4)\quad \mathcal{E}^\rho(h_1, h_2) = \mathcal{E}^*(h_1, h_2) - \int_{\mathbb{R}^d \times \Theta} \sum_{i,j=1}^d a_{ij} (\partial_i \rho)(\partial_j \rho) h_1 h_2 dx d\nu$$
$$+ \int_{\mathbb{R}^d \times \Theta} \sum_{i,j=1}^d a_{ij} ((\partial_i - D_i)\rho h_1)(\partial_j \rho) h_2 dx d\nu$$
$$- \int_{\mathbb{R}^d \times \Theta} \sum_{i,j=1}^d a_{ij} h_1 (\partial_i \rho)(\partial_j - D_j)(\rho h_2) dx d\nu.$$

Let I_i denote the i-th term in the right-hand side of (2.4). Then by (2.3)

$$(2.5)\qquad \begin{aligned} |I_2| &\le C_8 \|h_1\|_{L^2(\rho)} \|h_2\|_{L^2(\rho)} \\ |I_3| &\le C_9 \mathcal{E}^*(h_1, h_1)^{1/2} \|h_2\|_{L^2(\rho)} \\ |I_4| &\le C_9 \mathcal{E}^*(h_2, h_2)^{1/2} \|h_1\|_{L^2(\rho)}. \end{aligned}$$

Here $C_8 = C_3 C_7$ and $C_9 = C_3 (C_7/C_2)^{1/2}$. Taking $h_1 = h_2 = h$ in (2.4) we see

$$\begin{aligned} \mathcal{E}^\rho(h, h) &\ge \mathcal{E}^*(h, h) - C_8 \|h\|^2_{L^2(\rho)} - 2C_9 \|h\|_{L^2(\rho)} \mathcal{E}^*(h, h)^{1/2} \\ &\ge \mathcal{E}^*(h, h) - (C_8 + 4C_9^2) \|h\|^2_{L^2(\rho)} - \frac{1}{4} \mathcal{E}^*(h, h) \\ &= \frac{3}{4} \mathcal{E}^*(h, h) - (C_8 + 4C_9^2) \|h\|^2_{L^2(\rho)}. \end{aligned}$$

Hence $\mathcal{E}^\rho_\lambda$ are positive for all $\lambda > \lambda_0 = C_8 + 4C_9^2$;

$$(2.6)\qquad \mathcal{E}^\rho_\lambda(h, h) \ge \frac{3}{4} \mathcal{E}^*(h, h).$$

Let $\mathcal{E}^*_1 = \mathcal{E}^* + (\cdot, \cdot)_{L^2(\rho)}$ and set $C_{10} = C_8 + 2C_9$. Then by (2.4) and (2.5)

$$(2.7)\qquad |\mathcal{E}^\rho(h_1, h_2)| \le |\mathcal{E}^*(h_1, h_2)| + C_{10} \mathcal{E}^*_1(h_1, h_1)^{1/2} \mathcal{E}^*_1(h_2, h_2)^{1/2}.$$

The first statement in (1) follows from (2.6), (2.7) and the closability of $\mathcal{E}^*$. By (2.6) and (2.7) we have $\mathcal{D}^* = \mathcal{D}^\rho$. So quasi regularity of $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ follows from that of $(\mathcal{E}^*, \mathcal{D}^*, L^2(\rho))$.

Note that $\mathcal{E}_1^*(x^i \otimes 1, x^i \otimes 1) < \infty$ and $\mathcal{E}_1^*((x^i)^2 \otimes 1, (x^i)^2 \otimes 1) < \infty$. Combining this with $\mathcal{D}^* = \mathcal{D}^\rho$ yields (2). □

Let $\mathbb{E}_{x\theta}$ be the expectation with respect to $\mathbb{P}_{x\theta}$ and R_λ denote the λ-resolvent of $\mathbb{H}_{XY}$;

$$R_\lambda h(x, \theta) = \mathbb{E}_{x\theta}\left[\int_0^\infty e^{-\lambda t} h(X_t, Y_t) dt\right].$$

Since $\mathbb{H}_{XY}$ is associated with $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d \times \Theta, dx \times \mu)$, R_λ can be regarded as the resolvent of the Dirichlet space $(\mathcal{E}_{XY}, \mathcal{D}_{XY}, L^2(\mathbb{R}^d \times \Theta, dx \times \mu))$. Moreover, R_λ is associated with $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ in the sense that for $\lambda > 0$

$$\mathcal{E}_\lambda^\rho(R_\lambda h_1, h_2) = (h_1, h_2)_{L^2(\rho)} \quad \text{for all } h_1, h_2 \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y.$$

In the rest of this paper we fix $\lambda' > \lambda_0$ and set

$$\mathcal{E}' = \mathcal{E}_{\lambda'}^\rho.$$

By Lemma 2.1 $(\mathcal{E}', \mathcal{D}^\rho, L^2(\rho))$ is a Dirichlet space. So let R'_λ be the λ-resolvent of $(\mathcal{E}_{\lambda'}^\rho, \mathcal{D}^\rho, L^2(\rho))$. The relation between resolvents is given by the following.

$$R'_\lambda h_1 = R_{\lambda+\lambda'} h_1 \quad \text{for } h_1 \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y. \tag{2.8}$$

Accordingly, the same equality also holds for $h_1 \in L^2(\rho)$.

Let T_t and T'_t denote semigroups associated with $(\mathcal{E}_{XY}, \mathcal{D}_{XY}, L^2(\mathbb{R}^d \times \Theta, dx \times \mu))$ and $(\mathcal{E}', \mathcal{D}^\rho, L^2(\rho))$, respectively. Then it is easy to see

$$T_t h = e^{\lambda' t} T'_t h \quad \text{for } h \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y.$$

Hence T_t can be regarded as the strongly continuous semigroup on $L^2(\rho)$ such that

$$\|T_t\|_{L^2(\rho) \to L^2(\rho)} \leq e^{\lambda' t}.$$

Here $\|\cdot\|_{L^2(\rho) \to L^2(\rho)}$ means the operator norm on $L^2(\rho)$. We remark, if we regard T_t as the semigroup on $L^2(\rho)$, then T_t is not a contraction semigroup in general.

LEMMA 2.2. – *Let* $h_1 \in \mathcal{D}^\rho$ *and* $h_2 \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_Y$. *Then*

$$(1) \qquad \lim_{\lambda \to \infty} \lambda((I - \lambda R_\lambda)h_1, h_2)_{L^2(\rho)} = \mathcal{E}^\rho(h_1, h_2),$$

$$(2) \qquad \lim_{\lambda \to \infty} \lambda((I - \lambda R_\lambda)h_1, h_2)_{L^2(\mathbb{R}^d \times \Theta, dx \times \mu)} = \mathcal{E}_{XY}(h_1, h_2).$$

Proof. – By (2.8) we have

$$\begin{aligned}
&\lim_{\lambda \to \infty} \lambda((I - \lambda R_\lambda)h_1, h_2)_{L^2(\rho)} \\
&= \lim_{\lambda \to \infty} \lambda\big((I - \lambda R'_{\lambda - \lambda'})h_1, h_2\big)_{L^2(\rho)} \\
&= \lim_{\lambda \to \infty} \Big\{\lambda\big((I - (\lambda - \lambda')R'_{\lambda - \lambda'})h_1, h_2\big)_{L^2(\rho)} - \lambda'\lambda\big(R'_{\lambda - \lambda'}h_1, h_2\big)_{L^2(\rho)}\Big\} \\
&= \mathcal{E}'(h_1, h_2) - \lambda'(h_1, h_2)_{L^2(\rho)} \\
&= \mathcal{E}^\rho(h_1, h_2),
\end{aligned}$$

which implies (1). We obtain (2) by replacing h_2 by $\rho^{-2}h_2$ in (1). □

Recall that $\mathbb{H}_Y = (\Omega, \mathfrak{G}, \mathfrak{G}_t, Y_t, \{\mathbb{P}_\theta\}_{\theta \in \Theta})$ and $\mathbb{H}_{XY} = (\Omega, \mathfrak{H}, \mathfrak{H}_t, (X_t, Y_t), \{\mathbb{P}_{x\theta}\})$ are special standard processes associated with $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\Theta, \mu)$ and $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d \times \Theta, dx \times \mu)$, respectively. We remark $\{\mathbb{P}_{x\theta}\}$ is unique up to q.e. (x, θ).

LEMMA 2.3. – *Assume* (A.1)–(A.5). *Then* $\mathbb{H}_{XY}$ *is conservative. Moreover there exists a version of* $\{\mathbb{P}_{x\theta}\}$ *satisfying the following: For each* $\theta \in \Theta$

$$(2.9) \qquad \mathbb{P}_{x\theta} \circ (X_t - X_0, Y_t)^{-1} \text{ is independent of } x \in \mathbb{R}^d.$$

$$(2.10) \qquad \mathbb{P}_{x\theta} \circ Y^{-1} = \mathbb{P}_\theta \quad \text{for all } x \in \mathbb{R}^d.$$

Proof. – Let

$$\mathbb{D}[h_1, h_2] = \sum_{i,j=1}^d a_{ij}(D_i - \partial_i)h_1(D_j - \partial_j)h_2.$$

For $a \in \mathbb{R}^d$ let $\tau_a h(x, \theta) = h(x + a, \theta)$. Since $a_{ij}(\theta)$ are independent of x, we see

$$\tau_a(\mathbb{D}[h_1, \tau_{-a}h_2]) = \mathbb{D}[\tau_a h_1, h_2].$$

Since $dx \times \mu$ and $dx \times \nu$ are invariant under transformations $(x, \theta) \mapsto (x + a, \theta)$, we have

$$(2.11) \qquad \mathcal{E}_{XY,\lambda}(h_1, \tau_{-a}h_2) = \mathcal{E}_{XY,\lambda}(\tau_a h_1, h_2),$$

where $\mathcal{E}_{XY,\lambda} = \mathcal{E}_{XY} + \lambda(\cdot,\cdot)_{L^2(\mathbb{R}^d\times\Theta,dx\times\mu)}$. Let R_λ denote the λ-resolvent of $(\mathcal{E}_{XY},\mathcal{D}_{XY})$ on $L^2(\mathbb{R}^d\times\Theta, dx\times\mu)$ as before. Then by (2.11) we see for all $a \in \mathbb{R}^d$

$$\tau_a(R_\lambda h) = R_\lambda(\tau_a h) \qquad \text{a.s. } (x,\theta). \tag{2.12}$$

For $f' \in C_0^1(\mathbb{R}^d)$ and $g' \in \mathcal{D}_Y$ we have

$$\begin{aligned}
&\mathcal{E}_{XY,\lambda}(1\otimes R_{Y,\lambda}g, f'\otimes g')\\
&= \int_{\mathbb{R}^d} f'(x)dx\mathcal{E}_{Y,\lambda}(R_{Y,\lambda}g,g') - \int_{\mathbb{R}^d\times\Theta}\sum_{i,j=1}^d a_{ij}(D_iR_{Y,\lambda}g)\partial_j f'g'dxd\nu\\
&= \int_{\mathbb{R}^d} f'(x)dx\mathcal{E}_{Y,\lambda}(R_{Y,\lambda}g,g') - \sum_{i,j=1}^d \{\int_\Theta a_{ij}(D_iR_{Y,\lambda}g)g'd\nu\}\{\int_{\mathbb{R}^d}\partial_j f'dx\}\\
&= \int_{\mathbb{R}^d} f'(x)dx\mathcal{E}_{Y,\lambda}(R_{Y,\lambda}g,g') \qquad (\text{ by } \textstyle\int_{\mathbb{R}^d}\partial_j f'dx = 0 \text{ })\\
&= \int_{\mathbb{R}^d\times\Theta} g(y)f'(x)g'(y)dxd\mu,
\end{aligned}$$

where $\mathcal{E}_{Y,\lambda} = \mathcal{E}_Y + \lambda(*,\cdot)_{L^2(\Theta,\mu)}$ and $R_{Y,\lambda}$ is the λ-resolvent of $(\mathcal{E}_Y,\mathcal{D}_Y)$ on $L^2(\Theta,\mu)$. Then we easily see

$$R_\lambda(1\otimes g) = 1\otimes R_{Y,\lambda}g = R_{Y,\lambda}g. \tag{2.13}$$

Since $\mathbb{H}_Y$ is conservative, we have $R_{Y,\lambda}1 = 1/\lambda$. So by (2.13) we see $R_\lambda(1\otimes 1) = 1/\lambda$, which implies $\mathbb{H}_{XY}$ is conservative.

If $f \in C_0(\mathbb{R}^d)$, then $R_\lambda(f\otimes g)(x,\theta)$ is continuous in x for fixed θ. Indeed

$$\begin{aligned}
&\lim_{a\to 0}\mathbb{E}_{x+a,\theta}[\int_0^\infty e^{-\lambda t}f(X_t)g(Y_t)dt]\\
&= \lim_{a\to 0}\mathbb{E}_{x\theta}[\int_0^\infty e^{-\lambda t}f(X_t+a)g(Y_t)dt] \qquad \text{by (2.12)}\\
&= \mathbb{E}_{x\theta}[\int_0^\infty e^{-\lambda t}f(X_t)g(Y_t)dt].
\end{aligned}$$

Combining (2.12) with the continuity in x of $R_\lambda(f\otimes g)(x,\theta)$ yields (2.9). Combining (2.12) with the continuity of $R_\lambda(1\otimes g)(x,\theta)$ in x we obtain (2.10). □

3. MEAN FORWARD VELOCITY AND ENERGY

The purpose of this section is to prove the existence of the mean forward velocity of $\mathbb{H}_{XY}$. For this we will use a (non-symmetric) form $(\mathcal{E}^\rho, \mathcal{D}^\rho)$ on $L^2(\rho)$, introduced in Section 2, that is also associated with $\mathbb{H}_{XY}$. Recall that the existence of the mean forward velocity is clear if $x^i \otimes 1 \in \mathcal{D}_{XY}$. Unfortunately $x^i \otimes 1$ is not in $\mathcal{D}_{XY}$; we have however by Lemma 2.1 that $x^i \otimes 1 \in \mathcal{D}^\rho$, which is the reason we consider $(\mathcal{E}^\rho, \mathcal{D}^\rho)$ on $L^2(\rho)$.

Let $\varphi^i : \widetilde{\mathcal{D}}_0 \to \mathbb{R}$ be the linear functional given by

$$\varphi^i(\mathfrak{h}) = -\widetilde{\mathcal{E}}_Y(\mathrm{e}^i, \mathfrak{h}). \tag{3.1}$$

By (A.3), we see φ^i are bounded functional. If $\mathfrak{h} = h/\sim \in \widetilde{\mathcal{D}}_{00}$, then

$$\varphi^i(\mathfrak{h}) = -\int_\Theta \sum_{j=1}^d a_{ij} D_j h d\nu. \tag{3.2}$$

By abuse of notation we set $\varphi^i(h) = -\int_\Theta \sum_{j=1}^d a_{ij} D_j h d\nu$ for $h \in \mathcal{D}_Y$.

For a function u which has a quasi continuous modification $\hat{u}$, we denote by $A^{[u]}$ the additive functional given by $A[u] = \hat{u}(X_t, Y_t) - \hat{u}(X_0, Y_0)$. For an additive functional $A = A_t$ we set $A_\lambda = \int_0^\infty e^{-\lambda t} A_t dt$.

LEMMA 3.1. – (1) *Let* $f \in C_0^\infty(\mathbb{R}^d)$ *and* $g \in \mathcal{D}_Y$. *Then*

$$\lim_{\lambda\to\infty} \lambda^2 \Big(\mathbb{E}_{x\theta}[A_\lambda^{[x^i\otimes 1]}], f\otimes g \Big)_{L^2(\rho)} = -\int_{\mathbb{R}^d} f\rho^2 dx \cdot \varphi^i(g). \tag{3.3}$$

(2) *Let* $u \in \mathcal{D}_Y$. *Then* $1\otimes u \in \mathcal{D}^\rho$ *and*

$$\lim_{\lambda\to\infty} \lambda^2 \Big(\mathbb{E}_{x\theta}\Big[A_\lambda^{[1\otimes u]}\Big], f\otimes g \Big)_{L^2(\rho)} = \int_{\mathbb{R}^d} f\rho^2 dx \cdot \mathcal{E}_Y(u, g). \tag{3.4}$$

Proof. – By definition $\lambda^2 \mathbb{E}_{x\theta}[A_\lambda^{[x^i\otimes 1]}] = -\lambda(I - \lambda R_\lambda) x^i \otimes 1$. So by Lemma 2.1 (2) we see

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^2 \Big(\mathbb{E}_{x\theta}[A_\lambda^{[x^i\otimes 1]}], f\otimes g \Big)_{L^2(\rho)} \\ &= -\mathcal{E}_{XY}(x^i\otimes 1, f\rho^2 \otimes g) \\ &= -\mathcal{E}_{XY}^1(x^i\otimes 1, f\rho^2 \otimes g). \end{aligned} \tag{3.5}$$

Here we used $\mathcal{E}_{XY}^2(x^i\otimes 1, f\rho^2\otimes g) = \int_{\mathbb{R}^d} x^i f\rho^2 dx \mathcal{E}_Y^2(1, g) = 0$ for the third line. Since $a_{ij}(\theta)$ and $g(\theta)$ are independent of x, we see

$$\int_{\mathbb{R}^d\times\Theta} dx d\nu \sum_{j=1}^d a_{ij} \partial_j (f\rho^2) \cdot g = 0. \tag{3.6}$$

Hence

$$\begin{aligned}\mathcal{E}^1_{XY}(x^i\otimes 1, f\rho^2\otimes g) &= \int_{\mathbb{R}^d\times\Theta} dxd\nu\left(-\sum_{j=1}^d a_{ij}D_jg\cdot f\rho^2 + \sum_{j=1}^d a_{ij}\partial_j(f\rho^2)\cdot g\right)\\ &= -\int_{\mathbb{R}^d\times\Theta} dxd\nu\sum_{j=1}^d a_{ij}D_jg\cdot f\rho^2 \qquad (\text{ by } (3.6)\ )\\ &= \int_{\mathbb{R}^d} f\rho^2dx\cdot\varphi^i(g).\end{aligned}$$

Combining this with (3.5) yields (1). The proof of (2) is similar. So we omit it. □

Remark 3.1. – By (2.9) and Lemma 3.1 (2) we have

$$\lim_{\lambda\to\infty}\lambda^2\Big(\mathbb{E}_{x\theta}[A_\lambda^{[x^i\otimes 1]}], g\Big)_{L^2(\Theta,\mu)} = \varphi^i(g) \quad \text{for all } x\in\mathbb{R}^d.$$

In this sense $\varphi = (\varphi^i)$ is a mean forward velocity of X_t.

We next introduce energies of additive functionals of (X_t, Y_t) of $\mathbb{H}_{XY}$. For additive functionals $A = A_t$ and $B = B_t$, we set

$$\begin{aligned} e_x(A,B) &= \lim_{\lambda\to\infty}\frac{\lambda^2}{2}\int_0^\infty e^{-\lambda t}\mathbb{E}_{x\mu}[A_tB_tdt], \qquad (3.7)\\ e_\rho(A,B) &= \int_{\mathbb{R}^d} e_x(A,B)\rho^2dx.\end{aligned}$$

Here $\mathbb{E}_{x\mu} = \int_\Theta d\mu\mathbb{E}_{x\theta}$. Let $e_x(A) = e_x(A,A)$ and $e_\rho(A) = e_\rho(A,A)$.

LEMMA 3.2. – *Let* $u_i\in C^\infty(\mathbb{R}^d)$ *and* $v_i\in\mathcal{D}_Y$ *such that* $u_i\otimes v_i\in\mathcal{D}^\rho$ $(i = 1,2)$. *Set* $a_{ij}^{\text{sym}} = \{a_{ij}+a_{ji}\}/2$. *Then*

$$\begin{aligned} e_\rho(A^{[u_1\otimes v_1]}, A^{[u_2\otimes v_2]}) &=\\ \int_{\mathbb{R}^d\times\Theta}\sum_{i,j=1}^d a_{ij}^{\text{sym}}&(u_1D_iv_1 - v_1\partial_iu_1)(u_2D_jv_2 - v_2\partial_ju_2)\rho^2dxd\nu\\ &+\left\{\int_{\mathbb{R}^d\times\Theta}u_1u_2\ \rho^2dx\right\}\mathcal{E}_Y(v_1,v_2).\end{aligned}$$

Proof. – We only prove the case of $u = u_1 = u_2$ and $v = v_1 = v_2$.

$$e_\rho(A^{[u\otimes v]}) = \lim_{\lambda\to\infty} \frac{\lambda^2}{2} \int_{\mathbb{R}^d\times\Theta} \int_0^\infty e^{-\lambda t} \mathbb{E}_{x\theta}[(u(X_t)v(Y_t) - u(X_0)v(Y_0))^2] dt \rho^2 dx d\mu$$

$$= \lim_{\lambda\to\infty} \frac{\lambda}{2} \int_{\mathbb{R}^d\times\Theta} (\lambda R_\lambda u^2 \otimes v^2 - u^2 \otimes v^2) \rho^2 dx d\mu$$

$$+ \lim_{\lambda\to\infty} \lambda \int_{\mathbb{R}^d\times\Theta} (u\otimes v - \lambda R_\lambda u\otimes v) uv\rho^2 dx d\mu$$

$$= -\frac{1}{2}\mathcal{E}_{XY}(u^2\otimes v^2, \rho^2\otimes 1) + \mathcal{E}_{XY}(u\otimes v, u\rho^2\otimes v) \qquad \text{(by Lemma 2.2 (2))}$$

The statement follows from this immediately. □

4. PROOF OF THEOREM 1–3

In this section we prove Theorem 1–3. Let ψ^i_λ $(\lambda > 0, i = 1, \ldots, d)$ denote the unique solution of equation (4.1) in $\mathcal{D}^d_Y$:

$$\lambda(\psi^i_\lambda, g)_{L^2(\Theta,\mu)} + \mathcal{E}_Y(\psi^i_\lambda, g) = -\varphi^i(g) \text{ for all } g \in \mathcal{D}^d_Y. \tag{4.1}$$

Here as before $\varphi^i(g) = -\int_\Theta \sum_{j=1}^d a_{ij} D_j g d\nu$. Let $\tilde{\psi}^i_\lambda$ be the element of $\tilde{\mathcal{D}}$ whose representative is ψ^i_λ. Let ψ^i be the solution of (0.9) as before.

LEMMA 4.1. – [21, Proposition 2.2] *Let $\psi^i_\lambda, \tilde{\psi}^i_\lambda$ and ψ^i be as above. Then*

$$\lim_{\lambda\to 0} \tilde{\mathcal{E}}_Y(\tilde{\psi}^i_\lambda - \psi^i, \tilde{\psi}^i_\lambda - \psi^i) = 0, \tag{4.2}$$

$$\lim_{\lambda,\lambda'\to 0} \mathcal{E}_Y(\psi^i_\lambda - \psi^i_{\lambda'}, \psi^i_\lambda - \psi^i_{\lambda'}) = 0, \tag{4.3}$$

$$\lim_{\lambda\to 0} \lambda(\psi^i_\lambda, \psi^i_\lambda)_{L^2(\Theta,\mu)} = 0. \tag{4.4}$$

Let $\mathbb{M}$ be the collection of d-dimensional cadlag processes that are L^2-martingales on $(\Omega, \mathfrak{F}, \mathfrak{H}_t, \mathbb{P}_{\rho\mu})$ satisfying $M_0 = 0$ a.s. and have stationary increments. Then $\mathbb{M}$ is a complete metric space with the metric induced by $\|\cdot\|$, where

$$\|M\| = \sum_{n=1}^\infty 2^{-n} \min\{1, \mathbb{E}_{\rho\mu}[|M_n|^2]^{1/2}\}.$$

LEMMA 4.2. – *Let $M^\lambda = (M^{\lambda,i})$ be the d-dimensional additive functional given by*

$$M^{\lambda,i} = A^{[x^i\otimes 1 + 1\otimes\psi^i_\lambda]} - \lambda \int_0^\cdot \psi^i_\lambda(Y_s) ds.$$

Then $M^{\lambda} \in \mathbb{M}$.

Proof. – Let $N^i = A^{[x^i \otimes 1]} - \lambda \int_0^{\cdot} \psi_\lambda^i(Y_s)ds$. We will prove N^i satisfies the assumptions (1)–(4) in Lemma 5.2. (1) and (2) in Lemma 5.2 follow from $x^i \otimes 1 \in L^2(\rho)$ and the fact that T_t is a strongly continuous semigroup on $L^2(\rho)$. (3) in Lemma 5.2 is clear by definition. To prove (4) we observe for $f \otimes g \in C_0^\infty \otimes \mathcal{D}_Y$

$$\begin{aligned}
&\lambda \lim_{p\to\infty} \left(\mathbb{E}_{x\theta}\left[\int_0^\infty p^2 e^{-pt} \int_0^t \psi_\lambda^i(Y_s)dsdt \right], f\otimes g \right)_{L^2(\rho)} \\
&= \lambda(1\otimes\psi_\lambda^i, f\otimes g)_{L^2(\rho)} \\
&= \int_{\mathbb{R}^d} f\rho^2 dx \cdot \{-\mathcal{E}_Y(\psi_\lambda^i, g) - \varphi^i(g)\} \\
&= -\mathcal{E}^\rho(1\otimes\psi_\lambda^i, f\otimes g) - \int_{\mathbb{R}^d} f\rho^2 dx \cdot \varphi^i(g).
\end{aligned}$$

Here we used (4.1) for the third line, and (3.6) for the last line. By Lemma 3.1 (1) we have

$$\lim_{p\to\infty} (\mathbb{E}_{x\theta}\left[\int_0^\infty p^2 e^{-pt} A_t^{[x^i\otimes 1]} dt \right], f\otimes g)_{L^2(\rho)} = -\int_{\mathbb{R}^d} f\rho^2 dx \cdot \varphi^i(g).$$

Let $\mathcal{N}_p = \mathbb{E}_{x\theta}[\int_0^\infty e^{-pt} N_t dt]$. Combining these two representations we see

$$\lim_{p\to\infty} (p^2\mathcal{N}_p, f\otimes g)_{L^2(\rho)} = -\mathcal{E}^\rho(1\otimes\psi_\lambda^i, f\otimes g) \quad \text{for all } f\otimes g \in C_0^\infty\otimes\mathcal{D}_Y.$$

It is not difficult to see the above relation can be extended to the one for all $h \in \mathcal{D}^\rho$. Hence N satisfies Lemma 5.2 (4) with $u = \psi_\lambda^i$. $\square$

Since that $M = (M^i)_{1\le i\le d} \in \mathbb{M}$ has stationary increments under $\mathbb{P}_{\rho\mu}$, we have

$$\mathbb{E}_{\rho\mu}[M_t^i M_t^j] = t \cdot \mathbb{E}_{\rho\mu}[M_1^i M_1^j] = 2t \cdot e_\rho(M^i, M^j). \tag{4.5}$$

Here e_ρ is the energy introduced by (3.7). Hence $\mathbb{M}$ is a complete metric space with metric $e_\rho(M)^{1/2}$, where $e_\rho(M) = \sum_{i=1}^d e_\rho(M^i)$ for $M = (M^i) \in \mathbb{M}$.

LEMMA 4.3. – *There exists an* $M = (M^i) \in \mathbb{M}$ *such that* M^λ *under* $\mathbb{P}_{\rho\mu}$ *converges to* M *in* $\mathbb{M}$ *as* $\lambda \to 0$. *Moreover,*

$$e_\rho(M^i, M^j) = \widetilde{\mathcal{E}}_Y^{\text{sym}}(\mathbf{e}^i - \psi^i, \mathbf{e}^j - \psi^j). \tag{4.6}$$

Proof. – We first prove that M^λ is e_ρ-Cauchy. By definition we easily see

$$M_t^\lambda - M_t^{\lambda'} = A_t^{[1\otimes(\psi_\lambda - \psi_{\lambda'})]} - \lambda\int_0^t \psi_\lambda(Y_s)ds + \lambda'\int_0^t \psi_{\lambda'}(Y_s)ds.$$

Note that $e_\rho(\int_0^\cdot \psi_\lambda(Y_s)ds) = 0$. Combining this with Lemma 3.2 we have

$$e_\rho(M^\lambda - M^{\lambda'}) = e_\rho(A^{[1\otimes(\psi_\lambda - \psi_{\lambda'})]}) = \sum_{i=1}^d \mathcal{E}_Y(\psi_\lambda^i - \psi_{\lambda'}^i, \psi_\lambda^i - \psi_{\lambda'}^i).$$

Thus by (4.3) we see $\{M^\lambda\}$ is e_ρ-Cauchy, which yields the first claim. The second claim follows from:

$$\begin{aligned} &e_\rho(M^i, M^j) \\ &= \lim_{\lambda\to 0} e_\rho(M^{\lambda,i}, M^{\lambda,j}) \\ &= \lim_{\lambda\to 0} e_\rho(A^{[x^i\otimes 1 + 1\otimes\psi_\lambda^i]}, A^{[x^j\otimes 1 + 1\otimes\psi_\lambda^j]}) && \text{by } e_\rho(\int_0^\cdot \psi_\lambda(Y_s)ds) = 0 \\ &= \lim_{\lambda\to 0} \widetilde{\mathcal{E}}_Y^{\text{sym}}(\mathbf{e}^i - \widetilde{\psi}_\lambda^i, \mathbf{e}^j - \widetilde{\psi}_\lambda^j) && \text{by Lemma 3.2} \\ &= \widetilde{\mathcal{E}}_Y^{\text{sym}}(\mathbf{e}^i - \psi^i, \mathbf{e}^j - \psi^j) && \text{by (4.2).} \end{aligned}$$

Here ψ_λ^i is given by (2.1) and $\widetilde{\psi}_\lambda^i \in \widetilde{\mathcal{D}}_{00}$ is the element whose representative is ψ_λ^i. □

LEMMA 4.4. – *Assume that* (A.1)–(A.6) *hold. Then*

$$\lim_{\epsilon\to 0} P_\theta^\epsilon = \widehat{P}_\theta \quad \textit{in f.d.d. in } \mu\textit{-measure.}$$

Here $\widehat{P}_\theta$ is the one given by Theorem 1 with $\widehat{a}_{ij}(\theta)$ such that

$$\widehat{a}_{ij}(\theta) = \frac{1}{2}\lim_{\epsilon\to 0}\epsilon^{-2}\mathbb{E}_{\rho\theta}\left[M_{1/\epsilon^2}^i \cdot M_{1/\epsilon^2}^j\right].$$

Proof. – The proof is now routine and same as [8]; for the sake of completeness we give it here. Note that P_θ^ϵ is same as the distribution of $\epsilon A_{\cdot/\epsilon^2}^{[x\otimes 1]}$ under $\mathbb{P}_{\rho\theta}$. Let $N_t^\epsilon = \epsilon M_{t/\epsilon^2}^{\epsilon^2} - \epsilon M_{t/\epsilon^2}$, $R_t^\epsilon = \epsilon\{A_{t/\epsilon^2}^{[1\otimes\psi_{\epsilon^2}]} - \epsilon\int_0^{t/\epsilon^2}\psi_{\epsilon^2}(Y_s)ds\}$. Then

$$\epsilon X_{t/\epsilon^2} - \epsilon X_0 = \epsilon M_{t/\epsilon^2} + N_t^\epsilon + R_t^\epsilon. \tag{4.7}$$

Since M^λ $(\lambda > 0)$ under $\mathbb{P}_{\rho\mu}$ have stationary increments, so does M. Hence by Helland theorem in [H] and the ergodic theorem we obtain for a.s.θ with respect to μ

$$\epsilon M_{t/\epsilon^2} \text{ under } \mathbb{P}_{\rho\theta} \text{ converge to } \hat{X}_t \text{ in f.d.d.} \tag{4.8}$$

Here $\hat{X}_t$ is a d-dimensional continuous martingale such that $\hat{X}_0 = 0$

$$\langle \hat{X}^i, \hat{X}^j \rangle_t = t \cdot \lim_{\epsilon \to 0} \epsilon^{-2} \mathbb{E}_{\rho\theta}\left[M^i_{1/\epsilon^2} \cdot M^j_{1/\epsilon^2} \right].$$

Since $e_\rho(N^\epsilon) = e_\rho(M^{\epsilon^2} - M) \to 0$ as $\epsilon \to 0$, we obtain by (4.1) that

$$\lim_{\epsilon \to 0} \mathbb{E}_{\rho\mu}\left[|N^\epsilon_t|^2 \right] = 2t \cdot e_\rho(N^\epsilon) = 0 \quad \text{for all } t. \tag{4.9}$$

We next prove $\lim_{\epsilon \to 0} \mathbb{E}_{\rho\mu}[|R^\epsilon_t|^2] = 0$ for all t. By the definition of R^ϵ_t we have

$$|R^\epsilon_t| \le |\epsilon \hat{\psi}_{\epsilon^2}(Y_{t/\epsilon^2})| + |\epsilon \hat{\psi}_{\epsilon^2}(Y_0)| + \epsilon \cdot \epsilon^2 \int_0^{t/\epsilon^2} |\psi_{\epsilon^2}(Y_s)| ds. \tag{4.10}$$

By (A.2) and (2.10) we have

$$\mathbb{E}_{\rho\mu}\left[|\epsilon \psi_{\epsilon^2}(Y_{t/\epsilon^2})|^2 \right] = \mathbb{E}_{\rho\mu}\left[|\epsilon \psi_{\epsilon^2}(Y_0)|^2 \right] = \sum_{i=1}^d \epsilon^2 (\psi^i_{\epsilon^2}, \psi^i_{\epsilon^2})_{L^2(\Theta,\mu)}.$$

So by (4.4) we obtain $\lim_{\epsilon \to 0} \mathbb{E}_{\rho\mu}\left[|\epsilon \psi_{\epsilon^2}(Y_{t/\epsilon^2})|^2 \right] = 0$. Similarly we have

$$\mathbb{E}_{\rho\mu}\left[\{\epsilon^3 \int_0^{t/\epsilon^2} |\psi_{\epsilon^2}(Y_s)| ds\}^2 \right] \le \epsilon^6 \frac{t}{\epsilon^2} \mathbb{E}_{\rho\mu}\left[\int_0^{t/\epsilon^2} |\psi_{\epsilon^2}(Y_s)|^2 ds \right]$$
$$= \epsilon^4 t \int_0^{t/\epsilon^2} \mathbb{E}_{\rho\mu}\left[|\psi_{\epsilon^2}(Y_s)|^2 \right] ds = \epsilon^2 t^2 \sum_{i=1}^d (\psi^i_{\epsilon^2}, \psi^i_{\epsilon^2})_{L^2(\rho)} \to 0 \text{ as } \epsilon \to 0.$$

Combining these estimates with (4.10) yields $\lim_{\epsilon \to 0} \mathbb{E}_{\rho\mu}[|R^\epsilon_t|^2] = 0$.

By (4.8), (4.9) and $\lim_{\epsilon \to 0} \mathbb{E}_{\rho\mu}[|R^\epsilon_t|^2] = 0$ the one dimensional distributions of $\epsilon X_{t/\epsilon^2}$ converge to those of $\hat{X}_t$ weakly in μ-measure. The proof for the convergence of the k-dimensional distributions is standard. Hence we omit it. □

Proof of Theorem 1. – The first claim follows from Lemma 4.4. The second claim follows from Lemma 5.5. □

Proof of Theorem 2. – By the proof of Lemma 4.4 we have

$$\lim_{\epsilon\to 0}\epsilon \dot{X}_{t/\epsilon^2} = \lim_{\epsilon\to 0}\epsilon M_{t/\epsilon^2}.$$

So we want to identify the limit $\lim_{\epsilon\to 0}\epsilon M_{t/\epsilon^2}$. By the mean ergodic theorem we have

$$\int_\Theta d\mu \lim_{\epsilon\to 0}\epsilon^{-2}\mathbb{E}_{\rho\theta}\left[M^i_{1/\epsilon^2}\cdot M^j_{1/\epsilon^2}\right] = \mathbb{E}_{\rho\mu}\left[M^i_1\cdot M^j_1\right].$$

Hence $2\int_\Theta \widehat{a}_{ij}(\theta)d\mu = \mathbb{E}_{\rho\mu}\left[M^i_1\cdot M^j_1\right]$. Combining this with (4.5) and (4.6) completes the proof. □

Proof of Theorem 3. – Let $\psi_\xi[\mathcal{E}_Y] = \sum_{i=1}^d \xi_i\psi^i[\mathcal{E}_Y]$, where $\psi^i[\mathcal{E}_Y]$ is the solution of (0.9). Then by Theorem 2 we see

$$\sum_{i,j=1}^d \xi_i\xi_j\alpha_{ij}[\mathcal{E}_Y] = \widetilde{\mathcal{E}}_Y^{\text{sym}}(\xi-\psi_\xi[\mathcal{E}_Y],\xi-\psi_\xi[\mathcal{E}_Y]). \tag{4.11}$$

Meanwhile $\psi_\xi[\mathcal{E}_Y]\in\widetilde{\mathcal{D}}_0$ is the solution of

$$\widetilde{\mathcal{E}}_Y(\psi_\xi[\mathcal{E}_Y],\mathfrak{g}) = \widetilde{\mathcal{E}}_Y(\xi,\mathfrak{g}) \quad \text{for all } \mathfrak{g}\in\widetilde{\mathcal{D}}_0. \tag{4.12}$$

Recall that $\widetilde{\mathcal{D}}$ is a Hilbert space with inner product $\widetilde{\mathcal{E}}_Y^{\text{sym}}$ and $\widetilde{\mathcal{D}}_0$ is its closed subspace. Let $\mathbf{P}:\widetilde{\mathcal{D}}\to\widetilde{\mathcal{D}}_0$ be the orthogonal projection. In case of $\mathcal{E}_Y = \mathcal{E}_Y^{\text{sym}}$ we obtain by (4.12) that

$$\psi_\xi[\mathcal{E}_Y^{\text{sym}}] = \mathbf{P}(\xi).$$

Statement (2) follows from this and (4.11) immediately. We next prove (1):

$$\begin{aligned}
&\sum_{i,j=1}^d \xi_i\xi_j\alpha_{ij}[\mathcal{E}_Y^{\text{sym}}] \\
&= \inf\{\widetilde{\mathcal{E}}_Y^{\text{sym}}(\xi-\mathbf{g},\xi-\mathbf{g});\mathbf{g}\in\widetilde{\mathcal{D}}_0\} && \text{(by (2))}\\
&\le \widetilde{\mathcal{E}}_Y^{\text{sym}}(\xi-\psi_\xi[\mathcal{E}_Y],\xi-\psi_\xi[\mathcal{E}_Y]) && (\text{by } \psi_\xi[\mathcal{E}_Y]\in\widetilde{\mathcal{D}}_0)\\
&= \sum_{i,j=1}^d \xi_i\xi_j\alpha_{ij}[\mathcal{E}_Y] && \text{(by (4.11))}.
\end{aligned}$$

We thus complete the proof. □

5. APPENDIX

In this section we prepare some results from Dirichlet form theory. Most of these are originally for regular Dirichlet forms on locally compact metric spaces. It can be proved that they also hold for quasi-regular Dirichlet forms on a Hausdorff topological space satisfying $\mathfrak{B}(\Theta) = \sigma[C(\Theta)]$ by using the transfer method (see [3], [12]).

In the first half of this section we assume (A.1)–(A.5) and later we will assume (A.6) in addition. Recall that $\mathbb{H}_{XY}$ is conservative and satisfies (2.9) and (2.10) by Lemma 2.3. Taking (2.9) and (2.10) into account, we consider the following space of d-dimensional martingales. Let $\widetilde{\mathbb{M}}$ denote the collection of d-dimensional additive functionals of $\mathbb{H}_{XY}$ satisfying hypotheses (5.1)–(5.3): For each $t \geq 0$

(5.1) $\mathbb{E}_{\rho\mu}[M_t^2] < \infty$

(5.2) $\mathbb{E}_{x\theta}[M_t] = 0 \quad \mu\text{-a.e. } \theta \quad \text{for all } x.$

(5.3) M_t is $\mathcal{H}_t$-measurable, where $\mathcal{H}_t = \sigma[(X_s - X_0, Y_s); s \leq t]$.

By (5.3) and (2.9) we see for each θ,

$$\mathbb{P}_{x\theta} \circ M^{-1} \text{ is independent of } x. \tag{5.4}$$

LEMMA 5.1. – *Let* $M \in \widetilde{\mathbb{M}}$. *Then* $M \in \mathbb{M}$.

Proof. – Let $u_t(\theta) = \mathbb{E}_{x\theta}[M_t]$. By (5.4) u_t is independent of x. Let ϑ_t denote the time shift operator. By definition we see

$$\begin{aligned} \mathbb{E}_{\rho\mu}[M_{s+t}|\mathfrak{H}_s] &= \mathbb{E}_{\rho\mu}[M_s|\mathfrak{H}_s] + \mathbb{E}_{\rho\mu}[M_t(\vartheta_s\cdot)|\mathfrak{H}_s] \\ &= M_s + \mathbb{E}_{X_sY_s}[M_t] \quad \mathbb{P}_{\rho\mu}\text{-a.e.} \\ &= M_s + u_t(Y_s) \quad \mathbb{P}_{\rho\mu}\text{-a.e..} \end{aligned} \tag{5.5}$$

Since $u_t(\theta) = 0$ μ-a.e. by (5.2) and $\mathbb{P}_{\rho\mu} \circ Y_s^{-1} = \mathbb{P}_\mu \circ Y_s^{-1} = \mu$ by (2.10) and (A.2), we have $u_t(Y_s) = 0$ $\mathbb{P}_{\rho\mu}$-a.e.. Hence M is martingale under $\mathbb{P}_{\rho\mu}$. We see under $\mathbb{P}_{\rho\mu}$

$$\mathbb{P}_{\rho\mu} \circ (M_{t+s} - M_s)^{-1} = \mathbb{P}_{X_sY_s} \circ M_t^{-1} = \mathbb{P}_{\rho\mu} \circ M_t^{-1}.$$

So M has a stationary increments under $\mathbb{P}_{\rho\mu}$. $\square$

LEMMA 5.2. – *Let* $u \in \mathcal{D}_\mathcal{Y}$. *Let* $N = (N^i)_{1\leq i\leq d}$ *be a d-dimensional additive functional satisfying (1)–(4) below. Then* $A^{[1\otimes u]} - N \in \mathbb{M}$.

(1) $\mathbb{E}_{\rho\mu}\left[\int_0^\infty e^{-\lambda t}|N_t|^2\right] < \infty$ *for some* $\lambda < \infty$.

(2) $\mathbb{E}_{x\theta}[N_t]$ *is continuous in t in* $L^2(\rho)$.

(3) N_t *is* $\mathcal{H}_t$*-measurable.* ($\mathcal{H}_t$ *is given by* (5.3))

(4) $\lim_{p\to\infty}(p^2\mathcal{N}_p, h)_{L^2(\rho)} = -\mathcal{E}^\rho(1\otimes u, h)$ *for all* $h \in \mathcal{D}^\rho$.

Here we set $\mathcal{N}_p = \mathbb{E}_{x\theta}[\int_0^\infty e^{-pt} N_t dt]$ *for* $p \geq \lambda$.

Remark 5.1. – (1) This result is a modification of [3, Theorem 5.2.4] in symmetric case and [22, Theorem 5.2.5] in non-symmetric case. If $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ is a Dirichlet space, then Lemma 5.2 follows from the above mentioned results immediately; however, we easily see $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ is not a Dirichlet space in general. Indeed, $(\mathcal{E}^\rho, \mathcal{D}^\rho, L^2(\rho))$ does not satisfies the weak sector condition in general, and the dual semigroup is not necessary Markovian.

(2) Lemma 5.2 is an analogy of [21, Proposition 4.3]. In [21] we assumed inequality (1) hold for all $\lambda > 0$. Since we let $p \to \infty$ in (4), it is enough that inequality (1) holds for some λ. In [21] we missed assumption (2). We need this assumption at the final step of the proof.

Proof. – Hypotheses (5.1) and (5.3) follow from (1) and (3), respectively. For $dx \times \mu$-a.e.

$$R_q(p\mathcal{N}_p) = R_p(q\mathcal{N}_q) \quad \text{for all } p, q > \lambda. \tag{5.6}$$

This follows from the standard argument.

By definition $(1 \otimes u)(x, \theta) = u(\theta)$; we write $1 \otimes u$ when we emphasize $1 \otimes u$ is a function on $\mathbb{R}^d \times \Theta$, otherwise we simply write u. We next prove for $dx \times \mu$-a.e.

$$p\mathcal{N}_p = pR_p u - u \quad \text{for all } p > \lambda. \tag{5.7}$$

Let R_p^* denote the dual resolvent of R_p on $L^2(\rho)$. Then

$$\begin{aligned}
(p\mathcal{N}_p, h)_{L^2(\rho)} &= \lim_{q \to \infty} (p\mathcal{N}_p, qR_q^* h)_{L^2(\rho)} \\
&= \lim_{q \to \infty} (q^2 \mathcal{N}_q, R_p^* h)_{L^2(\rho)} \qquad \text{by (5.6)} \\
&= -\mathcal{E}^\rho(1 \otimes u, R_p^* h) \qquad \text{by (4)} \\
&= (pR_p u - u, h)_{L^2(\rho)}.
\end{aligned}$$

Hence for all p we have $p\mathcal{N}_p = pR_p u - u$. Since for fixed (x, θ) both sides are continuous in p, we obtain (5.7). By (5.7) for each $h \in L^2(\rho)$ we have

$$\int_0^\infty e^{-pt} (\mathbb{E}_{x\theta}[A_t^{[u]} - N_t], h)_{L^2(\rho)} dt = \frac{1}{p} (pR_p u - u - p\mathcal{N}_p, h)_{L^2(\rho)} = 0$$

for all p. Combining this with (2) we see $(\mathbb{E}_{x\theta}[A_t^{[u]} - N_t], h)_{L^2(\rho)} = 0$ for all $t \geq 0$ and $h \in L^2(\rho)$. Hence for all $t \geq 0$ we obtain

$\mathbb{E}_{x\theta}[A_t^{[u]} - N_t] = 0$ $dx{\times}\mu$-a.e.. Combining this with (2.9) and (2.10) yields for all x $\mathbb{E}_{x\theta}[A_t^{[u]} - N_t] = 0$ μ-a.e., which means (5.2). □

In the rest of this section we assume (A.6).

LEMMA. – 5.3 *Let $f \in C^\infty(\mathbb{R}^d)$. Then $A^{[f\otimes 1]}$ is a continuous additive functional.*

Proof. – Let $\mathcal{E}_{XY}^{\text{sym}}$ denote the symmetrization of $\mathcal{E}_{XY}$. Recall the decomposition of Dirichlet forms (see [7, (2.8)], [3]):

$$\mathcal{E}_{XY}^{\text{sym}} = \mathcal{E}_{XY}^{(c)} + \mathcal{E}_{XY}^{(j)} + \mathcal{E}_{XY}^{(k)}.$$

Here $\mathcal{E}_{XY}^{(c)}$, $\mathcal{E}_{XY}^{(j)}$, $\mathcal{E}_{XY}^{(k)}$ are local, jump, killing parts of $\mathcal{E}_{XY}^{\text{sym}}$, respectively. Let $h_n = \chi_n f\otimes 1$, where $\chi_n \in C_0^\infty(\mathbb{R}^d)$ are such that $\chi_n = 1$ for $|x| \le n+1$. Then $h_n \in \mathcal{D}_{XY}$ and $\mathcal{E}_{XY}^2(h_n, h_n) = 0$. Hence $\mathcal{E}_{XY}^{\text{sym}}(h_n, h_n) = \mathcal{E}_{XY}(h_n, h_n) = \mathcal{E}_{XY}^1(h_n, h_n)$. Combining this with (A.6) we have

$$\mathcal{E}_{XY}^{(j)}(h_n, h_n) + \mathcal{E}_{XY}^{(k)}(h_n, h_n) = 0 \qquad \text{for all } n.$$

This implies $A^{[h_n]}$ is continuous additive functionals for all n (see, e.g., [7, Theorem 4.9]). This completes the proof because $A_t^{[f\otimes 1]} = A_t^{[h_n]}$ for $t < \sigma_n = \inf\{t > 0; |X_t| \ge n\}$. □

Let $\mathcal{A}$ denote the set of (1-dimensional) continuous additive functionals of $\mathbb{H}_{XY}$ and $\mathcal{A}_c$ the subset of $\mathcal{A}$ consisting of continuous processes. We refer to [7] for the definition. We set

$$\begin{aligned}&\mathcal{M} = \{M \in \mathcal{A};\ \mathbb{E}_{x\theta}[M_t^2] < \infty,\ \mathbb{E}_{x\theta}[M_t] = 0 \text{ q.e.} \quad e_\rho(M) < \infty\},\\ &\mathcal{M}_c = \mathcal{M} \cap \mathcal{A}_c,\\ &\mathcal{N}_c = \{N \in \mathcal{A}_c; N \text{ is finite, } e_\rho(A) = 0,\ \mathbb{E}_{x\theta}[|A_t|] < \infty \text{ q.e.}\}.\end{aligned}$$

Then $\mathcal{M}$ (resp. $\mathcal{N}_c$) is the set of martingale additive functionals of finite energy (additive functionals of zero energy) of $\mathbb{H}_{XY}$.

We now localize $\mathcal{M}_c$ and $\mathcal{N}_c$. We remark here that $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ is not a local form; however, we can localize it in $\mathbb{R}^d$-direction because $X = X_t$ is continuous process by Lemma 5.3. Let $\sigma_n = \inf\{t > 0; |X_t| \ge n\}$ and set

$$\mathcal{M}_{c,loc} = \{M;\ \text{there exist } \{M^n\}_n \in \mathcal{M}_c \text{ such that } M_t^n = M_t \text{ for } t < \sigma_n\}.$$

We define $\mathcal{N}_{c,loc}$ similarly.

LEMMA 5.4. – *Let $f \in C^\infty(\mathbb{R}^d)$. Then $A^{[f\otimes 1]}$ is decomposed as follows:*

$$(5.8)\quad A^{[f\otimes 1]} = M^{[f\otimes 1]} + N^{[f\otimes 1]}, \quad M^{[f\otimes 1]} \in \mathcal{M}_{c,loc}, \quad N^{[f\otimes 1]} \in \mathcal{N}_{c,loc}.$$

Moreover

$$(5.9) \qquad \langle M^{[f\otimes 1]}, M^{[f\otimes 1]}\rangle_t = 2\int_0^t \sum_{i,j=1}^d a_{ij}(Y_s)\partial_i f(X_s)\partial_j f(X_s)ds$$

Proof. – Let h_n be as in the proof of Lemma 5.3. Then since $h_n \in \mathcal{D}_{XY}$, (5.8) and (5.9) holds for h_n (see [7, Theorem 4.8]). So by taking $n \to \infty$ we complete the proof. □

LEMMA 5.5. – *Assume the matrix $a = (a_{ij})$ and the bilinear form $\mathcal{E}_Y^2$ are symmetric. Then*

$$\{\epsilon A^{[x^i\otimes 1]}{}_{t/\epsilon^2}\}_{\epsilon>0} \text{ is tight in } C([0,\infty) \to \mathbb{R}^d) \text{ under } \mathbb{P}_{\rho\mu}.$$

Proof. – By Lemma 5.4 we see

$$\langle M^{[x^i\otimes 1]}, M^{[x^i\otimes 1]}\rangle_t = 2\int_0^t a_{ii}(Y_s)ds.$$

Hence we can apply the proof of [3, Theorem 5.7.1] to the present case. □

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