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The maximal variation of a bounded martingale and the central limit theorem

by

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ABSTRACT. – Mertens and Zamir's paper [3] is concerned with the asymptotic behavior of the maximal L^1-variation $\xi_n^1(p)$ of a $[0,1]$-valued martingale of length n starting at p. They prove the convergence of $\xi_n^1(p)/\sqrt{n}$ to the normal density evaluated at its p-quantile.

This paper generalizes this result to the conditional L^q-variation for $q \in [1,2)$.

The appearance of the normal density remained unexplained in Mertens and Zamir's proof: it appeared as the solution of a differential equation. Our proof however justifies this normal density as a consequence of a generalization of the central limit theorem discussed in the second part of this paper. © Elsevier, Paris

RÉSUMÉ. – L'article [3] de Mertens et Zamir s'intéresse au comportement asymptotique de la variation maximale $\xi_n^1(p)$ au sens L^1 d'une martingale de longueur n issue de p et à valeurs dans $[0,1]$. Ils démontrent que $\xi_n^1(p)/\sqrt{n}$ converge vers la densité normale évaluée à son p-quantile.

Ce résultat est ici étendu à la variation L^q- conditionnelle pour $q \in [1,2)$.

L'apparition de la loi normale reste inexpliquée au terme de la démonstration de Mertens et Zamir : elle y apparaît en tant que solution d'une équation différentielle. Notre preuve justifie l'occurrence de la densité

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normale comme une conséquence d'une généralisation du Théorème Central Limite présentée dans la deuxième partie de l'article.

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1. ON THE MAXIMAL VARIATION OF A MARTINGALE

Let $\mathcal{M}_n(p)$ denote the set of all $[0,1]$-valued martingales X of length n: $X = (X_1, \ldots, X_n)$ with $E[X_1] = p$. For a martingale X in $\mathcal{M}_n(p)$, we will refer to the quantity $V_n^q(X)$:

$$V_n^q(X) := E\left[\sum_{k=1}^{n-1} (E[|X_{k+1} - X_k|^q | X_1, \ldots, X_k])^{\frac{1}{q}}\right]$$

as the conditional L^q-variation of X. In case $q = 1$, $V_n^1(X)$ turns out to be equal to the classical L^1-variation of X: $\sum_{k=1}^{n-1} \|X_{k+1} - X_k\|_{L^1}$.

Let us still define $\xi_n^q(p)$ as:

$$\xi_n^q(p) := \sup\{V_n^q(X) | X \in \mathcal{M}_n(p)\}.$$

With these notations, the main result of this section is:

THEOREM 1. – *For q in $[1, 2)$, the limit of $\frac{\xi_n^q(p)}{\sqrt{n}}$, as n increases to ∞, is*

$$\Phi(p) := exp(-x_p^2/2)/\sqrt{2\pi},$$

where x_p is such that $p = \int_{-\infty}^{x_p} exp(-s^2/2)/\sqrt{2\pi} ds$. (i.e. $\Phi(p)$ is the normal density evaluated at its p-quantile.)

Mertens and Zamir proved this result in [3] for the particular case $q = 1$ and they applied it to repeated game theory in [2]. The heuristic underlying their proof is based on a recursive formula for ξ_n^1 that could be written formally as $\xi_{n+1}^1/\sqrt{n+1} = T_n(\xi_n^1/\sqrt{n})$, where T_n is the corresponding recurrence operator. If the sequence $\xi_n^1/\sqrt{n}$ were to converge to a limit Φ, we would have $T_n(\Phi) \approx \Phi$. By interpreting heuristically the last relation as $T_n(\Phi) - \Phi = O(n^{-3/2})$, they are led to a differential equation whose solution is the normal density evaluated at its p-quantile. In fact, their proof contains no probabilistic justification of this appearance of the normal density. Our argument is of a completely different nature and this normal density appears as a consequence of the generalization of the central limit theorem presented in the next section.

Proof of Theorem 1. – Let us first observe that $V_n^q(X)$ just depends on the joint distribution of the random vector $X_1, \dots, X_n$.

Let then $(u_1, \dots, u_n)$ be a system of independent random variables uniformly distributed on $[0,1]$ and let $\mathcal{G} := \{\mathcal{G}_k\}_{k=1}^n$ be the filtration generated by $(u_1, \dots, u_n)$: $\mathcal{G}_k := \sigma\{u_1, \dots, u_k\}$.

It is well known that if F_1 denotes the distribution function of X_1, then $X_1' := F_1^{inv}(u_1)$ has the same distribution as X_1, where $F_1^{inv}(u) := \inf\{x | F_1(x) \geq u\}$. Applying this argument recursively on the distribution of X_{k+1} conditional on $(X_1, \dots, X_k)$, we obtain a $\mathcal{G}$-adapted martingale X' inducing on $\mathbb{R}^n$ the same distribution as X, and thus $V_n^q(X) = V_n^q(X')$. As a consequence,

$$\xi_n^q(p) = \sup\{V_n^q(X) | X \in \mathcal{M}_n(\mathcal{G}, p)\},$$

where $\mathcal{M}_n(\mathcal{G}, p)$ denotes the set of $\mathcal{G}$-adapted martingales in $\mathcal{M}_n(p)$.

It follows from the above construction of X' that, for $k = 0, \cdots, n-1$, X_{k+1}' is measurable with respect to $\sigma\{X_1', \dots, X_k', u_{k+1}\}$. Thus,

$$E[|X_{k+1}' - X_k'|^q | \mathcal{G}_k] = E[|X_{k+1}' - X_k'|^q | X_1', \dots, X_k'].$$

This last relation implies then that $V_n^q(X) = V_n^q(X') = \tilde{V}_n^q(X')$, where $\tilde{V}_n^q(X')$ denotes the L^q-variation conditional on $\mathcal{G}$ of the $\mathcal{G}$-adapted martingale X':

$$\tilde{V}_n^q(X') := E\left[\sum_{k=1}^{n-1} \left(E[|X_{k+1}' - X_k'|^q | \mathcal{G}_k]\right)^{\frac{1}{q}}\right].$$

We then infer that $\xi_n^q(p) \leq \sup\{\tilde{V}_n^q(X) | X \in \mathcal{M}_n(\mathcal{G}, p)\}$. On the other hand, since $\sigma\{X_1, \dots, X_k\}$ is included in $\mathcal{G}_k$, it follows from Jensen's inequality that $\tilde{V}_n^q(X) \leq V_n^q(X)$, and we may conclude that

$$\xi_n^q(p) = \sup\{\tilde{V}_n^q(X) | X \in \mathcal{M}_n(\mathcal{G}, p)\}.$$

We now will prove that the term

$$E[(E[|X_{k+1} - X_k|^q | \mathcal{G}_k])^{\frac{1}{q}}]$$

in the definition of $\tilde{V}_n^q(X)$ can be replaced with

$$\sup\{E[(X_{k+1} - X_k) Y_{k+1}] | Y_{k+1} \in \mathcal{B}_{k+1}\},$$

where $\mathcal{B}_{k+1}$ denotes the set of $\mathcal{G}_{k+1}$-measurable random variables Y_{k+1} such that $E[|Y_{k+1}|^{q'} | \mathcal{G}_k]$ is a.s. less than 1, with q' fulfilling $1/q + 1/q' = 1$. (In the particular case $q = 1$, we define $\mathcal{B}_{k+1}$ as the set of $[-1,1]$-valued $\mathcal{G}_{k+1}$-measurable random variables.). Indeed, a conditional version of Holder's inequality indicates that

$$E[(X_{k+1} - X_k)Y_{k+1}|\mathcal{G}_k] \leq (E[|X_{k+1} - X_k|^q|\mathcal{G}_k])^{\frac{1}{q}}(E[Y_{k+1}^{q'}|\mathcal{G}_k])^{\frac{1}{q'}}.$$

Thus, for $Y_{k+1} \in \mathcal{B}_{k+1}$, we have

$$E[(X_{k+1} - X_k)Y_{k+1}] \leq E[(E[|X_{k+1} - X_k|^q|\mathcal{G}_k])^{\frac{1}{q}}].$$

Since the equality is satisfied in the last relation for

$$Y_{k+1} = \operatorname{sgn}(X_{k+1} - X_k)|X_{k+1} - X_k|^{\frac{q}{q'}}/E[|X_{k+1} - X_k|^q|\mathcal{G}_k]^{\frac{1}{q'}} \in \mathcal{B}_{k+1},$$

we then conclude as announced that

$$E[(E[|X_{k+1} - X_k|^q|\mathcal{G}_k])^{\frac{1}{q}}] = \sup\{E[(X_{k+1} - X_k)Y_{k+1}]|Y_{k+1} \in \mathcal{B}_{k+1}\}.$$

As a next step, let us remark that, since X is a martingale, we have

$$\begin{aligned} E[(X_{k+1} - X_k)Y_{k+1}] &= E[(X_{k+1} - X_k)(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])] \\ &= E[X_{k+1}(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])] \\ &= E[X_n(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])] \end{aligned}$$

We obtain therefore:

$$\tilde{V}_n^q(X) = \sup\left\{E\left[X_n \sum_{k=1}^{n-1}(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])\right]|Y_2 \in \mathcal{B}_2, \ldots, Y_n \in \mathcal{B}_n\right\}.$$

This expression of $\tilde{V}_n^q(X)$ just depends on the final value X_n of the martingale X. Furthermore, if, for a σ-algebra $\mathcal{A}$, $\mathcal{R}(\mathcal{A}, p)$ denotes the class of $[0,1]$-valued $\mathcal{A}$-measurable random variables R with $E[R] = p$, any R in $\mathcal{R}(\mathcal{G}_n, p)$ is the value X_n at time n of a martingale X in $\mathcal{M}_n(\mathcal{G}, p)$. We then conclude that

$$\text{(1)} \qquad \xi_n^q(p) = \sup\left\{E\left[R\sum_{k=1}^{n-1}(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])\right] |R \in \mathcal{R}(\mathcal{G}_n, p), Y_2 \in \mathcal{B}_2, \ldots, Y_n \in \mathcal{B}_n\right\}.$$

By hypothesis we have $q < 2$. This implies $q' > 2$. Therefore $E[Y_{k+1}^2|\mathcal{G}_k] \leq 1$ since $Y_k \in \mathcal{B}_k$. Hence, the terms $(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])$

appearing in the last formula have a conditional variance bounded by 1. The process S defined as $S_m := \sum_{k=1}^{m-1}(Y_{k+1} - E[Y_{k+1}|\mathcal{G}_k])$ belongs therefore to the class $\mathcal{S}_n^{q'}([0,1],2)$ of the martingales S of length n starting at 0 and whose increments $S_{k+1}-S_k$ have a conditional variance $E[(S_{k+1}-S_k)^2|\mathcal{G}_k]$ a.s. valued in the interval $[0,1]$ and a conditional q'-order moment bounded by $2^{q'}$.

So, we infer that

$$\frac{\xi_n^q(p)}{\sqrt{n}} \leq \sup_{S\in\mathcal{S}_n^{q'}([0,1],2)} \mu_p\left(\frac{S_n}{\sqrt{n}}\right),$$

where

$$\mu_p\left(\frac{S_n}{\sqrt{n}}\right) := \sup_{R\in\mathcal{R}(\mathcal{G}_n,p)} E\left[R\frac{S_n}{\sqrt{n}}\right].$$

Obviously the quantity $\mu_p(\frac{S_n}{\sqrt{n}})$ just depends on the distribution of $S_n/\sqrt{n}$ and not on the σ-algebra on which this random variable is defined.

According to Theorem 3, there exists a κ such that for all S in $\mathcal{S}_n^{q'}([0,1],2)$ we can claim the existence of a Brownian Motion β on a filtration $\mathcal{F}$, of a [0,1]-valued stopping time τ and of a $\mathcal{F}_\infty$-measurable random variable Y such that Y has the same distribution as $S_n/\sqrt{n}$ and $\|Y - \beta_\tau\|_{L^2} \leq 2\kappa n^{\frac{1}{q'\wedge 4}-\frac{1}{2}}$.

We then conclude that

$$\mu_p\left(\frac{S_n}{\sqrt{n}}\right) = \sup_{R\in\mathcal{R}(\mathcal{F}_\infty,p)} E[R\cdot Y] \leq \sup_{R\in\mathcal{R}(\mathcal{F}_\infty,p)} E[R\cdot\beta_\tau] + 2\kappa n^{\frac{1}{q'\wedge 4}-\frac{1}{2}}.$$

Due to the inequality $\tau \leq 1$, it follows that:

$$\begin{aligned}\sup_{R\in\mathcal{R}(\mathcal{F}_\infty,p)} E[R\cdot\beta_\tau] &= \sup_{R\in\mathcal{R}(\mathcal{F}_\infty,p)} E[E[R|\mathcal{F}_\tau]\cdot\beta_\tau] \\ &= \sup_{R\in\mathcal{R}(\mathcal{F}_\tau,p)} E[R\cdot\beta_\tau] \\ &= \sup_{R\in\mathcal{R}(\mathcal{F}_\tau,p)} E[R\cdot\beta_1] \\ &\leq \sup_{R\in\mathcal{R}(\mathcal{F}_1,p)} E[R\cdot\beta_1].\end{aligned}$$

We will now explicitly compute $\sup_{R\in\mathcal{R}(\mathcal{F}_1,p)} E[R\cdot\beta_1]$: if $\mathcal{H}$ denotes $\sigma\{\beta_1\}$, then

$$\sup_{R\in\mathcal{R}(\mathcal{F}_1,p)} E[R\cdot\beta_1] = \sup_{R\in\mathcal{R}(\mathcal{F}_1,p)} E[E[R|\mathcal{H}]\cdot\beta_1] = \sup_{R\in\mathcal{R}(\mathcal{H},p)} E[R\cdot\beta_1].$$

Since this optimization problem consists of maximizing a linear functional on the convex set $\mathcal{R}(\mathcal{H},p)$, we may restrict our attention to the the extreme points of $\mathcal{R}(\mathcal{H}, p)$, which are clearly the $\{0, 1\}$-valued random variables R in $\mathcal{R}(\mathcal{H}, p)$ since the normal density has no atoms. Now, in order to maximize $E[R \cdot \beta_1]$, the random variable $R(\beta_1)$ has to map the highest values of β_1 to 1, and the lowest values to 0, i.e. $R(\beta_1) = \mathbb{1}_{\beta_1 \geq v}$, where v is a constant such that $p = E[\mathbb{1}_{\beta_1 \geq v}] = \int_v^\infty e^{(-s^2/2)}/\sqrt{2\pi} ds$.

Thus

$$\sup_{R \in \mathcal{R}(\mathcal{F}_1, p)} E[R \cdot \beta_1] = E[\mathbb{1}_{\beta_1 \geq v} \beta_1]$$
$$= \int_v^\infty s e^{(-s^2/2)}/\sqrt{2\pi} ds$$
$$= e^{(-v^2/2)}/\sqrt{2\pi}.$$

Observing that $v = -x_p$, we get

$$\sup_{R \in \mathcal{R}(\mathcal{F}_1, p)} E[R \cdot \beta_1] = \Phi(p),$$

and the following inequality is proved:

$$\frac{\xi_n^q(p)}{\sqrt{n}} \leq \Phi(p) + 2\kappa n^{\frac{1}{q' \wedge 4} - \frac{1}{2}}.$$

To get the reverse inequality, let us come back to equation (1). Obviously, if Y_k is a system of independent random variables adapted to $\mathcal{G}$, with $Y_k = +1$ or -1 each with probability $1/2$, we get $Y_k \in \mathcal{B}_k$ and we infer that

$$\frac{\xi_n^q(p)}{\sqrt{n}} \geq \mu_p\left(\frac{S_n}{\sqrt{n}}\right),$$

where $S_m := \sum_{k=1}^{m-1} Y_{k+1}$. Since $(S_{k+1} - S_k)^2 = 1$, S belongs to $\mathcal{S}_n^4([1, 1], 2)$. According to Theorem 3, there exist a Brownian motion β on a filtration $\mathcal{F}$ and a $\mathcal{F}_\infty$-measurable random variable Y distributed as $S_n/\sqrt{n}$, with the property $\|Y - \beta_1\|_{L^2} \leq 2\kappa n^{-\frac{1}{4}}$. We then infer that

$$\mu_p\left(\frac{S_n}{\sqrt{n}}\right) \geq \sup_{R \in \mathcal{R}(\mathcal{F}_1, p)} E[R \cdot \beta_1] - 2\kappa n^{-\frac{1}{4}} = \Phi(p) - 2\kappa n^{-\frac{1}{4}},$$

as we wanted to prove. □

To continue this analysis of the maximal variation of a bounded martingale, let us prove the following result:

THEOREM 2. – *For $q > 2$ and for $0 < p < 1$, $\xi_n^q(p)/\sqrt{n}$ tends to ∞ as n increases.*

Proof. – For fixed n let $X^n = (X_1^n, \ldots, X_n^n)$ denotes the martingale starting from p defined by the following transitions: $X_k^n = X_{k+1}^n$ conditionally on $X_k^n \in \{0, 1\}$, and conditionally on $X_k^n = p$, X_{k+1}^n takes the value 0, p and 1 with respective probability $(1-p)/n$, $1-n^{-1}$ and p/n.

An easy computation indicates that

$$V_n^q(X^n) = \sum_{k=1}^{n-1}(1 - n^{-1})^{k-1} n^{-\frac{1}{q}} \lambda(p) = (1 - (1 - n^{-1})^n) n^{1-\frac{1}{q}} \lambda(p),$$

with $\lambda(p) := (p(1-p)^q + (1-p)p^q)^{\frac{1}{q}} > 0$. Since $(1 - n^{-1})^n$ converges to e^{-1} as n tends to ∞, we conclude that $V_n^q(X^n) = O(n^{1-\frac{1}{q}})$, and thus $V_n^q(X^n)/\sqrt{n}$ tends to ∞ as far as $\frac{1}{2} - \frac{1}{q} > 0$ i.e. $q > 2$. □

So the only unexplored case is the asymptotic behavior of $\xi_n^2(p)/\sqrt{n}$. The argument used above to prove Theorem 1 fails to work here. However, it can be proved that $\lim_{n\to\infty} \xi_n^2(p)/\sqrt{n} = \Phi(p)$: the argument of Mertens and Zamir's paper can be adapted to this case.

2. A GENERALIZATION OF THE CENTRAL LIMIT THEOREM

The central limit theorem deals with the limit distributions of $S_n/\sqrt{n}$, where S_n is the sum of n i.i.d. random variables. The next result dispenses with the i.i.d. hypothesis: It identifies the class of all possible limit distributions of $X_n/\sqrt{n}$, where X_n is the terminal value of a discrete time martingale X whose n increments $X_{k+1} - X_k$ have a conditional variance in a given interval $[A, B]$ and a conditional q-order moment uniformly bounded for a $q > 2$, as the weak closure of the set of distributions of a Brownian motion stopped at a $[A, B]$-valued stopping time. The classical central limit theorem, when stated for i.i.d. random variables with bounded q-order moment, appears then as a particular case of this result when $A = B$.

To be more formal, let $\mathcal{S}_n^q([A, B], C)$ denote the set of n-stages martingales S such that for all k, both relations hold:

$$A \leq E[|S_{k+1} - S_k|^2 | S_1, \ldots, S_k] \leq B,$$

and

$$E[|S_{k+1} - S_k|^q | S_1, \ldots, S_k] \leq C^q.$$

THEOREM 3. – *There exists a universal constant κ such that for all $n \in \mathbb{N}$, for all $q > 2$, for all $A, B, C \in \mathbb{R}$ with $0 \leq A \leq B \leq C$ and for all $X \in \mathcal{S}_n^q([A, B], C)$, there exist a filtration $\mathcal{F}$, an $\mathcal{F}$-Brownian motion β, an*

$[A,B]$-valued stopping time τ on $\mathcal{F}$ and a $\mathcal{F}_\infty$-measurable random variable Y whose marginal distribution coincides with that of $X_n/\sqrt{n}$ and such that

$$E[(Y-\beta_\tau)^2] \le \kappa^2 C^2 n^{\frac{2}{q\wedge 4}-1}$$

To prove this result, we will need the following Lemma which is obvious in case $p = 2$:

LEMMA 4. – *For $p \in [1,2]$, for all discrete martingale X with $X_0 = 0$, we have:*

$$E[|X_n|^p] \le 2^{2-p} \sum_{k=0}^{n-1} E[|X_{k+1} - X_k|^p].$$

Proof [1].

By a recursive argument, this follows from the relation:

$$E[|x+Y|^p] \le |x|^p + 2^{2-p} E[|Y|^p],$$

that holds for all x in $\mathbb{R}$ whenever Y is a centered random variable: Indeed,

$$|x+Y|^p - |x|^p = Y \int_0^1 p|x+sY|^{p-1}\mathrm{sgn}(x+sY)ds$$

Thus, since $E[Y] = 0$, we get

$$E[|x+Y|^p]-|x|^p = E\left[Y \int_0^1 p\big(|x+sY|^{p-1}\mathrm{sgn}(x+sY) - |x|^{p-1}\mathrm{sgn}(x)\big)ds\right]$$

A straightforward computation indicates that, for $1 \le p \le 2$ and a fixed a, the function $g(x) := ||x+a|^{p-1}\mathrm{sgn}(x+a) - |x|^{p-1}\mathrm{sgn}(x)|$ reaches its maximum at $x = -a/2$, implying $g(x) \le 2^{2-p}|a|^{p-1}$.

So, $E[|x+Y|^p] - |x|^p \le E\left[|Y| \int_0^1 2^{2-p} p |sY|^{p-1} ds\right] = 2^{2-p} E[|Y|^p]$, as announced. □

Proof of Theorem 3. – Let W be a standard 1-dimensional Brownian motion starting at 0 at time 0 and let $\mathcal{H}_s$ denote the completion of the

[1] As suggested by an anonymous referee, we could obtain a similar inequality for $p > 1$, as a consequence of Burkholder's square function inequality for discrete martingales, since $p/2 < 1$. The constant factor 2^{2-p} should then be replaced by C_p^p, where C_p denotes Burkholder's universal constant. However, as stated in Theorem 3.2 of Burkholder's paper [1], the optimal choice of this constant C_p is $O(p\sqrt{q})$, where $p^{-1} + q^{-1} = 1$ and is thus unbounded as p decreases to 1. This would completely alterate the nature of the bound of Theorem 3 above.

σ-algebra generated by $\{W_t, t \leq s\}$. The filtration $\mathcal{G} := \{\mathcal{G}_k\}_{k=1}^n$ defined as $\mathcal{G}_k = \mathcal{H}_{\frac{k}{n}}$ is rich enough to insure the existence of an adapted system $(u_1, \ldots, u_n)$ of independent random variables uniformly distributed on $[0, 1]$.

Let then X be in $\mathcal{S}_n^q([A, B], C)$. As we saw in the previous section, it is possible to create a $\mathcal{G}$-adapted martingale Z inducing on $\mathbb{R}^n$ the same distribution as X, with the property $E[Z_{k+1} - Z_k|\mathcal{G}_k] = E[Z_{k+1} - Z_k|Z_1, \ldots, Z_k]$.

In turn, Z_k is the value at time k/n of the process $S_t := E[Z_n|\mathcal{H}_t]$. As a particular property of the Brownian filtration $\mathcal{H}$, any such martingale can be represented as the Itô-integral $S_t = \int_0^t R_s dW_s$ of a progressively measurable process R with $E[\int_0^1 R_s^2 ds] \leq \infty$ (see Proposition (3.2), Chapter V in [4]).

Let us now define the process $r_t := R_t/\sqrt{n}$, if $t \leq 1$ and $r_t := 1$ if $t > 1$, let $\phi(t)$ denote $\phi(t) := B$ if $t \leq 1$ and $\phi(t) := A$ otherwise. Let us define the stopping times

$$\theta := \inf\left\{t \middle| \int_0^t r_s^2 ds \geq \phi(t)\right\}$$

and

$$T_u := \inf\left\{t \middle| \int_0^t r_s^2 ds > u\right\}.$$

Let finally ρ_t be $\int_0^t r_s dW_s$.

With these definitions, our proof is as follows: On one hand, $Y := \rho_1$ is equal to $S_1/\sqrt{n}$ and has thus the same distribution as $X_n/\sqrt{n}$. According to Dambis Dubins Schwarz's Theorem (see Theorem 1.6, Chapter V in [4]), the process $\beta_u := \rho_{T_u}$ is a Brownian motion with respect to the filtration $\{\mathcal{H}_{T_u}\}_{u \geq 0}$ and for all t, the random variable $U_t := \int_0^t r_s^2 ds$ is a stopping time on this filtration. In particular, $Y = \beta_{U_1}$ is $\mathcal{H}_{T_\infty}$-measurable.

On the other hand, $\tau := U_\theta$ is a stopping time on $\{\mathcal{H}_{T_u}\}_{u \geq 0}$. Indeed, for all u, $\{\tau \leq u\} = \{\theta \leq T_u\} \in \mathcal{H}_{T_u}$, according to 4.16, chapter I in [4]. Due to the definition of θ, τ is $[A, B]$-valued and it remains for us to prove that $||Y - \beta_\tau||_{L^2} = ||\rho_1 - \rho_\theta||_{L^2}$ is bounded.

Now $||\rho_1 - \rho_\theta||_{L^2}^2 = E[\int_{\theta \wedge 1}^{\theta \vee 1} r_s^2 ds] = E[\int_{\theta \wedge 1}^1 r_s^2 ds] + E[\int_1^{\theta \vee 1} r_s^2 ds]$. According to the definition of θ, on $\{\theta > 1\}$, we have $\int_0^\theta r_s^2 ds = A$ and thus $\int_1^{\theta \vee 1} r_s^2 ds = A - \int_0^1 r_s^2 ds$. Since the event $\{\theta > 1\}$ is just equal to $\{\int_0^1 r_s^2 ds < A\}$, we conclude that $E[\int_1^{\theta \vee 1} r_s^2 ds] = E[(A - \int_0^1 r_s^2 ds)^+]$.

Similarly, on $\{\theta < 1\}$, $\int_0^\theta r_s^2 ds = B$ and $\int_\theta^1 r_s^2 ds = \int_0^1 r_s^2 ds - B$. Furthermore, on $\{\theta = 1\}$, $\int_0^1 r_s^2 ds \leq B$. Hence, $E[\int_{\theta \wedge 1}^1 r_s^2 ds] = E[(\int_0^1 r_s^2 ds - B)^+]$.

All together, we find $\|\rho_1 - \rho_\theta\|_{L^2}^2 = E[|\int_0^1 r_s^2 ds - V|]$, where

$$V := \left(B \wedge \left(A \vee \int_0^1 r_s^2 ds\right)\right)$$

is the "truncation" to the interval $[A, B]$ of the random variable $\int_0^1 r_s^2 ds$.

Obviously, among the $[A, B]$-valued random variables, V is the best L^1-approximation of $\int_0^1 r_s^2 ds$.

Taking into account the condition $E[(X_{k+1} - X_k)^2 | X_1, \ldots, X_k] \in [A, B]$ we have $\hat{\zeta}_k := E[\zeta_k | \mathcal{H}_{\frac{k}{n}}] \in [A, B]$, where $\zeta_k := \int_{\frac{k}{n}}^{\frac{k+1}{n}} R_s^2 ds$. Therefore, $V' := \sum_{k=0}^{n-1} \hat{\zeta}_k / n$ is also an $[A, B]$-valued random variable and we may conclude:

$$E\left[\left|\int_0^1 r_s^2 ds - V\right|\right] \leq E\left[\left|\int_0^1 r_s^2 ds - V'\right|\right] = \frac{1}{n}\left\|\sum_{k=0}^{n-1}(\zeta_k - \hat{\zeta}_k)\right\|_{L^1}.$$

Finally, the conditional q-order moment condition

$$E[|X_{k+1} - X_k|^q | X_1, \ldots, X_k] \leq C^q$$

implies $E[|X_{k+1} - X_k|^{\tilde{q}} | X_1, \ldots, X_k] \leq C^{\tilde{q}}$, where $\tilde{q} = 4 \wedge q$. As a joint consequence of Burkholder Davis Gundy's inequality and Doob's one, this condition becomes

$$E[\zeta_k^{\frac{\tilde{q}}{2}} | \mathcal{H}_{\frac{k}{n}}] \leq (1/c_{\tilde{q}}) E[\sup_{t \in [\frac{k}{n}, \frac{k+1}{n}]} \{|S_t - S_{\frac{k}{n}}|^{\tilde{q}}\} | \mathcal{H}_{\frac{k}{n}}] \leq \left(\frac{\tilde{q}}{\tilde{q} - 1}\right)^{\tilde{q}} C^{\tilde{q}} / c_{\tilde{q}},$$

where $c_{\tilde{q}}$ is the Burkholder Davis Gundy universal constant (see theorem (4.1), Chapter IV in [4]). Since, by hypothesis, $q > 2$, we have $\tilde{q}/2 \in [1, 2]$ and me may apply Lemma 4 to conclude that

$$\left\|\sum_0^{n-1}(\zeta_k - \hat{\zeta}_k)\right\|_{L^{\tilde{q}/2}}^{\tilde{q}/2} \leq \left(\frac{\tilde{q}}{\tilde{q} - 1}\right)^{\tilde{q}} \frac{2^{2-\tilde{q}/2}}{c_{\tilde{q}}} C^{\tilde{q}} n,$$

and thus:

$$\|Y - \beta_\tau\|_{L^2}^2 \leq \frac{1}{n}\left\|\sum_0^{n-1}(\zeta_k - \hat{\zeta}_k)\right\|_{L^1} \leq \left(\frac{\tilde{q}}{\tilde{q} - 1}\right)^2 \frac{2^{4/\tilde{q}-1}}{c_{\tilde{q}}^{2/\tilde{q}}} C^2 n^{2/\tilde{q}-1}.$$

This terminates the proof of Theorem 2 since, for $\tilde{q} \in [2, 4]$, the constant $c_{\tilde{q}}$ is bounded away from 0. □

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