ANNALES DE L'I. H. P., SECTION B

FRAYDOUN REZAKHANLOU
JAMES E. TARVER

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Annales de l'I. H. P., section B, tome 33, nº 6 (1997), p. 753-796
<http://www.numdam.org/item?id=AIHPB_1997__33_6_753_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 6, 1997, p. 753-796 *Probabilités et Statistiques*

Boltzmann-grad limit for a particle system in continuum

by

Fraydoun REZAKHANLOU and James E. TARVER

Department of Mathematics,
University of California Berkeley, CA 94720.

ABSTRACT. – We examine a system of one-dimensional particles in which the particles travel deterministically in between stochastic collisions. The collision rates are chosen so that finitely many collisions occur in a unit interval of time. We prove the kinetic limit and subsequently derive the discrete Boltzmann equation.

RÉSUMÉ. – Nous étudions un système de particules unidimensionnel dans lequel les particules se déplacent de manière déterministe entre des collisions aléatoires. Les taux de collisions sont choisis de façon que seul un nombre fini de collisions se produit sur un intervalle de temps fini. Nous obtenons la limite cinétique et en déduisons l'équation de Boltzmann discrète.

1. INTRODUCTION

We center our attention upon a particle system model and prove that its distribution of particles converges weakly, as the number of particles converges to infinity, to the unique solution of the discrete Boltzmann equation. We consider the discrete Boltzmann equation for several reasons. First, the DBE is a simplification of the (full) Boltzmann equation that maintains its essential characteristics-the free streaming of particles in between collisions and the quadratic nature of the collision. Partial results for the proof of the kinetic limit for the full equation are available (see [7], [11], [8]), but a general global result is at the moment not known; by proving the Boltzmann-Grad limit for the DBE, we will hopefully gain new insights into how to go about doing the same for the full equation. This would completely validate its use as the model for the evolution of the mass density of a dilute gas over time. Second, the DBE has further applications in fluid dynamics [9] and is thus interesting in its own right. Readers interested in learning more on the DBE (or discrete velocity model, as it is also known) are directed to the surveys by Gatignol [5], Illner and Platkowski [10], and Bellomo and Gustafsson [12].

The particle system is roughly described as follows. Initially, N particles are scattered about the unit circle [1] according to a given law μ_L. Each particle is represented by a vector $q_i = (x_i, \alpha_i)$, where x_i denotes the *location* of the particle on the unit circle and α_i denotes the *label* of the particle. (Each label α_i corresponds to some velocity v_{α_i}.) A particle evolves deterministically according to its velocity until it encounters another particle, which it either ignores (with probability $1 - \hat{\varepsilon}$) or collides into (with probability $\hat{\varepsilon}$). $(\hat{\varepsilon})^{-1} = O(N)$. The choice of this particular stochastic collision plays the same role as does choosing the dilute-gas scaling for hard-sphere models in higher dimensions – it guarantees the constancy of the mean free path. If two particles with labels α and β collide, they yield particles with new labels γ and δ with rate $K(\alpha\beta, \gamma\delta)$. Let $\rho_\alpha(x, t)$ denote the macroscopic density of particles with label α; we will show that $\rho = (\rho_1, \dots, \rho_n)$ solves the system

$$\begin{cases} \dfrac{\partial \rho_\alpha}{\partial t} + v_\alpha \dfrac{\partial \rho_\alpha}{\partial x} = \displaystyle\sum_{\beta\gamma\delta} (K(\gamma\delta, \alpha\beta)\rho_\gamma\rho_\delta - K(\alpha\beta, \gamma\delta)\rho_\alpha\rho_\beta) \\ \rho_\alpha(x, 0) = \rho_\alpha^0(x), \end{cases}$$

$$\alpha = 1, 2, \dots, n. \tag{1.1}$$

where $\rho^0 = (\rho_0^0, \dots, \rho_n^0)$ denotes the initial density.

This article follows the recent paper by Rezakhanlou [12], in which he established the kinetic limit for a one-dimensional lattice gas for which the movement of each particle is a simple random walk between stochastic collisions. Our result improves upon his in three important ways. First, the particle motion is deterministic instead of random, which is more physically realistic than the random walk assumption. Second, we drop the assumption that momentum must be conserved. Finally, we prove the collision bound without resorting to an argument of Bony's [3]. Bony's bound is a strict

1. We state and prove all of our results in terms of the unit circle; a standard argument extends our results to the entire real line.

one- dimensional result, and Rezakhanlou's use of it prevents him from extending his result to higher dimensions; instead, we modeled our argument largely upon Tartar's existence proof for the discrete Boltzmann equation [13], [14]. Since Tartar's proof has been extended to higher dimensions [6], then presumably we can extend ours as well.

Caprino and Pulvirenti [4] have also considered a one-dimensional particle system model in which particles travel deterministically in between stochastic collisions. Their approach is considerably different from ours in that they make a detailed analysis of the BBGKY hierarchy for the N-particle distribution functions.

We now more precisely describe our model and our results. Let $I := \{1, 2, \dots, n\}$; I denotes the set of labels of the n different types of particles. The set S^1 denotes the interval $[0, 1]$ with the end points identified. Define the state space $E := (S^1 \times I)^N$; $\vec{q} \in E$ is the N-tuple $\vec{q} = (q_1, \dots, q_N)$, $q_i = (x_i, \alpha_i)$. $\vec{q}$ identifies the configuration of N particles whose i-th particle has position x_i and velocity v_{α_i}. $\vec{q}_t \equiv \vec{q}(t)$ is a Feller process with infinitesimal generator $\mathcal{A}^{(L)}$, where L denotes the length scale and

$$\mathcal{A}^{(L)} = \mathcal{A}_0 + \mathcal{A}_c. \tag{1.2}$$

We choose L such that $\frac{N}{L} = M$, where $M := \int (\sum_\alpha \rho_\alpha^0(x))dx$. We have, for any smooth function g,

$$\mathcal{A}_0 g(\vec{q}) = \sum_{k=1}^{N} v_{\alpha_k} \frac{\partial g}{\partial x_k}(\vec{q}) \tag{1.3}$$

$$\mathcal{A}_c g(\vec{q}) = \frac{1}{2} \sum_{i \neq j} V(L(x_i - x_j)) \sum_{\gamma\delta} K(\alpha_i \alpha_j, \gamma\delta)[g(S_{ij}^{\gamma\delta} \vec{q}) - g(\vec{q})], \tag{1.4}$$

where $V : \mathbb{R} \to [0, \infty)$ is a smooth, even function and $\int_{\mathbb{R}} V(x)dx = 1$; $S_{ij}^{\gamma\delta} \vec{q}$ is the configuration obtained from $\vec{q}$ by replacing α_i with γ and α_j with δ. $\mathcal{A}_0$ is the free-motion generator, and $\mathcal{A}_c$ generates the collisions.

We make the following assumptions on K.

(i) $K(\alpha\beta, \gamma\delta) \geq 0$

(ii) $K(\alpha\beta, \gamma\delta) = K(\beta\alpha, \gamma\delta) = K(\alpha\beta, \delta\gamma)$

(iii) $K(\alpha\beta, \gamma\delta) = 0$ if $v_\alpha = v_\beta$

(iv) $K(\alpha\beta, \gamma\delta) = 0$ if $v_\beta = v_\gamma = v_\delta$

(v) $\exists \Lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i > 0$, such that $\forall \alpha, \beta, \gamma, \delta \in I$,

$$K(\alpha\beta, \gamma\delta)\lambda_\alpha\lambda_\beta = K(\gamma\delta, \alpha\beta)\lambda_\gamma\lambda_\delta. \tag{1.5}$$

Since we are thinking of K as a collision rate function, K is necessarily nonnegative; (ii) states that the collision rates depend upon the labels only and are independent of the particle numbers; (iii) implies that only particles with different velocities can collide; (iv) holds if the model satisfies a microscopic conservation of momentum; (v) states that Λ is a *Maxwellian* (*i.e.* Λ is an equilibrium solution of our discrete Boltzmann equation).

We use this last assumption on K to help determine an invariant measure for our process $\vec{q}(t)$. Note that $\mathcal{A}_0$ leaves labels unchanged and $\mathcal{A}_c$ leaves locations unchanged; the invariant measure ν_L will therefore be a product measure $\nu_L = \nu_1 \times \nu_2$, where $\nu_1 \in \mathcal{M}(S^N)$ and $\nu_2 \in \mathcal{M}(I^N)$. It is clear that for any smooth function g, we have that

$$\int \cdots \int \sum_{k=1}^{N} v_{\alpha_k} \frac{\partial g}{\partial x_k}(\vec{q}) dx_1 \ldots dx_N = 0; \tag{1.6}$$

therefore, we take $\nu_1 = d^N x$ (*i.e.* ν_1 is N-dimensional Lebesgue measure). Since there exists a Maxwellian $\Lambda = (\lambda_1, \ldots, \lambda_n)$, we also have that

$$\sum_{\alpha_1} \cdots \sum_{\alpha_N} \mathcal{A}_c g(\vec{q}) \lambda_{\alpha_1} \ldots \lambda_{\alpha_N} = 0. \tag{1.7}$$

Without loss of generality, we may assume $\lambda_1 + \ldots + \lambda_n = 1$, so we define ν_2 to be the product of the weighted counting measures that gives the weight λ_α to the label α; *i.e.*

$$\nu_2((\alpha_1, \ldots, \alpha_N) = (\beta_1, \ldots, \beta_N)) = \lambda_{\beta_1} \ldots \lambda_{\beta_N}.$$

Let μ_L be a sequence of probability measures on E and let $\rho_\alpha^0 : S^1 \to [0, \infty)$ be a sequence of bounded measurable functions.

NOTATION 1. – We will say that $\mu_L \sim \rho^0 = (\rho_1^0, \ldots, \rho_n^0)$ if the following conditions hold for every test function J:

$$\begin{aligned}
&\text{(i)} \quad \lim_{L \to \infty} \int \left| \frac{1}{L} \sum_{i=1}^{N} J(x_i) 1\!\!1(\alpha_i = \alpha) \right. \\
&\qquad\qquad \left. - \int J(x) \rho_\alpha^0(x) dx \right| \mu_L(d\vec{q}) = 0 \\
&\text{(ii)} \quad \mu_L \ll \nu_L \text{ and } \frac{d\mu_L}{d\nu_L} = F_L(0) \\
&\text{(iii)} \quad \exists \text{ constants } p > 1 \text{ and } b > 0 \\
&\text{such that } \sup_L e^{-bL} \int (F_L(0))^p d\nu_L \leq 1.
\end{aligned} \tag{1.8}$$

For example, suppose we define $\mu_L(d\vec{q})$ as follows: for any continuous function $f : E \to \mathbb{R}$, we have

$$\int f(\vec{q})\mu_L(d\vec{q}) = \sum_{\alpha_1}\cdots\sum_{\alpha_N}\cdots\int\Bigg(f(x_1,\alpha_1,...,x_N,\alpha_N)$$
$$\cdot\frac{1}{M^N}\rho^0_{\alpha_1}(x_1)\cdots\rho^0_{\alpha_N}(x_N)\Bigg)\lambda_{\alpha_1}\cdots\lambda_{\alpha_N}dx_1\cdots dx_N,$$

Condition (i) is the law of large numbers; condition (ii) is obvious; condition (iii) follows since $\rho^0_\alpha(x)$ is bounded $\forall\alpha\in I$. Condition (i) states that the macroscopic density of α-particles at time $t = 0$ is given by $\rho^0_\alpha(x) = \rho_\alpha(x,0)$. We expect the same thing for later times as well. Let $\rho(x,t)$ be the solution to the system (1.1), where a solution $\rho : S^1\times[0,\infty) \to \mathbb{R}^n_+$ (where $\mathbb{R}_+$ denotes the set of nonnegative numbers) is understood as follows:

$$\begin{aligned}
&\text{(i)}\quad \rho_\alpha \in C([0,T];L^1(S^1))\\
&\text{(ii)}\quad \rho_\alpha\rho_\beta \in L^1([0,T]\times S^1)\ \forall T\geq 0 \text{ and } v_\alpha\neq v_\beta\\
&\text{(iii)}\quad \rho_\alpha(x,t) = \rho^0_\alpha(x-v_\alpha t) + \int_0^t Q_\alpha(\rho,\rho)(x-v_\alpha(t-s),s)ds\\
&\qquad \forall t\geq 0,\ \forall\alpha\in I,\ \text{and for almost all } x.
\end{aligned}\tag{1.9}$$

$$Q_\alpha(\rho,\rho) := \sum_{\beta\gamma\delta}(K(\gamma\delta,\alpha\beta)\rho_\gamma\rho_\delta - K(\alpha\beta,\gamma\delta)\rho_\alpha\rho_\beta).\tag{1.10}$$

NOTATION 2. – Let $\vec{q}_L(t) = (x_1(t),\alpha_1(t),\ldots,x_N(t),\alpha_N(t))$ denote the Feller process whose infinitesimal generator is $\mathcal{A}^{(L)}$. When the space scaling is clear, we will drop the subscript and refer to the process as $\vec{q}(t)$ or even as $\vec{q}_t$. Also, let P_L denote the probability measure uniquely determined by the process $\vec{q}_L(\cdot)$ when its initial distribution is $\mu_L(d\vec{q})$, and let E_L denote the expectation with respect to P_L.

We are now ready to state the main result.

THEOREM 1.1. – *Suppose $\mu_L \sim \rho^0$, where ρ^0 is bounded, measurable, and nonnegative. Then, for every continuous J and every $\alpha\in I$,*

$$\lim_{L\to\infty} E_L\left|\frac{1}{L}\sum_{i=1}^N J(x_i(t))\mathbb{1}(\alpha_i(t)=\alpha) - \int J(x)\rho_\alpha(x,t)dx\right| = 0,\tag{1.11}$$

where $\rho(x,t)$ is the unique solution to the system (1.1).

2. ENTROPY

We first establish a bound on the growth of the entropy. As we will see later, this bound will prove to be the key to demonstrating that the number of collisions remains finite as $N \to \infty$. Before proving the bound, however, we need to introduce some notation and make some definitions.

$$\mathcal{N}_t(I) := \sum_{i=1}^{N} 1\!\!1(x_i(t) \in I) \tag{2.1}$$

$$a_{i,L}(t) := \mathcal{N}_t\left(\left[\frac{i}{L}, \frac{i+1}{L}\right)\right) \tag{2.2}$$

$$\varphi(x) := x \log x - x + 1 \tag{2.3}$$

$$\Phi(\vec{q}) := \frac{1}{L} \sum_{i=0}^{L-1} \varphi(a_{i,L}). \tag{2.4}$$

We denote the solution of the forward equation by $F_L(t) \equiv F_L(\vec{q}_t)$; *i.e.*

$$\frac{\partial}{\partial t} F_L(t) = \mathcal{A}^{(L)*} F_L(t), \tag{2.5}$$

where the adjoint is taken with respect to the invariant measure ν_L. Let $\hat{\pi} := \frac{p-1}{p}$, where p is given in (1.8(iii)). Finally, the letter C will stand for various constants.

THEOREM 2.1. – *$\exists$ constant $\bar{C} > 0$ such that $\forall L$,*

$$E_L\left(\sup_{0 \leq s \leq T} \exp \frac{\hat{\pi}}{2} L \Phi(\vec{q}_s)\right) \leq e^{\bar{C}L}. \tag{2.6}$$

We break the proof up into three parts. We first prove a similar claim for the process at equilibrium; we then show that

$$\int (F_L(t))^p \nu_L(d\vec{q}) \leq e^{bL}, \tag{2.7}$$

where p and b are defined in (1.8 (iii)); finally, we tie everything together to prove (2.6).

LEMMA 2.1.

$$\int \exp L\Phi(\vec{q}) \nu_L(d\vec{q}) \leq e^{CL}. \tag{2.8}$$

Proof. – We first note that at equilibrium, the particles are distributed uniformly over the circle. Therefore

$$\begin{aligned}\nu_L(a_{0,L} = l_0, \ldots, a_{L-1,L} = l_{L-1})\\ = \frac{N!}{l_0! \ldots l_{L-1}!}\left(\frac{1}{L}\right)^N, \quad l_0 + \ldots + l_{L-1} = N.\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\int \exp L\Phi(\vec{q})\nu_L(d\vec{q})\\
&= \int \left(\prod_{i=0}^{L-1} \exp \varphi(a_{i,L})\right)\nu_L(d\vec{q}) \le e^L \int \left(\prod_{i=0}^{L-1} (a_{i,L})^{a_{i,L}}\right)\nu_L(d\vec{q})\\
&\le e^L \sum_{l_0+\ldots+l_{L-1}=N} l_0^{l_0} \ldots l_{L-1}^{l_{L-1}} \nu_L(a_{0,L} = l_0, \ldots, a_{L-1,L} = l_{L-1})\\
&= e^L \sum_{l_0+\ldots+l_{L-1}=N} l_0^{l_0} \ldots l_{L-1}^{l_{L-1}} \frac{N!}{l_0! \ldots l_{L-1}!}\left(\frac{1}{L}\right)^N\\
&\le e^L \sum_{n_0+\ldots+n_{L-1}=N} \sum_{l_0+\ldots+l_{L-1}=N} n_0^{l_0} \ldots n_{L-1}^{l_{L-1}} \binom{N}{l_0, \ldots, l_{L-1}}\left(\frac{1}{L}\right)^N\\
&= e^L \sum_{n_0+\ldots+n_{L-1}=N} (n_0 + \ldots + n_{L-1})^N\left(\frac{1}{L}\right)^N\\
&= e^L\left(\frac{N}{L}\right)^N\binom{N+L-1}{N}
\end{aligned} \tag{2.10}$$

Since $\frac{N}{L} = M$, a straightforward application of Stirling's formula finishes the proof of the claim.

LEMMA 2.2. – $\exists\, p > 1$ *and* $b > 0$ *such that* $\forall\, t \ge 0$ *we have*

$$\sup_L e^{-bL} \int (F_L(t))^p \nu_L(d\vec{q}) \le 1. \tag{2.11}$$

Proof. – Recall that we assumed that $\mu_L \sim \rho^0$, which implies that (2.11) holds for $p > 1$ and $b > 0$ at $t = 0$. Therefore, (2.11) immediately follows if we can show that

$$\frac{d}{dt} \int (F_L(t))^p \nu_L(d\vec{q}) \le 0, \tag{2.12}$$

To simplify notation, we will suppress L and t and define $h := pF^{p-1}$, so that when we differentiate F^p, we Obtain

$$\frac{\partial}{\partial t}F^p = h\frac{\partial}{\partial t}F = h\mathcal{A}^*F. \tag{2.13}$$

A simple calculation verifies the equality

$$\mathcal{A}^* = -\mathcal{A}_0 + \mathcal{A}_c; \tag{2.14}$$

therefore, (2.12) will follow immediately after showing

$$\int h(-\mathcal{A}_0 F) d\nu \leq 0 \text{ and } \int h(\mathcal{A}_c F) d\nu \leq 0.$$

$$\begin{aligned}
\int h(-\mathcal{A}_0 F) d\nu &= \int pF^{p-1}(-\mathcal{A}_0 F) d\nu \\
&= -\sum_{\alpha_1} \cdots \sum_{\alpha_N} \int \cdots \int \sum_{k=1}^{N} v_{\alpha_k} pF^{p-1} \frac{\partial F}{\partial x_k} \lambda_{\alpha_1} \ldots \lambda_{\alpha_N} dx_1 \ldots dx_N \\
&= -\sum_{\alpha_1} \cdots \sum_{\alpha_N} \int \cdots \int \sum_{k=1}^{N} v_{\alpha_k} \left(\frac{\partial}{\partial x_k} (F)^p \right) \\
&\quad \times \lambda_{\alpha_1} \ldots \lambda_{\alpha_N} dx_1 \ldots dx_N = 0.
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\int h(\mathcal{A}_c F) d\nu &= \int (\mathcal{A}_c h) F d\nu \\
&= \sum_{\alpha_1} \cdots \sum_{\alpha_N} \int \cdots \int \frac{1}{2} \sum_{i \neq j} \Big(V(L(x_i - x_j)) \sum_{\gamma\delta} K(\alpha_i \alpha_j, \gamma\delta) \\
&\quad \cdot [h(S_{ij}^{\gamma\delta} \vec{q}) - h(\vec{q})] F(\vec{q}) \Big) \lambda_{\alpha_1} \ldots \lambda_{\alpha_N} dx_1 \ldots dx_N.
\end{aligned} \tag{2.16}$$

Since everything else is nonnegative, the nonpositivity of (2.16) will follow if we can show that for fixed i and j, $i \neq j$, that

$$\sum_{\alpha_i \alpha_j} \sum_{\gamma\delta} K(\alpha_i \alpha_j, \gamma\delta)(h(S_{ij}^{\gamma\delta} \vec{q}) - h(\vec{q})) F(\vec{q}) \lambda_{\alpha_i} \lambda_{\alpha_j} \leq 0. \tag{2.17}$$

Note that the sum in (2.17) depends solely upon the labels of the i-th and j-th particles, and so we can effectively treat both h and F as functions of α_i and α_j only. Therefore (2.17) is an immediate consequence of the next lemma.

LEMMA 2.3. – *Suppose* $f, g : I \times I \to \mathbb{R}$ *such that* $g(\gamma\delta) \geq g(\alpha\beta) \Leftrightarrow f(\gamma\delta) \geq f(\alpha\beta) \forall \alpha, \beta, \gamma, \delta \in I$. *Then*

$$\sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) \lambda_\alpha \lambda_\beta (g(\gamma\delta) - g(\alpha\beta)) f(\alpha\beta) \leq 0. \tag{2.18}$$

Proof.

$$\begin{aligned}
&\sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta)\lambda_\alpha\lambda_\beta(g(\gamma\delta)-g(\alpha\beta))f(\alpha\beta)\\
&\quad=\sum_{\alpha\beta\gamma\delta} K(\gamma\delta,\alpha\beta)\lambda_\gamma\lambda_\delta(g(\gamma\delta)-g(\alpha\beta))f(\alpha\beta)\\
&\quad=\sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta)\lambda_\alpha\lambda_\beta(g(\alpha\beta)-g(\gamma\delta))f(\gamma\delta)\\
&\quad=-\frac{1}{2}\sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta)\lambda_\alpha\lambda_\beta(g(\gamma\delta)-g(\alpha\beta))(f(\gamma\delta)-f(\alpha\beta))\leq 0.
\end{aligned}\tag{2.19}$$

The first equality follows from a direct application of (1.5 (v)); the second equality results after a change of variables. The rest is obvious.

Proof of Theorem.

Step 1. By Hölder's inequality, (2.7) and (2.8) we have

$$\begin{aligned}
E_L(\exp\hat{\pi}L\Phi(\vec{q}_t)) &\leq \left(\int \exp L\Phi(\vec{q})\nu_L(d\vec{q})\right)^{\hat{\pi}}\left(\int (F_L(t))^p\nu_L(d\vec{q})\right)^{1/p}\\
&\leq e^{CL}.
\end{aligned}\tag{2.20}$$

Step 2. Define the exponential martingale $M_\lambda(t,s)$ (for arbitrary $\lambda>0$) as

$$M_\lambda(t,s):=\exp\left(\lambda L\Phi(\vec{q}_t)-\lambda L\Phi(\vec{q}_s)-\int_s^t e^{-\lambda L\Phi}\mathcal{A}^{(L)}e^{\lambda L\Phi}(\vec{q}_u)du\right).\tag{2.21}$$

Note, however, that $\mathcal{A}_c\Phi(\vec{q})=0$, so we can rewrite (2.21) as

$$M_\lambda(t,s)=\exp\left(\lambda L\Phi(\vec{q}_t)-\lambda L\Phi(\vec{q}_s)-\int_s^t e^{-\lambda L\Phi}\mathcal{A}_0e^{\lambda L\Phi}(\vec{q}_u)du\right).\tag{2.22}$$

We state the next lemma without proof, but the proof is not too difficult; it essentially boils down to applications of Doob's inequality, Cauchy-Schwarz, and the fact that the expectation of a martingale does not change over time.

LEMMA 2.4. – *Suppose $f:E\longrightarrow\mathbb{R}$ is a smooth function. Then, for any $S\geq 0$ and $\lambda>0$, we have*

$$E_L\left(\sup_{S\leq t\leq T}\exp\lambda(f(\vec{q}_t)-f(\vec{q}_S))\right)\leq P^{\frac{1}{8}}Q^{\frac{1}{4}},\tag{2.23}$$

where

$$P = E_L\left(\exp 2\int_s^T |e^{-4\lambda f(\vec{q}_u)}\mathcal{A}^{(L)}e^{4\lambda f(\vec{q}_u)}|du\right)$$

and

$$Q = E_L\left(\exp 8\int_s^T |e^{-\lambda f(\vec{q}_u)}\mathcal{A}^{(L)}e^{\lambda f(\vec{q}_u)}|du\right).$$

Before we continue, we need to address a small technical point. $\mathcal{A}_0$ is a differential operator, but $a_{i,L}(\vec{q})$ (and hence $\Phi(\vec{q})$) is not continuous. We get around this by replacing $a_{i,L}$ with a smooth sequence. Let ζ be a smooth nonnegative function of compact support that is identically 1 on the interval $[0,1]$ and $\zeta \leq 1$. Then, it is not hard to see

$$\Phi(\vec{q}) - 1 \leq \hat{\Phi}(\vec{q}) := \frac{1}{L}\sum_{u=0}^{L-1}\varphi\left(\sum_{j=1}^{N}\zeta(L(x_j - \frac{u}{L}))\right) \leq c_0\Phi(\vec{q}), \tag{2.24}$$

for some c_0, and

$$\begin{aligned} &e^{-\lambda L\hat{\Phi}}\mathcal{A}_0 e^{\lambda L\hat{\Phi}}(\vec{q}) \\ &= \lambda L\sum_{k=1}^{N} v_{\alpha_k}\frac{1}{L}\sum_{u=0}^{L-1}\left[\varphi'\left(\sum_{j=1}^{N}\zeta(L(x_j - \frac{u}{L}))\right)L\zeta'(L(x_k - \frac{u}{L}))\right]. \end{aligned} \tag{2.25}$$

A straightforward calculation yields

$$|e^{-\lambda L\hat{\Phi}}\mathcal{A}_0 e^{\lambda L\hat{\Phi}}(\vec{q})| \leq C_0\lambda L^2 N\log N \leq CL^4 \tag{2.26}$$

for some constants C_0 and C.

Step 3. Partition $[0,T]$ into L^k subintervals, so that $0 = t_0 < t_1 < \ldots < t_{L^k} = T$, and each subinterval has length at most $\frac{T}{L^k}$. Then by (2.24),

$$E_L(\exp\hat{\pi}L\Phi(\vec{q}_t)) \leq e^{CL} \Rightarrow E_L\left(\max_{0\leq i\leq L^k-1}\exp\hat{\pi}L\hat{\Phi}(\vec{q}_{t_i})\right) \leq L^k e^{c_0CL}. \tag{2.27}$$

$$\begin{aligned} &E_L\left(\sup_{0\leq t\leq T}\exp\frac{\hat{\pi}}{2}L\hat{\Phi}(\vec{q}_t)\right) \\ &= E_L\left(\max_{0\leq i\leq L^k-1}\sup_{t\in[t_i,t_{i+1})}\exp(\frac{\hat{\pi}}{2}L\hat{\Phi}(\vec{q}_t) - \frac{\hat{\pi}}{2}L\hat{\Phi}(\vec{q}_{t_i}) + \frac{\hat{\pi}}{2}L\hat{\Phi}(\vec{q}_{t_i}))\right) \\ &\leq \left(E_L\left(\max_{0\leq i\leq L^k-1}\sup_{t\in[t_i,t_{i+1})}\exp(\hat{\pi}L\hat{\Phi}(\vec{q}_t) - \hat{\pi}L\hat{\Phi}(\vec{q}_{t_i}))\right)\right)^{\frac{1}{2}} \\ &\quad\cdot\left(E_L\left(\max_{0\leq i\leq L^k-1}\exp\hat{\pi}L\hat{\Phi}(\vec{q}_{t_i})\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{2.28}$$

By (2.26) and Lemma 2.4 we have

$$E_L\Big(\sup_{t\in[t_i,t_{i+1})}\exp(\hat{\pi}L\hat{\Phi}(\vec{q}_t)-\hat{\pi}L\hat{\Phi}(\vec{q}_{t_i}))\Big)\le e^{CL^4L^{-k}}, \tag{2.29}$$

and

$$E_L\Big(\max_{0\le i\le L^k-1}\sup_{t\in[t_i,t_{i+1})}\exp(\hat{\pi}L\hat{\Phi}(\vec{q}_t)-\hat{\pi}\hat{\Phi}(\vec{q}_{t_i}))\Big)$$
$$\le L^k e^{CL^4L^{-k}}\le \text{const.}L^k \text{ for } k\ge 4. \tag{2.30}$$

$$\therefore E_L\Big(\sup_{0\le s\le T}\exp\frac{\hat{\pi}}{2}L\hat{\Phi}(\vec{q}_s)\Big)\le(\text{const.})L^k e^{CL}\le e^{\bar{C}L} \text{ for some } \bar{C}>0. \tag{2.31}$$

This and (2.24) imply (2.6).

COROLLARY 2.1. – *For each $k\in\mathbb{N}$ $\exists$ a constant $C_k>0$ such that*

$$\sup_L E_L\Big(\sup_{0\le s\le T}\Phi(\vec{q}_s)\Big)^k\le C_k. \tag{2.32}$$

3. COLLISION BOUND

The key to the proof of any kinetic limit is that as the number of particles converges to infinity, the mean free path remains fixed and thus the collisions remain bounded. This is the content of the next theorem.

THEOREM 3.1. – *There exists a continuous function $C_1(T)$ with $C_1(0)=0$ such that whenever $v^*TL>1$,*

$$E_L\int_0^T\frac{1}{L}\sum_{i,j}V(L(x_i(t)-x_j(t))\mathbb{1}(v_{\alpha_i}(t)\ne v_{\alpha_j}(t))dt\le C_1(T), \tag{3.1}$$

where $v_\alpha(t)\equiv v_{\alpha(t)}$.

Remarks 1. – The following notation will be used freely in the sequel:

$$v^*:=\max_{\alpha\in I}|v_\alpha| \tag{3.2}$$

$$A_L(\vec{q}_t,\alpha,\beta): =\frac{1}{L}\sum_{i\ne j}V(L(x_i(t)-x_j(t)))\mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta) \tag{3.3}$$

$$\hat{A}_L(\vec{q}_t):=\frac{1}{L}\sum_{i\ne j}V(L(x_i(t)-x_j(t)))\mathbb{1}(v_{\alpha_i}(t)\ne v_{\alpha_j}(t)). \tag{3.4}$$

2. $E^{\vec{q}}$ denotes the expectation with respect to the process $\vec{q}_L(\cdot)$ when its initial distribution is concentrated upon the single configuration $\vec{q}$.

3. Since $A_L(\vec{q}_t, \alpha, \beta) \leq \hat{A}_L(\vec{q}_t)$, we see that (3.1) implies that

$$\sup_L E_L \int_0^T A_L(\vec{q}_t, \alpha, \beta) dt \leq C_1(T), \tag{3.5}$$

whenever $v_\alpha \neq v_\beta$.

4. We will sometimes write the coordinates of $\vec{q}$ as $\vec{q} = (x_1, \alpha_1, \ldots, x_N, \alpha_N)$ and at other times as $\vec{q} = (x_1, x_2, \ldots, x_N, \alpha_1, \alpha_2, \ldots, \alpha_N)$. Since we use only Roman letters to refer to location and only Greek letters to refer to labels, this permutation should cause little confusion. The advantage of the second notation is that it allows us to write $(x_1 + v_{\alpha_1}(t-s), \alpha_1, \ldots, x_N + v_{\alpha_N}(t-s), \alpha_N)$ more compactly as $(\vec{x} + \vec{v}_\alpha(t-s), \vec{\alpha})$.

We now continue with a lemma.

LEMMA 3.1. – *Let* $h(\delta) := |1 + \log \delta|^{-1}$ *and assume that* I *is any subinterval of* $[0, 1]$. *Then for every* $\delta \in (0, \frac{1}{e})$

$$\sup_{|I| < \frac{\delta}{2}} \frac{\mathcal{N}_t(I)}{L} \leq 3h_L(\delta) \left[\sup_{0 \leq s \leq T} \left(1 + \Phi(\vec{q}_s)\right)\right], \tag{3.6}$$

where $h_L(\delta) := h(\delta_L)$ *and* $\delta_L := \delta \vee \frac{4}{L}$.

Proof. – $\delta_L > \frac{4}{L} \Rightarrow \exists k$ such that $\frac{k-1}{L} \leq \frac{\delta}{2} < \frac{k}{L}$, $k \in \{3, 4, \ldots, L\}$; therefore

$$\sup_{|I| < \frac{\delta}{2}} \frac{\mathcal{N}_t(I)}{L} \leq \sup_{0 \leq i \leq L-1} \frac{1}{L} \sum_{j=0}^{k} a_{i+j,L}. \tag{3.7}$$

Now, $\forall l > e$ we have

$$\begin{aligned} \frac{1}{L} \sum_{j=0}^{k} a_{i+j,L} &= \frac{1}{L} \sum_{j=0}^{k} a_{i+j,L} \mathbb{1}(a_{i+j,L} \leq l) + \frac{1}{L} \sum_{j=0}^{k} a_{i+j,L} \mathbb{1}(a_{i+j,L} > l) \\ &\leq \frac{(k+1)l}{L} + \frac{1}{L} \sum_{j=0}^{k} a_{i+j,L} \left(\frac{\log a_{i+j,L} - 1}{\log a_{i+j,L} - 1} \right) \mathbb{1}(a_{i+j,L} > l) \\ &\leq \frac{(k+1)l}{L} + \frac{1}{\log \frac{l}{e}} \left(\frac{1}{L} \sum_{j=0}^{k} \varphi(a_{i+j,L}) \mathbb{1}(a_{i+j,L} > l) \right) \end{aligned} \tag{3.8}$$

$$\Rightarrow \sup_i \left(\frac{1}{L} \sum_{j=0}^{k} a_{i+j,L} \right)$$
$$\leq \inf_{l \geq 1} \left[\frac{e(k+1)l}{L} + \frac{1}{\log l} \left(\frac{1}{L} \sum_{i=0}^{L-1} \varphi(a_{i,L}) \right) \right]. \tag{3.9}$$

Now, for any $c \in (0,1)$ and $K > 0$, we have

$$\inf_{l \geq 1} \left(cl + \frac{K}{\log l} \right) \leq \inf_{l \geq 1} \left(cl + \frac{K+1}{\log l} \right)$$
$$\leq 3|\log c|^{-1}(K+1), \tag{3.10}$$

as can easily be seen by choosing $l = \frac{K+1}{c|\log c|}$. Therefore, since $\frac{k+1}{L} \leq \delta_L$, we have (3.6). The case $\delta_L = \frac{4}{L}$ can be treated likewise.

Proof of Theorem. 3.1.

Step 1. Let $f : [0,T] \times E \to \mathbb{R}$ be smooth, and suppose $t \geq 0$ is fixed. Define $g : [0,T] \times E \to \mathbb{R}$ by $g(s,\vec{q}) = f(s, \vec{x} + \vec{v}_\alpha(t-s), \vec{\alpha})$. We then have

$$\begin{aligned}
\frac{d}{ds} E^{\vec{q}} g(s, \vec{q}_s) &= \left[E^{\vec{q}} \left(\frac{\partial}{\partial s} g(s, \vec{q}_s) \right) \right] + \left[E^{\vec{q}} \left(\mathcal{A}^{(L)} g(s, \vec{q}_s) \right) \right] \\
&= \left[E^{\vec{q}} \left(\frac{\partial}{\partial s} f(s, \vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s) \right) \right. \\
&\quad \left. + E^{\vec{q}} \left(-\mathcal{A}_0 f(s, \vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s) \right) \right] \\
&\quad + \left[E^{\vec{q}} \left(\mathcal{A}_0 f(s, \vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s) \right) + E^{\vec{q}} \left(\mathcal{A}_c g(s, \vec{x}_s, \vec{\alpha}_s) \right) \right] \\
&= E^{\vec{q}} \left(\frac{\partial}{\partial s} f(s, \vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s) \right) + E^{\vec{q}} \left(\mathcal{A}_c g(s, \vec{x}_s, \vec{\alpha}_s) \right).
\end{aligned} \tag{3.11}$$

Therefore

$$\begin{aligned}
E^{\vec{q}} \int_c^T f(t, \vec{x}_t, \vec{\alpha}_t) dt =& E^{\vec{q}} \int_c^T f(c, \vec{x}_c + \vec{v}_{\alpha_c}(t-c), \vec{\alpha}_c) dt \\
&+ E^{\vec{q}} \int_c^T \int_c^t \mathcal{A}_c g(s, \vec{x}_s, \vec{\alpha}_s) ds dt \\
&+ E^{\vec{q}} \int_c^T \int_c^t \frac{\partial}{\partial s} f(s, \vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s) ds dt.
\end{aligned} \tag{3.12}$$

Step 2. In particular, if $c = 0$ and $f(t,\vec{q}) = \frac{1}{L}\sum_{i\neq j} J(x_i+z,t)V(L(x_i+z-x_j))1\!\!1(\alpha_i=\alpha,\alpha_j=\beta)$, where $J \in C_c^\infty(S^1\times[0,\infty))$, we then have

$$\begin{aligned}&E^{\vec{q}}\int_0^T \frac{1}{L}\sum_{i\neq j} J(x_i(t)+z,t)V(L(x_i(t)\\&\quad + z - x_j(t)))1\!\!1(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)dt\\&\quad = I(z,J)+II(z,J)+III(z,J)+IV(z,J)+V(z,J),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}I(z,J) = E^{\vec{q}}\int_0^T \frac{1}{L}\sum_{i\neq j}\Big(&J(x_i(0)+z+v_\alpha t,0)V(L(x_i(0)\\&+z-x_j(0)+(v_\alpha - v_\beta)t))1\!\!1(\alpha_i(0)=\alpha,\alpha_j(0)=\beta)\Big)dt,\end{aligned} \tag{3.14}$$

$$\begin{aligned}II(z,J) = {}& E^{\vec{q}}\int_0^T\int_0^t \frac{1}{L}\sum_{\substack{k\neq i,j\\ i\neq j}}\\&\times\Big(J(x_i(s)+z+v_\alpha(t-s),s)V(L(x_i(s)-x_k(s)))\\&\cdot\sum_{\gamma\delta} K(\alpha_i(s)\alpha_k(s),\gamma\delta)[V(L(x_i(s)+z-x_j(s)+(v_\alpha-v_\beta)(t-s)))\\&\cdot(1\!\!1(\gamma=\alpha)-1\!\!1(\alpha_i(s)=\alpha))1\!\!1(\alpha_j(s)=\beta)]\Big)dsdt,\end{aligned} \tag{3.15}$$

$$\begin{aligned}III(z,J) = {}& E^{\vec{q}}\int_0^T\int_0^t \frac{1}{L}\sum_{\substack{k\neq i,j\\ i\neq j}}\\&\times\Big(J(x_i(s)+z+v_\alpha(t-s),s)V(L(x_j(s)-x_k(s)))\\&\cdot\sum_{\gamma\delta} K(\alpha_j(s)\alpha_k(s),\gamma\delta)[V(L(x_i(s)+z-x_j(s)+(v_\alpha-v_\beta)(t-s)))\\&\cdot(1\!\!1(\delta=\beta)-1\!\!1(\alpha_j(s)=\beta))1\!\!1(\alpha_i(s)=\alpha)]\Big)dsdt,\end{aligned} \tag{3.16}$$

$$IV(z,J) = E^{\vec{q}} \int_0^T \int_0^t \frac{1}{L} \sum_{i \neq j}$$
$$\times \Bigg(J(x_i(s) + z + v_\alpha(t-s), s) V(L(x_i(s) - x_j(s)))$$
$$\cdot \sum_{\gamma\delta} K(\alpha_i(s)\alpha_j(s), \gamma\delta)[V(L(x_i(s) + z - x_j(s) + (v_\alpha - v_\beta)(t-s)))$$
$$\cdot (\mathbb{1}(\gamma = \alpha, \delta = \beta) - \mathbb{1}(\alpha_i(s) = \alpha, \alpha_j(s) = \beta))] \Bigg) ds dt, \tag{3.17}$$

$$V(z,J) = E^{\vec{q}} \int_0^T \int_0^t \frac{1}{L} \sum_{i \neq j}$$
$$\times \Bigg(\frac{\partial}{\partial s} J(x_i(s) + z + v_\alpha(t-s), s)$$
$$\cdot V(L(x_i(s) + z - x_j(s) + (v_\alpha - v_\beta)(t-s))) \mathbb{1}(\alpha_i(0)$$
$$= \alpha, \alpha_j(0) = \beta) \Bigg) ds dt. \tag{3.18}$$

Remarks. – 1. By the Optional Sampling Theorem, we have that (3.13) holds when T is replaced by a stopping time τ. By averaging over all configurations, we also have that (3.13) holds when $E^{\vec{q}}$ is replaced with E_L.

2. In order to show (3.1), it suffices to consider the case when $z = 0$ and $J \equiv 1$. Note that in this case the fifth term $V(0,1) = 0$. Later on, when we prove the spatial regularity of the collision, we will need to consider more general z and J, and we will then have to take greater care with $V(z,J)$.

3. By first summing over all α and β such that $v_\alpha \neq v_\beta$ and then by recalling that the total number of labels is finite, we obtain the following inequality:

$$E_L \int_0^T \frac{1}{L} \sum_{i \neq j} V(L(x_i(t) - x_j(t))) \mathbb{1}(v_{\alpha_i}(t)$$
$$\neq v_{\alpha_j}(t)) dt \leq \hat{I} + \widehat{II} + \widehat{III} + \widehat{IV}, \text{ where} \tag{3.19}$$

$$\hat{I} = E_L \int_0^T \frac{1}{L} \sum_{i \neq j} V(L(x_i(0) - x_j(0)$$
$$+ (v_{\alpha_i}(0) - v_{\alpha_j}(0))t)) \mathbb{1}(v_{\alpha_i}(0) \neq v_{\alpha_j}(0)) dt, \tag{3.20}$$

$$\begin{aligned}\widehat{II} =& E_L \int_0^T \int_0^t \frac{C}{L} \sum_{\substack{k\neq i,j\\ i\neq j}} \Big(V(L(x_i(s) - x_k(s)))1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))\\ &\cdot [V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t-s)))1\!\!1(v_{\alpha_i}(s)\\ &\neq v_{\alpha_j}(s))]\Big) dsdt, \end{aligned} \tag{3.21}$$

$$\begin{aligned}\widehat{III} =& E_L \int_0^T \int_0^t \frac{C}{L} \sum_{\substack{k\neq i,j\\ i\neq j}} \Big(V(L(x_j(s) - x_k(s)))1\!\!1(v_{\alpha_j}(s) \neq v_{\alpha_k}(s))\\ &\cdot [V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t-s)))1\!\!1(v_{\alpha_i}(s)\\ &\neq v_{\alpha_j}(s))]\Big) dsdt, \end{aligned} \tag{3.22}$$

$$\begin{aligned}\widehat{IV} =& E_L \int_0^T \int_0^t \frac{C}{L} \sum_{i\neq j} \Big(V(L(x_i(s) - x_j(s)))1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_j}(s))\\ &\cdot [V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))\\ &\times (t-s)))1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_j}(s))]\Big) dsdt, \end{aligned} \tag{3.23}$$

Step 3. We begin with $\hat{I}$. By making the change of variables $z := x_j(0) - (v_{\alpha_i}(0) - v_{\alpha_j}(0))t$, we obtain

$$\begin{aligned}\hat{I} &\leq \frac{C}{L} \sum_{i=1}^{N} E_L \int V(L(x_i(0) - z)) \Big(\sum_{j=1}^{N} 1\!\!1(|x_j(0) - z| \leq 2Tv^*) \Big) dz\\ &\leq \frac{C}{L} \sum_{i=1}^{N} E_L \Big(\sup_{|I|\leq 2Tv^*} \frac{\mathcal{N}_0(I)}{L} \Big) \int LV(L(x_i(0) - z)) dz \end{aligned} \tag{3.24}$$

From this, we gather that if $2v^*T \leq \frac{1}{e}$,

$$\begin{aligned}\hat{I} &\leq \frac{CN}{L} E_L \Big(\sup_{|I|\leq 2v^*T} \frac{\mathcal{N}_0(I)}{L} \Big)\\ &\leq Ch_L(4v^*T) E_L \Big(\sup_{0\leq s\leq T} (1 + \Phi(\vec{q}_s)) \Big) \leq Ch_L(4Tv^*) \end{aligned} \tag{3.25}$$

The second inequality follows from Lemma 3.1, while the third is a consequence of Corollary 2.1.

Step 4. We now consider $\widehat{II}$. First, interchange the s-integral with the t-integral and then isolate the t-integral, so that we have

$$\widehat{II} = CE_L \int_0^T \frac{1}{L} \sum_{i \neq k} V(L(x_i(s) - x_k(s))) \mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot \left[\int_s^T \sum_{j \neq i,k} V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t - s))) \right.$$
$$\left. \mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_j}(s)) dt \right] ds. \tag{3.26}$$

We concentrate upon the t-integral. Make the change of variables $z = x_j(s) - (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t - s)$. From this we see that the t-integral is bounded by

$$C \int V(L(x_i(s) - z)) \Big(\sum_j \mathbb{1}(|x_j(s) - z| \leq 2v^* T) \Big) dz$$
$$\leq C \int LV(L(x_i(s) - z)) \Big(\sup_{|I| \leq 2v^* T} \frac{\mathcal{N}_s(I)}{L} \Big) dz. \tag{3.27}$$

Therefore, we have

$$\widehat{II} \leq CE_L \int_0^T \hat{A}_L(\vec{q}_s) \Big(\sup_{|I| \leq 2v^* T} \frac{\mathcal{N}_s(I)}{L} \Big) ds. \tag{3.28}$$

Step 5. Observe that $\widehat{III} = \widehat{II}$. As for $\widehat{IV}$, note that

$$\widehat{IV} = CE_L \int_0^T \int_s^T \frac{1}{L} \sum_{i \neq j} \Big(V(L(x_i(s) - x_j(s))) \mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_j}(s))$$
$$\cdot V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t - s))) \Big) dt ds$$
$$= CE_L \int_0^T \frac{1}{L} \sum_{i \neq j} \Big(V(L(x_i(s) - x_j(s))) \mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_j}(s))$$
$$\cdot \left[\int_s^T V(L(x_i(s) - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t - s))) dt \right] \Big) ds. \tag{3.29}$$

Since $v_{\alpha_i} \neq v_{\alpha_j}$, it is clear that the factor in brackets is of order $\frac{1}{L}$; thus

$$\widehat{IV} \leq \frac{C}{L}\left(E_L \int_0^T \frac{1}{L}\sum_{i\neq j} V(L(x_i(s)-x_j(s)))1\!\!1(v_{\alpha_i}(s)\neq v_{\alpha_j}(s))ds\right). \tag{3.30}$$

Final Step. After putting the various pieces together, we see that we have (for some constant $C_1 > 0$)

$$E_L \int_0^T \hat{A}_L(\vec{q}_t)dt \leq C_1 h_L(4v^*T) + C_1 E_L \int_0^T \hat{A}_L(\vec{q}_t)\left(\sup_{|I|\leq 2Tv^*} \frac{\mathcal{N}_t(I)}{L}\right)dt. \tag{3.31}$$

Since (3.13) holds whenever T is replaced by a stopping time, a similar argument shows that

$$E_L \int_0^\tau \hat{A}_L(\vec{q}_t)dt \leq C_1 h_L(4v^*T) + C_1 E_L \int_0^\tau A_L(\vec{q}_t)\left(\sup_{|I|\leq 2v^*\tau} \frac{\mathcal{N}_t(I)}{L}\right)dt, \tag{3.32}$$

where τ is the stopping time

$$\tau := T \wedge \inf\left\{t : 3C_1 h_L(4v^*t)\left(\sup_{0\leq s\leq t}(1+\Phi(\vec{q}_s))\right) \geq \frac{1}{2}, 4v^*T \leq \frac{2}{e}\right\}. \tag{3.33}$$

Now

$$\begin{aligned} &E_L \int_0^\tau \hat{A}_L(\vec{q}_t)C_1\left(\sup_{|I|\leq 2v^*\tau} \frac{\mathcal{N}_t(I)}{L}\right)dt \\ &\leq E_L \int_0^\tau \hat{A}_L(\vec{q}_t)\left(3C_1 h_L(4v^*\tau)\left(\sup_{0\leq s\leq \tau}(1+\Phi(\vec{q}_s))\right)\right)dt \\ &\leq \frac{1}{2}E_L \int_0^\tau \hat{A}_L(\vec{q}_t)dt; \end{aligned} \tag{3.34}$$

therefore,

$$E_L \int_0^\tau \hat{A}_L(\vec{q}_t)dt \leq 2C_1 h_L(4v^*T). \tag{3.35}$$

We take advantage of this by noting that

$$\begin{aligned} E_L \int_0^T \hat{A}_L(\vec{q}_t)dt &= E_L \int_0^T \hat{A}_L(\vec{q}_t)1\!\!1(\tau = T)dt \\ &\quad + E_L \int_0^T \hat{A}_L(\vec{q}_t)1\!\!1(\tau < T)dt \\ &\leq E_L \int_0^\tau \hat{A}_L(\vec{q}_t)dt + CLTP_L(\tau < T) \\ &\leq 2C_1 h_L(4v^*T) + CLTP_L(\tau < T). \end{aligned} \tag{3.36}$$

We now need to show that the second term is negligible for sufficiently small T. Suppose $4v^*T \le \frac{2}{e}$.

$$\begin{aligned}
P_L(\tau < T) &= P_L\left(\frac{1}{2} \le 3C_1 h_L(4v^*T)\left(\sup_{0\le s\le T}(1+\Phi(\vec{q}_s))\right)\right)\\
&= P_L\left(\frac{\hat{\pi}L}{12C_1 h_L(4v^*T)} \le \sup_{0\le s\le T}\frac{\hat{\pi}L}{2}(1+\Phi(\vec{q}_s))\right)\\
&\le \exp\left(\frac{-\hat{\pi}L}{12C_1 h_L(4v^*T)}\right)\exp{(\bar{C}+\frac{\hat{\pi}}{2})L} \qquad (3.37)
\end{aligned}$$

by Chebyshev's inequality and Theorem 2.1. We choose T_0 sufficiently small so that $2(\bar{C}+\frac{\hat{\pi}}{2}) < \frac{\hat{\pi}}{12C_1 h(4v^*T_0)}$. For large L we may choose T_0 so that $4v^*T_0 > \frac{4}{L}$. This implies that

$$P_L(\tau < T_0) \le e^{-\bar{C}L}. \qquad (3.38)$$

This finishes the proof for $T \le T_0$. For $T > T_0$, we bootstrap; *i.e.* we run the process up to time T_0, note that the distribution at time T_0 satisfies (1.8(iii)), and observe that our collision bound argument now works for $T \in [T_0, 2T_0]$ (and hence $\forall T \ge 0$). We define $C_1(T)$ as follows:

$$C_1(T) := \begin{cases} 3C_1 h(4v^*T) + C_0 T & T \le T_0 \\ \\ kC_1(T_0) + 3C_1 h(4v^*(T-kT_0)) & kT_0 \le T < (k+1)T_0. \end{cases}$$

where C_0 is an upper bound for $CLe^{-\bar{C}L}$.

COROLLARY 3.1

$$\sup_L E_L\left(\int_0^T \hat{A}_L(\vec{q}_t)dt\right)^k \le C_k(T)\,\forall k \in \mathbb{N}, \qquad (3.39)$$

whenever $v^*TL > 1$, *where* $C_k(T)$ *is a continuous function of* T *with* $C_k(0) = 0$.

Proof. We prove (3.39) inductively. Suppose (3.39) holds for all integers p, $1 \le p \le k$. Then the Markov property implies that

$$\begin{aligned}
&E_L\left(\int_0^T \hat{A}_L(\vec{q}_t)dt\right)^{k+1}\\
&= (k+1)!E_L \int\!\!\cdots\!\!\int_{0\le s_1\le\cdots\le s_{k+1}\le T} \prod_{i=1}^{k+1}\hat{A}_L(\vec{q}_{s_i})ds_1\ldots ds_{k+1}
\end{aligned}$$

$$= (k+1)!E_L \int\cdots\int_{0\le s_1\le\cdots\le s_k\le T} \prod_{i=1}^{k} \hat{A}_L(\vec{q}_{s_i}) \\ \times \left(E^{\vec{q}_{s_k}} \int_0^{T-s_k} \hat{A}_L(\vec{q}_s)ds\right) ds_1 \dots ds_k \tag{3.40}$$

Recalling that $\frac{N}{L} = M$, we have that (3.40) is bounded by

$$(k+1)!E_L \int\cdots\int_{0\le s_1\le\cdots\le s_k\le T} \prod_{i=1}^{k} \hat{A}_L(\vec{q}_{s_i}) \bigg[E^{\vec{q}_{s_k}} C_1 M \bigg(\sup_{|I|\le 2v^*T} \frac{\mathcal{N}_0(I)}{L}\bigg) \\ + C_1 E^{\vec{q}_{s_k}} \int_0^{T-s_k} \hat{A}_L(\vec{q}_s)\bigg(\sup_{|I|\le 2v^*(T-s_k)} \frac{\mathcal{N}_s(I)}{L}\bigg) ds\bigg] ds_1 \dots ds_k \\ \le C_1(k+1)M^2 C_k(T) \\ + E_L\bigg[3C_1 h(4v^*T)\bigg(\sup_{0\le t\le T}(1+\Phi(\vec{q}_t))\bigg)\bigg(\int_0^T \hat{A}_L(\vec{q}_s)ds\bigg)^{k+1}\bigg], \tag{3.41}$$

where the last inequality is a consequence of the induction hypothesis and Lemma 3.1. The remainder of the proof mimics the last step in the collision bound.

Define τ as in (3.33); we then have

$$E_L\bigg(\int_0^T \hat{A}_L(\vec{q}_s)ds\bigg)^{k+1} \\ = E_L\bigg(\int_0^T \hat{A}_L(\vec{q}_s)1\!\!1(\tau = T)ds + \int_0^T \hat{A}_L(\vec{q}_s)1\!\!1(\tau < T)ds\bigg)^{k+1} \\ \le 2^k E_L\bigg(\int_0^T \hat{A}_L(\vec{q}_s)1\!\!1(\tau = T)ds\bigg)^{k+1} \\ + 2^k E_L\bigg(\int_0^T \hat{A}_L(\vec{q}_s)1\!\!1(\tau < T)ds\bigg)^{k+1} \\ \le 2^k E_L\bigg(\int_0^\tau \hat{A}_L(\vec{q}_s)ds\bigg)^{k+1} \\ + C(LT)^{k+1} P_L(\tau < T). \tag{3.42}$$

By (3.38), the second term is negligible. As for the first term, we have

$$E_L\left(\int_0^\tau \hat{A}_L(\vec{q}_s)ds\right)^{k+1} \leq CC_k(\tau)(k+1)$$
$$+ E_L\left[3C_1 h(4v^*T)\left(\sup_{0\leq t\leq \tau}(1+\Phi(\vec{q}_t))\right)\right.$$
$$\left.\times\left(\int_0^\tau \hat{A}_L(\vec{q}_s)ds\right)^{k+1}\right] \tag{3.43}$$

and thus

$$E_L\left(\int_0^\tau \hat{A}_L(\vec{q}_s)ds\right)^{k+1} \leq C_{k+1}(T). \tag{3.44}$$

We combine (3.42) and (3.44) to get our result.

COROLLARY 3.2. – *Let σ denote a stopping time taking values in the interval $[0,T]$. Suppose $v^*TL > 1$. We have that*

$$\sup_{0\leq\sigma\leq T} E_L \int_\sigma^{T+\sigma} \hat{A}_L(\vec{q}_t)dt \leq \tilde{C}_1(T), \tag{3.45}$$

where $\tilde{C}_1(T) \to 0$ as $T \to 0$.

Proof. – The Strong Markov Property and a repetition of the proof of Theorem 3.1 give us that

$$E_L\int_\sigma^{T+\sigma} \hat{A}_L(\vec{q}_t)dt = E_L E^{\vec{q}_\sigma}\int_0^T \hat{A}_L(\vec{q}_t)dt$$
$$\leq \frac{CN}{L}E_L E^{\vec{q}_\sigma}\left(\sup_{|I|\leq 2v^*T}\frac{\mathcal{N}_0(I)}{L}\right)$$
$$+ CE_L E^{\vec{q}_\sigma}\int_0^T \hat{A}_L(\vec{q}_t)\left(\sup_{|I|\leq 2v^*T}\frac{\mathcal{N}_t(I)}{L}\right)dt$$

and thus

$$E_L\int_\sigma^{T+\sigma} \hat{A}_L(\vec{q}_t)dt \leq Ch(4v^*T)E_L\left(\sup_{0\leq s\leq T}(1+\Phi(\vec{q}_s))\right)$$
$$+ CM\left(\int_0^T \hat{A}_L(\vec{q}_t)dt\right) \leq \tilde{C}_1(T). \tag{3.46}$$

4. SPATIAL REGULARITY OF COLLISION

In order to carry out the sort of averaging necessary to prove the kinetic limit, we must be sure that shifting particles around somewhat does not dramatically alter the value of the collision. The next theorem guarantees that this does not happen.

THEOREM 4.1. – *Suppose $|z| < \varepsilon$, J is smooth, and $v_\alpha \neq v_\beta$. Then for every L*

$$\begin{aligned} E_L \Bigg| \int_0^T \frac{1}{L} \sum_{i \neq j} \Big[& V(L(x_i(t) + z - x_j(t)))J(x_i(t) + z, t) \\ & - V(L(x_i(t) - x_j(t)))J(x_i(t), t) \Big] 1\!\!1(\alpha_i(t) = \alpha, \alpha_j(t) = \beta) dt \Bigg| \\ \leq C(T, J) \sqrt{h_L(4\varepsilon)}. \end{aligned} \tag{4.1}$$

for some constant $C(T, J)$. (See Lemma 3.1 for the definition of h_L.)

Proof. – We begin by first proving a lemma whose statement resembles (4.1) except that the expectation is inside the absolute value. Moving the expectation inside allows us to employ the identity (3.12) for an appropriate function $f : [0, T] \times E \to \mathbb{R}$.

LEMMA 4.1. – *Suppose the conditions above hold. Then there exists a constant $C'(T, J)$ such that*

$$\begin{aligned} \Bigg| E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq j} \Big(\Big[& V(L(x_i(t) + z - x_j(t)))J(x_i(t) + z, t) \\ & - V(L(x_i(t) - x_j(t)))J(x_i(t), t) \Big] 1\!\!1(\alpha_i(t) = \alpha, \alpha_j(t) = \beta) \Big) dt \Bigg| \\ \leq C'(T, J) h_L(4\varepsilon) E^{\vec{q}} & \left[\left(\sup_{0 \leq s \leq T} (1 + \Phi(\vec{q}_s)) \right) \left(1 + \int_0^T \hat{A}_L(\vec{q}_t) dt \right) \right] \end{aligned} \tag{4.2}$$

Proof. – Let

$$\begin{aligned} f(t, \vec{x}(t), \vec{\alpha}(t)) = \frac{1}{L} \sum_{i \neq j} \Big[& V(L(x_i(t) + z - x_j(t)))J(x_i(t) + z, t) \\ & - V(L(x_i(t) - x_j(t)))J(x_i(t), t) \Big] 1\!\!1(\alpha_i(t) = \alpha, \alpha_j(t) = \beta). \end{aligned} \tag{4.3}$$

Substituting into (3.12), we obtain

$$E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq j} \Big[V(L(x_i(t) + z - x_j(t))) J(x_i(t) + z, t)$$
$$- V(L(x_i(t) - x_j(t))) J(x_i(t), t) \Big] \mathbb{1}(\alpha_i(t) = \alpha, \alpha_j(t) = \beta) dt$$
$$= \Delta_z I + \Delta_z II + \Delta_z III + \Delta_z IV + \Delta_z V, \tag{4.4}$$

where $\Delta_z I := \Big(I(z, J) - I(0, J)\Big)$ and $I(z, J), ..., V(z, J)$ are defined by (3.14),..., (3.18). $\Delta_z II, ..., \Delta_z V$ are defined similarly. Note that $\Delta_z III$ is comparable to $\Delta_z II$ and $\Delta_z IV$ is comparable to $\frac{1}{L}\Delta_z II$, so it suffices to bound $\Delta_z I$, $\Delta_z II$, and $\Delta_z V$. We also remark that we may assume $\frac{z}{v_\alpha - v_\beta} > 0$ without loss of generality.

We first consider $\Delta_z I$. We initially make the shift $t \mapsto t + \frac{z}{v_\alpha - v_\beta}$ to observe that

$$I(z, J) = E^{\vec{q}} \int_{\frac{z}{v_\alpha - v_\beta}}^{T + \frac{z}{v_\alpha - v_\beta}} \frac{1}{L} \sum_{i \neq j} \Bigg(V(L(x_i(0) - x_j(0) + (v_\alpha - v_\beta)t))$$
$$\cdot J\Bigg(x_i(0) + v_\alpha t - \frac{v_\beta z}{v_\alpha - v_\beta}, 0 \Bigg) \mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta) \Bigg) dt \tag{4.5}$$

Replace $J\Big(x_i(0) + v_\alpha t - \frac{v_\beta z}{v_\alpha - v_\beta}, 0\Big)$ with $J(x_i(0) + v_\alpha t, 0)$. The error generated by the replacement is bounded by

$$C\varepsilon \left\| \frac{\partial J}{\partial x} \right\|_\infty E^{\vec{q}} \int_{\frac{z}{v_\alpha - v_\beta}}^{T + \frac{z}{v_\alpha - v_\beta}} \frac{1}{L} \sum_{i \neq j} V(L(x_i(0) - x_j(0)$$
$$+ (v_\alpha - v_\beta)t)) \mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta) dt, \tag{4.6}$$

and so showing that the error is $O(\varepsilon)$ is equivalent to showing that

$$E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq j} V(L(x_i(0) + z - x_j(0)$$
$$+ (v_\alpha - v_\beta)t)) \mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta) dt \tag{4.7}$$

is bounded. First, let $y = x_j(0) - (v_\alpha - v_\beta)t$. Then (4.7) is bounded by a constant multiple of

$$E^{\vec{q}} \int \frac{1}{L} \sum_i LV(L(x_i(0) + z - y))$$
$$\times \Bigg(\frac{1}{L} \sum_j \mathbb{1}(|x_j(0) - y| \leq 2v^* T) \Bigg) dy \leq \left(\frac{N}{L} \right)^2 = M^2.$$

Therefore, we conclude that

$$\begin{aligned}\left|\Delta_z I\right| = \Bigg| E^{\vec{q}} \int_T^{T+\frac{z}{v_\alpha - v_\beta}} \frac{1}{L}\sum_{i\neq j}\Big(& V(L(x_i(0) - x_j(0) + (v_\alpha - v_\beta)t)) \\ & \cdot J(x_i(0) + v_\alpha t, 0)\mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta)\Big) dt \\ - E^{\vec{q}} \int_0^{\frac{z}{v_\alpha - v_\beta}} \frac{1}{L}\sum_{i\neq j}\Big(& V(L(x_i(0) - x_j(0) + (v_\alpha - v_\beta)t)) \\ & \cdot J(x_i(0) + v_\alpha t, 0)\mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta)\Big) dt\Bigg| + O(\varepsilon),\end{aligned} \tag{4.8}$$

and thus

$$\begin{aligned}\left|\Delta_z I\right| \leq \|J\|_\infty E^{\vec{q}} \int_T^{T+\frac{z}{v_\alpha - v_\beta}} \frac{1}{L}\sum_{i\neq j}\Big(& V(L(x_i(0) - x_j(0) + (v_\alpha - v_\beta)t)) \\ & \cdot \mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta)\Big) dt\end{aligned} \tag{4.9}$$

$$\begin{aligned}+ \|J\|_\infty E^{\vec{q}} \int_0^{\frac{z}{v_\alpha - v_\beta}} \frac{1}{L}\sum_{i\neq j}\Big(& V(L(x_i(0) - x_j(0) + (v_\alpha - v_\beta)t)) \\ & \cdot \mathbb{1}(\alpha_i(0) = \alpha, \alpha_j(0) = \beta)\Big) dt + O(\varepsilon).\end{aligned} \tag{4.10}$$

We first consider (4.9). Make the change of variables $w = x_j - (v_\alpha - v_\beta)(t - T)$ to observe that (4.9) is bounded by

$$\begin{aligned} & C\|J\|_\infty E^{\vec{q}} \frac{1}{L}\sum_i \int LV(L(x_i(0) - w \\ & \quad + (v_\alpha - v_\beta)T))\left(\frac{1}{L}\sum_j \mathbb{1}(|x_j - w| \leq \varepsilon)\right)\frac{dw}{|v_\alpha - v_\beta|} \\ & \leq C\|J\|_\infty E^{\vec{q}} \frac{1}{L}\sum_i \int LV(L(x_i(0) \\ & \quad - w + (v_\alpha - v_\beta)T))\left(\sup_{|I|\leq 2\varepsilon} \frac{\mathcal{N}_0(I)}{L}\right) dw \\ & \leq CM\|J\|_\infty h_L(4\varepsilon) E^{\vec{q}}\left(\sup_{0\leq s\leq T}(1 + \Phi(\vec{q}_s))\right)\end{aligned} \tag{4.11}$$

where the last inequality follows from Lemma 3.1. We obtain the same bound for (4.10) by making the change of variables $\tilde{w} = x_j - (v_\alpha - v_\beta)t$.

We now bound $\Delta_z V$. Since the argument is essentially the same as the one given for $\Delta_z I$, we will give only a brief sketch. Concentrate initially upon $V(z, J)$. Interchange the s-integral and the t-integral and isolate the t-integral. Next, make the shift $t \mapsto t + \frac{z}{v_\alpha - v_\beta}$; we then have

$$V(z,J) = E^{\vec{q}} \int_0^T \int_{s+\frac{z}{v_\alpha - v_\beta}}^{T+\frac{z}{v_\alpha - v_\beta}} \frac{1}{L} \sum_{i \neq j}$$

$$\times \Bigg(V(L(x_i(s) - x_j(s) + (v_\alpha - v_\beta)(t-s)))$$

$$\cdot \frac{\partial}{\partial s} J(x_i(s) + v_\alpha(t-s) - \frac{v_\beta}{v_\alpha - v_\beta}, s) \mathbb{1}(\alpha_i(s) = \alpha, \alpha_j(s) = \beta) \Bigg) dt ds. \tag{4.12}$$

Replace $\frac{\partial}{\partial s} J(x_i(s) + v_\alpha(t-s) - \frac{v_\beta z}{v_\alpha - v_\beta}, s)$ with $\frac{\partial}{\partial s} J(x_i(s) + v_\alpha(t-s), s)$; this yields, as we argued before, an error that is $O(\varepsilon)$. This implies that

$$\left|\Delta_z V\right| \leq \left\| \frac{\partial J}{\partial s} \right\|_\infty E^{\vec{q}} \int_0^T \int_T^{T+\frac{z}{v_\alpha - v_\beta}} \frac{1}{L} \sum_{i \neq j}$$

$$\times \Bigg(V(L(x_i(s) - x_j(s) + (v_\alpha - v_\beta)(t-s)))$$

$$\cdot \mathbb{1}(\alpha_i(s) = \alpha, \alpha_j(s) = \beta) \Bigg) dt ds \tag{4.13}$$

$$+ \left\| \frac{\partial J}{\partial s} \right\|_\infty E^{\vec{q}} \int_0^T \int_s^{s+\frac{z}{v_\alpha - v_\beta}} \frac{1}{L} \sum_{i \neq j}$$

$$\times \Bigg(V(L(x_i(s) - x_j(s) + (v_\alpha - v_\beta)(t-s)))$$

$$\cdot \mathbb{1}(\alpha_i(s) = \alpha, \alpha_j(s) = \beta) \Bigg) dt ds + O(\varepsilon) \tag{4.14}$$

We make the change of variables $w = x_j - (v_\alpha - v_\beta)(t - T)$ in (4.13); from this we observe that (4.13) is bounded from above by

$$C \left\| \frac{\partial J}{\partial s} \right\|_\infty E^{\vec{q}} \int_0^T \frac{1}{L} \sum_i \int LV(L(x_i(s) - w$$

$$+ (v_\alpha - v_\beta)(T-s))) \left(\sup_{|I| \leq 2\varepsilon} \frac{\mathcal{N}_s(I)}{L} \right) dw ds$$

$$\leq CTM \left\| \frac{\partial J}{\partial s} \right\|_\infty h_L(4\varepsilon) E^{\vec{q}} \left(\sup_{0 \leq s \leq T} (1 + \Phi(\vec{q}_s)) \right). \tag{4.15}$$

By making the change of variables $\tilde{w} = x_j - (v_\alpha - v_\beta)t$, we see that (4.14) has the same bound as (4.13); therefore, all that is left to consider is $\Delta_z II$.

The first two steps are precisely the same as in the previous two cases. First make the shift $t \mapsto t + \frac{z}{v_\alpha - v_\beta}$, and then replace $J(x_i(s) + v_\alpha(t-s) - \frac{v_\beta z}{v_\alpha - v_\beta}, s)$ with $J(x_i(s) + v_\alpha(t-s), s)$. Now, however, we have to argue more carefully, since the error bound depends upon the collision bound; that is, the error is of the form $\varepsilon C(J)$ times

$$E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq k} V(L(x_i(s) - x_k(s))) 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot \sum_j \int_s^T V(L(x_i(s) + z - x_j(s) + (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t-s)))$$
$$1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_j}(s)) dt ds. \tag{4.16}$$

It is not hard to see that for each j the t-integral is of size $O(L^{-1})$. This in turn implies that the j-sum in (4.16) is bounded above by a constant multiple of M. Therefore, the error has the form

$$\varepsilon C(J) \int_0^T \hat{A}_L(\vec{q}_t) dt; \tag{4.17}$$

this clearly conforms to the statement of the lemma. Therefore, $|\Delta_z II|$ is bounded by sum of small term $O(\varepsilon)$ and

$$C\|J\|_\infty E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq k} V(L(x_i(s) - x_k(s))) 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot \int_T^{T + \frac{z}{v_{\alpha_i} - v_{\alpha_j}}} V(L(x_i(s) - x_j(s)$$
$$+ (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t-s))) 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_j}(s)) dt ds \tag{4.18}$$

$$+ C\|J\|_\infty E^{\vec{q}} \int_0^T \frac{1}{L} \sum_{i \neq k} V(L(x_i(s) - x_k(s))) 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot \int_s^{s + \frac{z}{v_{\alpha_i} - v_{\alpha_j}}} V(L(x_i(s) - x_j(s)$$
$$+ (v_{\alpha_i}(s) - v_{\alpha_j}(s))(t-s))) 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_j}(s)) dt ds. \tag{4.19}$$

By making the change of variables $w = x_j - (v_{\alpha_i} - v_{\alpha_j})(t - T)$, it is not hard to see that (4.18) is bounded by

$$C\|J\|_\infty h_L(4\varepsilon) E^{\vec{q}} \left[\left(\sup_{0 \leq s \leq T} (1 + \Phi(\vec{q}_s)) \right) \left(\int_0^T \hat{A}_L(\vec{q}_t) dt \right) \right]. \tag{4.20}$$

Similarly, the change of variables $\tilde{w} = x_j - (v_{\alpha_i} - v_{\alpha_j})(t-s)$ yields the same bound for (4.19). Finally, combining (4.11), (4.15), (4.17), and (4.20) finishes the proof of the lemma.

Proof of Theorem 4.1. – Denote the integrand of (4.1) by $R_L(\vec{q}_t)$, with this notation, we can restate the claim as

$$\sup_L E_L \left| \int_0^T R_L(\vec{q}_t) dt \right| \le C(T,J)\sqrt{h_L(4\varepsilon)} \tag{4.21}$$

$$\begin{aligned} E_L\left(\int_0^T R_L(\vec{q}_t)dt\right)^2 &= 2E_L \int_0^T \int_s^T R_L(\vec{q}_t)R_L(\vec{q}_s)dtds \\ &= 2E_L \int_0^T R_L(\vec{q}_s)\left(E^{\vec{q}_s}\int_0^{T-s} R_L(\vec{q}_t)dt\right)ds \\ &\le 2E_L \int_0^T |R_L(\vec{q}_s)| \left|E^{\vec{q}_s}\int_0^{T-s} R_L(\vec{q}_t)dt\right| ds \\ &\le C'(T,J)h_L(4\varepsilon)E_L \int_0^T |R_L(\vec{q}_s)| \\ &\left[E^{\vec{q}_s}\left[\left(\sup_{0\le s\le T-s}(1+\Phi(\vec{q}_s))\right)\cdot\left(1+\int_0^{T-s}\hat{A}_L(\vec{q}_t)dt\right)\right]\right]ds. \end{aligned} \tag{4.22}$$

We now note that, in addition to (3.1), we also have

$$\begin{aligned} \sup_L E_L \int_0^T \frac{1}{L} \sum_{i\ne j} V(L(x_i(t) + z - x_j(t))) \\ \mathbb{1}(\alpha_i(t) = \alpha, \alpha_j(t) = \beta)dt \le C_1(T); \end{aligned} \tag{4.23}$$

the proof of (4.23) is identical to the proof of (3.1). Given (4.23), we then clearly have that

$$\sup_L E_L \left(\int_0^T |R_L(\vec{q}_t)|dt\right)^k \le C'_k(T), \tag{4.24}$$

for some constant $C'_k(T)$. Therefore,

$$\begin{aligned} E_L\left(\int_0^T R_L(\vec{q}_t)dt\right)^2 &\le C\|J\|_\infty h_L(4\varepsilon)E_L \\ &\left[\left(\sup_{0\le s\le T}(1+\Phi(\vec{q}_s))\right)\cdot\left(1+\int_0^T \hat{A}_L(\vec{q}_s)ds\right)\left(\int_0^T |R_L(\vec{q}_t)|dt\right)\right] \\ &\le C(T,J)h_L(4\varepsilon) \end{aligned} \tag{4.25}$$

where for the last inequality we used Hölder's inequality, (4.24) and (2.32). (4.21) clearly follows.

5. UNIFORM INTEGRABILITY

We begin with some definitions.

$$|\Omega| \equiv m\{x \in \Omega\} := \text{ the Lebesgue measure of the set } \Omega. \tag{5.1}$$

$$\psi(x) := \begin{cases} x(\log x)^{\frac{1}{2}} & x \geq e \\ e & x < e. \end{cases} \tag{5.2}$$

$$\begin{aligned} X_t(x) := \int_0^t \sum_{i \neq j} & V(L(x_i(s) - x_j(s))) \\ & 1\!\!1(\alpha_i(s) = \alpha, \alpha_j(s) = \beta) V(L(x_i(s) - x - vs))ds \end{aligned} \tag{5.3}$$

$$\tau(x) := \inf\{t : X_t(x) \geq l\}, \text{ where } l > 0 \text{ is fixed.} \tag{5.4}$$

THEOREM 5.1. – *There exists a constant $\hat{C}_1(T)$ such that for every $v = v_\gamma \neq v_\alpha, v_\beta$,*

$$\sup_L E_L \int \psi(X_T(x))dx \leq \hat{C}_1(T). \tag{5.5}$$

The next lemma holds the key to verifying (5.5).

LEMMA 5.1. – *There exists a constant $\hat{C}_2(T)$ such that for every $v = v_\gamma \neq v_\alpha, v_\beta$,*

$$E_L \int (X_T(x) - X_{T \wedge \tau(x)}(x))dx \leq \hat{C}_2(T)((\log l)^{-1} + L^{-1}). \tag{5.6}$$

We prove Lemma 5.1 below and omit the rest since the remainder of the proof is essentially identical to the proof of Theorem 7.1 in [12]. To begin, we need to justify a particular adaptation of Lemma 3.1.

LEMMA 5.2. – *Suppose $f \in L^\infty(S^1)$ and $\|f\|_{L^1(S^1)} \leq 1$. We then have*

$$\int \sum_{j=1}^N V(L(x_j - w))f(w)dw \leq \|f\|_\infty C\bar{h}(\|f\|_{L^1(S^1)})(1 + \Phi(\vec{q})). \tag{5.7}$$

The similarity between (3.6) and (5.7) becomes clearer if we take $f(x) = 1\!\!1(x \in I)$, $|I| < \delta$. $\bar{h}$ appears in (5.7) instead of h because

$h(x)$ was used for $x \in (0, \frac{2}{e})$ only, and we now need a bounded function defined everywhere; therefore, we modify h as follows:

$$\bar{h}(x) = \begin{cases} 5h(x) & x \leq e^{-4} \\ 1 & x > e^{-4} \end{cases}$$

The function $\bar{h}$ is chosen so that the functions $\bar{h}^4$ and $\bar{h}^2$ are concave. Such property will be used in the proof of Theorem 5.1.

Proof. – Let

$$\begin{aligned}\mathcal{N}^{(L)}(w) = \sum_{j=1}^{N} V(L(x_j - w)), \text{ so that for } l > e, &\int \mathcal{N}^{(L)}(w) f(w) dw \\ = &\int \mathcal{N}^{(L)}(w) \mathbb{1}(\mathcal{N}^{(L)}(w) \leq l) f(w) dw \\ &+ \int \mathcal{N}^{(L)}(w) \mathbb{1}(\mathcal{N}^{(L)}(w) > l) f(w) dw \\ \leq &\, l \|f\|_1 + \frac{\|f\|_\infty}{\log l - 1} \int \varphi(\mathcal{N}^{(L)}(w)) dw \\ \leq &\, l \|f\|_1 + \frac{C \|f\|_\infty}{\log l} \left(\frac{1}{L} \sum_{u=0}^{L-1} \varphi(a_{u,L}) \right). \end{aligned} \tag{5.8}$$

$$\inf_{l \geq 1} \left(el \|f\|_1 \bar{h}(\|f\|_1)(1 + C\|f\|_\infty \Phi(\vec{q}) \right). \tag{5.9}$$

provided $\|f\|_1 \leq e^{-5}$. Finally note that the inequality (5.7) trivially holds for a suitable C if $\|f\|_1 > e^{-5}$.

Proof of Lemma 5.1. – Let $\sigma(x) := T \wedge \tau(x)$; $\tau(x)$ is a stopping time implies that $\sigma(x)$ is one as well. Note that if we denote the integrand of (5.3) by $g(s, x)$, then we can restate (5.6) in the following way:

$$E_L \int \int_{\sigma(x)}^{T} g(s, x) ds dx \leq \hat{C}_2(T)((\log l)^{-1} + L^{-1}). \tag{5.10}$$

This suggests that we again attempt to use (3.12). Let $f(\vec{x}, \vec{\alpha}_t, x)$ be

$$\sum_{i \neq j} V(L(x_i + z_1 - x_j - z_2)) \cdot \mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta) V(L(x_i + z_1 - x - vt)), \tag{5.11}$$

where v is different from both v_α and v_β. (Note that if $z_1 = z_2 = 0$, then $g(t, x) \equiv f(\vec{x}_t, \vec{\alpha}_t, x)$.)

Claim.

$$
\begin{aligned}
&E_L \int \int_{\sigma(x)}^{T} f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \\
&= E_L \int \int_{\sigma(x)}^{T} f(\vec{x}_{\alpha(x)} + \vec{v}_{\alpha_{\sigma(x)}}(t - \sigma(x)), \vec{\alpha}_{\sigma(x)}, x) dt dx \\
&+ E_L \int \int_{\sigma(x)}^{T} \int_{\sigma(x)}^{t} \mathcal{A}_c f(\vec{x}_s + \vec{v}_{\alpha_s}(t-s), \vec{\alpha}_s, x) ds dt dx. \qquad (5.12)
\end{aligned}
$$

If $\sigma(x)$ were constant, then the claim would follow immediately from (3.12). We can readily show (5.12) holds for discrete stopping times by an application of the Markov property, and (5.12) follows for arbitrary stopping times by approximation with discrete stopping times. So, in particular, we have

$$
\begin{aligned}
&E_L \int \int_{\sigma(x)}^{T} \sum_{i \neq j} \Big(V(L(x_i(t) + z_1 - x_j(t) - z_2)) \\
&\cdot 1\!\!1(\alpha_i(t) = \alpha, \alpha_j(t) = \beta) V(L(x_i(t) + z_1 - x - vt)) \Big) \\
&\times dt dx = I + II + III + IV, \qquad (5.13)
\end{aligned}
$$

where

$$
\begin{aligned}
I = &E_L \int \int_{\sigma(x)}^{T} \\
&\sum_{i \neq j} \Big(V(L(x_i(\sigma(x)) + z_1 - x_j(\sigma(x)) - z_2 + (v_\alpha - v_\beta)(t - \sigma(x)))) \\
&\cdot V(L(x_i(\sigma(x)) + z_1 + v_\alpha(t - \sigma(x)) - x - vt)) \\
&\cdot 1\!\!1(\alpha_i(\sigma(x)) = \alpha, \alpha_j(\sigma(x)) = \beta) \Big) dt dx \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
II = &E_L \int \int_{\sigma(x)}^{T} \int_{\sigma(x)}^{t} \sum_{\substack{k \neq i,j \\ i \neq j}} \\
&\times \Big(V(L(x_i(s) - x_k(s))) \sum_{\gamma\delta} K(\alpha_i(s)\alpha_k(s), \gamma\delta) \\
&\cdot \Big[(1\!\!1(\gamma = \alpha) - 1\!\!1(\alpha_i(s) = \alpha)) 1\!\!1(\alpha_j(s) = \beta) \Big] V(L(x_i(s) \\
&+ z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta)(t - s))) \\
&\cdot V(L(x_i(s) + z_1 + v_\alpha(t - s) - x - vt)) \Big) ds dt dx \qquad (5.15)
\end{aligned}
$$

$$III = E_L \int \int_{\sigma(x)}^{T} \int_{\sigma(x)}^{t} \sum_{\substack{k \neq i,j \\ i \neq j}}$$
$$\times \Bigg(V(L(x_j(s) - x_k(s))) \sum_{\gamma\delta} K(\alpha_j(s)\alpha_k(s), \gamma\delta)$$
$$\cdot \Big[(\mathbb{1}(\delta = \beta) - \mathbb{1}(\alpha_j(s) = \beta))\mathbb{1}(\alpha_i(s) = \alpha) \Big] V(L(x_i(s)$$
$$+ z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta)(t - s)))$$
$$\cdot V(L(x_i(s) + z_1 + v_\alpha(t - s) - x - vt)) \Bigg) dsdtdx \tag{5.16}$$

$$IV = E_L \int \int_{\sigma(x)}^{T} \int_{\sigma(x)}^{t} \sum_{i \neq j}$$
$$\times \Bigg(V(L(x_i(s) - x_j(s))) \sum_{\gamma\delta} K(\alpha_i(s)\alpha_j(s), \gamma\delta)$$
$$\cdot V(L(x_i(s) + z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta)(t - s)))V(L(x_i(s) + z_1$$
$$+ v_\alpha(t - s) - x - vt))$$
$$\cdot \Big[\mathbb{1}(\gamma = \alpha, \delta = \beta) - \mathbb{1}(\alpha_i(s) = \alpha, \alpha_j(s) = \beta) \Big] \Bigg) dsdtdx. \tag{5.17}$$

As before, it suffices to bound I and II, since III is comparable to II and IV is comparable to $\frac{1}{L} \cdot II$. We first bound II. Define $\Omega = \Omega_{L,l} = \{x : X_T(x) \geq l\} = \{x : \tau(x) \leq T\}$. Since $x \notin \Omega \Rightarrow \sigma(x) = T$, we have

$$II \leq CE_L \int \int_{\sigma(x)}^{T} \int_{\sigma(x)}^{t} \sum_{\substack{k \neq i,j \\ i \neq j}}$$
$$\times \Bigg(V(L(x_i(s) - x_k(s)))\mathbb{1}(x \in \Omega)\mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot V(L(x_i(s) + z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta)(t - s)))$$
$$\cdot V(L(x_i(s) + z_1 + v_\alpha(t - s) - x - vt)) \Bigg) dsdtdx$$

$$\leq CE_L \int \int_{0}^{T} \int_{0}^{t} \sum_{\substack{k \neq i,j \\ i \neq j}}$$
$$\Bigg(V(L(x_i(s) - x_k(s)))\mathbb{1}(x \in \Omega)\mathbb{1}(v_{\alpha_i}(s) \neq v_{\alpha_k}(s))$$
$$\cdot V(L(x_i(s) + z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta)(t - s)))$$
$$\cdot V(L(x_i(s) + z_1 + v_\alpha(t - s) - x - vt)) \Bigg) dsdtdx \tag{5.18}$$

Make the substitution $y = x + vt - v_\alpha(t-s) - z_1$ and then change the order of integration for s and t to obtain

$$\begin{aligned} II \leq & CE_L \int \int_0^T \\ & \times \sum_{i \neq k} V(L(x_i(s) - x_k(s)))V(L(x_i(s) - y))1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s)) \\ & \times \left(\int_s^T \sum_j V(L(x_i(s) + z_1 - x_j(s) - z_2 + (v_\alpha - v_\beta) \right. \\ & \left. \times (t-s)))1\!\!1_\Omega(y - vt + v_\alpha(t-s) + z_1)dt \right) dsdy. \end{aligned} \tag{5.19}$$

We concentrate on the t-integral.

Let $w = x_i(s) + z_1 - z_2 + (v_\alpha - v_\beta)(t-s)$. With this substitution, the t-integral has the form

$$C \int \sum_j V(L(x_j(s) - w))p(w)dw, \tag{5.20}$$

where

$$p(w) = 1\!\!1_\Omega \left(y + (v_\alpha - v) \left(\frac{w - x_i(s) - z_1 + z_2}{v_\alpha - v_\beta} \right) - vs + z_1 \right).$$

Since

$$||p||_{L^1(S^1)} = \left| \frac{v_\alpha - v_\beta}{v_\alpha - v} \right| |\Omega|,$$

we know that the integral in (5.19) is bounded above by a constant multiple of

$$\bar{h}(|\Omega|) \left(\sup_{0 \leq t \leq T} (1 + \Phi(\vec{q}_t)) \right)$$

by an application of Lemma 5.2. Therefore

$$\begin{aligned} II \leq & CE_L \left[\bar{h}(|\Omega|) \left(\sup_{0 \leq t \leq T} (1 + \Phi(\vec{q}_t)) \right) \int \int_0^T \sum_{i \neq k} \left(V(L(x_i(s) - x_k(s))) \right. \right. \\ & \left. \left. \cdot 1\!\!1(v_{\alpha_i}(s) \neq v_{\alpha_k}(s)) V(L(x_i(s) - y)) \right) dsdy \right] \\ \leq & CE_L \left[\bar{h}(|\Omega|) \left(\sup_{0 \leq t \leq T} (1 + \Phi(\vec{q}_t)) \right) \left(\int_0^T \hat{A}_L(\vec{q}_t)dt \right) \right]. \end{aligned} \tag{5.21}$$

After applying Hölder's inequality and the corollaries (2.6) and (3.6), we see that

$$II \leq C(E_L \bar{h}^4(|\Omega|))^{\frac{1}{4}} \leq C\bar{h}(E_L|\Omega|). \tag{5.22}$$

(The last inequality follows from the concavity of $\bar{h}^4$ and Jensen's inequality.) Now

$$\begin{aligned}
E_L|\Omega| &= E_L(m\{x : X_T(x) \geq l\}) \leq E_L\frac{1}{l}\int X_T(x)dx \\
&\leq \frac{1}{l}E_L\int\int_0^T \sum_{i\neq j} V(L(x_i(s) - x_j(s)))\mathbb{1}(\alpha_i(s) \\
&= \alpha, \alpha_j(s) = \beta)V(L(x_i - x - vs))dsdx \\
&\leq \frac{1}{l}E_L\int_0^T \frac{1}{L}\sum_{i\neq j} V(L(x_i(s) - x_j(s)))\mathbb{1}(\alpha_i(s) \\
&= \alpha, \alpha_j(s) = \beta)ds \leq \frac{C_1(T)}{l}.
\end{aligned} \tag{5.23}$$

A problem arises when we try to bound I in a similar manner. Recall that in the course of bounding II we made the change of variable $y = x + vt - z_1 - v_\alpha(t - s)$; the corresponding change-of-variable in the case of I would be $y = x + vt - z_1 - v_\alpha t + v_\alpha\sigma(x)$, which clearly cannot work because of $\sigma(x)$. We must therefore find another method of proof.

$$\begin{aligned}
&E_L\int \mathbb{1}(\sigma(x) \leq \lambda)\int_\lambda^T f(\vec{x}_t, \vec{\alpha}_t, x)dtdx \\
&\quad = E_L\int \mathbb{1}(\sigma(x) \leq \lambda)\int_\lambda^T f(\vec{x}_\lambda + \vec{v}_{\alpha_\lambda}(t - \lambda), \vec{\alpha}_\lambda, x)dtdx \\
&\quad + E_L\int \mathbb{1}(\sigma(x) \leq \lambda)\int_\lambda^T\int_\lambda^t \mathcal{A}_c f(\vec{x}_s + \vec{v}_{\alpha_s}(t - s), \vec{\alpha}_s, x)dsdtdx \\
&\quad = I(\lambda) + II(\lambda).
\end{aligned} \tag{5.24}$$

Note that $\forall\lambda \in [0, T]$, we have

$$|II(\lambda)| \leq E_L\int\int_{\sigma(x)}^T\int_{\sigma(x)}^t |\mathcal{A}_c f(\vec{x}_s + \vec{v}_{\alpha_s}(t - s), \vec{\alpha}_s, x)|dsdtdx, \tag{5.25}$$

which we have already bounded. We therefore concentrate on $I(\lambda)$.

Note that the set $\{\sigma(x) \leq \lambda < T\} \subseteq \{x \in \Omega\}$; thus

$$\begin{aligned} I(\lambda) \leq & E_L \int \mathbb{1}(x \in \Omega) \int_\lambda^T \\ & \times \sum_{i \neq j} \Big(V(L(x_i(\lambda) + z_1 - x_j(\lambda) - z_2 + (v_\alpha - v_\beta)(t - \lambda))) \\ & \cdot \mathbb{1}(\alpha_i(\lambda) = \alpha, \alpha_j(\lambda) = \beta) V(L(x_i(\lambda) \\ & + z_1 + v_\alpha(t - \lambda) - x - vt)) \Big) dt dx. \end{aligned} \tag{5.26}$$

Now we can make the change of variables $y = x + vt - z_1 - v_\alpha(t - \lambda)$. Then

$$\begin{aligned} I(\lambda) \leq & E_L \int \sum_i V(L(x_i(\lambda) - y)) \mathbb{1}(\alpha_i(\lambda) = \alpha, \alpha_j(\lambda) = \beta) \\ & \times \left(\int_\lambda^T \mathbb{1}_\Omega (y - vt - z_1 - v_\alpha(t - \lambda)) \right. \\ & \left. \cdot \sum_j V(L(x_i(\lambda) + z_1 - x_j(\lambda) - z_2 + (v_\alpha - v_\beta)(t - \lambda))) dt \right) dy. \end{aligned} \tag{5.27}$$

Let $w = x_i + z_1 - z_2 + (v_\alpha - v_\beta)(t - \lambda)$. The integral in parentheses is then of the form

$$C \int \sum_j V(L(x_j - w)) p(w) dw,$$

and so by Lemma 5.2,

$$\begin{aligned} I(\lambda) & \leq C E_L \Big(\bar{h}(|\Omega|) \Big(\sup_{0 \leq s \leq T} (1 + \Phi(\vec{q}_s)) \Big) \Big) \\ & \leq C \Big(E_L (\bar{h}(|\Omega|))^2 \Big)^{\frac{1}{2}} \Big(E_L \Big(\sup_{0 \leq s \leq T} (1 + \Phi(\vec{q}_s)) \Big)^2 \Big)^{\frac{1}{2}} \\ & \leq C \bar{h}(E_L |\Omega|) \leq C |\log l|^{-1}. \end{aligned} \tag{5.28}$$

(5.28) holds uniformly in λ; therefore

$$\int \int_0^T \int_\lambda^T \mathbb{1}(\sigma(x) \leq \lambda) f(\vec{x}_t, \vec{\alpha}_t, x) dt d\lambda dx \leq CT |\log l|^{-1} =: C_l(T). \tag{5.29}$$

We can rewrite (5.29) as

$$E_L \int \int_{\sigma(x)}^T (t - \sigma(x)) f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \leq C_l(T). \tag{5.30}$$

This inequality holds forall T; in particular, it holds for $\tilde{T} = T + \delta$. We then have

$$E_L \int \int_{\sigma_\delta(x)}^{T+\delta} (t - \sigma_\delta(x)) f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \leq C_l(T + \delta), \tag{5.31}$$

where $\sigma_\delta(x) := \tau(x) \wedge (T + \delta)$. Equivalently,

$$E_L \int \mathbb{1}(\tau(x) \leq T + \delta) \int_{\tau(x)}^{T+\delta} (t - \tau(x)) f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \leq C_l(T + \delta). \tag{5.32}$$

Note that this implies that

$$\begin{aligned} &E_L \int \mathbb{1}(\tau(x) \leq T) \int_{\tau(x)+\delta}^{T+\delta} \delta f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \\ &\leq E_L \int \mathbb{1}(\tau(x) \leq T + \delta) \int_{\tau(x)}^{T+\delta} (t - \tau(x)) f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \leq C_l(T + \delta). \end{aligned} \tag{5.33}$$

Therefore we have

$$E_L \int \mathbb{1}(\tau(x) \leq T) \int_{\tau(x)+\delta}^{T+\delta} f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \leq \frac{C_l(T + \delta)}{\delta}. \tag{5.34}$$

$\sigma(x)$ is a stopping time implies that $\sigma(x) + \delta$ is a stopping time; thus

$$\begin{aligned} &E_L \int \int_{\sigma(x)+\delta}^{T+\delta} f(\vec{x}_t, \vec{\alpha}_t, x) dt dx \\ &= E_L \int \int_{\sigma(x)+\delta}^{T+\delta} f(\vec{x}_{\sigma(x)} + \vec{v}_{\alpha_{\sigma(x)}}(t - \sigma(x)), \vec{\alpha}_{\sigma(x)}, x) dt dx \\ &\quad + E_L \int \int_{\sigma(x)+\delta}^{T+\delta} \int_{\sigma(x)}^{t} \mathcal{A}_c f(\vec{x}_s + \vec{v}_{\alpha_s}(t - s), \vec{\alpha}_s, x) ds dt dx \\ &= I(\delta) + II(\delta) \leq \frac{C_l(T + \delta)}{\delta}. \end{aligned} \tag{5.35}$$

Since $|II(\delta)| \leq C_l(T + \delta)$, we necessarily have $I(\delta) \leq (1 + \delta^{-1}) C_l(T + \delta)$; *i.e.*

$$\begin{aligned} &E_L \int \int_{\sigma(x)+\delta}^{T+\delta} \sum_{i \neq j} \\ &\times \Big(V(L(x_i(\sigma(x)) - x_j(\sigma(x)) + (v_\alpha - v_\beta)(t - \sigma(x)) + z_1 - z_2)) \\ &\cdot \mathbb{1}(\alpha_i(\sigma(x)) = \alpha, \alpha_j(\sigma(x)) = \beta) V(L(x_i(\sigma(x)) \\ &+ z_1 + v_\alpha(t - \sigma(x)) - x - vt)) \Big) dt dx \leq \frac{C_l(T + \delta)}{\delta}. \end{aligned} \tag{5.36}$$

This holds uniformly in z_1 and z_2; therefore, let $z_1 := (v_\alpha - v)\delta$; let $z_2 := (2v_\alpha - v_\beta - v)\delta$; and let $\tilde{t} := t - \delta$; (5.36) then implies that

$$\begin{aligned}
E_L \int \int_{\sigma(x)}^{T} & \\
\sum_{i \neq j} \Bigg(& V(L(x_i(\sigma(x)) - x_j(\sigma(x)) + (v_\alpha - v_\beta)(\tilde{t} - \sigma(x)))) \\
& \cdot 1\!\!1(\alpha_i(\sigma(x)) = \alpha, \alpha_j(\sigma(x)) = \beta) V(L(x_i(\sigma(x)) \\
& + v_\alpha(\tilde{t} - \sigma(x)) - x - v\tilde{t})) \Bigg) d\tilde{t} dx \leq \frac{C_l(T+\delta)}{\delta}.
\end{aligned} \tag{5.37}$$

The left-hand side is I. This finishes the proof of Lemma 5.1.

6. THE KINETIC LIMIT

First, we need to make a few definitions.

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\Big(\frac{x}{\varepsilon}\Big), \tag{6.1}$$

where η is a nonnegative smooth function of compact support with $\int_{\mathbb{R}} \eta(x)dx = 1$.

Define the new process $F^{(L)}$ to be

$$F_\alpha^{(L)}(x,t) := \sum_{i=1}^{N} V(L(x_i(t) - x)) 1\!\!1(\alpha_i(t) = \alpha). \tag{6.2}$$

The map $\vec{q} \mapsto F^{(L)}$ induces a probability measure on $D([0,T]; L^1(S^1 \times I))$, which we refer to as P_L. We can now restate Theorem 1.1 as follows:

THEOREM 6.1. – *$P_L \Rightarrow P$, where P is concentrated upon the single function ρ that solves the system (1.1); i.e. $\forall J \in C_c^\infty(\mathbb{R} \times [0,\infty))$ and for each $\alpha \in I$,*

$$\begin{aligned}
\int \Bigg| \int_0^\infty \int \Bigg[& \Big(\frac{\partial J}{\partial t} + v_\alpha \frac{\partial J}{\partial x} \Big) f_\alpha \\
& + J Q_\alpha(f,f) \Bigg] dx dt + \int J(x,0)\rho_\alpha^0(x)dx \Bigg| P(df) = 0.
\end{aligned} \tag{6.3}$$

Proof of Theorem 6.1. – We first show that a claim similar to (6.3) holds as $L \to \infty, \varepsilon \to 0$ when we replace P with P_L and the quadratic term $Q_\alpha(f,f)$ with $Q_\alpha(f*\eta_\varepsilon, f*\eta_\varepsilon)$. We then prove that the family of measures $\{P_L\}$ is tight and thus relatively compact by Prohorov's theorem, so that the claim holds for every limit point P of the sequence $\{P_L\}$. We next cite a result of Rezakhanlou that states that the family of products $(f_\alpha * \eta_\varepsilon)(f_\beta * \rho_\varepsilon)$ is uniformly integrable in ε if $v_\alpha \neq v_\beta$, and then we finally complete the proof.

LEMMA 6.1. – *Suppose $J \in C_c^\infty(\mathbb{R}\times[0,\infty))$ and $\alpha \in I$.*

$$\lim_{\varepsilon\to 0}\lim_{L\to\infty}\int\left|\int_0^\infty\int\left[\left(\frac{\partial J}{\partial t}+v_\alpha\frac{\partial J}{\partial x}\right)f_\alpha + JQ_\alpha(f*\eta_\varepsilon, f*\eta_\varepsilon))\right]dxdt\right.$$
$$\left.+\int J(x,0)\rho_\alpha^0(x)dx\right|P_L(df)=0. \tag{6.4}$$

Proof.. – Let

$$G_1(t,\vec{q}):=\frac{1}{L}\sum_{i=1}^N J(x_i,t)1\!\!1(\alpha_i=\alpha). \tag{6.5}$$

$$\begin{aligned}G_2(t,\vec{q}) :&= \sum_{i=1}^N\int V(L(x_i-x))J(x,t)1\!\!1(\alpha_i=\alpha)dx\\ &=\int F_\alpha^{(L)}(x,t)J(x,t)dx.\end{aligned} \tag{6.6}$$

Since J is smooth, we clearly have $G_2(t,\vec{q}) = G_1(t,\vec{q})+O(L^{-1})$. On the other hand, standard Markov theory implies that for the function G_1, the processes M and N are martingales, where

$$M_t := G_1(t,\vec{q}_t) - G_1(0,\vec{q}_0) - \int_0^t\left(\frac{\partial G_1}{\partial\theta}+\mathcal{A}^{(L)}G_1\right)(\theta,\vec{q}_\theta)d\theta \tag{6.7}$$

$$N_t := (M_t)^2 - \int_0^t(\mathcal{A}^{(L)}G_1^2 - 2G_1\mathcal{A}^{(L)}G_1)(\theta,\vec{q}_\theta)d\theta. \tag{6.8}$$

It is then straightforward to show that,

$$\mathcal{A}_0G_1^2 - 2G_1\mathcal{A}_0G_1 = 0 \tag{6.9}$$

and

$$|\mathcal{A}_cG_1^2 - 2G_1\mathcal{A}_cG_1| \leq \frac{C}{L^2}\|J\|_\infty^2\sum_{i\neq j}V(L(x_i-x_j))1\!\!1(v_{\alpha_i}\neq v_{\alpha_j}). \tag{6.10}$$

Note also that $E_L N_t = 0\ \forall t \geq 0$. We then have

$$E_L\left(\sup_{0\leq t\leq T}(M_t)^2\right) \leq 4E_L(M_T)^2$$

$$= 4E_L\int_0^T (\mathcal{A}^{(L)}G_1^2 - 2G_1\mathcal{A}^{(L)}G_1)(t,\vec{q}_t)dt \leq \frac{C}{L}\|J\|_\infty^2 C_1(T) \qquad (6.11)$$

by Doob's inequality and Theorem 3.1. Choose T large enough so that it lies outside the support of J. (Note that this implies that $G_1(T,\vec{q}_T) = 0$.)

Claim.

$$E_L|M_T^1| = \int\left|\int_0^T\int\left[\left(\frac{\partial J}{\partial t} + v_\alpha\frac{\partial J}{\partial x}\right)f_\alpha + JQ_\alpha(f*\eta_\varepsilon, f*\eta_\varepsilon)\right]dxdt\right.$$

$$\left.+\int J(x,0)\rho_\alpha^0(x)dx\right|P_L(df) + o(1). \qquad (6.12)$$

Note that extablishing the claim will finish the proof of the lemma.

Proof of claim. – Recall that we assumed initially that $\mu_L \sim \rho^0$. Since

$$\int_0^T\int\left(\frac{\partial J}{\partial t} + v_\alpha\frac{\partial J}{\partial x}\right)F_\alpha^{(L)}(x,t)dxdt$$

$$= -\int_0^T\left(\frac{\partial G_1}{\partial t} + \mathcal{A}_0 G_1\right)dt + O(L^{-1}), \qquad (6.13)$$

establishing the claim reduces to demonstrating that

$$E_L\left|\int_0^T \mathcal{A}_c G_1(t,\vec{q}_t)dt\right.$$

$$\left.-\int_0^T\int JQ_\alpha\Big(F^{(L)}*\eta_\varepsilon, F^{(L)}*\eta_\varepsilon\Big)(x,t)dxdt\right| = o(1). \quad (6.14)$$

It is not hard to see that $\mathcal{A}_c G_1(t,\vec{q}_t)$ equals to

$$\frac{1}{L}\sum_{i\neq j} V(L(x_i(t) - x_j(t)))$$

$$\times\sum_{\gamma\delta} K(\alpha_i(t)\alpha_j(t),\gamma\delta)J(x_i(t),t)[1\!\!1(\gamma=\alpha) - 1\!\!1(\alpha_i(t)=\alpha)].$$

We observe that

$$\sum_{\gamma\delta} K(\alpha_i\alpha_j,\gamma\delta)(1\!\!1(\gamma=\alpha) - 1\!\!1(\alpha_i=\alpha))$$

$$= \sum_{\beta\gamma\delta}\Big[K(\gamma\delta,\alpha\beta)(1\!\!1(\alpha_i=\gamma)1\!\!1(\alpha_j=\delta))$$

$$- K(\alpha\beta,\gamma\delta)(1\!\!1(\alpha_i=\alpha)(1\!\!1(\alpha_j=\beta))\Big],$$

so establishing the claim reduces to demonstrating that

$$E_L\Big|\int_0^T \frac{1}{L}\sum_{i\neq j} V(L(x_i(t)-x_j(t)))J(x_i(t),t)\mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)dt$$
$$-\int_0^T\int J(x,t)\Big(F_\alpha^{(L)}*\eta_\varepsilon\Big)(x,t)\Big(F_\beta^{(L)}*\eta_\varepsilon\Big)(x,t)dxdt\Big| = o(1). \tag{6.15}$$

According to Theorem 4.1, if $|z_1|, |z_2| \leq \varepsilon$, then

$$\begin{aligned} E_L\Big|&\int_0^T \frac{1}{L}\sum_{i\neq j}\\ &\times V(L(x_i(t)-x_j(t)))J(x_i(t),t)\mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)dt\\ &-\int_0^T \frac{1}{L}\sum_{i\neq j} V(L(x_i(t)-z_1-x_j(t)+z_2))J(x_i(t)-z_1+z_2,t)\\ &\times \mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)dt\Big|\\ &\leq C(T,J)\sqrt{h(4\varepsilon)}. \end{aligned} \tag{6.16}$$

We first replace the second integral in (6.16) with

$$\int_0^T\int\int\frac{1}{L}\sum_{i\neq j}\Big(V(L(x_i(t)-z_1-x_j(t)+z_2))J(x_i(t)-z_1+z_2,t)$$
$$\cdot\,\mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)\xi(z_1)\xi(z_2)\Big)dz_1dz_2dt, \tag{6.17}$$

where the support of ξ is less than ε and $\|\xi\|_{L^1(S^1)} = 1$; the replacement is valid since the bound in (6.16) is uniform in z_1 and z_2. We next replace $J(x_i(t)-z_1+z_2,t)$ with $J(x_i(t)-z_1,t)$; according to Theorem 3.1, the error is $O(\varepsilon)$. After a change of variables, (6.17) becomes

$$\int_0^T\int\int\frac{1}{L}\sum_{i\neq j}\Big(V(L(z_1-z_2))J(z_1,t)\mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta)$$
$$\cdot\,\xi(x_i(t)-z_1)\xi(x_j(t)-z_2)\Big)dz_1dz_2dt. \tag{6.18}$$

Next, define $\xi(w) := \int LV(L(w-x))\eta_\varepsilon(x)dx$; (6.18) becomes

$$\begin{aligned}
&\int_0^T \int\int LV(L(z_1-z_2))J(z_1,t)\\
&\quad\times\Big(\int\sum_i V(L(x_i(t)-y_1))1\!\!1(\alpha_i(t)=\alpha)\eta_\varepsilon(y_1-z_1)dy_1\Big)\\
&\quad\cdot\Big(\int\sum_j V(L(x_j(t)-y_2))1\!\!1(\alpha_j(t)=\beta)\eta_\varepsilon(y_2-z_2)dy_2\Big)dz_1dz_2dt\\
&=\int_0^T\int\int LV(L(z_1-z_2))J(z_1,t)\\
&\quad\times\Big(F_\alpha^{(L)}*\eta_\varepsilon\Big)(z_1,t)\Big(F_\beta^{(L)}*\eta_\varepsilon\Big)(z_2,t)dz_1dz_2dt \qquad (6.19)
\end{aligned}$$

Finally, we replace $\Big(F_\beta^{(L)}*\eta_\varepsilon\Big)(z_2,t)$ with $\Big(F_\beta^{(L)}*\eta_\varepsilon\Big)(z_1,t)$. Since the error generated is $O(\frac{1}{L\varepsilon})$, this finishes the proof of the claim.

LEMMA 6.2. – *The sequence* $\{P_L\}$ *is tight.*

Proof. – Let $M(S^1 \times I)$ denote the space of nonnegative measure vectors $(\nu_\alpha : \alpha \in I)$ with $\sum_\alpha \nu_\alpha(S^1) = M$. We regard $L^1(S^1 \times I)$ as a subspace of $M(S^1 \times I)$. Note that the space $M(S^1 \times I)$ is a complete separable metric space. We define $\hat{D} = D([0,T]; M(S^1 \times I))$ which is also a complete separable metric space. We regard P_L as a sequence of probability measures on $\hat{D}$. Since $\hat{D}$ is a complete separable metric space, we can appeal to Prohorov's theorem to assert that the sequence $\{P_L\}$ is relatively compact if it is tight.

Claim . Any limit point of $\{P_L\}$ is concentrated on the space $D([0,T], L^1(S^1 \times I))$.

Proof of Claim – To see this first note that the function φ is covex and grows faster that the linear function. As a result, the space of vector functions $(F_\alpha : \alpha \in I)$ with

$$\sup_{0\le t\le T}\sum_\alpha \int \varphi(F_\alpha(x,t))dx \le k \qquad (6.20)$$

is weakly closed in $\hat{D}$, for any given k. Furthermore, by Corollary 2.1, it is not hard to show

$$\sup_L E_L \sup_{0\le t\le T}\sum_\alpha \int \varphi(F_\alpha^{(L)}(x,t))dx < \infty. \qquad (6.21)$$

This completes the proof of the claim. (6.11) and (6.13) guarantee that the following is valid:

$$\lim_{L\to\infty} E_L \sup_{0\le s\le t\le T} \left| \int J(x)(F_\alpha^{(L)}(x,t) - F_\alpha^{(L)}(x,s))dx \right.$$
$$\left. - \int_s^t \int (v_\alpha J'(x) F_\alpha^{(L)}(x,\theta) dx d\theta + \int_s^t Y_L(\theta)d\theta \right| P_L(df) = 0, \tag{6.22}$$

where $Y_L(\theta)$ equals to

$$\frac{1}{L}\sum_{i\neq j} V(L(x_i(\theta) - x_j(\theta)))$$
$$\times \sum_{\gamma\delta} K(\alpha_i(\theta)\alpha_j(\theta),\gamma\delta) J(x_i(\theta))[\mathbb{1}(\gamma=\alpha) - \mathbb{1}(\alpha_i(\theta)=\alpha)]. \tag{6.23}$$

We also have

$$\sup_L E_L \sup_{0\le t-s\le\delta} \left| \int_s^t J'(x) F_\alpha^{(L)}(x,\theta) dx d\theta \right| \le C\|J'\|_\infty \delta \tag{6.24}$$

$$\lim_{L\to\infty} \sup_{0\le\sigma\le T} E_L \left| \int_\sigma^{\sigma+\delta} Y_L(\theta)d\theta \right| \le C\|J\|_\infty \tilde{C}_1(\delta), \tag{6.25}$$

where σ ranges over all stopping times taking values in the interval $[0,T]$. (6.24) is obvious from the definition of $F^{(L)}$ and (6.25) is a consequence of Corollary 3.2. Therefore, the processes

$$\int_0^t \int J'(x) F_\alpha^{(L)}(x,\theta) dx d\theta$$

and

$$\int_0^t Y_L(\theta)d\theta$$

are tight. (See [1] for the tightness of the second process). (6.22), (6.24), and (6.25) together imply that the sequence $\{P_L\}$ is tight.

LEMMA 6.3. – *Let P be any limit point of P_L. Then for any $J \in C_c^\infty(\mathbb{R}\times[0,\infty))$ and any $\alpha \in I$,*

$$\lim_{\varepsilon\to 0} \int \left| \int_0^\infty \int \left[\left(\frac{\partial J}{\partial t} + v_\alpha \frac{\partial J}{\partial x} \right) f_\alpha + J Q_\alpha(f*\eta_\varepsilon, f*\eta_\varepsilon) \right] dx dt \right.$$
$$\left. + \int J(x,0)\rho_\alpha^0(x)dx \right| P(df) = 0. \tag{6.26}$$

Proof.. – Let

$$\mathcal{F}_{\alpha,\varepsilon}(f) := \left| \int_0^\infty \int \left[\left(\frac{\partial J}{\partial t} + v_\alpha \frac{\partial J}{\partial x} \right) f_\alpha + J Q_\alpha (f * \eta_\varepsilon, f * \eta_\varepsilon) \right] dx dt \right.$$
$$\left. + \int J(x,0) \rho_\alpha^0(x) dx \right|. \tag{6.27}$$

The functional $\mathcal{F}_{\alpha,\varepsilon}$ is both bounded and continuous with respect to the weak topology, since $F_\alpha^{(L)} * \eta_\varepsilon \leq C\varepsilon^{-1} \Rightarrow ||\mathcal{F}_{\alpha,\varepsilon}|| \leq \frac{C}{\varepsilon^2}$. Therefore,

$$\lim_{L\to\infty} \int \mathcal{F}_{\alpha,\varepsilon}(f) P_L(df) = \int \mathcal{F}_{\alpha,\varepsilon}(f) P(df). \tag{6.28}$$

Our result follows from (6.4) and (6.28).

LEMMA 6.4. – *Let P be any limit point of $\{P_L\}$, and suppose T_0 is arbitrary but fixed. Then P is concentrated on the set of f for which*

$$\sum_{v_\alpha \neq v_\beta} \sup_\varepsilon \int_0^{T_0} \int \psi[(f_\alpha * \eta_\varepsilon)(f_\beta * \eta_\varepsilon)] dx dt < \infty. \tag{6.29}$$

Proof. – See [12], Section 8, Lemma 8.4.

(Note: it is only here that we use the assumption 1.5(iv)) on K and Theorem 5.1.)

Proof of the Kinetic Limit. – We can restate (6.26) as

$$\lim_{\varepsilon\to 0} \int \mathcal{F}_{\alpha,\varepsilon}(f) P(df) = 0; \tag{6.30}$$

this implies there exists a subsequence $\{\varepsilon_m\}$ such that

$$\lim_{m\to\infty} \mathcal{F}_{\alpha,\varepsilon_m}(f) = 0 \qquad P- \text{ a.s.} \tag{6.31}$$

We want to replace $f * \eta_\varepsilon$ with f; it suffices to show that for some subsequence ε_m and whenever $v_\alpha \neq v_\beta$,

$$\lim_{m\to\infty} \int_0^T \int J(f_\alpha * \eta_{\varepsilon_m})(f_\beta * \eta_{\varepsilon_m}) dx dt = \int_0^T \int J f_\alpha f_\beta dx dt \qquad P- \text{ a.s.} \tag{6.32}$$

Now, since both f_α and f_β are in L^1, we have that

$$\lim_{\varepsilon\to 0} (f_\alpha * \eta_\varepsilon)(f_\beta * \eta_\varepsilon) = f_\alpha f_\beta \text{ for a.a. } (x,t); \tag{6.33}$$

since $J(f_\alpha * \eta_\varepsilon)(f_\beta * \eta_\varepsilon)$ is uniformly integrable by Lemma 6.4, we know that (6.32) holds $P-$ a.s. Thus P is concentrated on the set of functions

f that satisfies (1.1) in distribution, so once we show that $f_\alpha f_\beta \in L^1$, we will be finished. It suffices to show that

$$\int \left(\sum_{v_\alpha \neq v_\beta} \int_0^T \int f_\alpha f_\beta dx dt \right) P(df) < \infty. \tag{6.34}$$

By Fatou's Lemma

$$\int \Big(\sum_{v_\alpha \neq v_\beta} \int_0^T \int f_\alpha f_\beta dx dt \Big) P(df)$$
$$\leq \liminf_{\varepsilon_m \downarrow 0} \int \Big(\sum_{v_\alpha \neq v_\beta} \int_0^T \int (f_\alpha * \eta_\varepsilon)(f_\beta * \eta_\varepsilon) dx dt \Big) P(df) < \infty \tag{6.35}$$

by Lemma 6.4. This completes the proof of the kinetic limit.

ACKNOWLEDGEMENTS

The work of F. R. is partially supported by a Sloan Foundation fellowship and NSF grant DMS 9424270.

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(Manuscript received June 12, 1996;
Revised January 8, 1997.)