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The one dimensional annealed δ-Lyapounov exponent

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ABSTRACT. – We study a one dimensional Brownian motion moving among Poisson points of constant intensity $\nu > 0$. We introduce the "annealed δ - Lyapounov exponent" $\beta_\delta(c)$. Here "annealed" refers to the fact that averages are both taken with respect to the path and environment measures. The exponent $\beta_\delta(c)$ measures how costly it is for the Brownian motion to reach a remote location while it receives a penalty "proportional" to $c \in (0, \infty)$ for spending too much time at Poisson points, and when the particle can pick its own time to perform the displacement.

We derive a formula for $\beta_\delta(c)$, which shows that for all $c \in (0, \infty)$, $\beta_\delta(c) < \nu$. We conjecture that in general this is also true for the one dimensional "annealed Lyapounov exponent", introduced by Sznitman, which is an analogue object to $\beta_\delta(c)$. © Elsevier, Paris

RÉSUMÉ. – Nous étudions un mouvement brownien en une dimension se déplaçant entre des points de répartition poissonnienne d'intensité constante $\nu > 0$. Nous introduisons « l'exposant moyenné δ - Lyapounov » $\beta_\delta(c)$. Ici « moyenné » signifie que les moyennes sont prises sur la trajectoire et les mesures d'environnement. L'exposant $\beta_\delta(c)$ mesure combien cela coûte au mouvement brownien d'atteindre un endroit éloigné alors qu'il reçoit une pénalité proportionnelle à $c \in (0, \infty)$ en restant trop longtemps sur les

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points de répartition poissonnienne et si la particule peut choisir son propre temps pour effectuer ce déplacement.

Nous prouvons une formule pour $\beta_\delta(c)$, qui montre que pour tout $c \in (0,\infty)$, $\beta_\delta(c) < \nu$. Nous conjecturons qu'en général ceci est également vrai pour "l'exposant moyenné Lyapounov" en une dimension, introduit par Sznitman, qui est un objet analogue à $\beta_\delta(c)$.

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0. INTRODUCTION

The main goal of the present article is to introduce and characterize what we call the one dimensional "δ - Lyapounov exponent". Namely we let $Z_.$ denote a canonical one dimensional Brownian motion, P_0 the Wiener measure on $C(\mathbb{R}_+, \mathbb{R})$ and $\mathbb{P}$ the law of a Poisson point process of constant intensity $\nu > 0$ on the space Ω of simple pure point measures on $\mathbb{R}$. For $c \in (0, \infty)$ and $x \in \mathbb{R}$ we define:

$$f_{\delta,c}(x) = \mathbb{E} \otimes E_0 \left[\exp\left\{ -c \int L_{H(x)}(y)\, \omega(dy) \right\} \right], \tag{0.1}$$

where $\omega \in \Omega$ and $L_{H(x)}(y)$ denotes the Brownian local time of the point $y \in \mathbb{R}$ up to the first hitting time of x: $H(x) = \inf\{t \geq 0;\ Z_t = x\}$.

The "δ - Lyapounov exponent" will be the rate of exponential decay of $f_{\delta,c}(x)$ as x tends to infinity, see Theorem 1 below. But let us first start with some comment on $f_{\delta,c}(x)$.

The term $\exp\{-c \int L_{H(x)}(y)\, \omega(dy)\}$ in (0.1) represents a penalty, depending on $c \in (0, \infty)$, for Brownian motion stopped at its first hitting time of x, for spending too much time at Poisson points. In this context $f_{\delta,c}(x)$ should be viewed as the analogous object to

$$f_W(x) = \mathbb{E} \otimes E_0 \left[\exp\left\{ - \int_0^{H(x)} V(Z_s, \omega)\, ds \right\} \right], \tag{0.2}$$

introduced in [6], where the Poisson potential $V(\cdot, \omega)$ is defined as: $V(x, \omega) = \int W(x-y)\, \omega(dy)$, and the "shape function" $W \geq 0$, is bounded measurable, compactly supported, not a.s. equal to zero and $a = a(W) > 0$ is the smallest number such that $W(\cdot)$ is zero outside $[-a, a]$. Indeed, in view of the "occupation times formula", see for instance [7], Section 3, p. 27,

$$\int_0^t f(Z_s)\,ds = \int_{\mathbb{R}} f(x)L_t(x)\,dx, \tag{0.3}$$

where f is a bounded Borel function on $\mathbb{R}$, $t \geq 0$, $x \in \mathbb{R}$, we can formally write

$$f_{\delta,c}(x) = \mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^{H(x)} V_\delta^c(Z_s,\omega)\,ds\right\}\right], \tag{0.4}$$

where $V_\delta^c(x,\omega) = \int W_\delta^c(x-y)\,\omega(dy)$, and $W_\delta^c(x) = c\delta(x)$, the "$\delta$ - shape function".

Our main purpose is to show in Section 1 the following.

THEOREM 1. – *Let $c \in (0,\infty)$ and denote by $\lambda_1(c) = \inf \operatorname{spec}(-\frac{1}{2}\Delta + U_c)$ where Δ denotes the Laplacian in $\mathbb{R}^2$ and for $x \in \mathbb{R}^2$: $U_c(x) = -\nu\exp\{-c|x|^2\}$. We then have*

$$\beta_\delta(c) \stackrel{\text{Def}}{=} -\lim_{|x|\to\infty} \frac{1}{|x|}\log f_{\delta,c}(x) = \nu - |\lambda_1(c)| \in (0,\nu). \tag{0.5}$$

Let us mention here that the fact that $\lambda_1(c) \in (-\nu, 0)$, which then implies that $\beta_\delta(c) \in (0,\nu)$, is an application of Theorem 3.4 in [4], see the proof of (1.3) in Section 1.

This result should be compared to the case where the shape function W has a nondegenerate support. Indeed, among other things it was shown in [6], specialized to $d = 1$, see Theorem 1.3 and Theorem 1.4 in [6], that with $f_W(x)$ from (0.2)

$$-\lim_{|x|\to\infty} \frac{1}{|x|}\log f_W(x) \stackrel{\text{Def}}{=} \beta \in (0,\nu]. \tag{0.6}$$

The interesting fact stemming out of Theorem 1 is now that for all $c \in (0,\infty)$ we have that $\beta_\delta(c) < \nu$. Indeed, this has the following probabilistic interpretation. One possible "strategy" for the Brownian motion which travels to the distant point x while it receives a penalty "proportional" to $c > 0$ for spending too much time at Poisson points is for instance to stay until time $H(x)$ in an interval of length of order $|x|$, which contains the starting point 0 and the terminal location x and receives no Poisson points. The cost for such a strategy would be of leading order $\exp\{-\nu|x|\}$. But as Theorem 1 shows $\beta_\delta(c) < \nu$, hence it is more favorable for the process to go through the Poisson points rather quickly, no matter how big $c \in (0,\infty)$ is. This should be contrasted to the case of "hard obstacles". By this we mean that we replace $f_{\delta,c}(x)$ by $f_{\delta,\infty}(x) = \mathbb{E} \otimes E_0[T > H(x)]$, where T denotes the hitting time of the Poisson points, which is interpreted as the killing time of the process. Observe that this formally corresponds to putting in (0.1): $c = \infty$. But then because $f_{\delta,\infty}(x) = \mathbb{E} \otimes E_0[T > H(x)] = E_0[\exp\{-\nu|C_{H(x)}|\}]$, where $C_{H(x)}$ denotes the support of the path up to time $H(x)$, one easily sees that $\beta_\delta(\infty) = \nu$. This corresponds to the fact that in the presence of "hard obstacles" the particle gets killed once it hits a Poisson point.

The result of Theorem 1 leads to the natural question whether for all shape functions $W(\cdot)$ having a nondegenerate support, we also have, like in the "δ - case", that $\beta < \nu$, c.f. (0.6). Indeed, this question was one of our main motivations for the present work. The problem of determining whether in general $\beta < \nu$, originates from our previous work [2] on the large deviation principle in the critical scale $t^{1/3}$ for the position at time t of an "annealed" one dimensional Brownian motion moving in a Poisson potential. The "annealed" weighted measure was defined as

$$Q_t = \frac{1}{S_t} \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\} P_0(dw)\,\mathbb{P}(d\omega), \qquad (0.7)$$

and S_t is the normalization.

We have shown in [2], Theorem 2.2, that $t^{-1/3} Z_t$ satisfies under Q_t a large deviation principle at rate $t^{1/3}$ with rate function $J_1(\cdot)$ which was characterized in [2], (0.3).

Roughly speaking, the essence of Theorem 2.2 in [2] is that for $y \in \mathbb{R}$, $t \to \infty$:

$$Q_t\Big(Z_t \sim y t^{1/3}\Big) = \exp\Big\{-t^{1/3} J_1(y)(1+o(1))\Big\} \qquad (0.8)$$

and that the function $J_1(\cdot)$ exhibits three regimes. This indicates that there are qualitatively three different scenarios of how Brownian motion under Q_t performs until large times t an excursion of the order $yt^{1/3}$. The structure of $J_1(\cdot)$ is such that for $|y| \in [0, c_0]$ and $|y| \in [c_0, c_1]$ in (0.8), where $0 < c_0 = (\frac{\pi^2}{\nu})^{1/3} < (\frac{\pi^2}{\nu-\beta})^{1/3} = c_1$, we are in the first two regimes. If $|y| \geq c_1$, the third regime of $J_1(\cdot)$ takes over. At this point observe that if $\nu = \beta$, we would have $c_1 = \infty$, and $J_1(\cdot)$ would only exhibit two different regimes. For a more detailed explanation on this, we refer the reader to the introduction of [2].

However, we are going to show in Section 2 the following

PROPOSITION 2.

a) *Let $c, a \in (0,\infty)$ and denote by β the annealed Lyapounov exponent corresponding to $W(x) = c1_{[-a,a]}(x)$. We have:*

$$\beta \geq \nu - |\lambda_1(2ca)| \in (0,\nu), \tag{0.9}$$

where $\lambda_1(2ca) \in (-\nu, 0)$ was defined in Theorem 1.

b) *Let $\|W\|_\infty = \sup_{-a\leq x\leq a} W(x) \in (0,\infty)$, where a is the smallest number such that $W(\cdot) = 0$ on $[-a,a]^c$. Denote by β the corresponding annealed Lyapounov exponent. We then have:*

For given $\nu > 0$, if $\|W\|_\infty a > 0$ is sufficiently small: $\beta < \nu$.

c) *Denote by $\beta(\nu, c, a)$ the annealed Lyapounov exponent corresponding to $W(x) = c1_{[-a,a]}(x)$, $c, a > 0$. We then have for all $\lambda > 0$:*

$$\beta\left(\nu\lambda, c\lambda^2, \frac{a}{\lambda}\right) = \lambda\beta(\nu, c, a). \tag{0.10}$$

Let us give some comments on the above results.

First observe that the function $W(x) = c1_{[-a,a]}(x)$ models the "soft obstacles" giving the "biggest penalty" for the Brownian motion in the Poisson potential. In other words, Proposition 2 a), together with Theorem 1 leads to the conjecture, although untouched, that in general we expect $\beta < \nu$. Let us recall at this point that it was shown in Theorem 1.4 of [6], specialized to $d = 1$, that there exists a constant $\gamma(\nu, W) > 0$, such that $\beta \geq \max(\sqrt{2}, \gamma(\nu, W)) > 0$. This bound shows that the exponent β which measures the exponential decay of $f_W(x)$ as $|x| \to \infty$ is nondegenerate, c.f. (0.6). However, the purpose of Proposition 2 a) is to provide in the "worst case scenario" a lowerbound on β which explicitly contains the parameters of the corresponding function $W(\cdot)$. In our context the important point of (0.9) is that this lowerbound is known to be always strictly smaller than ν.

Proposition 2 b), which is very much in the spirit of Proposition 1.2 in [2], shows that at least if the "area" of the shape function is sufficiently small, we have $\beta < \nu$.

Finally, Proposition 2 c) provides a scaling realtion which states that the result of Proposition 2 b) isn't enough to settle the general case.

The article is organized as follows. In Section 1, we give the proof of Theorem 1 and explain the strategy we use. Section 2 contains the proof of Proposition 2.

1. THE PROOF OF THEOREM 1

The goal of this section is to prove Theorem 1 from the introduction, which defines the one dimensional "δ-Lyapounov exponent".

Before we begin with the proof of Theorem 1, let us briefly explain the strategy we use. For simplicity we assume $x > 0$ in (0.5). The main tool is the Ray - Knight Theorem, see for instance RK1, RK2 in [7] Section 3, p. 28, which enables us to identify the local time $L_{H(1)}(y)$ as an inhomogeneous Markov process in the spatial variable, the law of which can be described precisely.

Indeed, $L_{H(1)}(1-y)$, $0 \leq y \leq 1$, is identical in law to a two dimensional squared Bessel process starting in 0, restricted to the time interval $[0,1]$. Furthermore $L_{H(1)}(1-y)$, $y \geq 1$, conditioned on $L_{H(1)}(1-y)$, $0 \leq y \leq 1$, is identical in law to a zero dimensional squared Bessel process starting at time $y = 1$ from $L_{H(1)}(0)$.

To obtain our desired asymptotic upperbound on $f_{\delta,c}(x)$ we will then use scaling and the characterization of $L_{H(1)}(1-y)$, $0 \leq y \leq 1$.

To obtain the correct lowerbound on $f_{\delta,c}(x)$, the idea is that the leading asymptotic behaviour of $f_{\delta,c}(x)$ should come from paths which go "relative directly" from the starting point to x, in the sense that $\min_{0 \leq s \leq H(x)} Z_s$ should not be to small, and which spend not too much time in zero. Applying then RK1 and RK2 and using the Markov property, we find up to correction terms our desired lowerbound on $f_{\delta,c}(x)$. We are now ready to begin with the

Proof of Theorem 1

Pick $c \in (0,\infty)$. Thanks to symmetry of Brownian motion it is enough to show (0.5) for $x > 0$.

We have to give suitable asymptotic upper and lower bounds on $f_{\delta,c}(x)$ defined in (0.1).

Indeed, the claim of Theorem 1 follows once we show

$$\overline{\lim}_{x\to\infty} \frac{1}{x} \log f_{\delta,c}(x) \leq -\nu - \lambda_1(c), \tag{1.1}$$

resp.

$$\underline{\lim}_{x\to\infty} \frac{1}{x} \log f_{\delta,c}(x) \geq -\nu - \lambda_1(c) \tag{1.2}$$

and that for all $c \in (0,\infty)$:

$$\lambda_1(c) \in (-\nu, 0) \tag{1.3}$$

We start with the proof of (1.1).

Denote by $m_x = \min_{0\le s\le H(x)} Z_s$. Since $L_{H(x)}(y) = 0$ for $y \le m_x$, $(y \ge x)$, we have

$$f_{\delta,c}(x) = E_0\left[\exp\left\{-\nu\int_{m_x}^{x}[1-\exp\{-cL_{H(x)}(y)\}]\,dy\right\}\right]$$
$$\le \exp\{-\nu x\}E_0\left[\exp\left\{\nu\int_0^x \exp\{-cL_{H(x)}(y)\}\,dy\right\}\right], \qquad (1.4)$$

where the first equality follows from using Fubini and performing the $\mathbb{E}$ expectation in (0.1).

From the scaling property of Brownian motion and the occupation times formula from (0.3) we find for any $z, y \in \mathbb{R}, \alpha > 0$: (see for instance Chap. VI, (2.11) in [3])

$$L_{H(z)}(y) \overset{\text{law}}{=} \frac{1}{\alpha}L_{H(\alpha z)}(\alpha y). \qquad (1.5)$$

By using (1.5) with $\alpha = x$ in RK2 from [7], Section 3.1, p. 28, we easily conclude that

$$\left(L_{H(x)}(x-s)\right)_{0\le s\le x} \overset{\text{law}}{=} \left(|Z_s|^2\right)_{0\le s\le x} \qquad (1.6)$$

where $(Z_s)_{0\le s\le x}$ is a two dimensional Brownian motion starting from zero.

Denoting by E_0^2 the two dimensional Wiener measure starting from 0 we then find after an obvious change of variables, applying (1.6) in (1.4) that

$$f_{\delta,c}(x) \le \exp\{-\nu x\}E_0^2\left[\exp\left\{\nu\int_0^x \exp\{-c|Z_s|^2\}\,ds\right\}\right]$$
$$= \exp\{-\nu x\}E_0^2\left[\exp\left\{-\int_0^x U_c(Z_s)\,ds\right\}\right], \qquad (1.7)$$

where $U_c(x) = -\nu\exp\{-c|x|^2\}$, $x \in \mathbb{R}^2$.

Since $U_c(\cdot)$ is continuous and $|U_c(y)| \le \nu$ we know from Remark II.3.10 in [1] that

$$-\lambda_1(c) = \lim_{x\to\infty}\frac{1}{x}\log E_0^2\left[\exp\left\{-\int_0^x U_c(Z_s)\,ds\right\}\right], \qquad (1.8)$$

where we recall that $\lambda_1(c) = \inf \operatorname{spec}(-\frac{1}{2}\Delta + U_c)$ and Δ denotes the Laplacian in $\mathbb{R}^2$.

Combining this with (1.7) we find

$$\overline{\lim}_{x\to\infty}\frac{1}{x}\log f_{\delta,c}(x) \le -\nu - \lambda_1(c) \qquad (1.9)$$

which shows (1.1).

We continue with showing (1.2).

To this end by again performing the $\mathbb{E}$ expectation in (0.1) and then changing variables in an obvious way, we get

$$f_{\delta,c}(x) = E_0\left[\exp\left\{-\nu\int_0^\infty [1-\exp\{-cL_{H(x)}(x-y)\}]\,dy\right\}\right]. \quad (1.10)$$

Observe however that $1-\exp\{-cL_{H(x)}(x-y)\} = 0$, for $y \geq x+|m_x|$ and m_x was introduced before (1.4).

We are now going to apply the Ray - Knight Theorems to $L_{H(x)}(x-y)$, $y \geq 0$, in (1.10). Indeed, by again using the scaling relation (1.5) with $\alpha = x$ in RK1 and RK2 from [7] Section 3.1, p. 28, we know, applying RK1 with the starting point equal to $L_{H(x)}(0)$, that

$$\left(L_{H(x)}(x-s)\right)_{s\geq 0} \overset{\text{law}}{=} (Y_s)_{s\geq 0}\,, \quad (1.1)$$

where $Y_.$ is an inhomogeneous Markov process starting in 0, which is the square of a two dimensional Bessel process for $0 \leq s \leq x$, and a square of a zero dimensional Bessel process for $s \geq x$, starting in Y_x. For a definition of these processes see Chap. XI, Def. 1.1 in [3].

Recall also from Proposition 1.5, Chapter XI in [3], that for a squared zero dimensional Bessel process, the point 0 is absorbing.

If we now denote by $E_{x=0,t=0}$ the expectation with respect to the path measure of the diffusion $(Y_s)_{s\geq 0}$ we find in view of (1.11) that

$$f_{\delta,c}(x) = E_{0,0}\left[\exp\left\{-\nu\int_0^\infty [1-\exp\{-cY_s\}]\,ds\right\}\right]. \quad (1.12)$$

Denote now for $y \geq 0$ by Q_y^0 the path measure of a zero dimensional squared Bessel process $(X_s)_{s\geq 0}$ starting from y. As before we also denote by E_0^2 the expectation with respect to the two dimensional Wiener measure starting from 0, and by $(Z_s)_{s\geq 0}$ the corresponding two dimensional Brownian motion. Applying the Markov property in (1.12) we find in view of the discussion after (1.11) that

$$f_{\delta,c}(x) = E_0^2\left[\exp\left\{-\nu\int_0^x [1-\exp\{-c|Z_s|^2\}]\,ds\right\}\right.$$
$$\left. Q^0_{|Z_x|^2}\left[\exp\left\{-\nu\int_0^\infty [1-\exp\{-cX_s\}]\,ds\right\}\right]\right]. \quad (1.13)$$

Pick now $R > 0$ and define $B(0,R) = \{x \in \mathbb{R}^2;\ |x| < R\}$. Denoting by $T_{B(0,R)}$ the exit time from $B(0,R)$, *i.e.* $T_{B(0,R)} = \inf\{t \geq 0;\ Z_t \in B(0,R)^c\}$ we find in regard of (1.13) that

$$f_{\delta,c}(x) \geq E_0^2\left[\exp\left\{-\nu \int_0^x [1-\exp\{-c|Z_s|^2\}]\, ds\right\},\ T_{B(0,R)} > x\right] \inf_{0\leq y\leq R^2} Q_y^0\left[\exp\left\{-\nu \int_0^\infty [1-\exp\{-cX_s\}]\, ds\right\}\right]. \tag{1.14}$$

We are now going to give a suitable lowerbound on the expectation involving Q_y^0 in (1.14). To this end recall that 0 is absorbing for $(X_s)_{s\geq 0}$. Therefore we find for all $y \in [0, R^2]$:

$$\begin{aligned} Q_y^0&\left[\exp\left\{-\nu \int_0^\infty [1-\exp\{-cX_s\}]\, ds\right\}\right] \\ &\geq Q_y^0\left[X_R = 0, \exp\left\{-\nu \int_0^R [1-\exp\{-cX_s\}]\, ds\right\}\right] \\ &\geq \exp\{-\nu R\} Q_y^0[X_R = 0]. \end{aligned} \tag{1.15}$$

Using now in (1.15) the formula, see for instance Chap. XI, Corollary 1.4 in [7],

$$Q_y^0[X_R = 0] = \exp\{-y/2R\}, \tag{1.16}$$

we get in view of (1.14) that

$$\begin{aligned} f_{\delta,c}(x) &\geq \exp\{-R(\nu+1/2)\} \\ &\quad\times E_0^2\left[\exp\left\{-\nu \int_0^x [1-\exp\{-c|Z_s|^2\}]\, ds\right\}, T_{B(0,R)} > x\right] \\ &= \exp\{-\nu x - R(\nu+1/2)\} \\ &\quad\times E_0^2\left[\exp\left\{-\int_0^x U_c(Z_s)\, ds\right\}, T_{B(0,R)} > x\right], \end{aligned} \tag{1.17}$$

where again $U_c(x) = -\nu\exp\{-c|x|^2\}$, $x \in \mathbb{R}^2$.

If we now denote by $\lambda_1^c(B(0,R))$ the first Dirichlet eigenvalue of $-\frac{1}{2}\Delta + U_c$ in $B(0,R)$, we find as in the proof of (1.1) that: (See for instance the discussion after Remark II.3.10 in [1])

$$-\lambda_1^c(B(0,R)) = \lim_{x\to\infty} \frac{1}{x} \log E_0^2\left[\exp\left\{-\int_0^x U_c(Z_s)\, ds\right\}, \acute{T}_{B(0,R)} > x\right]. \tag{1.18}$$

Combining this with (1.17) we get for all $R > 0$:

$$\underline{\lim}_{x\to\infty} \frac{1}{x} \log f_{\delta,c}(x) \geq -\nu - \lambda_1^c(B(0,R)). \tag{1.19}$$

Since $\lambda_1^c(B(0,R)) \longrightarrow \lambda_1(c)$, as R tends to infinity, and because $R > 0$ was arbitrary, this shows (1.2).

It finally remains to show (1.3).

The first point is that for all $c \in (0,\infty)$, $\lambda_1(c)$ is an eigenvalue with $\lambda_1(c) < 0$.

To show this, we recall the fact, see for instance Theorem II.4.3 in [1], that since $U_c(\cdot)$ is continuous and $\lim_{|x|\to\infty} U_c(x) = 0$, we have $\sigma_{\text{ess}}(-\frac{1}{2}\Delta + U_c) = \sigma_{\text{ess}}(-\frac{1}{2}\Delta) = [0,\infty)$, where $\sigma_{\text{ess}}(\cdot)$ denotes the essential spectrum. Since $\int |U_c(x)|^2\, dx < \infty$, resp. $\int (1+|x|^2)|U_c(x)|\, dx < \infty$ and $U_c(x) < 0$, for all $x \in \mathbb{R}^2$, we can apply Theorem 3.4 in [4] to conclude that for all $c \in (0,\infty)$: $\sigma_{\text{d}}(-\frac{1}{2}\Delta + U_c) \neq \emptyset$, where $\sigma_{\text{d}}(\cdot)$ denotes the discrete spectrum. It follows that $\lambda_1(c) \in [-\nu, 0)$ is an eigenvalue.

To show that $\lambda_1(c) > -\nu$, we denote by ϕ_1 the corresponding normalized eigenfunction. From Corollary 25.7 in [5] we know that $\phi_1 \in C^1$. Since $\phi_1(\cdot)$ decays exponentially fast as $|x| \to \infty$, see Corollary 25.12 in [5], we get, using integration by parts,

$$\begin{aligned} \lambda_1(c) &= \int U_c(x)\phi_1(x)^2\, dx + \frac{1}{2}\int |\nabla\phi_1(x)|^2\, dx \\ &\geq -\nu + \frac{1}{2}\int |\nabla\phi_1(x)|^2\, dx \\ &> -\nu \end{aligned} \tag{1.20}$$

which shows (1.3) and finishes the proof of Theorem 1. □

2. THE PROOF OF PROPOSITION 2

In this section, we give the proof of Proposition 2 from the introduction. In view of Proposition 2 a), observe that for shape functions $W_i(\cdot)$, $i = 1, 2$, with the property that for all $x \in \mathbb{R}$: $W_1(x) \leq W_2(x)$, we have that $\beta_1 \leq \beta_2$, where β_i denotes the Lyapounov exponent coresponding to $W_i(\cdot)$, $i = 1, 2$. In particular, the result of Proposition 2 a) together with Theorem 1 leads to the conjecture, untouched here, that in any case we expect that $\beta < \nu$.

The proof of Proposition 2 a) hinges on the observation that after performing in (0.2) the $\mathbb{E}$ expectation, using the occupation times formula (0.3) and applying Jensens inequality to the integral involving the local time, in order to give a lowerbound on β, the situation can be reduced to the "δ-case" treated in Section 1. The proof of Proposition 2 b) is almost similar to the proof of Proposition 1.2 from [2].

Proof of Proposition 2

We begin with the proof of Proposition 2 a).

As in the proof of Theorem 1.1 it is enough to consider $f_W(x)$ from (0.2) for $x > 0$.

Let now $c, a \in (0, \infty)$. Using Fubini and performing the $\mathbb{E}$ expectation in (0.2) we find with $W(x) = c1_{[-a,a]}(x)$:

$$f_W(x) = E_0\left[\exp\left\{-\nu\int\left[1-\exp\left\{-c\int_0^{H(x)} 1_{[y-a,y+a]}(Z_s)ds\right\}\right]dy\right\}\right]. \tag{2.1}$$

Applying the occupation times formula from (0.3) in (2.1) we see that

$$f_W(x) = E_0\left[\exp\left\{-\nu\int\left[1-\exp\left\{-c\int_{y-a}^{y+a} L_{H(x)}(z)\,dz\right\}\right]dy\right\}\right]. \tag{2.2}$$

Now Jensens inequality implies that

$$\exp\left\{-2ca\,\frac{1}{2a}\int_{y-a}^{y+a} L_{H(x)}(z)\,dz\right\} \le \frac{1}{2a}\int_{y-a}^{y+a} \exp\left\{-2ca\,L_{H(x)}(z)\right\}dz \tag{2.3}$$

and using this in (2.2) we get

$$f_W(x) \le E_0\left[\exp\left\{-\nu\int dy\,\frac{1}{2a}\int_{y-a}^{y+a} dz\,[1-\exp\{-2ca\,L_{H(x)}(z)\}]\right\}\right]. \tag{2.4}$$

Finally, applying Fubini in (2.4) we find

$$\begin{aligned} f_W(x) &\le E_0\left[\exp\left\{-\nu\int[1-\exp\{-2ca\,L_{H(x)}(z)\}]\,dz\right\}\right] \\ &= g_W(x) \end{aligned} \tag{2.5}$$

In view of Theorem 1, the claim of Proposition 2 a) follows.

We continue with the proof of Proposition 2 b).

We know from (1.30) of Proposition 1.2 in [2] that

$$\beta \le \min\{F_{1_{[-a,a]}(\cdot)}(\|W\|_\infty), \nu\}, \tag{2.6}$$

where

$$F_{1_{[-a,a]}(\cdot)}(\|W\|_\infty) = -\log E_0\left[\exp\left\{-\nu\int[1-\exp\{-\|W\|_\infty \times \int_0^{H(1)} 1_{[y-a,y+a]}(Z_s)\,ds\}]\,dy\right\}\right]$$

$$= -\log E_0\left[\exp\left\{-\nu\int_{m_1-a}^{1+a}[1-\exp\{-\|W\|_\infty \times \int_{y-a}^{y+a} L_{H(1)}(x)\,dx\}]\,dy\right\}\right], \qquad (2.7)$$

with $m_1 = \min_{0\le s\le H(1)} Z_s$, and where we used the occupation times formula in the second equality.

Fix now $\nu > 0$ and observe that if $\|W\|_\infty a = 0$ we have $F_{1_{[-a,a]}(\cdot)}(\|W\|_\infty) = 0$. In view of (2.6) the claim of Proposition 2 b) now follows from a continuity argument exactly in the spirit of the proof of Proposition 1.2 from [2].

Finally the proof of Proposition 2 c) easily follows from scaling. □

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REFERENCES

[1] R. CARMONA and J. LACROIX, *Spectral theory of random Schrödinger operators*, Birkhäuser, 1990.
[2] T. POVEL, Critical large deviations of one dimensional annealed Brownian motion in a Poissonian Potential, to appear in *Ann. Probab.*
[3] D. REVUZ and M. YOR, *Continous martingales and Brownian motion*, Springer, 1991.
[4] B. SIMON, The bound state of weakly coupled Schrödinger operators in one and two dimensions, *Ann. Physics*, Vol. **97**, 1976, pp. 279-288.
[5] B. SIMON, *Functional integration and quantum physics*, Academic Press, 1979.
[6] A. S. SZNITMAN, Annealed Lyapounov exponents and large deviations in a Poissonian potential I, *Ann. Scient. Ec. Norm. Sup.*, 4^e Serie, t. 28, 1995, pp. 345-370.
[7] M. YOR, *Some aspects of Brownian motion*, Lectures in Mathematics ETH Zürich, Birkhäuser, 1992.

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