ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 33, nº 6 (1997), p. 727-751
<http://www.numdam.org/item?id=AIHPB_1997__33_6_727_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 6, 1997, p. 727-751 *Probabilités et Statistiques*

Occurrence of rare events in ergodic interacting spin systems

by

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ABSTRACT. – We give asymptotics for the occurrence time of rare events in infinite spin systems whose invariant measure satisfies a Logarithmic Sobolev inequality. We then describe the typical paths close to this occurrence time. Finally, in the case of non-interacting spins, we obtain sharper estimates for the expected value of the occurrence time.

Key words: Interacting spin systems, Large Deviations.

RÉSUMÉ. – Nous établissons des estimations asymptotiques de la distribution du temps de réalisation d'évènements rares dans des systèmes de spins sur $\mathbb{Z}^d$ dont la mesure invariante satisfait une inégalité logarithmique de Sobolev. Aussi, nous caractérisons les trajectoires typiques dans un intervalle de temps précédant la réalisation de l'évènement considéré. Enfin, dans le cas de spins indépendants, nous obtenons des estimations plus précises de l'espérance du temps de réalisation.

1. INTRODUCTION

We consider stationary ergodic spin-flip systems on the infinite lattice. We estimate the occurrence time of an event E having a small probability with respect to the stationary measure.

Heuristics based on the ergodic theorem suggest that the occurrence time of a rare event should be roughly the inverse of the probability of the event itself. Moreover, provided that time correlations decay sufficiently fast, one expects the system to perform several 'independent' trials before entering E, which suggests that the occurrence time should be close to an exponential time. The mathematical problem is to show that these heuristics are true. Also, it is of interest to describe the typical behavior of the system just before E occurs for the first time (*hitting path*); this would allow, for example, to predict from observed data the occurrence of E.

Answers to these questions have been given for several classes of stochastic systems. Among the first results on the subject we mention [9], where Markov chains with strong recurrence properties were considered. Later, the Freidlin and Wentzell theory [8] permitted a good understanding of occurrence times for small random perturbations of finite dimensional deterministic systems.

Quite related to [8] is the study of spin systems on a finite periodic lattice with mean field interaction, in the limit where the mesh of the lattice goes to zero. Results on the escape time from metastable states have been obtained for such systems with or without conservation of the particle number (*see* [1] or [2] respectively). Escape time from the basin of attraction of a metastable state has also been studied, with different techniques, for stochastic Ising models ([15, 16, 17]) at very low temperature. In the case of finite Markov chains, deep results on occurrence time of rare events have been recently obtained in [19, 20]. Closer in spirit to our paper are the results in [12] and [6, 7], where non conservative and conservative spin systems, respectively, are studied.

In this paper, the rare event E is expressed in terms of the empirical process (*see* (2.3)), so that it may depend on any macroscopic observables.

Most of our results are proved under the assumption that the invariant measure for the system satisfies a Logarithmic Sobolev (L-S) inequality. The L-S inequality implies good mixing properties for the system, so that the asymptotics for the hitting time can be obtained by arguments close to the ones in [12], where attractiveness is instead assumed. More precisely, if τ_n is the first time the empirical process R_n enters E, a rare event for the invariant measure ν, then we show that $E^{\nu}(\tau_n) = \exp(o(n^d))/\nu(R_n \in E)$.

On the other hand, estimates of the type

$$E^{\nu}(\tau_n) = \frac{1}{\nu(R_n \in E)} e^{O(1)}. \tag{1.1}$$

have been proved for the one dimensional symmetric simple exclusion process ([7]), assuming $\{R_n \in E\}$ is the event that the sites $\{1, \ldots, n\}$ are all occupied. In the last Section of this paper we exploit ideas of [7] to get sharp estimates for independent spin flips, assuming $\{R_n \in E\}$ is the event that the empirical density in a cube of side n is above a given value. In particular, we show that (1.1) is false in this simple case. The correct answer differs by a factor of the order of $\sqrt{n}$.

A different approach is needed to describe the features of the hitting path. In the spirit of Freidlin and Wentzell theory, we study the large deviations on the path space of the process. Thus, we characterize the asymptotics of the empirical process (and so of any macroscopic observable) any time t before the system enters E.

We now outline the structure of the paper. In Section 2 and 3 we describe the model and state our results on the hitting time. Proofs are postponed to Section 4. In Section 5 we develop the techniques needed in Section 6 to describe the hitting path. Finally, Section 7 is devoted to the computation of sharp estimates in the case of independent spin flips.

2. MODEL AND NOTATIONS

We consider a spin flip system on $\mathbb{Z}^d$, *i.e.* a Markov process with state space $S = \{-1, 1\}^{\mathbb{Z}^d}$. The evolution is determined by the generator

$$Lf(\sigma) = \sum_{i \in \mathbb{Z}^d} c(i, \sigma) \nabla_i f(\sigma). \tag{2.1}$$

with f a function depending on a finite number of spins (*local*), $\sigma \in S$ and $\nabla_i f(\sigma) = f(\sigma^i) - f(\sigma)$, where σ^i is obtained from σ by flipping the i-th spin. We assume the *rates* $c(i, \sigma)$ to be translation invariant, *i.e.* $c(i, \sigma) = c(\theta_i \sigma)$, where $c(\cdot)$ is a given positive local function, and θ_i is the shift on $\mathbb{Z}^d$. We denote by P^{μ} the law of this process when starting from the initial measure μ. We write P^{ξ} rather than $P^{\delta_{\xi}}$, for $\xi \in S$. Expectation with respect to P^{μ} is denoted by E^{μ}. We remark that E^{μ} will also denote μ-expectation in S.

For $\sigma \in S$, σ_i denotes the spin value at $i \in \mathbb{Z}^d$. If f is a real valued local function, let supp$(f) = \{i : \nabla_i f \neq 0\}$. For $\Lambda \subset \mathbb{Z}^d$, let

$\partial\Lambda = \{i \in \Lambda^c : \text{dist}(i, \Lambda) = 1\}$ be its boundary, and $|\Lambda|$ its cardinality. Moreover, we define $\mathcal{F}_\Lambda$ to be the σ-field in S generated by $\{\sigma_i : i \in \Lambda\}$. We write $\mathcal{F}_n$ when $\Lambda = V_n = [0, n-1]^d$. We denote by $\mathcal{M}_s$ the set of probability measures on S that are θ_i-invariant for all $i \in \mathbb{Z}^d$. This space is provided with the weak topology. For $\mu, \mu' \in \mathcal{M}_s$ the *free energy* or *specific relative entropy* of μ w.r.t. μ' is defined by

$$h(\mu|\mu') = \limsup_{n\to\infty} \frac{1}{n^d} E^\mu \left\{ \log \frac{d\mu}{d\mu'}|_{\mathcal{F}_n} \right\} \tag{2.2}$$

where $\frac{d\mu}{d\mu'}|_{\mathcal{F}_n}$ denotes the Radon-Nykodim derivative of the restrictions of μ and μ' to $\mathcal{F}_n$. The value of $h(\mu|\mu')$ is infinite if $\mu|_{\mathcal{F}_n}$ is singular with respect to μ' for some n. Of special interest among the elements of $\mathcal{M}_s$ are the Gibbs measures. We say that $\nu \in \mathcal{M}_s$ is a Gibbs measure if there exists a family of functions

$$\Phi_\Lambda : \{-1, 1\}^\Lambda \to \mathbb{R}$$

indexed by the finite subsets of $\mathbb{Z}^d$, such that

$$\sum_\Lambda ||\Phi_\Lambda||_\infty < \infty$$

and

$$\nu(\sigma_0|\mathcal{F}_{\mathbb{Z}^d\setminus\{0\}}) = \frac{1}{Z} \exp[\sum_{\Lambda\ni 0} \Phi_\Lambda(\sigma)]$$

where Z is a normalization factor depending on $\{\sigma_i : i \neq 0\}$. Now, given $\sigma \in S$ we consider the associated empirical process

$$R_n(\sigma) = \frac{1}{n^d} \sum_{i\in V_n} \delta_{\theta_i\sigma^{(n)}} \tag{2.3}$$

where $\sigma^{(n)}$ is the configuration obtained by repeating periodically the restriction of σ to the sites in V_n. Note that $R_n(\sigma) \in \mathcal{M}_s$, so that the empirical process can be thought of as a measure valued random variable. We will use later the fact that the laws of R_n under ν satisfy a Large Deviation Principle with rate function $h(\cdot|\nu)$. For the definition of Large Deviation Principle and a proof of this result we refer to [18].

The path space for a spin flip system is the space of cadlag functions $\Omega = D([0, +\infty), S)$, provided with the product Skorohod topology. For $\omega \in \Omega$, $\omega_i(t)$ denotes the i-th spin at time t. We also let, for $T > 0$, $\Omega_T = D([0, T), S)$. Similarly to above, $\mathcal{G}_\Lambda$ denotes the σ-field in Ω_T

generated by $\{\omega_i : i \in \Lambda\}$, and we write $\mathcal{G}_n$ for $\mathcal{G}_{V_n}$. The shifts θ_i act naturally on Ω_T, so that we define $\mathcal{M}_s(\Omega_T)$ to be the space of shift invariant probability measures on Ω_T. The specific relative entropy between two such measures is defined as in (2.2), with $\mathcal{G}_n$ replacing $\mathcal{F}_n$. We use the same notation for the relative entropy in $\mathcal{M}_s(\Omega_T)$ and $\mathcal{M}_s$, the ambiguity being eliminated by the fact that elements of $\mathcal{M}_s(\Omega_T)$ will be denoted by capital letters $P, Q, \ldots$

To a given $\omega \in \Omega_T$ we associate the empirical process

$$\rho_n(\omega) = \frac{1}{n^d} \sum_{i \in V_n} \delta_{\theta_i \omega^{(n)}} \tag{2.4}$$

where the shifts θ_i act in the obvious way on Ω_T, and $\omega^{(n)}$ is the periodization of ω. Note that $\rho_n \in \mathcal{M}_s(\Omega_T)$.

3. ASYMPTOTICS FOR THE HITTING TIME

Most results in this paper will be proved under the following assumptions.

A1. The system has an invariant measure ν which is a Gibbs measure.

A2. The measure ν satisfies a Log-Sobolev inequality

$$E^\nu(f \log f) \le \alpha \sum_i E^\nu(\nabla_i \sqrt{f})^2$$

where $f \ge 0$, $E^\nu f = 1$ and $\alpha > 0$.

Assumption A2 says basically that the correlations decay exponentially in time. There is a large literature linking rate of decay of correlations to Log-Sobolev inequality: *see* [21] for a fundamental result and [14] for the best results in two dimensions for finite range interactions.

Now, for $A \subset \mathcal{M}_s$ measurable we consider the sequence of stopping times

$$\tau_n(\omega) = \inf\{t \ge 0 : R_n(\omega(t)) \in A\}.$$

In what follows, for $B \subset \mathcal{M}_s$, we define $h(B|\nu) = \inf\{h(\mu|\nu) : \mu \in B\}$. We assume the set A to be such that $h(\bar{A}|\nu) > 0$ (but possibly infinite).

PROPOSITION 1. – *Under A1, for every* $\epsilon > 0$

$$\lim_{n\to\infty} P^\nu\{\tau_n \ge \exp[n^d(h(\bar{A}|\nu) - \epsilon)]\} = 1.$$

PROPOSITION 2. – *Under A1 and A2 there is a constant $c > 0$ such that*

$$E^{\nu}(\tau_n) \leq \frac{cn^d}{\nu\{R_n \in A\}}. \tag{3.1}$$

Moreover, for every $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{\xi\in S} P^{\xi}\{\tau_n \leq \exp[n^d(h(\mathring{A}\,|\nu) + \epsilon)]\} = 1.$$

Now let $\beta_n > 0$ be defined by

$$P^{\nu}\{\tau_n > \beta_n\} = e^{-1}.$$

Note that Propositions 1 and 2 imply

$$h(\bar{A}|\nu) \leq \liminf_n \frac{1}{N} \log \beta_n \leq \limsup_n \frac{1}{N} \log \beta_n \leq h(\mathring{A}\,|\nu).$$

PROPOSITION 3. – *Under A1, A2*

$$\lim_{n\to\infty} P^{\nu}\{\tau_n \geq \beta_n t\} = e^{-t}.$$

PROPOSITION 4. – *Under A1, A2*

$$\lim_{n\to\infty} \frac{\beta_n}{E^{\nu}(\tau_n)} = 1.$$

4. PROOFS OF PROPOSITIONS 1-4

Proof of Proposition 1. – The proof is identical to the one of Prop. 1 in [12], and is based on the large deviations upper bound for the ν-law of the empirical process (*see* [18]). ■

Before giving the proofs of the other propositions, we state two lemmas. The first one is close to a classical result [10] which deals with dependence of the law of the process on the initial condition. Its proof is standard and we refer the reader to [10]. We prove the second one at the end of this section.

LEMMA 1. – *For $T > 0$ and $\Lambda \subset \mathbb{Z}^d$ finite, let $\mathcal{A}_T^{\Lambda}$ be the family of space-time events depending on $\{\omega_i(t) : i \in \Lambda,\ 0 \leq t \leq T\}$ only. Define*

$\Lambda_t = \{i \in \mathbb{Z}^d : \text{dist}(i, \Lambda) \le t\}$. *Then there is a* $C > 0$ *such that for all* M *large enough (independently of* T*)*

$$\sup\{|P^\eta(A) - P^\xi(A)| : A \in \mathcal{A}_T^\Lambda, \xi_i = \eta_i \ \ \forall i \in \Lambda_{MT}\} \le C|\Lambda|e^{-MT/2}.$$

Remark 1. – The following variation of Lemma 1 will also be used later. For Λ a finite subset of $\mathbb{Z}^d$, define

$$c_{\Lambda,i} = E^\nu\{c(\theta_i\cdot)|\mathcal{F}_\Lambda\}.$$

It is easy to check that the Markov chain on $\{-1,1\}^\Lambda$ having generator

$$L^\Lambda f = \sum_{i\in\Lambda} c_{\Lambda,i}\nabla_i f$$

has the restriction of ν to $\mathcal{F}_\Lambda$ as unique invariant measure. Denote by P_Λ^μ the law of this Markov chain when starting from the initial measure μ. Then

$$\sup_\mu \sup_{A\in\mathcal{A}_T^\Lambda} |P^\mu_{\Lambda_{MT}}(A) - P^\mu(A)| \le C|\Lambda|e^{-MT/2}$$

for M large enough, where the supremum in μ is over all probability measures on $\{-1,1\}^\Lambda$.

In the next Lemma we show that Assumption A2 implies fast convergence to equilibrium. Let S_t be the semigroup associated to L, *i.e.* $S_t f(\eta) = E^\eta(f(\omega(t)))$.

LEMMA 2. – *Under A2, there exist constants* $C, k > 0$ *such that for every* $t > 0$ *and any local function* g

$$\|S_t g - E^\nu(g)\|_\infty \le C\|g\|_\infty|\text{supp}(g)|e^{-kt}. \tag{4.1}$$

Proof of Proposition 2. – Let c be any positive constant.

$$\begin{aligned} E^\nu(\tau_n) &= c\int_0^\infty P^\nu(\tau_n \ge ct)dt \\ &\le 2cn^d \sum_{k\ge0} P^\nu(\tau_n \ge 2cn^d k) \\ &\le 2cn^d \sum_{k\ge0} (\sup_\eta P^\eta(\tau_n \ge 2cn^d))^k \\ &\le \frac{2cn^d}{\inf_\eta P^\eta(\tau_n \le 2cn^d)}. \end{aligned} \tag{4.2}$$

But, by using Lemma 2, and choosing c large enough

$$\inf_\eta P^\eta(\tau_n \le 2cn^d) \ge \inf_\eta P^\eta(R_n(\omega(cn^d)) \in A)$$
$$\ge \nu(R_n \in A) - Cn^d e^{-kcn^d} \ge \frac{1}{2}\nu(R_n \in A)$$

that, plugged in (4.2), gives the desired bound. ■

Proof of Proposition 3. – It is enough to show that for any $s, t > 0$

$$\lim_{n\to\infty} |P^\nu\{\tau_n > (s+t)\beta_n\} - P^\nu\{\tau_n > t\beta_n\}P^\nu\{\tau_n > s\beta_n\}| = 0.$$

If we define, for $0 \le t < t'$,

$$\chi[t,t'] = \mathbf{1}_{\{R_n(\sigma(s))\notin A \ \forall s\in[t,t']\}}$$

then $\{\tau_n > t\} = \{\chi[0,t] > 0\}$. Thus we need to show

$$\lim_{n\to\infty} |E^\nu\{\chi[0,(s+t)\beta_n]\} - E^\nu\{\chi[0,t\beta_n]\}E^\nu\{\chi[0,s\beta_n]\}| = 0.$$

Take any sequence λ_n going to infinity, such that $\lambda_n \le e^{\lambda n^d}$ for some $\lambda < h(\bar{A}|\nu)$. By Proposition 1, for such λ_n, $E^\nu(\chi[0,\lambda_n])$ goes to zero as $n \to \infty$. Thus, by stationarity

$$|E^\nu\{\chi[0,(s+t)\beta_n]\} - E^\nu\{\chi[0,t\beta_n]\chi[t\beta_n+\lambda_n,(t+s)\beta_n]\}|$$
$$\le E^\nu\{\chi[0,\lambda_n]\} \to 0$$

as $n \to \infty$. Similarly

$$|E^\nu\{\chi[0,s\beta_n]\} - E^\nu\{\chi[\lambda_n,s\beta_n]\}| \to 0$$

as $n \to \infty$. Therefore we only have to show that

$$|E^\nu\{\chi[0,t\beta_n]\chi[t\beta_n+\lambda_n,(t+s)\beta_n]\} - E^\nu\{\chi[0,t\beta_n]\}E^\nu\{\chi[\lambda_n,s\beta_n]\}| \to 0$$

as $n \to \infty$. By the Markov property:

$$E^\nu\{\chi[0,t\beta_n]\chi[t\beta_n+\lambda_n,(t+s)\beta_n]\} = E^\nu\{\chi[0,t\beta_n]E^{\omega(t\beta_n)}\{\chi[\lambda_n,s\beta_n]\}.$$

Now we let

$$f_n(\xi) = E^\xi\{\chi[0,s\beta_n-\lambda_n]\}.$$

By Lemma 1 we can find a local function $\hat{f}_n$ such that $\text{supp}(\hat{f}_n) \subset V_{n+Ms\beta_n}$ for some M large, and

$$\|f_n - \hat{f}_n\|_\infty \leq Cn^d e^{-Ms\beta_n}.$$

Thus, using Lemma 2,

$$\begin{aligned}
&|E^\nu\{\chi[0, t\beta_n]\chi[t\beta_n + \lambda_n, (t+s)\beta_n]\} - E^\nu\{\chi[0, t\beta_n]\}E^\nu\{\chi[\lambda_n, s\beta_n]\}| \\
&\quad \leq E^\nu\left\{\left|E^\xi(f_n(\omega(\lambda_n))) - \int f_n d\nu\right|\right\} \\
&\quad \leq \left\|E^\xi(f_n(\omega(\lambda_n))) - \int f_n d\nu\right\|_\infty \\
&\quad \leq Cn^d e^{-Ms\beta_n} + C'(n + Ms\beta_n)^d e^{-c\lambda_n}
\end{aligned}$$

fore some $c > 0$. The last expression goes to zero if for example $\lambda_n = n^{d+1}$. This completes the proof. ■

Proof of Proposition 4. – We first show that, for n large enough,

$$P^\nu\{\tau_n > k\beta_n\} \leq h(k) \tag{4.3}$$

for a decreasing $h(k)$ satisfying $\sum_k h(k) < \infty$. Now let $\lambda_n = n^{d+1}$. By the argument in the proof of Proposition 3, and the definition of β_n,

$$\sup_\xi E^\xi\{\chi[\lambda_n, \beta_n]\} \leq E^\nu\{\chi[0, \beta_n]\} + o(1) = e^{-1} + o(1).$$

Therefore, by the Markov property,

$$\begin{aligned}
P^\nu\{\tau_n > k\beta_n\} &= E^\nu\{\chi[0, (k-1)\beta_n]E^{\omega((k-1)\beta_n)}(\chi[0, \beta_n])\} \\
&\leq E^\nu\{\chi[0, (k-1)\beta_n]E^{\omega((k-1)\beta_n)}(\chi[\lambda_n, \beta_n])\} \\
&\leq P^\nu\{\tau_n > (k-1)\beta_n\}(e^{-1} + o(1))
\end{aligned}$$

and so

$$P^\nu\{\tau_n > k\beta_n\} \leq (e^{-1} + o(1))^k$$

which proves (4.3). Now, as in [12], we observe that

$$\frac{E^\nu(\tau_n)}{\beta_n} = \int_0^\infty P^\nu(\tau_n > \beta_n t)dt. \tag{4.4}$$

Now, let $n \to \infty$ in (4.4). By (4.3) we can use Dominated Convergence Theorem in the r.h.s. of (4.4), which, together with Proposition 3, completes the proof. ■

Proof of Lemma 2. – The proof relies on approximations with finite volume dynamics. Let Λ be a finite subset of $\mathbb{Z}^d$, and define c_Λ and L^Λ as in Remark 1. It is shown in [5] that Log-Sobolev inequality implies exponential decay of relative entropy:

$$h_\Lambda((S_t^\Lambda)^*\mu|\nu) \leq h_\Lambda(\mu|\nu)e^{-2t/\alpha'} \tag{4.5}$$

for some $\alpha' > 0$, where

$$h_\Lambda(\mu|\nu) = E^\mu\left\{\log\left(\frac{d\mu}{d\nu}\bigg|_{\mathcal{F}_\Lambda}\right)\right\},$$

S^Λ is the semigroup generated by L^Λ and μ is any probability measure on S.

Now let g be a given local function. For $t \geq 0$ we construct an increasing family $\{\Lambda(t)\}$ of finite subsets on $\mathbb{Z}^d$ such that

$$||S_t^{\Lambda(t)}g - E^\nu(g)||_\infty \leq A||g||_\infty|\text{supp}(g)|e^{-at} \tag{4.6}$$

and

$$||S_t^{\Lambda(t)}g - S_t g||_\infty \leq B||g||_\infty|\text{supp}(g)|e^{-bt} \tag{4.7}$$

for some $A, a, B, b > 0$, that would conclude the proof.

Note that (4.7) is easily implied by what stated in Remark 1 once we let $\Lambda = \text{supp}(g)$, and $\Lambda(t) = \Lambda_{Mt}$ (defined as in Lemma 1) for M large enough.

To show that, with the same choice of $\Lambda(t)$ also (4.6) holds, let

$$\partial_R\Lambda_{Mt} = \{i \in \Lambda_{Mt} : \text{dist}(i, \Lambda_{Mt}^c) \leq R\}$$

where R is, as above, the diameter of supp(g). By letting $||\cdot||_{TV}$ denote the total variation norm in the space of signed measures, using (4.5) and Csisar's inequality, we obtain

$$\begin{aligned}||S_t^{\Lambda_{Mt}}g - E^\nu(g)||_\infty &\leq ||g||_\infty \sup_\eta ||(S_t^{\Lambda_{Mt}})^*\delta_\eta - \nu|_{\mathcal{F}_{\Lambda_t}}||_{TV} \\ &\leq 2||g||_\infty \sup_\eta (h_\Lambda((S_t^{\Lambda_{Mt}})^*\delta_\eta|\nu))^{1/2} \\ &\leq \text{const.}||g||_\infty|\Lambda_{Mt}|^{1/2}e^{-t/\alpha}\end{aligned}$$

which proves (4.6), since $\Lambda_0 = \text{supp}(g)$ and $|\Lambda_{Mt}|$ grows like t^d. ■

5. SOME TOOLS FROM LARGE DEVIATIONS THEORY

In analogy with [8] we use large deviation estimates on the paths of the process to analyze how the system enters the rare event $\{R_n \in A\}$.

In what follows, given $\mu \in \mathcal{M}_1(S)$ = space of probability measures on S, we let $\bar{P}^\mu$ be the law of the spin flip process with rates $c(\cdot) \equiv 1$ and initial condition μ. In the following theorem, we prove a large deviation principle for the empirical process ρ_n on the path space.

THEOREM 1. – *Let $\mu \in \mathcal{M}_s(S)$ be a Gibbs measure. Then the sequence of measures $P^\mu \circ \rho_n^{-1} \in \mathcal{M}_1(\mathcal{M}_s(\Omega_T))$ satisfies a n^d-large deviation principle with a good rate function $H_T^\mu(\cdot)$. Moreover*

$$H_T^\mu(Q) = h(Q|\bar{P}^\mu) - E^Q\left\{\int_0^T [1-c(\omega_t)]dt + \int_0^T \log c(\omega_{t^-})dN_0(t)\right\} \quad (5.1)$$

$$= h(Q|P^\mu) \quad (5.2)$$

where $N_0(t)$ is the point process counting the jumps of $\omega_0(t)$.

Proof. – The proof is an application of Varadhan's Lemma. Several superexponential estimates are required. Since all technicalities are straightforward adaptations of ideas in [3, 4], we only sketch the proof, and refer there for details.

Step 1. – Proof of the LDP for $\bar{P}^\mu \circ \rho_n^{-1}$: This comes from the fact that $\bar{P}^\mu$ is a Gibbs measure on Ω_T.

Step 2. – A perturbation argument. Let

$$Z_n(\omega) = \exp\left[n^d \int \rho_n(\omega)(d\eta)\left\{\int_0^T [1-c(\eta_t)]dt + \int_0^T \log c(\eta_{t^-})dN_0(t)\right\}\right].$$

By the same proof as in [3], Lemma 7.3 and [4] Corollary 4.4, we have that, for every sequence $A_n \in \mathcal{G}_n$

$$\limsup_n\,(\inf)\frac{1}{n^d}\log E^{\bar{P}^\mu}\{Z_n(\omega)\mathbf{1}_{A_n}\} = \limsup_n\,(\inf)\frac{1}{n^d}\log P^\mu(A_n). \quad (5.3)$$

Then note that Z_n is of the form $Z_n(\omega) = \exp[n^d F(\rho_n(\omega))]$. The function F here is not continuous, due to the unboundedness of the stochastic integral $\int_0^T \log c(\omega_{t^-})dN_0(t)$. Nevertheless, by the approximation argument in [3], Lemma 7.8, one can exploit (5.3) and Varadhan's Lemma to show that the sequence $P^\mu \circ \rho_n^{-1}$ satisfies a n^d-LDP with rate function

$$H_T^\mu(Q) = h(Q|\bar{P}^\mu) - E^Q(F),$$

that proves (5.1). To show (5.2) it is enough to observe that by the same proof as in [4], Proposition 4.7, it is shown that for every $\rho \in \mathcal{M}_1(S)$ (in particular for $\rho = \mu$) and for every $Q \in \mathcal{M}_s(\Omega_T)$ with $h(Q|\bar{P}^\mu) < \infty$

$$\lim_{n\to\infty} \frac{1}{n^d} E^Q\left\{\log\left(\frac{dP^\rho}{d\bar{P}^\rho}\bigg|_{\mathcal{G}_n}\right)\right\}$$
$$= E^Q\left\{\int_0^T [1 - c(\omega_t)]dt + \int_0^T \log c(\omega_{t-})dN_0(t)\right\}. \tag{5.4}$$

■

For technical reasons, we will use a slightly stronger version of Assumption A1.

A1$'$. Same as A1 plus either one of the following conditions:
i) The system is reversible w.r.t. ν;
ii) ν is Gibbsian for a finite range potential.

We do not believe that these new conditions are necessary. As will be seen later, to avoid the use of A1' we would need to prove Theorem 1 for a system with nonlocal rates (but with dependence on distant spins decaying sufficiently fast). This should be possible, but we prefer to avoid this issue here.

Now define

$$V_\nu(\mu) = \inf\{H_T^\nu(Q) : \Pi_T Q = \mu\}$$

where $\Pi_t Q$ is the projection of Q at time t. The function V_ν plays the role of the *quasipotential* in the Freidlin-Wentzell theory [8].

PROPOSITION 5. – *Assume A1$'$. Then, for every $\mu \in \mathcal{M}_s(S)$*

$$V_\nu(\mu) = h(\mu|\nu).$$

Moreover, assuming $h(\mu|\nu) < \infty$, we have that a measure Q with $\Pi_T Q = \mu$ is such that $H_T^\nu(Q) = h(\mu|\nu)$ if and only if $h(Q|\int \mu(d\xi)P_\xi^\nu) = 0$, where P_ξ^ν is the regular conditional probability distribution (r.c.p.d.) of P^ν w.r.t. $\sigma\{\omega(T)\}$.

Proof. – The inequality $V_\nu(\mu) \geq h(\mu|\nu)$ is clear, since relative entropy decreases under projection. To prove the reverse inequality we first assume that the system is reversible (A1$'$ i)). By applying time reversal, we only need to show that, under the condition $\Pi_0 Q = \mu$, $H_T^\nu(Q) = h(\mu|\nu)$ is equivalent to $h(Q|P^\mu) = 0$. Notice that

$$\begin{aligned}
&\frac{1}{n^d}E^Q\left\{\log\left(\frac{dQ}{dP^\nu}\bigg|_{\mathcal{G}_n}\right)\right\} \\
&= \frac{1}{n^d}E^Q\left\{\log\left(\frac{dQ}{dP^\mu}\bigg|_{\mathcal{G}_n}\right)\right\} + \frac{1}{n^d}E^Q\left\{\log\left(\frac{dP^\mu}{d\bar{P}^\mu}\bigg|_{\mathcal{G}_n}\right)\right\} \\
&- \frac{1}{n^d}E^Q\left\{\log\left(\frac{dP^\nu}{d\bar{P}^\nu}\bigg|_{\mathcal{G}_n}\right)\right\} + \frac{1}{n^d}E^Q\left\{\log\left(\frac{d\bar{P}^\mu}{d\bar{P}^\nu}\bigg|_{\mathcal{G}_n}\right)\right\}. \qquad (5.5)
\end{aligned}$$

By (5.4)

$$\lim_{n\to\infty}\left(\frac{1}{n^d}E^Q\left\{\log\left(\frac{dP^\mu}{d\bar{P}^\mu}\bigg|_{\mathcal{G}_n}\right)\right\} - \frac{1}{n^d}E^Q\left\{\log\left(\frac{dP^\nu}{d\bar{P}^\nu}\bigg|_{\mathcal{G}_n}\right)\right\}\right) = 0. \qquad (5.6)$$

Moreover

$$\frac{d\bar{P}^\mu}{d\bar{P}^\nu}\bigg|_{\mathcal{G}_n} = \frac{d\mu}{d\nu}\bigg|_{\mathcal{F}_n}. \qquad (5.7)$$

Letting $n \to \infty$ in (5.5), using (5.6), (5.7), and $\Pi_0 Q = \mu$, we get

$$h(Q|P^\nu) = h(Q|P^\mu) + h(\mu|\nu) \qquad (5.8)$$

and the conclusion follows.

We now turn to the nonreversible case (A1′ ii)). Denote by $R[\cdot]$ the time reversal operator acting on $\mathcal{M}_s(\Omega_T)$. Since the relative entropy is invariant under time reversal

$$V_\nu(\mu) = \inf\{h(Q|R[P^\nu]) : \Pi_0 Q = \mu\}.$$

By a result in [11], $R[P^\nu]$ is a spin-flip system with "local" spin-flip rates (the fact that the potential in finite range is used here). Thus Theorem 1 holds if we replace P^ν by $R[P^\nu]$, and the argument above can be repeated. ∎

6. THE HITTING PATH

Consider an arbitrary fixed time T. We want to characterize the typical paths in the time window $[\tau_n - T, \tau_n]$. For $K \subset \mathcal{M}_s(\Omega_T)$ measurable, we introduce the stopping times referring to the path space

$$\hat{\tau}_n^K(\omega) = \inf\{t \geq T : \rho_n(\omega(\cdot + t - T)) \in K\}$$

which will be more convenient for the proof that follows. It is clear that the arguments used for Propositions 1 and 2 can be repeated to obtain that, under A1 and A2

$$\lim_{n\to\infty} P^\nu\{e^{n^d[H_T^\nu(\bar{K})-\epsilon]} \le \hat{\tau}_n^K \le e^{n^d[H_T^\nu(\overset{\circ}{K})+\epsilon]}\} = 0 \tag{6.1}$$

for all $\epsilon > 0$. Now, for $K = A_T = \{Q \in \mathcal{M}_s(\Omega_T) : \Pi_T Q \in A\}$, we write $\hat{\tau}_n$ for $\hat{\tau}_n^K$. Note that $\hat{\tau}_n = \tau_n$ on $\{\omega : \tau_n(\omega) \ge T\}$. Note also that, by Proposition 5, $H_T^\nu(\bar{A}_T) = h(\bar{A}|\nu)$, and $H_T^\nu(\overset{\circ}{A}_T) = h(\overset{\circ}{A}|\nu)$. From now on, besides $h(\bar{A}|\nu) > 0$, we shall assume $h(\overset{\circ}{A}|\nu) < \infty$.

In what follows, for $\epsilon > 0$, we let

$$A_T^\epsilon = \{Q \in \mathcal{M}_s(\Omega_T) : \exists\, P \in \bar{A}_T \text{ with } H_T^\nu(P) \le h(\overset{\circ}{A}|\nu) + \epsilon \text{ and } d(Q,P) < \epsilon\},$$

where $d(\cdot,\cdot)$ is the Prohorov metric. This is the ϵ-neighborhood of the set of the elements in $\bar{A}_T$ whose specific relative entropy with respect to P^ν is not much larger that $h(\overset{\circ}{A}|\nu)$.

PROPOSITION 6. – *Under A1′ and A3, for every* $\epsilon > 0$

$$\lim_{n\to\infty} P^\nu\{\rho_n(\omega(\cdot + \tau_n - T)) \notin A_T^\epsilon\} = 0.$$

Proof. – Define

$$\begin{aligned}\hat{\tau}_n^\epsilon &= \inf\{t \ge T : \rho_n(\omega(\cdot + t - T)) \in \bar{A}_T \setminus A_T^\epsilon\}\\ &= \hat{\tau}_n^K \qquad \text{with } K = \bar{A}_T \setminus A_T^\epsilon.\end{aligned}$$

Note that $\bar{A}_T \setminus A_T^\epsilon$ is closed, and

$$\inf\{H_T^\nu(Q) : Q \in \bar{A}_T \setminus A_T^\epsilon\} \ge h(\overset{\circ}{A}|\nu) + \epsilon.$$

Thus, by (6.1)

$$\lim_{n\to\infty} P^\nu\{\hat{\tau}_n^\epsilon \ge e^{n^d[h(A^\circ|\nu)+3\epsilon/4]}\} = 1. \tag{6.2}$$

But we also have

$$\lim_{n\to\infty} P^\nu\{\hat{\tau}_n \le e^{n^d[h(A^\circ|\nu)+\epsilon/4]}\} = 1. \tag{6.3}$$

The conclusion now follows from (6.2), (6.3) and Proposition 1, after having observed that

$$\{\rho_n(\omega(\cdot + \tau_n - T)) \notin A_T^\epsilon\} \subset \{\tau_n \ne \hat{\tau}_n\} \cup \{\hat{\tau}_n^\epsilon \le \hat{\tau}_n\}.$$

∎

Under some regularity of the rare set A, a better description of the hitting path can be given.

PROPOSITION 7. – *Suppose A is a continuity set for h,* i.e. $h(\bar{A}|\nu) = h(\overset{\circ}{A}|\nu)$. *Define*

$$M = \{\mu \in \bar{A} : h(\mu|\nu) = h(\bar{A}|\nu)\},$$

$$M_T = \left\{ Q \in \bar{A}_T : h\left(Q \middle| \int P^\nu_\xi \mu(d\xi)\right) = 0 \text{ for some } \mu \in M \right\},$$

and let M^ϵ_T be the ϵ-neighborhood of M_T in Prohorov metric. Then, for every $\epsilon > 0$

$$\lim_{n\to\infty} P^\nu\{\rho_n(\omega(\cdot + \tau_n - T)) \in M^\epsilon_T\} = 1.$$

The proof of Proposition 7 is an easy application of Propositions 5 and 6.

Example 1. – We illustrate here Proposition 7 in a simple example. Suppose $d = 1$, and that the system has rates $c(\cdot) \equiv 1$ (independent spin flips). The system is reversible with respect to the symmetric Bernoulli measure ν. Let

$$A = \{\mu \in \mathcal{M}_s : E^\mu(\sigma_0\sigma_1) \geq \alpha\}$$

where $0 < \alpha < 1$. Note that $E^\nu(\sigma_0\sigma_1) = 0$, and that A is closed. In what follows we let μ_β be the ferromagnetic nearest neighbor Ising measure with inverse temperature β, where β is chosen so that $E^{\mu_\beta}(\sigma_0\sigma_1) = \alpha$. Note that μ_β is unique, since in one dimension there is no phase transition for finite range Gibbs measures.

Fact 1. – μ_β is the unique element of A satisfying $h(\mu_\beta|\nu) = h(A|\nu)$.

This follows from the following standard facts. Let $\mathcal{T}$ be the tail σ-field in S. Then it is known that, for every $\mu \in \mathcal{M}_s$,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{d\mu_\beta}{d\nu}\bigg|_{\mathcal{F}_n} = \beta E^\mu(\sigma_0\sigma_1|\mathcal{T}) + p(\beta)$$

μ-a.s. and in $L^1(\mu)$, where the pressure, $p(\beta)$, is a continuous function of β. Thus, for $\mu \in A$,

$$\lim_{n\to\infty} \frac{1}{n} E^\mu\left(\log \frac{d\mu_\beta}{d\nu}\bigg|_{\mathcal{F}_n}\right) \geq \beta\alpha + p(\beta)$$

Moreover

$$\frac{1}{n}E^{\mu}\left(\log\frac{d\mu}{d\nu}\bigg|_{\mathcal{F}_n}\right)=\frac{1}{n}E^{\mu}\left(\log\frac{d\mu}{d\mu_\beta}\bigg|_{\mathcal{F}_n}\right)+\frac{1}{n}E^{\mu}\left(\log\frac{d\mu_\beta}{d\nu}\bigg|_{\mathcal{F}_n}\right).$$

When n goes to infinity in the above equality we get that $h(\mu|\nu) = \beta\alpha + p(\beta)$ if and only if $h(\mu|\mu_\beta) = 0$. Due to the absence of phase transition, this is equivalent to $\mu = \mu_\beta$. To show that A is a continuity set, observe that the β for which $E^{\mu_\beta}(\sigma_0\sigma_1) = \alpha$ is an increasing continuous function of α, and so $h(\mu_\beta|\nu)$ is a continuous function of α.

Fact 2. – Recall that R is the time reversal action on $\mathcal{M}_s(\Omega_T)$,

$$\lim_{n\to\infty}\rho_n(\omega(\cdot+\tau_n-T))=R[P^{\mu_\beta}] \quad P^{\nu}\text{-a.s.} \tag{6.4}$$

To show (6.4), observe that, using the notations in Proposition 7, $M = \{\mu_\beta\}$ and $M_T = \{R[P^{\mu_\beta}]\}$, where, for this last equality, we use the fact that P^{μ_β} is a finite range, one dimensional Gibbs field. Thus, all we have to show is that $P^{\nu}(\rho_n(\omega(.+\tau_n-T)) \notin M_T^{\epsilon})$ goes to zero exponentially fast in n. By examining the argument leading to Proposition 7, it is easy to show that such exponential convergence is implied by

$$\inf\{h(Q|P^{\nu}) : Q\in A_T\cap(M_T^{\epsilon})^c\} > h(P^{\mu_\beta}|P^{\nu}) = h(\mu_\beta|\nu). \tag{6.5}$$

To show this, first observe that the infimum in (6.5) is attained at some $Q^*\in A_T\cap(M_T^{\epsilon})^c$. By (5.8)

$$h(Q^*|P^{\nu}) = h(Q^*|P^{\mu_\beta}) + h(\mu_\beta|\nu).$$

So (6.5) follows from the fact that $h(Q^*|P^{\mu_\beta}) > 0$.

Fact 3

$$\lim_{n\to\infty}R_n(\omega(\tau_n-t))=\Pi_t P^{\mu_\beta} \quad P^{\nu}\text{-a.s.} \tag{6.6}$$

This follows immediately from Fact 2. In particular, we can compute the path described by $\frac{1}{n}\sum_0^{n-1}\sigma_i^{(n)}(t)\sigma_{i+1}^{(n)}(t)$ close to the hitting time τ_n. In fact, by (6.6) and easy calculations

$$\lim_{n\to\infty}\frac{1}{n}\sum_0^{n-1}\sigma_i^{(n)}(\tau_n-t)\sigma_{i+1}^{(n)}(\tau_n-t) = E^{P^{\mu_\beta}}(\omega_0(t)\omega_1(t)) = \alpha e^{-4t} \quad P^{\nu}\text{-a.s.}$$

7. SHARP ASYMPTOTICS FOR INDEPENDENT SPIN FLIPS

We consider a one dimensional non interacting system with flip rates 1. Also for convenience, our spin variable ξ has values in $\{0, 1\}$. Define

$$A_{n,k} = \left\{\xi : \sum_{i=1}^{n} \xi_i = k\right\},$$

$$A_n = \bigcup_{k=l}^{n} A_{n,k} \qquad \text{and} \qquad \partial_i A_n = \{\eta_i = 0\} \cap \left\{\sum_{j=1}^{n} \eta_j = l - 1\right\}.$$

$\partial_i A_n$ will be often denoted by ∂A if the indices are not important. We think of l as $[\rho n]$, for $\rho > 1/2$. Let $\tau_n = \inf\{t : \xi(t) \in A_n\}$. The connection with the formalism of the previous section is obvious. Indeed, the set A of Section 3 is

$$A = \{\mu \in \mathcal{M}_s : E^{\mu}(\sigma_0) \geq \rho\}.$$

Our main result is

PROPOSITION 8. – *There are two positive constants a and b such that*

$$(n\nu(\partial A))E^{\nu}\tau_n \in [a, b]$$

for all $n \geq 1$.

It is easy to see that A is a continuity set of the specific relative entropy, thus by Propositions 1 and 2 we already know that

$$E^{\nu}\tau_n = [(2\rho)^{\rho}(2 - 2\rho)^{1-\rho}]^{n+o(n)},$$

however, we see by Proposition 8 and Estimate 1 below that

$$E^{\nu}\tau_n = \sqrt{n}[(2\rho)^{\rho}(2 - 2\rho)^{1-\rho}]^{n+o(1)}.$$

The proof will require several lemmas, but first we introduce a key quantity: the number of excursions into A_n during the time $[0, t]$

$$X[0, t] = I_{A_n} + \sum_{i=1}^{n} \int_0^t I_{\partial_i A_n}(\xi_s) dJ_i(s)$$

where $\{J_i : i = 1, \ldots, n\}$ are independent Poisson processes with intensity 1. Also, define $T_n(t) = t/(n\nu(\partial A))$.

LEMMA 3. – *For any* $t \geq 0$

$$\nu(\tau_n \geq T_n(t)) \geq 1 - t - \nu(A_n). \tag{7.1}$$

LEMMA 4. – *For any* δ *in* $(0, 1)$

$$E^\nu (X[0, T_n(t)])^2 \leq Ct^2 + C't + (n\nu(\partial A))^{1-\delta}. \tag{7.2}$$

As a corollary of Lemma 4 (*see* [FGL])

$$\begin{aligned}\nu(\tau_n \geq T_n(t)) &\leq 1 - \frac{(E^\nu X[0, T_n(t)])^2}{E^\nu X^2[0, T_n(t)]} \\ &\leq 1 - \frac{(t + \nu(A_n))^2}{Ct + C't^2 + (n\nu(\partial A))^{1-\delta}}.\end{aligned} \tag{7.3}$$

Proof of Proposition 8. – We have seen in Section 3 that if β_n is defined by

$$\nu(\tau_n \geq \beta_n) = e^{-1}, \qquad \text{then} \qquad \lim \frac{E^\nu \tau_n}{\beta_n} = 1.$$

Now, $\nu(A_n)$ and $n\nu(\partial A)$ can be made arbitrarily small as n goes to infinity. It follows from Lemma 1 that there is t_0 such that

$$\nu(\tau_n \geq T_n(t_0)) \geq e^{-1},$$

or in other words $T_n(t_0) \leq \beta_n$. Now inequality (7.3) means that for t_1 small enough

$$\nu(\tau_n \geq T_n(t_1)) \leq 1 - 2\frac{t_1}{C} \leq e^{-2t_1/C}.$$

On the other hand for any $\alpha > 0$, $\lim \nu(\tau_n \geq \alpha\beta_n) = e^{-\alpha}$ so for n large enough and

$$\alpha = 2\frac{t_1}{C} \qquad \text{then} \qquad \nu\left(\tau_n \geq 2\frac{t_1}{C}\beta_n\right) \geq \nu(\tau_n \geq T_n(t_1)).$$

This means that

$$\frac{t_1}{n\nu(\partial A)} \geq 2\frac{t_1}{C}\beta_n \Longrightarrow \frac{C}{2n\nu(\partial A)} \geq \beta_n.$$

■

Proof of Lemma 3. – We rewrite $X[0,t]$ in terms of a mean 0 martingale M_t

$$X[0,t] = I_{A_n} + \sum_{i=1}^{n} \int_0^t I_{\partial_i A_n}(\xi_s) ds + M_t. \tag{7.4}$$

Now using the stationarity of X,

$$E^\nu X[0, T_n(t)] = \nu(A_n) + T_n(t) \sum_{i=1}^{n} \nu(\partial_i A) = \nu(A_n) + t.$$

Now the result follows from the bound $\nu(\tau_n \leq t) \leq E^\nu X[0,t]$. ■

Proof of Lemma 4. – First define

$$p(t) = \frac{1 + e^{-2t}}{2} \qquad \text{and} \qquad q(t) = 1 - p(t).$$

From equation (7.2) and Cauchy's inequality

$$E^\nu X^2[0, T_n(t)]$$
$$\leq 3\left(\nu(A_n) + \sum_{i,j} E^\nu \int_0^{T_n(t)} \int_0^{T_n(t)} I_{\partial_i A_n} I_{\partial_j A_n} ds ds' + E^\nu M^2_{T_n(t)} \right).$$

Now, $E^\nu M_t^2 = E^\nu X[0,t]$ so that only the middle term has to be evaluated.
The middle term can be decomposed as follows

$$\sum_{i,j} \nu(\partial_i A) \iint_0^{T_n(t)} p_u(\xi, \partial_j A) du ds$$
$$= \sum_i \frac{1}{2} \nu(A_{n-1,l-1}) \iint_0^{T_n(t)} p(u) p_u(\xi^1, A_{n-1,l}) du ds$$
$$+ \sum_{i \neq j} \iint_0^{T_n(t)} \left[\frac{1}{2^2} \nu(A_{n-2,l-1}) p(u)^2 p_u(\xi^2, A_{n-2,l-1}) \right.$$
$$+ \frac{1}{2^2} \nu(A_{n-2,l-2}) q(u)^2 p_u(\xi^3, A_{n-2,l-2})$$
$$\left. + 2 \frac{1}{2^3} \nu(A_{n-3,l-2}) q^2(u) p(u) p_u(\xi^4, A_{n-3,l-2}) \right] du ds, \tag{7.5}$$

where $\xi^1 \in A_{n-1,l-1}$, $\xi^2 \in A_{n-2,l-1}$, $\xi^3 \in A_{n-2,l-2}$ and $\xi^4 \in A_{n-3,l-2}$. The first term in the last sum corresponds to configurations where both i and j make an even number of flips, in the second they both make an odd number of flips, while in the last only one makes an odd number of flips. The last term concerns all pairs $(\eta, \xi) \in \partial_i A \times \partial_j A$ such that $\eta(j) \neq \xi(i)$, in which case there is necessarily a site x such that $\eta(j) + \eta(x) = \xi(i) + \xi(x)$.

It is easy to see that we only need for $\xi \in A_{n,l}$

$$n^2 \nu(\partial A) \int_0^{T_n(t)} \int_0^s p_u(\xi, A_{n,l}) du ds \leq Ct^2 + C't. \tag{7.6}$$

It is then enough to show that the derivative in t of (7.6) is bounded

$$n^2 \nu(\partial A) \frac{1}{n\nu(\partial A)} \int_0^{T_n(s)} p_u(\xi, A_{n,l}) du \leq 2Cs + C'.$$

On the other hand, for small s, say $s \leq t_s = (n\nu(\partial A))^{1-\delta}$ with $1 > \delta > 0$, we can use the trivial bound $p_u(\xi, A_{n,l}) \leq 1$ and then

$$n^2 \nu(\partial A) \frac{1}{n\nu(\partial A)} \int_0^{T_n(s)} p_u(\xi, A_{n,l}) du \leq \frac{s}{\nu(\partial A)},$$

thus, integrating a second time up to t_s

$$\int_0^{t_s} n^2 \nu(\partial A) \frac{1}{n\nu(\partial A)} \int_0^{T_n(s)} p_u(\xi, A_{n,l}) du \leq (n\nu(\partial A))^{1-2\delta}.$$

PROPOSITION 9. – *For $\xi \in A_{n,l}$, and $s \geq (n\nu(\partial A))^{1-\delta}$ with $\delta \in (0,1)$*

$$n \int_0^{T_n(s)} p_u(\xi, A_{n,l}) du \leq Cs + C'. \tag{7.7}$$

Proof. – We distinguish the configurations of $A_{n,l}$ according to the number of mismatches, say $2k$, with ξ. It is clear that

$$p_u(\xi, A_{n,l}) = \sum_{k=0}^{n-l} \binom{l}{k} \binom{n-l}{k} p(u)^{n-2k} q(u)^{2k}.$$

The first task is the evaluation of

$$X_{n,2k} = \int_0^{T_n(s)} p(u)^{n-2k} q(u)^{2k} du. \tag{7.8}$$

By using that $1 = q(u) + p(u)$ and expanding $(1 - p(u))^{2k}$

$$X_{n,2k} = \sum_{i=0}^{2k} \binom{2k}{i} (-1)^i \int_0^{T_n(s)} p(u)^{n-2k+i} du = \sum_{i=0}^{2k} \binom{2k}{i} (-1)^i X_{n-2k+i,0}.$$

We start by integrating

$$
\begin{aligned}
X_{j,0} = & \int_0^{T_n} \left(\frac{1+e^{-2t}}{2} \right)^j dt = \frac{1}{2^j} \left[T_n + \sum_{i=1}^{j} \binom{j}{i} \frac{1-e^{-2iT_n}}{2i} \right] \\
= & X_j^1 + X_j^2 + X_j^3,
\end{aligned}
$$

$$
\text{with} \qquad X_j^1 = T_n/2^j, \qquad X_j^2 = \frac{1}{2^j} \sum_{i=1}^{j} \frac{\binom{j}{i}}{2i}, \tag{7.9}
$$

and X_j^3 is the remaining term. Terms with superscript i in X_j^i, contributes to $X_{n,2k}$ an amount that we will denote $X_{n,2k}^i$. It is easy to see that

$$
\begin{aligned}
X_{n,2k}^1 = & \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j \frac{T_n}{2^{n-2k+j}} = \frac{T_n}{2^n}, \\
X_{n,2k}^2 = & \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j \sum_{i=1}^{n-2k+j} \binom{n-2k+j}{i} \frac{1}{2i} \\
= & \frac{1}{2^n} \int_0^1 \frac{(1+x)^{n-2k}(1-x)^{2k} - 1}{x} dx.
\end{aligned} \tag{7.10}
$$

Once the alternating series $\{X_{n,2k}^i, k = 0, \ldots, n-l\}$ are bounded, we will provide the following estimates

$$
\text{for } i = 1, 2, 3, \qquad n \sum_{k=0}^{n-l} \binom{l}{k} \binom{n-l}{k} X_{n,2k}^i \leq C_i s + C_i'.
$$

To this very effect, we will need the following estimate.

LEMMA 5. – *If $l = [\rho n]$, with $\rho > 1/2$ then for n large*

$$
\sum_{k=0}^{n-l} \frac{\binom{l}{k}\binom{n-l}{k}}{\binom{n}{2k}} \leq \frac{2}{1 - 4\rho(1-\rho)}. \tag{7.11}
$$

Proof. – Define

$$
a_k = \frac{\binom{l}{k}\binom{n-l}{k}}{\binom{n}{2k}},
$$

then

$$
a_{k+1} = a_k \frac{(k+\frac{1}{2})(l-k)(n-l-k)}{(k+1)(\frac{n}{2}-k)(\frac{n}{2}-k-\frac{1}{2})} \leq a_k \frac{(l-k)(n-l-k)}{(\frac{n}{2}-k)(\frac{n}{2}-k-\frac{1}{2})}.
$$

A simple calculation shows that

$$\frac{(l-k)(n-l-k)}{(\frac{n}{2}-k)(\frac{n}{2}-k-\frac{1}{2})}$$

decreases with k, thus

$$\frac{(l-k)(n-l-k)}{(\frac{n}{2}-k)(\frac{n}{2}-k-\frac{1}{2})} \leq \gamma_n = \frac{(l)(n-l)}{(\frac{n}{2})(\frac{n}{2}-\frac{1}{2})}.$$

So we have seen that $a_{k+1} \leq \gamma_n a_k$, and the conclusion follows because $a_0 = 1 > 2a_1$ and

$$\lim_{n \to \infty} \gamma_n = 4\rho(1-\rho).$$

■

Remark 2. – We have seen that the a_k are bounded by a geometric series. This implies that

$$\sum_{k=0}^{n-l}(k+1)\frac{\binom{l}{k}\binom{n-l}{k}}{\binom{n}{2k}} \leq \left(\frac{2}{1-4\rho(1-\rho)}\right)^2. \tag{7.12}$$

The following estimate is a simple consequence of Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-\epsilon_n} \qquad \text{with} \qquad \frac{1}{12n+1} < \epsilon_n < \frac{1}{12n}.$$

Estimate 1. – If $l = [n\rho]$, with $1 > \rho > 1/2$

$$n\nu(\partial A) \sim \rho\sqrt{\frac{\rho(1-\rho)}{2\pi}}\sqrt{n}\alpha^n \quad \text{with} \quad \alpha = \left(\frac{1}{2\rho}\right)^\rho \left(\frac{1}{2(1-\rho)}\right)^{1-\rho} < 1.$$

Estimate 2. – The contribution of X_j^3 to the left hand side of (7.7) goes to 0 as n goes to infinity.

Proof. – We take care of the last term of (7.9). For this term we will have a strong estimate using absolute values.

$$|X_j^3| = \frac{1}{2^j}\sum_{i=1}^{j}\binom{j}{i}\frac{e^{-2iT_n}}{2i} \leq e^{-2T_n} \qquad \text{implies that}$$

$$|X_{n,k}^3| \leq 2^{2k}e^{-2T_n} \tag{7.13}$$

and

$$n\sum_{k=0}^{n-l}\binom{l}{k}\binom{n-l}{k}|X_{n,k}^3| = n\sum_{k=0}^{n-l}\frac{\binom{l}{k}\binom{n-l}{k}}{\binom{n}{2k}}\binom{n}{2k}2^{2k}e^{-2T_n}$$
$$\leq n(1+2\gamma_n)^n e^{-2T_n}. \tag{7.14}$$

Now, because of estimate 1, we see that for $t > (n\nu(\partial A))^{1-\delta}$, this term goes to 0. ■

Estimate 3:

$$u_{n,k} = \frac{1}{2^n}\int_0^1 \frac{(1+x)^{n-k}(1-x)^k - 1}{x}dx = \frac{1}{2}(u_{n-1,k} + v_{n-1,k})$$
$$= \frac{1}{2}v_{n-1,k} + \frac{1}{2^2}v_{n-2,k} + \ldots + \frac{1}{2^{n-2k}}(v_{2k,k} + u_{2k,k}), \tag{7.15}$$

where we have called

$$v_{n,k} = \frac{1}{2^n}\int_0^1 (1+x)^{n-k}(1-x)^k dx.$$

It will convenient to introduce

$$w_{n,k} = \frac{1}{2^n}\int_{-1}^1 (1+x)^{n-k}(1-x)^k dx = \frac{2}{(n+1)\binom{n}{k}}.$$

Indeed, another way of writing Lemma 5 is

$$n\sum_{k=0}^{n-l}\binom{l}{k}\binom{n-l}{k}w_{n,2k} \leq \frac{1}{1-4\rho(1-\rho)}.$$

The obvious fact $u_{2k,k} < 0$ implies that

$$u_{n,k} \leq \frac{1}{2}w_{n-1,k} + \ldots + \frac{1}{2^{n-2k-1}}w_{2k+1,k} + \frac{1}{2^{n-2k}}v_{2k,k}. \tag{7.16}$$

We want to see that

$$\frac{w_{n-i-1,k}}{2w_{n-i,k}} \leq 1-\epsilon,$$

actually, this ratio decreases as i increases, so we only need to check it for $i = n-2k-2$, that is

$$\frac{w_{2k+1,k}}{2w_{2k+2,k}} = \frac{2k+3}{2k+4} = \frac{k+3/2}{k+2} = 1-\epsilon$$

with $\epsilon = 1/(2k+4)$. Now, to evaluate the last term of the series, note that $2w_{2k+1,k} = 2w_{2k,k}$. Thus, we obtain

$$u_{n,k} \leq 2(k+2)w_{n-1,k}.$$

As a consequence of Lemma 5

$$n \sum_{k=0}^{n-l} \binom{l}{k} \binom{n-l}{k} u_{n,2k} \leq \frac{1}{(1-4\rho(1-\rho))^2}.$$

■

Estimate 4. – The contribution of X_j^1 to $X_{n,k}$ is exactly $T_n(s)/2^n$. So we need to see that

$$n\frac{T_n(s)}{2^n} \leq C_1 s,$$

which is obvious. ■

ACKNOWLEDGEMENTS

We are grateful to J.L.Lebowitz for suggestions and encouragement. P.D.P. was partially supported by a "CNR fellowship 203.01.63" and would like to thank the Mathematics Department of Rutgers University for the warm hospitality.

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(Manuscript received June 10, 1996;
Revised November 21, 1996.)