ANNALES DE L'I. H. P., SECTION B

THIERRY BODINEAU

Interface for one-dimensional random Kac potentials

Annales de l'I. H. P., section B, tome 33, nº 5 (1997), p. 559-590
<http://www.numdam.org/item?id=AIHPB_1997__33_5_559_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 5, 1997, p. 559-590. *Probabilités et Statistiques*

Interface for one-dimensional random Kac potentials

by

Thierry BODINEAU

DMI, Ecole Normale Supérieure,
45, rue d'Ulm, 75005 Paris, France.

ABSTRACT. – We consider a ferromagnetic Ising system with impurities where interaction is given by a Kac potential of positive scaling parameter γ. The random position of the magnetic atoms is described by a quenched variable y. In the Lebowitz Penrose limit, as γ goes to 0, we prove that the quenched Gibbs measure obeys a large deviation principle with rate function depending on y. We then show that for almost all y the magnetization is locally approximately constant. However, interfaces occur and magnetization change for almost all y at a distance of the order of $\exp(\frac{\Phi}{\gamma})$, where Φ is a constant given by a variational formula.

Key words and phrases: Gibbs fields, interface, metastability, random interactions.

RÉSUMÉ. – On considère un modèle d'Ising ferromagnétique unidimensionnel dont les interactions aléatoires sont définies par un potentiel de Kac $\gamma J(\gamma r)$. Le système décrit un alliage contenant des atomes ferromagnétiques et des particules non ferromagnétiques dont la répartition aléatoire est représentée par une variable y. Quand γ tend vers 0, on prouve un principe de grandes déviations pour la mesure de Gibbs dont la fonction de taux dépend de y. On en déduit que localement la magnétisation est proche d'une constante. Cependant quand on observe le système sur des distances exponentielles, la magnétisation change pour presque tout y. En particulier une interface apparaît à une distance de l'ordre de $\exp(\frac{\Phi}{\gamma})$, où Φ est une constante obtenue par une formule variationnelle.

A.M.S. : Classification: 60 F 10, 82 B 44.

1. INTRODUCTION

In the Statistical Mechanics formulation of van der Waals theory, the long range attractive forces between molecules are described by Kac potentials that depend on a positive scaling parameter γ [13]. In dimension one, Cassandro, Orlandi and Presutti [5] proved a metastability property for Ising spin systems with ferromagnetic Kac potentials (see also Bodineau [2]). In this paper we generalize their results to a disordered system which describes an alloy of magnetic (eg Fe) and non-magnetic (eg Au) materials. To investigate the behavior of magnetization, we prove large deviations estimates for the quenched Gibbs measure. Large deviations for conditionally independent identically distributed (iid) lattice systems were introduced by Comets in [6] and a general overview of large deviations methods is given by Seppäläinen in [14]. Kac potentials have already been introduced in the context of disordered systems by Bovier, Gayrard and Picco [3] who proved for the Kac version of the Hopfield model a Lebowitz Penrose theorem for the distribution of the overlap parameters.

The quenched variable y is a sequence of random variables $y = (y_i)_{i \in \mathbb{Z}}$ which take values in $\{0, 1\}$. At site i, there will be a ferromagnetic particle if the occupation number y_i equals 1 and a non-ferromagnetic particle otherwise. Therefore the Hamiltonian of such a system has the following structure

$$H(S) = -\sum_{i,j} J_\gamma(i-j) y_i y_j S_i S_j.$$

We consider an infinite lattice and we study the behavior of the system in the limit as the scaling parameter γ goes to 0. Working with an infinite volume is one of the major problems because we do not have an accurate expression of the Gibbs measure as in the paper written by Eisele and Ellis [8] which treats the case of a finite region. New problems arise in the non-homogeneous case, for example the quenched Gibbs measure is no longer shift invariant and the invariance of the measure by spin flip is also lost, so that the symmetry of the system cannot be used as in the deterministic case. In particular, compared to the deterministic case, here the critical values depend on the dilution. To overtake the difficulty of the infinite volume in this non-homogeneous case, we generalize the methods developed in [5], [2]. This enables us to establish as γ goes to 0 a large deviation principle for the quenched Gibbs measure with a rate function depending on y.

As γ goes to 0 the range of interactions goes to infinity and we recover the mean field theory. We prove that, below the critical temperature, there are two distinct equilibrium values denoted by $\pm m_p$ which depend on the temperature and on some parameter p which rules the distribution of the occupation numbers $\{y_i\}_{i\in\mathbb{Z}}$. Furthermore, for almost all y, the magnetization is close to one of the equilibrium values $\pm m_p$ on blocks of size $[-\gamma^{-k}, \gamma^{-k}]$, for any integer k. Difficulties occur when we want to compute the magnetization on much longer spatial intervals, because the dilution exhibits large deviations which will modify the behavior of the system. By analogy with dynamical systems we will prove that on exponential distances the local magnetization performs a transition from one equilibrium value to the other. The location of this interface could be interpreted in terms of exit time of a stochastic process from a neighborhood of a stable equilibrium position. Many attempts have been made in this direction and we refer the reader to Freidlin and Wentzell [10] for the original theory and to Galves, Olivieri and Vares [11] for a precise description of the phenomenon of metastability. The Markov property of the Gibbs measure, which is peculiar to the one dimension, enables us to adapt Freidlin-Wentzell theory.

From Erdös-Rényi law we observe at exponential distances some intervals which contain a small percentage of magnetic atoms. Intuitively, we guess that the interface occurs more easily in the dilute case because the strength of interactions decreases at locations where the dilution is important. In this paper, we prove that for almost all y the first interface is located at a distance of the order of $\exp(\frac{\Phi}{\gamma})$. The computation of Φ leads us to minimize the sum of two action functionals. The first one rules the presence of non-magnetic particles and the other one could be interpreted as a cost of a leap from one stable value to the other; so that the whole minimization problem is the correct balance between these two rare events.

In section 2 we present the model and state the main results. We establish in section 3 some asymptotic properties for product measures. In section 4 we state a large deviation principle for the Gibbs measure that will be used in section 5 to estimate the location where the first interface occurs.

After this paper was completed, we were advised by A. Bovier, V. Gayrard and P. Picco that they obtained independently results similar to those proven here in the case of Kac-Hopfield model [4]. This model, which has been defined in [3], has different properties than the one we study, in particular they show that the magnetization jumps on a much smaller scale (γ^{-2}).

2. NOTATION AND MAIN RESULTS

2.1. Description of the system

We consider a one-dimensional system such that at each site $i \in \mathbb{Z}$ there is either a spin (with values ± 1) or a non-ferromagnetic particle. In order to describe the random position of the magnetic atoms in terms of occupation numbers, we introduce random variables y_i which take values 0 or 1. We suppose for simplicity that the variables y_i are iid with law given by a Bernoulli measure $\nu = p\delta_1 + (1-p)\delta_0$, where p is a constant in $]0,1]$. The set of quenched variables is denoted by $Y = \{0,1\}^{\mathbb{Z}}$, we put on Y the product measure $\mathbb{P} = \otimes_Z \nu$. Let $\Omega = \{-1,0,1\}^{\mathbb{Z}}$ be the space of configurations $S = \{S_i\}_{\mathbb{Z}}$. A configuration S is a sequence of iid random variables whose distributions depend on the quenched variable y; for each sequence y in Y we introduce $\tilde{\rho}^y = \otimes_{\mathbb{Z}} \rho^{y_i}$ a product measure on Ω, where ρ^{y_i} is defined by

$$\rho^{y_i} = \begin{cases} \delta_0 & \text{if } y_i = 0, \\ \frac{1}{2}(\delta_{-1} + \delta_1) & \text{if } y_i = 1. \end{cases} \tag{2.11}$$

The long range attractive forces between molecules are described by Kac potentials [13] that depend on a positive scaling parameter γ which controls the strength and the range of the potential. Let J be a smooth, even and non negative function supported by $[-1,1]$ such that

$$\int_{\mathbb{R}} J(r)\, dr = 1.$$

DEFINITION 2.1. – *A family of Kac potentials is a family of functions J_γ depending on the scaling parameter γ. These functions are defined in terms of J by the rule:*

$$\textit{for all } r \textit{ in } \mathbb{R}, \qquad J_\gamma(r) = \gamma J(\gamma r). \tag{2.1.2}$$

If Δ is a finite subset of $\mathbb{Z}$, the energy of the configuration $S_\Delta = \{S_i;\ i \in \Delta\}$ given the external condition $\xi = \{\xi_i;\ i \in \Delta^c\}$ is

$$H_\gamma^\Delta(S_\Delta \,|\, \xi) = -\frac{1}{2} \sum_{\substack{i \neq j \\ i,j \in \Delta}} J_\gamma(i-j) S_i S_j - \sum_{\substack{i \in \Delta \\ j \in \Delta^c}} J_\gamma(i-j) S_i \xi_j. \tag{2.1.3}$$

The probability ρ^{y_i} (see (2.1.1)) can be regarded as the distribution of a fictive particle at site i with spin $y_i S_i$. Therefore, we recover interactions of the form

$$\sum J_\gamma(i-j) y_i y_j S_i S_j$$

which have been presented in the introduction.

For each γ, we define at the temperature $\frac{1}{\beta}$ a Gibbs measure on Ω.

DEFINITION 2.2. – *Let y be in Y. For each finite subset Δ in $\mathbb{Z}$ we introduce a probability measure on $\{-1,0,1\}^{\Delta}$*

$$\tilde{\mu}^{y}_{\beta,\gamma,\Delta}(S_\Delta|\xi) = \frac{1}{Z^{y}_{\beta,\gamma}(\xi)} \exp(-\beta H^{\Delta}_{\gamma}(S_\Delta \mid \xi))\Pi_{i\in\Delta}\rho^{y_i}(dS_i), \tag{2.1.4}$$

where $Z^{y}_{\beta,\gamma}(\xi)$ is the normalization factor

$$Z^{y}_{\beta,\gamma}(\xi) = \int \exp(-\beta H^{\Delta}_{\gamma}(S_\Delta \mid \xi))\Pi_{i\in\Delta}\rho^{y_i}(dS_i).$$

The measure $\tilde{\mu}^{y}_{\beta,\gamma,\Delta}(S_\Delta|\xi)$ is called the Gibbs distribution in Δ with boundary condition $\xi = \{\xi_i \mid i \in \Delta^c\}$. In our case, there is a unique measure $\tilde{\mu}^{y}_{\beta,\gamma}$ on Ω which is defined by the conditional probabilities above (Dobrushin-Landford-Ruelle equations).

Rescaling by a factor γ, we will define a continuous version of this system.

DEFINITION 2.3. – *We denote by E the space of magnetic profiles and by $\mathcal{Y}$ the space of environment profiles. The sets E and $\mathcal{Y}$ are subsets of $\mathcal{L}^{\infty}(\mathbb{R}, dr)$, the space of the bounded measurable functions*

$$E = \{\sigma \in \mathcal{L}^{\infty}(\mathbb{R}, dr) \mid \|\sigma\|_{\infty} \leq 1\},$$

and

$$\mathcal{Y} = \{q \in \mathcal{L}^{\infty}(\mathbb{R}, dr) \mid 0 \leq q \leq 1\}.$$

Henceforth, for any bounded interval I in $\mathbb{R}$, we denote by E_I (resp $\mathcal{Y}_I$) the set which contains the restriction at the interval I of the profiles in E (resp $\mathcal{Y}$).

DEFINITION 2.4. – *Let κ_γ be the function from Ω to E which maps the configuration S to the piecewise constant function σ_γ in E defined by*

$$\textit{for all } r \textit{ in } \mathbb{R}, \qquad \sigma_\gamma(r) = S_{[\frac{r}{\gamma}]}. \tag{2.1.5}$$

We also introduce the mapping π_γ from Y to $\mathcal{Y}$ such that

$$\textit{for all } r \textit{ in } \mathbb{R}, \qquad \pi_\gamma(y)(r) = y_{[\frac{r}{\gamma}]}. \tag{2.1.6}$$

We equip these spaces with a weak topology.

DEFINITION 2.5. – *We consider the $\mathcal{L}^2_{loc}$ weak topology τ on E which satisfies*

$$m_n \rightarrow_\tau m \iff \forall L \in \mathbb{R},$$

$$lim_{n\rightarrow 0} m_n|_{[-L,L]} = m|_{[-L,L]} \quad weakly\ in\ \mathcal{L}^2([-L,L], dr)$$

Similarly, we define by τ' the $\mathcal{L}^2_{loc}$ weak topology on $\mathcal{Y}$.

The set E_I (resp $\mathcal{Y}_I$) is endowed with the restriction of the weak topology on $\mathcal{L}^2(I, dr)$.

We say that W is a weak neighborhood of 0 in E (resp $\mathcal{Y}$) defined by the family of functions $\{f_i\}_{i\leq N}$ (with supports included in a compact I) and the parameter ε if

$$f \in W \iff \forall i \leq N, \qquad |< f, f_i >_I| \leq \varepsilon, \tag{2.1.7}$$

where $<,>_I$ denotes the duality bracket of $\mathcal{L}^2(I, dr)$. We say that a set of E (resp $\mathcal{Y}$) has a compact basis C if it is defined by test functions with supports included in C.

DEFINITION 2.6. – *We denote by T the translation operator on E*

$$for\ L\ in\ \mathbb{R},\ for\ \sigma\ in\ E, \qquad T_L(\sigma) = \sigma(. - L). \tag{2.1.8}$$

We will use the same notation for the translation operator on $\mathcal{Y}$.

DEFINITION 2.7. – *We denote by $\mathbb{P}_\gamma$ the image law on $\mathcal{Y}$ of the measure $\mathbb{P}$ under the mapping π_γ.*

For each y in Y we define also by ρ^y_γ (resp $\mu^y_{\beta,\gamma}$) the image law on E of $\tilde{\rho}^y$ (resp $\tilde{\mu}^y_{\beta,\gamma}$) under the mapping κ_γ.

The projection of ρ^y_γ (resp $\mu^y_{\beta,\gamma}$) on E_I is denoted by $\rho^y_{I,\gamma}$ (resp $\mu^y_{\beta,\gamma,I}$).

2.2. Main results

In [8] it is proven that in the limit $\gamma \rightarrow 0^+$, below the critical temperature, there are two distinct thermodynamic phases with different magnetizations. The hypothesis made on $\mathbb{P}$ imply that the environment profiles are locally close to p. So that the mean field equation is

$$m = p \tanh(\beta m). \tag{2.2.1}$$

When β is greater than $\beta_c = \frac{1}{p}$ the above equation has two distinct solutions denoted by $\pm m_p$. Henceforth we fix β greater than β_c.

THEOREM 2.1. – *Let V be a sufficiently small weak neighborhood of 0 in E and (a_γ) be a sequence which satisfies the rule*

$$\lim_{\gamma\to 0} \gamma \ln a_\gamma = 0 \quad \text{and} \quad \lim_{\gamma\to 0} a_\gamma = \infty.$$

We introduce the open sets

$$\lambda = \pm 1, \quad A_\gamma^\lambda = \bigcap_{\substack{-a_\gamma < l < a_\gamma \\ l\in\mathbb{Z}}} (T_l V + \lambda m_p).$$

As β is greater than β_c there is a subset $\mathbb{Y}$ of Y with $\mathbb{P}$-probability one such that

$$\text{(2.2.2)} \qquad \lambda = \pm 1, \quad \lim_{\gamma\to 0} \inf_{y\in\mathbb{Y}} \mu_{\beta,\gamma}^y (A_\gamma^\lambda) = \frac{1}{2}.$$

This Theorem tells us that $\mathbb{P}$-a.s. local magnetization will stay close to one of the equilibrium state $\pm m_p$. We recover for the quenched Gibbs measure properties already proved in the deterministic case [5]. Locally the model behaves like a mean field model, however on larger distances a new phenomenon appears. Indeed, magnetization performs jumps from one equilibrium value to the other.

Let V be a weak neighborhood of 0 for the topology τ (see definition 2.5) which is a cylinder set with basis $[-1, 0]$. We denote by $\mathcal{L}_\gamma^V$, the function from E into $\gamma\mathbb{Z} = \{n\gamma\}_{n\in\mathbb{Z}}$ which associates to each magnetic profile of E the position of the first interface after 0. More explicitly $\mathcal{L}_\gamma^V$ is defined by

DEFINITION 2.8.

$$\mathcal{L}_\gamma^V(\sigma) = \inf\{l \in \gamma\mathbb{Z} \mid \exists l' \in \gamma\mathbb{Z},\ l > l' \geq 0,$$
$$\exists \lambda \in \{\pm 1\},\ \sigma \in (T_{l'}V + \lambda m_p) \cap (T_l V - \lambda m_p)\}.$$

First the profile enters in a neighborhood of the state λm_p and at location $\mathcal{L}_\gamma^V(\sigma)$ the profile hits a neighborhood of $-\lambda m_p$.

THEOREM 2.2. – *Let β be greater than β_c. There is a constant Φ positive and a subset $\mathbb{Y}$ of Y with $\mathbb{P}$-probability one such that for each ε positive*

$$\text{(2.2.3)} \qquad \lim_{\gamma\to 0} \inf_{y\in\mathbb{Y}} \mu_{\beta,\gamma}^y \left(\exp(\frac{\Phi - \varepsilon}{\gamma}) \leq \mathcal{L}_\gamma^V \leq \exp(\frac{\Phi + \varepsilon}{\gamma}) \right) = 1.$$

This statement gives an estimate $\mathbb{P}$-a.s. of the location of the first interface. The constant Φ will be computed in terms of a variational formula which depends on two action functionals. The proof of the previous Theorem involves the following large deviations estimates.

We denote by K the log-Laplace transform of $\nu = p\delta_1 + (1-p)\delta_0$

$$\text{(2.2.4)} \qquad \text{for } t \text{ in } \mathbb{R}, \qquad K(t) = \ln(\nu(\exp(ty))).$$

Well known arguments imply that the family of measures $\mathbb{P}_\gamma$ obeys a large deviation principle (see for instance Baldi [1])

THEOREM 2.3. – *We define the action functional $\mathcal{K}$ by*

$$\text{(2.2.5)} \qquad \textit{for } q \textit{ in } \mathcal{Y}, \qquad \mathcal{K}(q) = \int K^*(q(r))\, dr,$$

where K^* is the Legendre transform of K (2.2.4).

For any subset A of $\mathcal{Y}$ with compact basis, we have

$$-\inf_{q \in A^\circ} \mathcal{K}(q) \le \liminf_{\gamma \to 0} \gamma \ln \mathbb{P}_\gamma(A) \le \limsup_{\gamma \to 0} \gamma \ln \mathbb{P}_\gamma(A) \le -\inf_{q \in \bar{A}} \mathcal{K}(q),$$

where $\overset{\circ}{A}$ is the interior of A and $\bar{A}$ its closure.

From this Theorem, we deduce that environment profiles are close to the constant profile p on intervals of length γ^{-k}, for any integer k. However when observed on intervals of exponential length dilution is not constant, this causes a change of critical values which plays a key role in the proof of Theorem 2.2. As it will become clear from the proofs of section 5, we use only the hypothesis that $\mathbb{P}_\gamma$ obeys a large deviation principle. Thus we can generalize the proofs to a wider class of measures on Y.

We will state now large deviations for the quenched Gibbs measure. Let Λ be the log-Laplace transform of the measure $\frac{1}{2}(\delta_1 + \delta_{-1})$ and Λ^* be the Legendre transform of Λ. We introduce the "entropy" which depends on the dilution parameter q

$$h(q,m) = \begin{cases} 0 & \text{if } m = q = 0, \\ q\Lambda^*(\frac{m}{q}) & \text{if } |m| \le q, \\ \infty & \text{otherwise.} \end{cases}$$

By analogy with the Curie-Weiss model, we define for $\delta > 0$ and q in $[0,1]$

$$\text{(2.2.7)} \qquad \text{for } m \text{ in } [-1,1], \qquad f_q^\delta(m) = \min\left(\frac{1}{\delta}, h(q,m)\right) - \frac{\beta}{2} m^2,$$

Let q be an environment profile, we define an action functional depending on q

DEFINITION 2.9. – *For all δ positive, the δ-rate function is*

$$\text{(2.2.8)} \qquad \text{for } \sigma \text{ in } E, \qquad \mathcal{G}^{q,\delta}(\sigma) = \mathcal{F}^{q,\delta}(\sigma) - \inf_{\sigma \in E} \mathcal{F}^q(\sigma),$$

where

$$\mathcal{F}^{q,\delta}(\sigma) = \int_{\mathbb{R}} \Big(f^{\delta}_{q(r)}(\sigma(r)) - f^{\delta}_{q(r)}(m_q(r)) \Big)\, dr + \frac{\beta}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} J(r - r')(\sigma(r) - \sigma(r'))^2 \, dr dr',$$

where m_q is a function which will be defined in section 4.1.

The constant δ has been introduced for technical reasons. We simply write $\mathcal{F}^q$ or $\mathcal{G}^q$ if $\delta = 0$. The quenched Gibbs measure obeys a large deviation principle

THEOREM 2.4. – *Let ε be a positive constant and q be an environment profile such that $\mathcal{K}(q)$ is finite. For any closed set F in E with compact basis, there is W a neighborhood of q in $\mathcal{Y}$ with compact basis depending on q, F and ε such that*

$$\text{(2.2.9)} \qquad \limsup_{\gamma \to 0} \gamma \ln \Big(\sup_{y \in \pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(F) \Big) \leq - \inf_{\sigma \in F} \mathcal{G}^{q,\varepsilon}(\sigma) + \varepsilon.$$

THEOREM 2.5. – *Let ε be a positive constant and q be an environment profile such that $\mathcal{K}(q)$ is finite. For any open O in E with compact basis, there is W a weak neighborhood of q in $\mathcal{Y}$ with compact basis depending on q, O and ε such that*

$$\text{(2.2.10)} \qquad \liminf_{\gamma \to 0} \gamma \ln \Big(\inf_{y \in \pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(O) \Big) \geq - \inf_{\sigma \in O} \mathcal{G}^q(\sigma) - \varepsilon.$$

Locally the dilution is almost constant and the action functional is $\mathbb{P}$-a.s. equal to $\mathcal{G}^p$. On exponential distances different environment profiles appear, so that we have to consider the functional $\mathcal{G}^q$ for any q. We can now detail the formula for Φ. Let $\mathcal{S}$ be the set of magnetic profiles which jump from one equilibrium value to the other

$$\text{(2.2.11)} \qquad \mathcal{S} = \{\sigma \in E \mid \lim_{r \to -\infty} \sigma(r) = -m_p \text{ and } \lim_{r \to \infty} \sigma(r) = m_p\}.$$

We set

$$\text{(2.2.12)} \qquad \Phi = \inf_{q \in \mathcal{Y}} \Big(\mathcal{K}(q) + \inf_{\sigma \in \mathcal{S}} \mathcal{G}^q(\sigma) \Big),$$

The constant Φ minimizes the cost of a leap over all the disordered configurations.

3. LARGE DEVIATION ESTIMATE FOR THE INDEPENDENT CASE

In this section we give results to control the asymptotic of the measure ρ_γ^y as $\pi_\gamma(y)$ is localized in a subset of $\mathcal{Y}$. Throughout this section we fix I a bounded interval in $\mathbb{R}$, q a profile in $\mathcal{Y}_I$ and ε a positive constant.

3.1. Preliminary result for $\rho_{I,\gamma}^y$

THEOREM 3.1. – *We denote by W a weak neighborhood of 0 in $\mathcal{Y}$ defined by ε and the family of functions $(\Lambda(f_i))_{i\leq N}$. For any environment y in $\pi_\gamma^{-1}(W+q)$, we get as γ goes to 0*

$$\forall i \leq N,$$

$$\left|\gamma \ln \rho_{I,\gamma}^y\left(\exp(\frac{< f_i, \sigma >_I}{\gamma})\right) - \int_I \Lambda(f_i(r))\, q(r)\, dr\right| \leq O(\varepsilon) + O(\gamma),$$

where $\rho_{I,\gamma}^y$ was introduced in definition 2.7 and $O(\gamma)$ converges to 0 as γ goes to 0 and does not depend on y.

In the sequel, notation $O(\cdot)$ means that the function O vanishes as its parameter goes to 0.

Proof. – Let $f = f_i$. Noticing that for y_0 in $\{0,1\}$, we get $\tilde{\rho}^{y_0}(f(S)) = \frac{1}{2}(\delta_1 + \delta_{-1})[f(y_0 S)]$, we get

$$\gamma \ln \rho_{I,\gamma}^y\left(\exp(\frac{< f, \sigma >_I}{\gamma})\right) = \int_I \Lambda\big(f_\gamma(r)\pi_\gamma(y)(r)\big)\, dr, \tag{3.1.1}$$

where

$$\forall r \in \mathbb{R}, \qquad f_\gamma(r) = \sum_{\gamma i \in I} \frac{1}{\gamma}\left(\int_{i\gamma}^{(i+1)\gamma} f(x)\, dx\right) 1_{[i\gamma,(i+1)\gamma[}(r). \tag{3.1.2}$$

Furthermore, we assume for the sake of simplicity that the parameter γ takes values in the set $\{2^{-n} \mid n \in \mathbb{N}\}$; by slight modification of the proof this condition can be dropped.

We note that $\pi_\gamma(y)$ is a piecewise constant function which takes only values 0 or 1. As $\Lambda(0) = 0$ we deduce from (3.1.1) that

$$\gamma \ln \rho_{I,\gamma}^y\left(\exp(\frac{< f, \sigma >_I}{\gamma})\right) = \int_I \Lambda(f_\gamma(r))\pi_\gamma(y)(r)\, dr. \tag{3.1.3}$$

It is obvious to see that the sequence (f_γ) converges to f strongly in $\mathcal{L}^1(I, dr)$. Therefore noticing that $\pi_\gamma(y)$ is bounded for the supremum norm and Λ is Lipschitz continuous, we get for any y

$$\gamma \ln \rho^y_{I,\gamma}\left(\exp(\frac{< f, \sigma >_I}{\gamma})\right) = \int_I \Lambda(f(r))\, \pi_\gamma(y)(r)\, dr + O(\gamma).$$

Since y is in the set $\pi_\gamma^{-1}(W + q)$, the Theorem is complete. □

Before getting into the details, we introduce some notation

DEFINITION 3.1. – *For any bounded set I, the Legendre transform of the functional* $f \to \int_I \Lambda(f(r))\, q(r)\, dr$ *is*

$$\mathcal{I}^q_I(\sigma) = \sup_f(< f, \sigma >_I - \int_I \Lambda(f(r))\, q(r)\, dr),$$

and more precisely

$$\mathcal{I}^q_I(\sigma) = \int_I h(q(r), \sigma(r))\, dr, \tag{3.1.4}$$

where h is defined in (2.2.6).

Later on we will use a truncation of $\mathcal{I}^q_I$. For each δ positive, we define

$$\mathcal{I}^{q,\delta}_I(\sigma) = \int_I \min\left(\frac{1}{\delta}, h(q(r), \sigma(r))\right) dr. \tag{3.1.5}$$

From now on, for any functional $\mathcal{F}$ on E (resp $\mathcal{Y}$) and any subset O of E (resp $\mathcal{Y}$) we denote by $\mathcal{F}(O)$ the infimum of $\mathcal{F}$ over all the elements in O.

3.2. Large deviation estimate for closed cylinders

THEOREM 3.2. – *Let q be any environment profile in $\mathcal{Y}_I$, then for any ε positive and for all closed set F in E_I, there is a weak neighborhood W of q in $\mathcal{Y}_I$ depending on F and ε such that*

$$\limsup_{\gamma \to 0} \gamma \ln\left(\sup_{y \in \pi_\gamma^{-1}(W)} \rho^y_{I,\gamma}(F)\right) \leq -\mathcal{I}^{q,\varepsilon}_I(F) + \varepsilon. \tag{3.2.1}$$

As we are working on the finite volume I, we do not need to impose conditions on the behavior of q at infinity.

Proof. – We denote $\mathcal{I}^{q,\varepsilon}_I(F)$ by λ. Due to the definition of $\mathcal{I}^{q,\varepsilon}_I$, for each σ in F there is a function f in $\mathcal{L}^2(I, dr)$ such that,

$$< f, \sigma >_I - \int_I \Lambda(f(r)) q(r)\, dr > \lambda - \varepsilon.$$

The set E_I is compact for the weak topology of $\mathcal{L}^2(I, dr)$, therefore, we cover the compact F with a finite family of open sets $(O_i)_{i\leq N}$ which are defined by

$$O_i = \{\sigma \mid < f_i, \sigma >_I - \int_I \Lambda(f_i(r))q(r)\, dr > \lambda - \varepsilon\}.$$

Noticing that $\rho^y_{I,\gamma}(F) \leq \sum_{i=1}^N \rho^y_{I,\gamma}(O_i)$, we get

$$\rho^y_{I,\gamma}(F) \leq \exp\left(\frac{\varepsilon - \lambda}{\gamma}\right) \sum_{i=1}^N \rho^y_{I,\gamma} \times \left(\exp \frac{1}{\gamma}[< f_i, \sigma >_I - \int_I \Lambda(f_i(r))q(r)\, dr]\right).$$

Let W be a weak neighborhood of q defined by ε and $(\Lambda(f_i))_{i\leq N}$. From Theorem 3.1 we deduce the upper bound

$$\sup_{y\in\pi_\gamma^{-1}(W)} \rho^y_{I,\gamma}(F) \leq N \exp\left(\frac{-\lambda + 2\varepsilon + O(\gamma)}{\gamma}\right).$$

By taking the limit as γ goes to 0, we derive (3.2.1). □

Now we must deal with the case of open sets.

3.3. Large deviation estimate for open sets

THEOREM 3.3. – *Let q be any environment profile in $\mathcal{Y}_I$, then for any ε positive and for each open set O in E_I, there exists a weak neighborhood W of q in $\mathcal{Y}_I$ depending on O and ε such that*

$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \rho^y_{I,\gamma}(O)\right) \geq -\mathcal{I}^q_I(O) - \varepsilon. \tag{3.3.1}$$

Proof. – It is enough to suppose that $\mathcal{I}^q_I(O)$ is finite. Let σ_0 be an element in O such that $\mathcal{I}^q_I(O) \geq \mathcal{I}^q_I(\sigma_0) - \varepsilon$. We denote by U a neighborhood of σ_0 included in O.

As $\mathcal{I}^q_I$ is strictly convex there is a function f such that for any σ different from σ_0 (in the sense of the Lebesgue measure),

$$\mathcal{I}^q_I(\sigma) > \mathcal{I}^q_I(\sigma_0) + < f, \sigma - \sigma_0 >_I .$$

To simplify the notation we write

$$\int_I \Lambda(f(r))q(r)\, dr =< \Lambda(f), q >_I .$$

Combining definition 3.1 and the inequality above, we check that

$$< \Lambda(f), q >_I = \sup_{\sigma \in E_I} (< \sigma, f >_I - \mathcal{I}_I^q(\sigma)) = < \sigma_0, f >_I - \mathcal{I}_I^q(\sigma_0).$$

We define a new probability measure by

$$d\tilde{\rho}_{f,y} = \exp\left(\frac{1}{\gamma}(< f, \sigma >_I - < \Lambda(f_\gamma), \pi_\gamma(y) >_I)\right) d\rho_{I,\gamma}^y(\sigma),$$

where f_γ has been introduced in (3.1.2). For simplicity we omit the dependence on γ in the notation above. Noticing that U is included in O and using the definition of $\tilde{\rho}_{f,y}$ we get

$$\rho_{I,\gamma}^y(O) \geq \int_U \exp\left(-\frac{1}{\gamma}(< f,\sigma >_I - < \Lambda(f_\gamma), \pi_\gamma(y) >_I)\right) d\tilde{\rho}_{f,y}(\sigma),$$

this leads to

$$\rho_{I,\gamma}^y(O) \geq \tilde{\rho}_{f,y}(U) \min_{\sigma \in U} \left[\exp\left(-\frac{1}{\gamma}(< f,\sigma >_I - < \Lambda(f_\gamma), \pi_\gamma(y) >_I)\right)\right].$$

At this point, we intend to introduce some conditions on W and U

$$\forall y \in \pi_\gamma^{-1}(W), \qquad < \Lambda(f), \pi_\gamma(y) - q >_I \geq -\varepsilon,$$

and we suppose that

$$\forall \sigma \in U, \qquad < f, \sigma >_I \leq < f, \sigma_0 >_I + \varepsilon.$$

From the previous inequalities we derive

$$\forall \sigma \in U, \qquad < f, \sigma >_I - < \Lambda(f), \pi_\gamma(y) >_I \leq \mathcal{I}_I^q(\sigma_0) + 2\varepsilon,$$

therefore for any y in $\pi_\gamma^{-1}(W)$

$$\tilde{\rho}_{f,y}(O) \geq \tilde{\rho}_{f,y}(U) \exp\left(-\frac{\mathcal{I}_I^q(\sigma_0) + 2\varepsilon}{\gamma}\right). \tag{3.3.2}$$

It remains to prove that $\tilde{\rho}_{f,y}(U^c)$ tends to 0 as γ goes to 0. We get

$$\gamma \ln \tilde{\rho}_{f,y}\left(\exp(\frac{1}{\gamma} < g, \sigma >_I)\right)$$
$$= < \Lambda(f_\gamma + g_\gamma), \pi_\gamma(y) >_I - < \Lambda(f_\gamma), \pi_\gamma(y) >_I .$$

The Legendre transform of $g \to < \Lambda(f + g), q >_I - < \Lambda(f), q >_I$ is

$$\mathcal{I}_f(\sigma) = \mathcal{I}_I^q(\sigma) - < f, \sigma >_I + < \Lambda(f), q >_I .$$

From the definition of f we see that $\mathcal{I}_f(\sigma)$ is positive as soon as σ is different from σ_0 (in the sense of the Lebesgue measure). As U^c is compact and $\mathcal{I}_f$ is lower semi-continuous, we check that $\mathcal{I}_f(U^c)$ is positive. We are dealing here with a situation like the one discussed in Theorem 3.2. Thus there is W a weak neighborhood in $\mathcal{Y}_I$ depending on ε and U such that

$$\lim_{\gamma \to 0} \inf_{y \in \pi_\gamma^{-1}(W)} \tilde{\rho}_{f,y}(U) = 1.$$

By using (3.3.2) the statement follows. □

4. LARGE DEVIATION PRINCIPLE FOR THE GIBBS MEASURE

In this section, we prove Theorems 2.4 and 2.5.

4.1. Preliminary results

First we compute for any environment profile q the magnetization profiles which minimize the functional $\mathcal{F}^q$ (see definition 2.9). We denote by m_q the non negative profile in E which minimizes $f^{\delta}_{q(r)}$ for all r. More explicitly, the profile m_q satisfies the equation below

$$\text{(4.1.1)} \qquad \text{for } r \text{ in } \mathbb{R}, \qquad m_q(r) = q(r)\tanh(\beta m_q(r)).$$

If q equals p we recover the mean field equation (2.2.1), however dilution is not usually constant and there are not two constant equilibrium values as in the classical mean field model (see Eisele and Ellis [8]). If $\beta q(r) \leq 1$, we check that $m_q(r)$ equals 0 and in the other case, equation (4.1.1) has two distinct solutions $\pm m_q(r)$.

For any pair $\lambda = (\lambda^+, \lambda^-)$ in $\{-1,1\}^2$, we introduce the profile

$$\text{(4.1.2)} \qquad m_q^{\lambda} = \lambda^+ m_q 1_{\mathbb{R}^+} + \lambda^- m_q 1_{\mathbb{R}^-}.$$

We denote by $\sigma_I \otimes \xi$ the extension of the profile σ by the profile ξ outside I.

DEFINITION 4.1. – *Let δ be a positive constant and q a profile in $\mathcal{Y}_{\mathcal{I}}$. For any bounded interval I we define the functional $\mathcal{F}_I^{q,\delta}$ on E_I by*

$$\text{(4.1.3)} \qquad \textit{for } \sigma \textit{ in } E_I, \qquad \mathcal{F}_I^{q,\delta}(\sigma) = \mathcal{I}_I^{q,\delta}(\sigma) - \frac{\beta}{2} < J * \sigma, \sigma >_I,$$

where $\mathcal{I}_I^{q,\delta}$ has been introduced in (3.1.5).

LEMMA 4.1. – *Let q be an environment profile such that $q - p$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$, then we have for each (λ^-, λ^+) and δ smaller than 1*

$$\text{(4.1.4)} \quad \mathcal{F}^{q,\delta}(\sigma_I \otimes m_q^{\lambda}) - \mathcal{F}^q(m_q) = \mathcal{F}^{q,\delta}_{I+\delta I}(\sigma_I \otimes m_q^{\lambda}) - \mathcal{F}^q_{I+\delta I}(m_q).$$

The proof is a straightforward computation and is left to the reader.

LEMMA 4.2. – *Let q be an environment profile such that $\mathcal{K}(q)$ is finite (see Theorem 2.3). For any positive ε, there is some constant a sufficiently large such that*

$$\text{(4.1.5)} \qquad \inf_{\sigma \in E} \mathcal{F}^q(\sigma_{[-a,a]} \otimes m_q) \leq \mathcal{F}^q(E) + \varepsilon.$$

Proof. – We fix σ_0 a profile in E such that $\mathcal{F}^q(\sigma_0) \le \mathcal{F}^q(E) + \frac{\varepsilon}{2}$. Noticing that $\mathcal{F}^q(\sigma_0) \ge \mathcal{F}^q(|\sigma_0|)$, we can suppose that $\sigma_0 \ge 0$. As $\mathcal{K}(q)$ is finite, we check that $p - q$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$. We introduce α such that $(p-\alpha)\beta > 1$. There is a constant d such that for any q which satisfies $|q - p| \le \alpha$ we have

$$\text{(4.1.6)} \qquad \forall x \in \mathbb{R}, \qquad f_q(x) - f_q(m_q) \ge d(|x| - m_q)^2,$$

where f_q was defined in (2.2.7). The preceding inequality tells us that $\sigma_0 - m_q$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$. This remark ensures that $\mathcal{F}^q(\sigma_0|_{[-a,a]} \otimes m_q)$ converges to $\mathcal{F}^q(\sigma_0)$ as a goes to infinity. The Lemma follows. □

As a consequence of Theorems 3.2 and 3.3, we get

THEOREM 4.1. – *Let I be a bounded interval in $\mathbb{R}$. We fix q in $\mathcal{Y}_I$ and ε a positive constant. For each closed subset F in E_I, there is W a weak neighborhood of q in $\mathcal{Y}_I$ such that*

$$\limsup_{\gamma\to 0} \gamma \ln\left(\sup_{y\in\pi_\gamma^{-1}(W)} \rho^y_{I,\gamma}\left(1_F \exp\left(\frac{\beta}{2\gamma} < J * \sigma_I, \sigma_I >\right)\right)\right) \le -\mathcal{F}_I^{q,\varepsilon}(F) + \varepsilon,$$

where W depends on q, F and ε.

For each open subset O in E_I, there exists W a weak neighborhood of q in $\mathcal{Y}_I$ such that

$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \rho^y_{I,\gamma}\left(1_O \exp(\frac{\beta}{2\gamma} < J * \sigma_I, \sigma_I >)\right)\right) \ge -\mathcal{F}_I^q(O) - \varepsilon,$$

where W depends on q, O and ε.

The proof is similar to the one of Varadhan's Theorem (see for instance Deuschel and Stroock [7] Theorem 2.1.10).

4.2. Large deviation estimate for closed sets (proof of Theorem 2.4)

Before going on, we introduce a family of weak neighborhoods in E. We note that $\mathcal{H} = \{J * \sigma 1_{[0,1]} \mid \sigma \in E\}$ is a subset in $\mathcal{C}([0,1])$, bounded for the uniform norm. Noticing that J is continuous and each σ in E is uniformly bounded, we check that $\mathcal{H}$ is uniformly equicontinuous. From this and from the Ascoli's theorem, there exists, for each ε positive, a finite set of continuous functions $\{g_i\}_{i\le N}$ which satisfy the condition

$$\mathcal{H} \subset \bigcup_{i\le N} \{f \mid \forall x \in [0,1], \qquad |f(x) - g_i(x)| < \varepsilon\}.$$

We introduce a class of subsets of E which depend on the functions $\{g_i\}_{i\leq N}$

DEFINITION 4.2. – *Let V_ε be the weak neighborhood of 0 in E defined by*

$$f \in V_\varepsilon \iff \forall i \leq N, \quad |< g_i, f >| < \varepsilon. \tag{4.2.1}$$

Throughout this section, we fix an environment profile q in $\mathcal{Y}$ such that $\mathcal{K}(q)$ is finite (see Theorem 2.3).

DEFINITION 4.3. – *For any (λ^+, λ^-) in $\{-1, 1\}^2$ and any pair of integers (l^+, l^-) we introduce the closed set $D_{\lambda^+,\lambda^-}^{l^+,l^-}(\varepsilon)$ which contains the profiles close to $\lambda^+ m_q$ around the location l^+ and close to $\lambda^- m_q$ around the location $-l^-$*

$$D_{\lambda^+,\lambda^-}^{l^+,l^-}(\varepsilon) = (T_{(l^+)}\bar{V}_\varepsilon + m_q^\lambda) \cap (T_{(-l^- -1)}\bar{V}_\varepsilon + m_q^\lambda),$$

where $\bar{V}_\varepsilon$ is the closure of V_ε and m_q^λ is defined in (4.1.2).

Proof of Theorem 2.4. – Let a be a positive constant, we suppose that F has a basis included in $[-a, a]$.

Step 1 : For the moment, we fix two positive integers a and L ($a < L$). Let l^+, l^- be in $[a, L]$ and (λ^+, λ^-) be in $\{-1, 1\}^2$. Let D be a shorthand of $D_{\lambda^+,\lambda^-}^{l^+,l^-}(\varepsilon)$.

We will follow the method used by Cassandro et al [5]. For any γ the uniqueness of the Gibbs measure implies that $\mu_{\beta,\gamma}^y$ is the limit of the Gibbs distribution with free boundary conditions (see definition 2.2).

$$\mu_{\beta,\gamma}^y(F \cap D) = \lim_{I \to \mathbb{R}} \mu_{\beta,\gamma,I}^y(F \cap D).$$

For any bounded interval I, we denote by S_I the discrete configuration $\{S_i \mid \gamma i \in I\}$ and by σ_I the associated profile $\kappa_\gamma(S_I)$. Let Δ be $[-l^-, l^+]$, we can write a continuous version of the Hamiltonian $H_\gamma^{\gamma\Delta}$ introduced in (2.1.3) (where $\gamma\Delta$ is the discrete set $\{i \in \mathbb{Z} \mid \gamma i \in \Delta\}$)

$$\bar{H}_\gamma(\sigma_\Delta|\sigma_{\Delta^c}) = -\frac{1}{2\gamma} < J * \sigma_\Delta, \sigma_\Delta + 2\sigma_{\Delta^c} > .$$

The difference between $H_\gamma^{\gamma\Delta}(S_{\gamma\Delta}|S_{(\gamma\Delta)^c})$ and $\bar{H}_\gamma(\sigma_\Delta|\sigma_{\Delta^c})$ is bounded by $|\Delta|^2$. By using the continuous version of the energy, we get

(4.2.2)
$$\mu^y_{\beta,\gamma,I}(F\cap D) \leq \frac{1}{Z_I} \max_{(S_{\Delta^c}\in D)} \rho^y_{\Delta,\gamma}\Big(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, \sigma_{\Delta^c}))\Big)$$
$$\rho^y_{I/\Delta,\gamma}\Big(1_{S_{I/\Delta}\in D}\exp(-\beta H_\gamma(S_{I/\Delta}))\Big)\exp(|\Delta|^2).$$

Using the properties of D, we modify the external conditions with a slight error

(4.2.3)
$$\mu^y_{\beta,\gamma,I}(F\cap D) \leq \frac{1}{Z_I} \rho^y_{\Delta,\gamma}\Big(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m^\lambda_q))\Big)$$
$$\rho^y_{I/\Delta,\gamma}\Big(1_{S_{I/\Delta}\in D}\exp(-\beta H_\gamma(S_{I/\Delta}))\Big)\exp(\frac{O(\varepsilon)}{\gamma}+|\Delta|^2).$$

As a consequence of the finite range interactions we see that configurations $S_{]-\infty,-l^-]}$ and $S_{[l^+,\infty[}$ do not interact. This implies

(4.2.4)
$$\sum_{S_{I/\Delta}\in D_{\lambda^+,\lambda^-}} \exp(-\beta H_\gamma(S_{I/\Delta})) = \sum_{S_{I/\Delta}\in D_{\lambda^+,\lambda^+}} \exp(-\beta H_\gamma(S_{I/\Delta})),$$

where we denote $D^{l^+,l^-}_{\lambda^+,\lambda^+}(\varepsilon)$ by D_{λ^+,λ^+}.

Therefore, combining (4.2.4) and (4.2.3) we get

$$\mu^y_{\beta,\gamma,I}(F\cap D) \leq \tfrac{1}{Z_I} \rho^y_{\Delta,\gamma}\Big(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m^\lambda_q))\Big)$$
$$\rho^y_{I/\Delta,\gamma}\Big(1_{S_{I/\Delta}\in D_{\lambda^+,\lambda^+}}\ \exp(-\beta H_\gamma(S_{I/\Delta}))\Big)\exp\left(\tfrac{O(\varepsilon)}{\gamma}+|\Delta|^2\right),$$

this leads to

$$\mu^y_{\beta,\gamma,I}(F\cap D)$$
$$\leq \mu^y_{\beta,\gamma,I}(D_{\lambda^+,\lambda^+})\frac{\rho^y_{\Delta,\gamma}(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m^\lambda_q)))}{\rho^y_{\Delta,\gamma}(\exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q)))}\exp\left(\frac{O(\varepsilon)}{\gamma}+|\Delta|^2\right).$$

By taking the limit as I goes to $\mathbb{R}$, we get

(4.2.5)
$$\mu^y_{\beta,\gamma}(F\cap D)$$
$$\leq \mu^y_{\beta,\gamma}(D_{\lambda^+,\lambda^+})\frac{\rho^y_{\Delta,\gamma}(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m^\lambda_q)))}{\rho^y_{\Delta,\gamma}(\exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q)))}\exp\left(\frac{O(\varepsilon)}{\gamma}+|\Delta|^2\right).$$

Noticing that $\mu^y_{\beta,\gamma}(D_{\lambda^+,\lambda^+}) \leq 1$, we have reduced the problem to the computation of

$$\rho^y_{\Delta,\gamma}(1_F \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q^\lambda))) \qquad \text{and} \qquad \rho^y_{\Delta,\gamma}(\exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q))).$$

Theorem 4.1 implies that there is a neighborhood W of q in $\mathcal{Y}$ with basis Δ such that

$$\limsup_{\gamma \to 0} \gamma \ln \left(\sup_{y \in \pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(F \cap D) \right) \leq - \inf_F \mathcal{F}^{q,\varepsilon}_\Delta(\sigma \otimes m_q^\lambda) + \inf \mathcal{F}^q_\Delta(\sigma \otimes m_q) + 3\varepsilon,$$

by applying Lemma 4.1, we see that

(4.2.6)
$$\limsup_{\gamma \to 0} \gamma \ln \left(\sup_{y \in \pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(F \cap D) \right) \leq - \inf_{\sigma \in F} \mathcal{F}^{q,\varepsilon}(\sigma_\Delta \otimes m_q^\lambda) + \inf_{\sigma \in E} \mathcal{F}^q(\sigma_\Delta \otimes m_q) + 3\varepsilon.$$

Step 2 : We define $U_{a,L}$ by

$$U_{a,L} = \left(\bigcup_{\substack{a < l^+ < L \\ a < l^- < L}} \bigcup_{\lambda^+,\lambda^-} D^{\lambda^+,\lambda^-}_{l^+,l^-} \right)^c$$

We note that the profiles in $U_{a,L}$ are not close to the profiles $\pm m_q$ on intervals $[a, L]$ and $[-L, -a]$. By applying (4.2.6), we get for a suitable constant C

$$\text{(4.2.7)} \qquad \limsup_{\gamma \to 0} \gamma \ln \left(\sup_{y \in \pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(U_{a,L}) \right) \leq -\mathcal{F}^{q,\varepsilon}(U_{a,L}) + C.$$

where W is a neighborhood of q with basis $[-L, L]$.

LEMMA 4.3. – *When a is fixed and $\mathcal{K}(q)$ is finite, we get*

$$\lim_{L \to \infty} \mathcal{F}^{q,\varepsilon}(U_{a,L}) = \infty.$$

The proof is analogous to the one given in [2].

Step 3 :

Collecting all the previous bounds we will derive the statement (2.2.9).

$$\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(F)\Big)$$
$$\leq \max\Bigg(\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(U_{a,L})\Big),$$
$$\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(F\cap U^c_{a,L})\Big)\Bigg),$$

where

$$\mu^y_{\beta,\gamma}(F\cap U^c_{a,L}) = \sum_{\substack{a<l^+<L\\ a<l^-<L}} \sum_{\lambda^+,\lambda^-} \mu^y_{\beta,\gamma}(F\cap D^{\lambda^+,\lambda^-}_{l^+,l^-}).$$

Combining the results of step 1 and step 2, we have

$$\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W_L)} \mu^y_{\beta,\gamma}(F)\Big) \leq \max\Bigg(-\mathcal{F}^{q,\varepsilon}(U_{a,L}) + C, \varepsilon$$
$$+ \max_{\substack{a\leq l^+\leq L\\ a\leq l^-\leq L}} \Big[-\inf_F \mathcal{F}^{q,\varepsilon}(\sigma_{[-l^-,l^+]}\otimes m^\lambda_q) + \inf \mathcal{F}^q(\sigma_{[-l^-,l^+]}\otimes m_q)\Big]\Bigg),$$

where W_L is a neighborhood of q in $\mathcal{Y}$ of basis $[-L, L]$.

When the constant a is sufficiently large, we deduce from the equation above and from Lemma 4.2 that

$$\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W_L)} \mu^y_{\beta,\gamma}(F)\Big)$$
$$\leq \max\left(-\mathcal{F}^{q,\varepsilon}(U_{a,L}) + C, -\mathcal{F}^{q,\varepsilon}(F) + \mathcal{F}^q(E)\right) + 2\varepsilon.$$

When a is fixed, Lemma 4.3 implies the existence of a suitable constant L_0 such that

$$\limsup_{\gamma\to 0} \gamma \ln\Big(\sup_{y\in\pi_\gamma^{-1}(W_{L_0})} \mu^y_{\beta,\gamma}(F)\Big) \leq -\mathcal{F}^{q,\varepsilon}(F) + \mathcal{F}^q(E) + 2\varepsilon.$$

Therefore, Theorem 2.4 follows. □

4.3. Large deviation estimate for open sets (proof of Theorem 2.5)

Proof. – In this section we exploit the same methods as the ones used before to prove the large deviation inequality for open sets. As in Theorem 2.4 we suppose that q is an environment profile such that $\mathcal{K}(q)$ is finite.

Step 1 :

First we assume that O is a neighborhood of the profile $\sigma_{[-R,R]} \otimes m_q^\lambda$ ($R > 0$). Let L be greater than R, we denote $[-L, L]$ by Δ and $\sigma_{[-R,R]} \otimes m_q^\lambda$ by σ_R. Let D° be the interior of $D^{R,R}_{\lambda^+,\lambda^-}(\varepsilon)$ (see definition 4.3). We proceed as before and deduce from (4.2.5)

(4.3.1)

$$\mu^y_{\beta,\gamma}(O \cap \overset{\circ}{D}) \geq \mu^y_{\beta,\gamma}(\overset{\circ}{D}_{\lambda^+,\lambda^+}) \frac{\rho^y_{\Delta,\gamma}\Big(1_O \exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q^\lambda))\Big)}{\rho^y_{\Delta,\gamma}\Big(\exp(-\beta \bar{H}_\gamma(\sigma_\Delta \,|\, m_q))\Big)} \exp(\frac{O(\varepsilon)}{\gamma} - |\Delta|^2),$$

where $D^\circ_{\lambda^+,\lambda^+}$ is the interior of $D^{R,R}_{\lambda^+,\lambda^+}(\varepsilon)$.

We will prove now that there is W a neighborhood of q in $\mathcal{Y}$ and a positive constant c such that

$$\liminf_{\gamma\to 0}\left(\inf_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(\overset{\circ}{D}_{\lambda^+,\lambda^+})\right) \geq c. \tag{4.3.2}$$

In the deterministic case the above statement follows immediately from the large deviation principle, whereas in the non deterministic case, it could be that

$$\mathcal{F}^q[(\overset{\circ}{D}_{\lambda^+,\lambda^+} \cup \overset{\circ}{D}_{\lambda^-,\lambda^-})^c] = 0$$

if q equals 0 on some interval. We need to use FKG inequality to avoid this difficulty (for an overview of moment inequalities see for instance Ellis [9]). We introduce the subset V of E

$$V = \{\sigma \,|\, <\sigma - \frac{m_p}{2}>_{[0,1]} > 0\}.$$

We can assume without any restriction that $D^\circ_{\lambda^+,\lambda^+}$ is included in the set $T_R V \cap T_{-R} V$.

Noticing that $q - p$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$, we deduce from Theorem 2.4 that for R sufficiently large there is a positive constant c' such that

$$\liminf_{\gamma\to 0}\left(\inf_{y\in\pi_\gamma^{-1}(W)} \min\left[\mu^y_{\beta,\gamma}(T_R V), \mu^y_{\beta,\gamma}(T_{-R} V)\right]\right) \geq c', \tag{4.3.3}$$

where W is some neighborhood of q.

Noticing that J is a summable ferromagnetic interaction on $\mathbb{Z}$ and that the indicator functions of $T_{\pm R}V$ are non decreasing, the FKG inequality leads to

$$\mu^y_{\beta,\gamma}(T_R V \cap T_{-R}V) \geq \mu^y_{\beta,\gamma}(T_R V)\mu^y_{\beta,\gamma}(T_{-R}V). \tag{4.3.4}$$

This enables us to deal with the product of the probabilities of two events which we can control with the large deviation principle. By applying Theorem 2.4, we get

$$\lim_{\gamma\to 0} \sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}\left((T_R V \cap T_{-R}V) \cap (\overset{\circ}{D}_{\lambda^+,\lambda^+})^c\right) = 0, \tag{4.3.5}$$

hence, we deduce the statement (4.3.2) from inequality (4.3.4).

We apply Theorem 4.1 to inequality (4.3.1) and we check that for some neighborhood W of q, we have

$$\begin{aligned}\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(O)\right) \\ \geq -\inf_O \mathcal{F}^q_\Delta(\sigma'_\Delta \otimes m^\lambda_q) + \inf \mathcal{F}^{q,\varepsilon}_\Delta(\sigma'_\Delta \otimes m_q) - \varepsilon,\end{aligned}$$

this implies

$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(O)\right) \geq -\mathcal{G}^q(\sigma_R) - \varepsilon. \tag{4.3.6}$$

Step 2 :

Let O be any cylinder set. Since $\mathcal{K}(q)$ is finite, we check the following Lemma (see for instance [2])

LEMMA 4.4. – *For any ε positive there is a profile σ in O such that $\sigma - m^\lambda_q$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$ for some λ in $\{-1,1\}^2$ and σ satisfies*

$$\mathcal{G}^q(\sigma) \leq \mathcal{G}^q(O) + \varepsilon.$$

As in the first step, we introduce the profile σ_R (σ is defined in the above Lemma). Since R is sufficiently large σ_R belongs to O so we get from (4.3.6)

$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(O)\right) \geq -\mathcal{G}^q(\sigma_R) - \varepsilon. \tag{4.3.7}$$

By taking the limit as R tends to infinity we have (because of Lemma 4.4)

$$\lim_{R\to\infty} \mathcal{G}^q(\sigma_R) = \mathcal{G}^q(\sigma).$$

The Theorem 2.5 follows. □

4.4. Shape of the profiles on small regions (Theorem 2.1)

Proof. – We will check that at the scale a_γ the environment profiles are approximately constant and equal to p. This remark will imply that locally the Gibbs measure $\mu^y_{\beta,\gamma}$ obeys $\mathbb{P}$-a.s. a large deviation principle with rate function $\mathcal{G}^p$.

We denote $(A^1_\gamma \cup A^{-1}_\gamma)^c$ by B_γ. As the system is symmetric under spin flip, it suffices to prove

$$\text{(4.4.1.)} \qquad \lim_{\gamma\to 0} \sup_{y\in Y} \mu^y_{\beta,\gamma}(B_\gamma) = 0.$$

We partition B_γ into two subsets

$$\begin{cases} B^1_\gamma = \{\sigma \mid \exists l \in \mathbb{Z},\ |l| < a_\gamma, \qquad \sigma \in (T_l V - m_p)^c \cap (T_l V + m_p)^c\} \\ B^2_\gamma = \{\sigma \mid \exists \lambda \in \{\pm 1\}, \exists l \in \mathbb{Z}, |l| < a_\gamma, \\ \qquad\qquad\qquad \sigma \in (T_l V + \lambda m_p) \cap (T_{(l+1)} V - \lambda m_p)\}. \end{cases}$$

First we treat the case of B^1_γ. We have

$$\mathbb{P}\left[\mu^y_{\beta,\gamma}(B^1_\gamma)\right] \le \sum_{l=-a_\gamma}^{a_\gamma} \mathbb{P}\left[\mu^y_{\beta,\gamma}((T_l V + m_p)^c \cap (T_l V - m_p)^c)\right].$$

From Theorem 2.4, there is W a weak neighborhood of p with compact basis such that

$$\text{(4.4.2)} \quad \forall \varepsilon > 0,\ \exists \gamma_0 > 0 \text{ such that } \forall \gamma < \gamma_0,$$
$$\sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}((V + m_p)^c \cap (V - m_p)^c) \le \exp\left(-\frac{\mathcal{G}^p((V + m_p)^c \cap (V - m_p)^c) - \varepsilon}{\gamma}\right),$$

and $\mathcal{G}^p((V + m_p)^c \cap (V - m_p)^c)$ is positive.

On the other hand, we get from the shift invariance of the measure $\mathbb{P}_\gamma$

$$\mathbb{P}\left(\left\{\pi_\gamma(y) \in \bigcup_{l=-a_\gamma}^{a_\gamma} T_l W^c\right\}\right) \le 2a_\gamma \mathbb{P}_\gamma[W^c],$$

recall that $\mathbb{P}_\gamma$ is the image law of the probability $\mathbb{P}$ on $\mathcal{Y}$.

By applying Theorem 2.3 we check that

$$1 - \mathbb{P}\left(\left\{\pi_\gamma(y) \in \bigcap_{l=-a_\gamma}^{a_\gamma} T_l W\right\}\right) \le \exp\left(-\frac{c_0}{\gamma}\right), \tag{4.4.3}$$

where c_0 is a positive constant.

Combining (4.4.2) and (4.4.3) we check that $\mathbb{P}(\mu^y_{\beta,\gamma}(B^1_\gamma))$ converges exponentially fast to 0.

Noticing that for each element σ in B^2_γ there is some constant r in $[-a_\gamma, a_\gamma]$ such that σ is not in $T_r V \pm m_p$. we derive as in the previous case

$$\exists c > 0,\ \exists \gamma_0 > 0 \text{ such that } \forall \gamma < \gamma_0, \qquad \mathbb{P}[\mu^y_{\beta,\gamma}(B^2_\gamma)] \le \exp(-\frac{c}{\gamma}).$$

Collecting the previous bounds we deduce that the event

$$\{y \text{ such that } \mu^y_{\beta,\gamma}(B_\gamma) > \delta\}$$

occurs with exponential small $\mathbb{P}$-probability. By applying Borel-Cantelli lemma we get (4.4.1) from which the Theorem follows. □

5. LOCATION OF THE INTERFACE (THEOREM 2.2)

5.1. Notation

The location of the first interface is given by the function $\mathcal{L}^V_\gamma$ (see definition 2.8). We also define the location where a profile begins to leap

$$\mathcal{L}^{V\prime}_\gamma(\sigma) = \sup\{l \in \gamma\mathbb{Z} \,|\, \mathcal{L}^V_\gamma > l \ge 0,\ \exists\lambda,\ \sigma \in (T_l V + \lambda m_p) \cap (T_{\mathcal{L}^V_\gamma} V - \lambda m_p)\}.$$

The change of phases begins at $\mathcal{L}^{V\prime}_\gamma$ and ends at $\mathcal{L}^V_\gamma$. The functions $\mathcal{L}^V_\gamma$ and $\mathcal{L}^{V\prime}_\gamma$ depend on V, but to keep the notation simple this dependence is suppressed. We suppose that V is sufficiently small and henceforth we fix it.

DEFINITION 5.1. – *Let $\mathcal{M}$ be the set of the profiles which begin to leap at location 0*

$$\mathcal{M} = \{\sigma \in E \mid \mathcal{L}'_\gamma(\sigma) = 0,\ \mathcal{L}_\gamma(\sigma) < \infty\}, \tag{5.1.1}$$

and for each positive constant R *we introduce the subset of* $\mathcal{M}$

$$\mathcal{M}_R = \{\sigma \in E \mid \mathcal{L}'_\gamma(\sigma) = 0,\ \mathcal{L}_\gamma(\sigma) < R\}. \tag{5.1.2}$$

We check that

$$\Phi = \inf_{q \in \mathcal{Y}} (\mathcal{K}(q) + \mathcal{G}^q(\mathcal{M})), \tag{5.1.3}$$

where Φ was defined in (2.2.12). Finally we introduce

$$\mathcal{A}_\Phi = \{q \in \mathcal{Y} \text{ such that } \mathcal{K}(q) \le \Phi + 1\},$$

noticing that $\mathcal{K}$ is a good rate function we deduce that $\mathcal{A}_\Phi$ is a compact of $\mathcal{Y}$.

5.2. Estimate of $\mathcal{L}_\gamma$

Proof of Theorem 2.2.

Throughout this proof we fix a positive constant ε.

Step 1 : The lower bound will be complete once we show that

$$\lim_{\gamma \to 0} \sup_{y \in Y} \mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \le \exp\left(\frac{\Phi - \varepsilon}{\gamma}\right)\right) = 0. \tag{5.2.1}$$

First we will derive an estimate of a leap as the environment profile is in a neighborhood of a profile in $\mathcal{A}_\Phi$ denoted by q. From the definition of $\mathcal{A}_\Phi$, we note that $q - p$ belongs to $\mathcal{L}^2(\mathbb{R}, dr)$. The profiles which are in the set $\{\mathcal{L}'_\gamma = 0,\ \mathcal{L}_\gamma > R\}$ are far from $\pm m_p$ on interval $[0, R]$. Therefore combining Lemma 4.3 and inequality (4.2.7), there is $\gamma_0(q)$, a constant $R(q)$ sufficiently large and a neighborhood $W(q)$ of q such that

$$\forall \gamma < \gamma_0(q), \qquad \sup_{y \in \pi_\gamma^{-1}(W(q))} \mu^y_{\beta,\gamma}(\mathcal{L}'_\gamma = 0,\ \mathcal{L}_\gamma > R(q)) \le \exp\left(-\frac{\Phi + 1}{\gamma}\right). \tag{5.2.2}$$

For a given $R(q)$, the set $\{\mathcal{L}'_\gamma = 0,\ \mathcal{L}_\gamma \le R(q)\}$ is a cylinder closed set. Thus, we deduce from the large deviation principle that there exists $\gamma_0'(q)$ and $W'(q)$ such that

(5.2.3) $\forall \gamma < \gamma_0'(q)$,

$$\sup_{y \in \pi_\gamma^{-1}(W'(q))} \mu^y_{\beta,\gamma}(\mathcal{L}'_\gamma = 0,\ \mathcal{L}_\gamma \le R(q)) \le \exp\left(\frac{-\mathcal{G}^q(\mathcal{M}_{R(q)}) + \frac{\varepsilon}{4}}{\gamma}\right).$$

Without any restriction, we choose the same constant $\gamma_0(q)$ and the same set $W(q)$ for the preceding inequalities. Noticing that $\mathcal{K}$ is lower semi-continuous, we impose the condition

$$\mathcal{K}(W(q)) \geq \mathcal{K}(q) - \frac{\varepsilon}{5},$$

this implies that

$$\Phi \leq \mathcal{K}(W(q)) + \mathcal{G}^q(\mathcal{M}) + \frac{\varepsilon}{5}. \tag{5.2.4}$$

We iterate the procedure for any profile in $\mathcal{A}_\Phi$ and we cover the compact set $\mathcal{A}_\Phi$ with a finite number N of neighborhoods $\{W_i\}_{i\leq N}$ with compact basis (where $W_i = W(q_i)$). Set

$$B_\gamma = \Big\{ y \in Y \text{ such that } \forall l \in \gamma\mathbb{N},\ l \leq \exp\Big(\frac{\Phi - \varepsilon}{\gamma}\Big), \quad T_{-l}(\pi_\gamma(y)) \in \bigcup_{i\leq N} W_i \Big\}.$$

Before going on, we want to estimate the probability of B_γ^c

$$\mathbb{P}(B_\gamma^c) \leq \sum_{l\leq \exp(\frac{\Phi-\varepsilon}{\gamma})} \mathbb{P}_\gamma\Big(\bigcap_{i\leq N} W_i^c\Big),$$

because of Theorem 2.3, we have

$$\mathbb{P}(B_\gamma^c) \leq \frac{1}{\gamma} \exp\Big(-\frac{1}{\gamma}\Big). \tag{5.2.5}$$

As the set $\{\mathcal{L}'_\gamma = l,\ \mathcal{L}_\gamma < \infty\}$ is included in $T_l\mathcal{M}$, we derive

$$\mu^y_{\beta,\gamma}\Big(\mathcal{L}_\gamma \leq \exp(\frac{\Phi - \varepsilon}{\gamma})\Big) \leq 1_{B_\gamma^c}(y) + \sum_{l\leq \exp(\frac{\Phi-\varepsilon}{\gamma})} 1_{B_\gamma}(y)\ \mu^y_{\beta,\gamma}(T_l\mathcal{M}),$$

therefore

$$\mu^y_{\beta,\gamma}\Big(\mathcal{L}_\gamma \leq \exp\Big(\frac{\Phi - \varepsilon}{\gamma}\Big)\Big) \leq 1_{B_\gamma^c}(y) + \sum_{l\leq \exp(\frac{\Phi-\varepsilon}{\gamma})} \sum_{i=1}^{N} 1_{W_i}(T_{-l}\pi_\gamma(y))\, \mu^y_{\beta,\gamma}(T_l\mathcal{M}).$$

This leads to

$$\mathbb{P}\left[\mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \leq \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\right)\right]$$
$$\leq \mathbb{P}(B^c_\gamma) + \sum_{l \leq \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)} \sum_{i=1}^{N} \mathbb{P}_\gamma(W_i) \sup_{y \in \pi_\gamma^{-1}(W_i)} \mu^y_{\beta,\gamma}(\mathcal{M}),$$

combining inequalities (5.2.2) and (5.2.3) we deduce for γ sufficiently small that

$$\text{(5.2.6)} \quad \mathbb{P}\left[\mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \leq \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\right)\right] \leq \frac{1}{\gamma}\exp\left(-\frac{1}{\gamma}\right)$$
$$+ \sum_{i=0}^{N} \frac{1}{\gamma}\exp\left(\frac{\Phi - \mathcal{K}(W_i) - \frac{\varepsilon}{2}}{\gamma}\right)\exp\left(\frac{-\mathcal{G}^{q_i}(\mathcal{M}) + \frac{\varepsilon}{4}}{\gamma}\right).$$

This could be interpreted by saying that "environment q_i" appears $\exp(\frac{\Phi-\mathcal{K}(q_i)-\varepsilon/2}{\gamma})$ times on interval $[0, \exp(\frac{\Phi-\varepsilon}{\gamma})]$. As (5.2.4) holds, we derive that there is a positive constant γ_0 such that

$$\gamma < \gamma_0, \qquad \mathbb{P}\left[\mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \leq \exp(\frac{\Phi-\varepsilon}{\gamma})\right)\right] \leq \exp\left(-\frac{\varepsilon}{10\gamma}\right).$$

Hence for any δ positive, the event

$$\{y \in Y \text{ such that } \quad \mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \leq \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\right) \geq \delta\}$$

occurs with exponential small $\mathbb{P}$-probability. By applying Borel-Cantelli lemma, we prove the statement (5.2.1).

Step 2 :

Before going on to the estimate of the upper bound we need to state an extension of the large deviation principle

LEMMA 5.1. – *Let O be an open cylinder in E with compact basis in $\mathbb{R}^+$. We suppose in addition that O is symmetric* i.e.

$$\text{(5.2.7)} \qquad O = \{-\sigma \,|\, \sigma \in O\}.$$

For any profile q in $\mathcal{A}_\Phi$ and for all ε positive there is a weak neighborhood W of q with compact basis such that

$$\text{(5.2.8)} \quad \begin{cases} \exists R > 0,\ \exists \gamma_0 > 0 \text{ such that } \forall \gamma < \gamma_0, \\ \inf_{y \in \pi_\gamma^{-1}(W)} \inf_\omega \mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\omega) \geq \exp\left(-\frac{\mathcal{G}^q(O)+\varepsilon}{\gamma}\right), \end{cases}$$

where $\mathcal{B}_{]-\infty,-R[}$ is the σ-field generated by the profiles in $E_{]-\infty,-R[}$.

In the notation above and in the following we adopt the convention that the infimum is taken over $\mu^y_{\beta,\gamma}$-a.s. configurations. Assumption (5.2.7) on O is relevant; in fact, a mixing property for general sets will never hold on finite distance. Before proving Lemma 5.1 we show how to conclude step 2. We have to check that

$$\lim_{\gamma\to 0}\sup_{y\in Y}\mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma \geq \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right) = 0. \tag{5.2.9}$$

From (5.1.3), we deduce that there exists an integer N, a magnetic profile σ and an environment profile q which satisfy the property below

$$\begin{cases} \mathcal{G}^q(\sigma) \leq \Phi - \mathcal{K}(q) + \frac{\varepsilon}{5} \\ \sigma(r) = -m_p \qquad \text{for } r \in [-1, 0] \\ \sigma(r) = m_p \qquad \text{for } r \in [N-1, N]. \end{cases} \tag{5.2.10}$$

The environment profiles which satisfies the condition above are the typical environment patterns in the region connecting the two phases. Although σ and q are not unique, from now on we fix them to satisfy the property above.

We introduce $O = O(\sigma)\cup O(-\sigma)$, where $O(\sigma)$ is a neighborhood of σ defined by

$$O(\sigma) = (V+\sigma)\cap(T_N V + \sigma).$$

From the definition of σ we can suppose without any restriction that

$$\mathcal{G}^q(O) \leq \mathcal{G}^q(\mathcal{M}) + \frac{\varepsilon}{5}.$$

Hence, by the symmetry of O and Lemma 5.1, there exists an integer R and a neighborhood W of q with compact basis of length L which satisfy

(5.2.11) $\exists\gamma_0$ such that $\forall\gamma < \gamma_0$,

$$\inf_{y\in\pi_\gamma^{-1}(W)}\inf_{\omega}\mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\omega) \geq \exp\left(-\frac{\mathcal{G}^q(\mathcal{M})+\frac{\varepsilon}{4}}{\gamma}\right).$$

Intuitively, one is led to predict that a magnetic profile will perform more easily a jump from one equilibrium position to another when the environment is close to q. Moreover, Equation (5.2.11) tells us that the probability of leaping is always greater than a constant c_γ. If the "environment q" appears n_γ times on the interval $[0, \exp(\frac{\Phi+\varepsilon}{\gamma})]$ we deduce that the probability of leaping will be greater than $n_\gamma c_\gamma$. We just have to check that n_γ is large enough.

To make this heuristic argument more explicit we introduce

$$r_\gamma = \left[\frac{1}{L+R}\exp(\frac{\mathcal{K}(q)+\frac{\varepsilon}{2}}{\gamma})\right] \qquad \text{and} \qquad n_\gamma = \left[\exp(\frac{\Phi - \mathcal{K}(q)+\frac{\varepsilon}{2}}{\gamma})\right],$$

where $[x]$ denotes the closest integer to x. We drop the dependence on γ and we write simply n and r.

Let C be a subset of $\mathcal{Y}$ defined by

$$C = \bigcap_{i=0}^{n} \bigcup_{l=ir}^{(i+1)r} T_{l(L+R)} W.$$

The profiles in C are close to q at least once on each interval $[ir, (i+1)r]$. We have

$$\mathbb{P}\left[\mu_{\beta,\gamma}^{y}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right]$$
$$\leq \mathbb{P}\left[1_C(\pi_\gamma(y))\ \mu_{\beta,\gamma}^{y}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right] + \mathbb{P}_\gamma(C^c).$$

First we want to get rid of $\mathbb{P}_\gamma(C^c)$. By using the translation invariance of the measure $\mathbb{P}_\gamma$ it is enough to prove that

$$\lim_{\gamma \to 0} n\ \mathbb{P}_\gamma\left[\bigcap_{l=0}^{r} T_{l(L+R)} W^c\right] = 0. \tag{5.2.12}$$

As the random variables y_i are independent and identically distributed, we have

$$\mathbb{P}_\gamma\left[\bigcap_{l=0}^{r-1} T_{l(L+R)} W^c\right] = (1 - \mathbb{P}_\gamma(W))^r.$$

The large deviation principle for the measures $\mathbb{P}_\gamma$ tells us that for γ sufficiently small

$$\mathbb{P}_\gamma(W) \geq \exp\left(-\frac{\mathcal{K}(q)+\frac{\varepsilon}{4}}{\gamma}\right).$$

Combining the preceding equations we derive (5.2.12). It remains to check that

$$\lim_{\gamma \to 0} \mathbb{P}\left[1_C(\pi_\gamma(y))\ \mu_{\beta,\gamma}^{y}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right] = 0. \tag{5.2.13}$$

We note that

$$\left\{\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right\} \subset \bigcap_{i=0}^{n} T_{k_i}\, O^c, \tag{5.2.14}$$

where $k = \{k_i\}_{i \leq n}$ belongs to the set $\mathbb{K}$ defined by

$$\mathbb{K} = \{(k_1, \ldots, k_n) \text{ such that } \forall i \leq n,\ k_i \in [ir, (i+1)r[\}.$$

This remark leads to

$$\mathbb{P}\left[1_C(\pi_\gamma(y))\, \mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right] \leq \sup_{k\in\mathbb{K}} \sup_{y\in\pi_\gamma^{-1}(\cap_i T_{k_i} W)} \mu^y_{\beta,\gamma}\left(\bigcap_{i=0}^{N} T_{k_i} O^c\right).$$

Because of the Markov property of the Gibbs measure, we have

$$\mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma > \exp(\frac{\Phi+\varepsilon}{\gamma})\right) \leq \mu^y_{\beta,\gamma}\left[\bigcap_{i=0}^{n-1} T_{k_i}\, O^c\ \mu^y_{\beta,\gamma}(T_{k_n}\, O^c \mid \mathcal{B}_{]-\infty,k_n-R]})\right].$$

We derive from (5.2.11)

$$\inf_{y\in\pi_\gamma^{-1}(\cap_i T_{k_i} W)} \inf_\omega \mu^y_{\beta,\gamma}(T_{k_n}\, O \mid \mathcal{B}_{]-\infty,k_n-R]})(\omega) \geq \exp\left(-\frac{\mathcal{G}^q(\mathcal{M}) + \frac{\varepsilon}{4}}{\gamma}\right),$$

we iterate the procedure to get

$$\mathbb{P}\left[1_C(\pi_\gamma(y))\, \mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right] \leq \left(1 - \exp\left(-\frac{\mathcal{G}^q(\mathcal{M}) + \frac{\varepsilon}{4}}{\gamma}\right)\right)^n,$$

and more precisely

$$\mathbb{P}\left[1_C(\pi_\gamma(y))\, \mu^y_{\beta,\gamma}\left(\mathcal{L}_\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right)\right] \leq \exp\left(-\exp\left(\frac{1}{\gamma}\left(\Phi - \mathcal{K}(q) - \mathcal{G}^q(\mathcal{M}) + \frac{\varepsilon}{4}\right)\right)\right).$$

By taking the limit as γ goes to 0, we then derive (5.2.13). In the same way as in the first step, the Borel-Cantelli lemma enables us to complete (5.2.9). □

We end this section by proving Lemma 5.1.

Proof. – We fix ε positive. We recall that the conditional probability $\mu^y_{\beta,\gamma,[0,k]}(A\,|\,\sigma,\sigma')$ with boundary conditions σ on $]-\infty,0[$ and σ' on $]k,\infty[$ was introduced in definition 2.2. Let ω and ω' be two block spins which are deduced from configurations in Ω by the mapping κ_γ (see definition 2.4). We suppose that ω' is in the set $\omega + T_{-1}V_\varepsilon$ (with V_ε as in definition 4.2), we check for any cylinder set A of basis included in $\mathbb{R}^+$ that for any k, the following holds $\mu^y_{\beta,\gamma}$-a.s.

$$\forall \gamma < \gamma_0, \quad \mu^y_{\beta,\gamma,[0,k]}(A\,|\,\omega',\sigma) = \mu^y_{\beta,\gamma,[0,k]}(A\,|\,\omega,\sigma)\exp\left(\frac{O(\varepsilon)}{\gamma}\right),$$

where $O(\varepsilon)$ do not depend on k,γ and σ. By taking the limit as k goes to infinity, we get

(5.2.15)
$$\forall \gamma < \gamma_0, \qquad \mu^y_{\beta,\gamma}(A \mid \mathcal{B}_{\mathbb{R}^-})(\omega') = \mu^y_{\beta,\gamma}(A \mid \mathcal{B}_{\mathbb{R}^-})(\omega)\exp\left(\frac{O(\varepsilon)}{\gamma}\right).$$

For any profile ω in E, we define G_ω by $\omega + T_{(-1)}V_\varepsilon$ (see definition 4.2). Let ω be given and denote G_ω by G. Since $\mathcal{F}^q(G)$ is finite we check that the set G is regular, *i.e.*, it satisfies the property

$$\mathcal{F}^q(G) = \mathcal{F}^q(\bar{G}), \tag{5.2.16}$$

where $\bar{G}$ is the closure of G.

We suppose that G satisfies the hypothesis below

$$\forall R_0 > 0, \exists R > R_0, \qquad \mathcal{F}^q(T_{-R}G) < \infty, \tag{5.2.17}$$

then, we will prove that

(5.2.18)
$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \inf_{\omega\in T_{-R}G}\left[\mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\omega)\right]\right) \geq -\mathcal{G}^q(O)+\varepsilon,$$

where the constant R and the set W will be fixed later.

Noticing that

$$\mu^y_{\beta,\gamma}(T_{-R}G \cap O) = \int d\mu^y_{\beta,\gamma}(\sigma)\; 1_{T_{-R}G}(\sigma)\; \mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\sigma),$$

by equation (5.2.15), we have for any subset W of $\mathcal{Y}$

(5.2.19)
$$\inf_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(T_{-R}G\cap O) \leq \inf_{y\in\pi_\gamma^{-1}(W)} \inf_{\omega\in T_{-R}G} \mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\omega)$$
$$\sup_{y\in\pi_\gamma^{-1}(W)} \mu^y_{\beta,\gamma}(T_{-R}G)\exp(\frac{O(\varepsilon)}{\gamma}).$$

Since G satisfies (5.2.17) then $T_{-R}G$ is regular and we derive from the large deviation Theorem and (5.2.19)

$$\liminf_{\gamma\to 0} \gamma \ln\left(\inf_{y\in\pi_\gamma^{-1}(W)} \inf_{\omega\in T_{-R}G} \left[\mu^y_{\beta,\gamma}(O \mid \mathcal{B}_{]-\infty,-R[})(\omega)\right]\right) \geq -\mathcal{F}^q(T_{-R}G\cap O) + \mathcal{F}^{q,\varepsilon}(T_{-R}G) + O(\varepsilon),$$

where W is a weak neighborhood of q with compact basis.

As $\mathcal{F}^q(T_{-R}G)$ is finite we can choose ε sufficiently small in order to replace $\mathcal{F}^{q,\varepsilon}$ by $\mathcal{F}^q$. The proof will be complete once we show that $\mathcal{F}^q(T_{-R}G\cap O)$ is almost equal to $\mathcal{F}^q(T_{-R}G) + \mathcal{F}^q(O)$ for a suitable constant R.

We supposed that q belongs to $\mathcal{A}_\Phi$, thus for any λ in $\{-1,1\}$ the symmetry of O enables us to find σ in O and a constant d such that

$$\begin{cases} \mathcal{F}(\sigma) \leq \mathcal{F}^q(\mathcal{O}) + \frac{\varepsilon}{4} \\ \sigma(r) = \lambda m_p \quad \text{for } r < -d. \end{cases}$$

We shift G in order to find an element σ' in $T_{-R}G$ which satisfies the property

$$\begin{cases} \mathcal{F}^q(\sigma') \leq \mathcal{F}^q(T_{-R}G) + \frac{\varepsilon}{4} \\ \sigma'(r) = \lambda m_p \quad \text{for } r \in [-d-1,-d], \end{cases}$$

where λ is in $\{-1,1\}$ depending on σ'.

Therefore the profile $\sigma'\otimes m_p\otimes\sigma$ belongs to $T_{-R}G\cap O$ and we get

$$\mathcal{F}^q(T_{-R}G) - \mathcal{F}^q(T_{-R}G\cap O) \geq -\mathcal{G}^q(\sigma) - \frac{\varepsilon}{2} \geq -\mathcal{G}^q(O) - \varepsilon.$$

This implies (5.2.18).

The compactness of E enables us to cover E with the finite family $\{G_{\omega_i}\}_{i\leq N}$. Furthermore, iterating the previous argument, we check that there is a suitable family such that (5.2.18) holds with the same constant R for any set G_i and that $\cup T_{-R}G_i$ covers E. □

ACKNOWLEDGMENTS

This paper is a part of my Ph.D. thesis. I thank F. Comets for suggesting this problem to me and for his guidance.

REFERENCES

[1] P. Baldi, Large deviations and stochastic homogenization, *Ann. Mat. Pura Applic.*, **132,** 1988.

[2] T. Bodineau, Interface in a one-dimensional Ising spin system, *Stoch. Proc. Appl.*, **61,** 1996, pp. 1-23.

[3] A. Bovier, V. Gayard and P. Picco, Large deviation principles for the Hopfield model and the Kac-Hopfield model., *Prob. Theory Relat. Fields*, **101,** 1995, pp. 511-546.

[4] A. Bovier, V. Gayard and P. Picco, Distribution of overlap profiles in the one-dimensional Kac-Hopfield model. Preprint 1996.

[5] M. Cassandro, E. Orlandi and E. Presutti, Interfaces and typical Gibbs configurations for one-dimensional Kac potentials, *Prob. Theo. Relat. Fields*, **96,** 1993, pp. 57-96.

[6] F. Comets, Large deviation estimates for a conditional probability distribution. Applications to random interaction Gibbs measures., *Prob. Theo. Relat. Fields*, **80,** 1989, pp. 407-432.

[7] J. D. Deuschel and D Stroock, *Large deviations*, San Diego, Academic Press, 1989.

[8] T Eisele and R Ellis, Symmetry breaking and random walks for magnetic systems on a circle, *Z wahr. Verw. Geb.*, **63,** 1983, pp. 297-348.

[9] R. Ellis, *Entropy large deviations and stastical mechanics*, Springer-Verlag, 1985.

[10] M. I. Freidlin and A. D. Wentzell, *Random perturbations of dynamical systems*, Springer-Verlag, 1983.

[11] A. Galves, E. Olivieri and M. E. Vares, Metastability for a class of dynamical systems subject to small random perturbations, *Ann. Prob.*, 1987, **87,** pp. 1288-1305.

[12] R. Georgii, *Gibbs measures and phase transitions*, studies in mathematics, De Gruyter, 1988.

[13] J. Lebowitz and O. Penrose, Rigourous treatment of the Van der Waals-Maxwell theory of the liquid vapor transition, *J. Math. Phys.*, **7,** 1996, pp. 98-113.

[14] T Seppäläinen, Entropy, limit theorems and variational principles for disordered lattice systems, *Comm. math. Phys.*, **171,** 1995, pp. 233-277.

[15] B. Zegarlinski, Interactions and pressure functionals for disordered lattice systems., *Comm. Math. Phys.*, **139,** 1991, pp. 305-339.

[16] B. Zegarlinski, Spin systems with long-range interactions, *Reviews in Math. Phys.*, **6,** 1994, pp. 115-134.

(Manuscript received April 9, 1996;
Revised November 29, 1996)