ANNALES DE L'I. H. P., SECTION B

DAYUE CHEN

Average properties of random walks on Galton-Watson trees

Annales de l'I. H. P., section B, tome 33, nº 3 (1997), p. 359-369
<http://www.numdam.org/item?id=AIHPB_1997__33_3_359_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 3, 1997, p. 359-369. *Probabilités et Statistiques*

Average properties of random walks on Galton-Watson trees

by

Dayue CHEN

Department of Probability and Statistics
Peking University, Beijing, 100871, China.

ABSTRACT. – We study the λ-biased random walk on Galton-Watson trees by the Dirichlet principle and a formula of mean exit time of a Markov chain. We prove that the average of escaping probability and mean exit time are bounded by the counterparts of the corresponding random walks on $\{0, 1, 2, \cdots\cdots\}$. In particular we partially verified the recent conjecture of Lyons, Pemantle and Peres on the upper bound of the speed of λ-biased random walk on Galton-Watson trees.

RÉSUMÉ. – Nous étudions la marche aléatoire de biais λ sur un arbre de Galton-Watson. Nous démontrons que la probabilité de fuite et le temps de sortie en moyenne sont bornés par ceux de la marche aléatoire correspondante sur $\{0, 1, 2, \cdots\cdots\}$. En particulier nous confirmons partiellement une conjecture de Lyons, Pemantle et Peres sur la limite supérieure de vitesse de la marche aléatoire de biais λ sur un arbre de Galton-Watson

1. INTRODUCTION

For a given tree T, a vertex is selected as the *root* and is denoted by o.

Supported in part by a grant from the NSF of China.

The distance from vertex v to o is the minimum number of edges linking o and v, and is denoted by $|v|$. It is called the *level* or *generation* of v. For vertex v other than root o ($i.e$, $|v| > 0$), there is a unique adjacent vertex which is of level $|v| - 1$. This unique adjacent vertex is called the *parent* of v, and is denoted by v_*. Other adjacent vertices of v are all of level $|v| + 1$, and are called *children* of v. Let k_v be the number of children of v. It is also known as the *branching number* of v. Children of v are denoted by v_i, $i = 1, 2, \cdots k_v$.

For positive number λ, *λ-biased random walk* on T is a Markov chain $\{X_n\}$ on the vertices of T with transition probability

$$p(v, v_*) = \frac{\lambda}{\lambda + k_v}, \qquad p(v, v_i) = \frac{1}{\lambda + k_v}, \qquad v \neq o. \tag{1}$$

The transition probability at o is different slightly in accordance with the lack of o_*. Let k_o be the branching number of o and o_i a child of o. We define $p(o, o_i) = 1/k_o$ in addition to (1). Note that (1) is also well defined for $\lambda = 0$ if $k_v \geq 1$ for all vertices v's of T. Let

$$\tau_s = \min\{n \geq 0; |X_n| = s\}; \tag{2}$$
$$\tau_o = \min\{n \geq 1; X_n = o\};$$
$$\gamma(T) = \lim_{s \to \infty} P(\tau_s < \tau_o | X_0 = o). \tag{3}$$

Tree T is called a Galton-Watson tree if it is a realization of a Galton-Watson process. Namely, k_v's are $i.i.d.$ random variables. Assume that the offspring distribution satisfies that

$$P(k = 0) = 0; \quad P(k = i) \geq 0, \quad \sum_{i=1}^{\infty} P(k = i) = 1. \tag{4}$$

The offspring distribution induces naturally a probability measure in the collection $\mathbf{T}$ of all Galton-Watson trees. Let $E_{\mathbf{T}}$ be the expectation according to that probability measure on $\mathbf{T}$. Define

$$m = \sum_i iP(k = i); \qquad \frac{1}{m'} = \sum_i \frac{1}{i} P(k = i). \tag{5}$$

Certainly $m \geq m' \geq 1$. *λ-biased random walk on random trees* is defined in two steps. First, take a Galton-Watson tree T according to the probability measure in $\mathbf{T}$. Then, define a random walk X_n on T according to (1) starting at root o. Thus a point in the big probability space has two components: a random tree and a random path. The offspring distribution and parameter λ determine a unique probability measure in this big space. In the following Theorem 2, the double expectation $E_{\mathbf{T}}E$ is the average first over all random walks on a fixed tree starting at root o, then over all Galton-Watson trees.

THEOREM 1. – *If $P(k = 0) = 0$ and $\lambda \leq m < \infty$, then*

$$1 - \frac{\lambda}{m} \geq E_{\mathbf{T}}\gamma(T) \geq 1 - \frac{\lambda}{m'}.$$

The equalities hold if and only if $m = m'$, i.e., *m is an integer and $P(k = m) = 1$.*

THEOREM 2. – *Assume that $P(k = 0) = 0$. Then*

$$\lim_{s\to\infty} E_T E \frac{\tau_s}{s} \geq \frac{m+\lambda}{m-\lambda} \quad \text{if } \lambda < m < \infty;$$
$$\lim_{s\to\infty} E_T E \frac{\tau_s}{s} \leq \frac{m'+\lambda}{m'-\lambda} \quad \text{if } \lambda < m'.$$

The equalities hold if and only if $m = m'$, i.e., *m is an integer and $P(k = m) = 1$.*

Random walk on random trees has been an active subject in recent years. It is shown in [4] that the random walk on random trees is transient $a.s.$ in the *big* space if $\lambda < m$. The *speed,* or the *rate of escape,* of the random walk is defined to be $\liminf_{n\to\infty} |X_n|/n$. Lyons, Pemantle and Peres proved recently in [5] that for a fixed λ ($\lambda < m$) and for $a.e.$ Galton-Watson tree T,

$$\lim_{n\to\infty} \frac{|X_n|}{n} \tag{6}$$

exists and is a positive constant, denoted by $speed(\lambda)$. $speed(\lambda)$ depends only on λ and the offspring distribution. For the case $\lambda = 1$, they computed the speed explicitly in [6].

$$speed(1) = \sum_i P(k = i)\frac{i-1}{i+1}. \tag{7}$$

On the other hand, consider the random walk on $\{0, 1, 2, 3, \cdots\}$ (which is the simplest tree) with the following transition probabilities.

$$p(0,1) = 1; \quad p(j, j-1) = \frac{\lambda}{\lambda + m}, \quad p(j, j+1) = \frac{m}{\lambda + m}, j \geq 1. \tag{8}$$

One can easily verify that $speed(\lambda) = (m - \lambda)/(m + \lambda)$ in this case. Comparing with (7) we see that when $\lambda = 1$ the random walk on random trees is slower than the corresponding random walk on $\{0, 1, 2, 3, \cdots\}$. It is often observed that a random walk is slowed down in random environments. A related example can be found in [8]. It is conjectured in [7] that

$$\lim_{n\to\infty} \frac{|X_n|}{n} \leq \frac{m - \lambda}{m + \lambda} \qquad a.s. \text{ if } \lambda < m.$$

We are motivated by this conjecture, and verify it partially.

COROLLARY 3. – *If* $P(k = 0) = 0$, $\lambda \leq 1$ *and* $m < \infty$, *then*

$$\frac{m' - \lambda}{m' + \lambda} \leq \lim_{n\to\infty} \frac{|X_n|}{n} \leq \frac{m - \lambda}{m + \lambda} \qquad a.s.$$

The equality holds if and only if $m = m'$, i.e., $P(k = m) = 1$ *for some integer* m.

By (7) and the convexity of function $(x - 1)/(x + 1)$, Corollary 3 holds for $\lambda = 1$. For $\lambda < 1$, one can show by coupling that τ_s is bounded above by that of a random walk on $\{0, 1, 2, 3, \cdots\}$ with transition probabilities

$$p(0, 1) = 1; \qquad p(j, j - 1) = \frac{\lambda}{\lambda + 1}, \quad p(j, j + 1) = \frac{1}{\lambda + 1}, j \geq 1.$$

Hence τ_s/s is uniformly integrable in the *big space*. By Proposition 5.112 of [1], we can exchange the integration and the limit, *i.e.*, the last equality, in the following derivation.

$$\frac{1}{speed(\lambda)} = \lim_{s\to\infty} \frac{\tau_s}{s} = E_T E \lim_{s\to\infty} \frac{\tau_s}{s} = \lim_{s\to\infty} E_T E \frac{\tau_s}{s}.$$

The corollary now follows from Theorem 2. The next two sections are devoted to the proof of Theorems 1 and 2 respectively.

2. PROOF OF THEOREM 1

For computing $P(\tau_s < \tau_o | X_0 = o)$ on a fixed Galton-Watson tree T, it suffices to consider $T_{[s]}$, the subtree of generations $0, 1, 2, \cdots, s$ of T. On $T_{[s]}$ define a random walk $\{X_n\}$ according to

$$\begin{aligned} p(v, v_*) &= \frac{\lambda}{\lambda + k_v}, \qquad & p(v, v_i) &= \frac{1}{\lambda + k_v}, \quad \text{if } 1 \leq |v| < s; \\ p(o, o_i) &= \frac{1}{k_o}; & p(v, v_*) &= 1 \quad \text{if } |v| = s. \end{aligned} \tag{9}$$

Then the random walk so defined is reversible in the sense $\pi_x p(x,y) = \pi_y p(y,x)$ for any vertices x, y (not necessarily adjacent) of T, and

$$\pi_o = k_o; \qquad \pi_x = \frac{\lambda + k_x}{\lambda^{|x|}} \quad \text{if } 1 \le |x| < s; \qquad \pi_v = \frac{1}{\lambda^{s-1}} \quad \text{if } |v| = s.$$

Let H be the collection of all functions h on the vertices of $T_{[s]}$ such that

$$0 \le h(x) \le 1; \quad h(o) = 1; \quad h(y) = 0 \text{ if } |y| = s.$$

Then, by the *Dirichlet principle* (page 99 of [3]),

$$\pi_o P(\tau_s < \tau_o | X_0 = o) = \inf_{h \in H} \sum_{x,y} \frac{1}{2} \pi_x p(x,y)[h(x) - h(y)]^2.$$

Consequently,

$$P(\tau_s < \tau_o | X_0 = o) = \inf_{h \in H} \frac{1}{k_o} \sum_{|x|<s} \frac{1}{\lambda^{|x|}} \sum_{i=1}^{k_x} [h(x) - h(x_i)]^2. \tag{10}$$

Upper bound. Define the decreasing sequence

$$c_n = \frac{\sum_{l=n}^{s-1} \left(\frac{\lambda}{m}\right)^l}{\sum_{l=0}^{s-1} \left(\frac{\lambda}{m}\right)^l} \qquad n = 0, 1, 2, \cdots, s-1; \quad \text{and } c_s = 0.$$

Take $h \in H$ such that $h(x) = c_{|x|}$. Then

$$\begin{aligned}
&P(\tau_s < \tau_o | X_0 = o) \\
&\le \frac{1}{k_o} \sum_{|x|<s} \frac{1}{\lambda^{|x|}} \sum_{i=1}^{k_x} [c_{|x|} - c_{|x|+1}]^2 \\
&= \frac{1}{k_o} \sum_{l=0}^{s-1} \frac{\text{number of vertices of level } (l+1)}{\lambda^l} [c_l - c_{l+1}]^2. \\
&E_{\mathbf{T}} P(\tau_s < \tau_o | X_0 = o) \\
&\le E_{\mathbf{T}} \frac{1}{k_o} \sum_{l=0}^{s-1} \frac{\text{number of vertices of level } (l+1)}{\lambda^l} [c_l - c_{l+1}]^2 \\
&= \sum_{l=0}^{s-1} \frac{m^l}{\lambda^l} [c_l - c_{l+1}]^2 = \frac{1}{\sum_{l=0}^{s-1} \left(\frac{\lambda}{m}\right)^l} = \frac{1 - \frac{\lambda}{m}}{1 - \left(\frac{\lambda}{m}\right)^s}.
\end{aligned}$$

Since $P(\tau_s < \tau_o | X_0 = o)$ is decreasing in s, converges to $\gamma(T)$, and is bounded,

$$E_{\mathbf{T}}\gamma(T) = E_{\mathbf{T}} \lim_{s\to\infty} P(\tau_s < \tau_o | X_0 = o)$$
$$= \lim_{s\to\infty} E_T P(\tau_s < \tau_o | X_0 = o) \leq 1 - \frac{\lambda}{m}.$$

Lower bound. Given a tree T, consider the simple *forward* random walk which chooses randomly (uniformly) among the children of the present vertex as the next vertex. Let $\mu(x)$ be the probability that the random walk starting at root o will visit vertex x. If k_{ix}'s are the branching numbers of the vertices along the shortest path from root o to x, then $\mu(x) = (k_o k_{1x} k_{2x} \cdots k_{x_*})^{-1}$. This is the *visibility measure* of the set of *rays* emanating from root o and passing vertex x. See §2 of [6] for the details.

By the Cauchy-Schwarz inequality, for any $h \in H$,

$$\left(\sum_{|x|<s} \sum_{i=1}^{k_x} \frac{1}{\lambda^{|x|}} [h(x) - h(x_i)]^2 \right)^{\frac{1}{2}} \left(\sum_{|x|<s} \sum_{i=1}^{k_x} \lambda^{|x|} (\mu(x_i))^2 \right)^{\frac{1}{2}}$$
$$\geq \sum_{|x|<s} \sum_{i=1}^{k_x} \mu(x_i)[h(x) - h(x_i)]$$

Since $\sum_{i=1}^{k_x} \mu(x_i) = \mu(x)$, the right hand side of the above inequality actually is equal to

$$\sum_{l=0}^{s-1} \sum_{|x|=l} \left[\mu(x)h(x) - \sum_{i=1}^{k_x} \mu(x_i)h(x_i) \right]$$
$$= \sum_{l=0}^{s-1} \left[\sum_{|x|=l} \mu(x)h(x) - \sum_{|y|=l+1} \mu(y)h(y) \right] = 1.$$

Thus by (10),

$$P(\tau_s < \tau_o | X_0 = o) \geq \frac{1}{k_o \sum_{|x|<s} \sum_{i=1}^{k_x} \lambda^{|x|} (\mu(x_i))^2}$$
$$= \left[k_o \sum_{|x|<s} \sum_{i=1}^{k_x} \frac{\lambda^{|x|}}{(k_o k_{1x} k_{2x} \cdots k_x)^2} \right]^{-1};$$

and

$$E_{\mathbf{T}}P(\tau_s < \tau_o | X_0 = o) \geq \left[E_{\mathbf{T}} k_o \sum_{|x|<s} \sum_{i=1}^{k_x} \frac{\lambda^{|x|}}{(k_o k_{1x} k_{2x} \cdots k_x)^2} \right]^{-1}$$

$$= \left[1 + \frac{\lambda}{m'} + \left(\frac{\lambda}{m'}\right)^2 + \cdots + \left(\frac{\lambda}{m'}\right)^{s-1} \right]^{-1} = \frac{1 - \frac{\lambda}{m'}}{1 - (\frac{\lambda}{m'})^s}.$$

Letting $s \to \infty$ we obtain the other half of Theorem 1. □

It is shown in the proof of Corollary 3.5 of [5] that

$$E_{\mathbf{T}}\gamma(T) \geq \frac{\lambda - 1}{2\lambda}(1 - q_\lambda)$$

where q_λ is the smallest nonnegative number satisfying

$$\sum_{j=0}^{\infty} P(k = j)(1 - \lambda^{-1}(1 - q_\lambda))^j = q_\lambda.$$

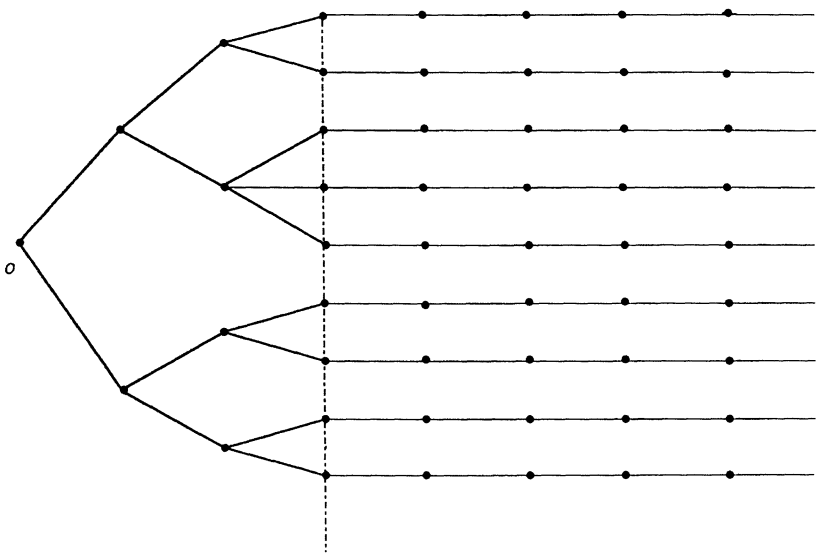
The lower bound of Theorem 1 is simpler and works better when $\lambda < 1$. $\gamma(T)$ is called the escaping probability. If tree T is thought as an electrical network, and if the resistance of an edge linking vertices of level l and $(l+1)$ is λ^l, then the total resistance between vertex o and the infinity is $1/\gamma(T)$. In deriving the lower bound we actually proved a stronger statement.

Corollary 4. – *If $P(k = 0) = 0$ and $\lambda \leq m' < \infty$, then the total resistance between root o and the infinity has a finite mean over all Galton-Watson trees. Namely,*

$$E_{\mathbf{T}} \frac{1}{\gamma(T)} \leq \frac{m'}{m' - \lambda}.$$

3. PROOF OF THEOREM 2

Choose $l \in [0, s]$. Take the subtree $T_{[l]}$ of the first l levels of a Galton-Watson tree and extend it by *pipes* (*see* Figure). In our earlier notation the tree is characterized by $k_v = 1$ for $|v| \geq l$. The collection of all such infinite trees with pipes at level l is denoted by $\mathbf{T}(l)$. The offspring distribution induces a probability measure on $\mathbf{T}(l)$ for every l. In the following Lemma 5, $E_{\mathbf{T}(l)}$ is the expectation taken with respect to this



induced measure on $\mathbf{T}(l)$. Restricting attention only to the first l levels, a subset of $\mathbf{T}(l)$ can be regarded also as a subset of $\mathbf{T}(l+1)$ and it has the same probability measure in both $\mathbf{T}(l)$ and $\mathbf{T}(l+1)$. This consistence of induced measures on $\mathbf{T}(l)$'s is used in the proofs of Lemma 5 and Theorem 2 below.

Run a random walk $\{X_n\}$ on $T \in \mathbf{T}(l)$ with transition probabilities

$$\begin{aligned} p(v, v_*) &= \frac{\lambda}{\lambda + k_v}, \quad p(v, v_i) = \frac{1}{\lambda + k_v} \quad \text{if } 0 < |v| < l; \\ p(v, v_*) &= \frac{\lambda}{\lambda + m}, \quad p(v, v_1) = \frac{m}{\lambda + m} \quad \text{if } 1 \le l \le |v|. \end{aligned} \tag{11}$$

Some obvious change is needed if $l = 0$ or $v = o$. Let $E_x \tau_s$ be the mean of the first hitting time of level s by the random walk defined by (11) starting at vertex x.

LEMMA 5. – $E_{\mathbf{T}(l+1)} E_o \tau_s \ge E_{\mathbf{T}(l)} E_o \tau_s$ *for* $0 \le l \le s - 1$.

Proof 1.1. – Suppose that tree $T' \in \mathbf{T}(l+1)$. That is, from level $(l+1)$ on there is only one child for each vertex. Suppose that u is a vertex of T', $|u| = l$ and k_u is the branching number of u. Notice that there are k_u pipes emanating from u and the transition probabilities along these pipes are identical. So we combine these pipes together as one *combined* *pipe*. Let u_1 be the only child of u after this combination, and change the transition probability at u as

$$p(u,u_*) = \frac{\lambda}{\lambda + k_u}, \quad p(u,u_1) = \frac{k_u}{\lambda + k_u}. \tag{12}$$

The randomness of the branching number of u is converted to the randomness of transition probability at u. The distribution of τ_s is preserved after this modification. In particular, we have

$$E_u\tau_s = 1 + \frac{\lambda}{\lambda + k_u} E_{u_*}\tau_s + \frac{k_u}{\lambda + k_u} E_{u_1}\tau_s. \tag{13}$$

In general

$$E_x\tau_s = 1 + \frac{\lambda}{\lambda + k_x} E_{x_*}\tau_s + \sum_{i=1}^{k_x} \frac{1}{\lambda + k_x} E_{x_i}\tau_s \qquad \text{if } 1 \le |x| \le l, x \ne u;$$

$$E_x\tau_s = 1 + \frac{\lambda}{\lambda + m} E_{x_*}\tau_s + \frac{m}{\lambda + m} E_{x_1}\tau_s \qquad \text{if } l+1 \le |x| \le s-1;$$

$$E_o\tau_s = 1 + \sum_{i=1}^{k_o} \frac{1}{k_o} E_{o_i}\tau_s; \qquad \text{and} \quad E_x\tau_s = 0 \quad \text{if } |x| = s.$$

Replacing (13) by

$$(\lambda + k_u)E_u\tau_s = (\lambda + k_u) + \lambda E_{u_*}\tau_s + k_u E_{u_1}\tau_s$$

and solving the system of linear equations by the Cramer rule, we see that $E_o\tau_s$ is the quotient of two determinants. Notice that k_u appears only in the last equation. Thus each determinant is a linear function of k_u and

$$E_o\tau_s = \frac{ak_u + b}{ck_u + d} \tag{14}$$

where a, b, c and d are independent of k_u.

Function $f(x) = (ax+b)/(cx+d)$ is convex if and only if $f(0) \ge f(\infty)$. However, $f(0)$ is $E_o\tau_s$ when $k_u = 0$, or in other words, $p(u,u_1) = 0$, $p(u,u_*) = 1$; and $f(\infty)$ is $E_o\tau_s$ when $p(u,u_1) = 1$, $p(u,u_*) = 0$. Define two random walks $\{Y_n\}$ and $\{Z_n\}$, both starting at root o, with the same transition probability everywhere except at u. For $\{Y_n\}$, $p(u,u_1) = 0$, $p(u,u_*) = 1$; for $\{Z_n\}$, $p(u,u_1) = 1$, $p(u,u_*) = 0$. Notice that the combined pipe and other pipes of the tree are symmetric beyond level

$(l+1)$, including level $(l+1)$. So $|Y_n| \leq |Z_n|$ by the method of coupling. It is follows from this fact that $f(0) > f(\infty)$ (unless $s = 1$).

We have demonstrated that $E_o\tau_s$ is a convex function of k_u. By the Jensen's inequality, the average of $E_o\tau_s$ over all possible k_u is greater than or equal to $(am + b)/(cm + d)$. This is exactly the mean hitting time of level s by the random walk with deterministic transition probability at u,

$$p(u, u_*) = \frac{\lambda}{\lambda + m}, \quad p(u, u_1) = \frac{m}{\lambda + m}.$$

The above argument can be applied to other vertices of level l one by one to decrease the mean hitting time of level s. What we have proved is that for $T \in \mathbf{T}(l)$, $E_o\tau_s$ is less than or equal to the average of $E_o\tau_s$ over those trees of $\mathbf{T}(l+1)$ whose subtree of first l levels is T. The equality holds if and only if $P(k = m) = 1$ for some integer m. The statement of this lemma then follows by taking the average of random trees of $\mathbf{T}(l)$. Namely, take $E_{\mathbf{T}(l)}$. $\square$

Remark. – This simplied proof is kindly suggested to the author by Professor R. Lyons. The original proof is lengthy and uses a cumbersome formula of the mean exit time from [2].

Proof of Theorem 2. – The distribution of first hitting time τ_s of level s is determined by the subtree of first s levels. By the consistence of induced measures on $\mathbf{T}(s)$ and $\mathbf{T}$, and by Lemma 5, we have that

$$E_{\mathbf{T}} E\tau_s = E_{\mathbf{T}(s)} E_o\tau_s \geq E_{\mathbf{T}(0)} E_o\tau_s. \tag{15}$$

However, there is only one member of $\mathbf{T}(0)$. The right hand side of (15) further reduces to $E_0\tau_s$, the mean of the first hitting time τ_s of s by the random walk on $\{0, 1, 2, 3, \cdots\}$ starting at 0 with transition probabilities given by (8). This can be calculated by solving a system of linear equations.

$$E_0\tau_s = s\frac{m + \lambda}{m - \lambda} - \frac{2m\lambda}{(m - \lambda)^2} + \left(\frac{\lambda}{m}\right)^{s-1} \frac{2\lambda^2}{(m - \lambda)^2}. \tag{16}$$

The first half of Theorem 2 is now an easy consequence of (15) and (16).

For the second half, rewrite (14) as

$$E_o\tau_s = \frac{a + b/k_u}{c + d/k_u}$$

which is a concave function of $1/k_u$. Taking the average over k_u we get

$$E_{k_u} E_o\tau_s \leq \frac{a + bE(1/k_u)}{c + dE(1/k_u)} = \frac{a + b/m'}{c + d/m'} = \frac{am' + b}{cm' + d}.$$

The remaining argument is identical with that of the first half. $\square$

Remark. – It is for simplicity that we assume throughout this paper that $P(k = 0) = 0$. This assumption is needed in the half involving m' of both theorems; but is not required for the other half (involving m).

ACKNOWLEDGMENTS

The author would like to thank Professor Y. Peres for private communications and encouragement, and Professor R. Lyons for valuable comments that simplify greatly the proof of Lemma 5. This work is done while the author is visiting UCLA. The author wishes to thank Professor Tom Liggett for his hospitality, and for discussions related to this research.

REFERENCES

[1] L. Breiman, *Probability*, SIAM, Philadelphia, 1992.
[2] D. Chen, J. Feng and M. P. Qian, The metastability of exponentially perturbed Markov chains, *Science in China*, Vol. **39**, 1996, pp. 7-28.
[3] T. M. Ligget, *Interacting Particle Systems*, Springer-Verlag, New York, 1985.
[4] R. Lyons, Random walks and percolation on trees, *Ann. Prob.*, Vol. **18**, 1990, pp. 931-958.
[5] R. Lyons, R. Pemantle and Y. Peres, Biased random walks on Galton-Watson trees, *Probability Theory and Related Fields*, Vol. **106**, 1996, pp. 249-264.
[6] R. Lyons, R. Pemantle and Y. Peres, Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure, *Ergod. Th. & Dynam. Sys.*, Vol. **15**, 1995, pp. 1-27.
[7] R. Lyons, R. Pemantle and Y. Peres, *Unsolved problems concerning random walks on trees*. In "Classical and Modern Branching Processes", K.B. Athreya and P. Jagers (editors), Springer-Verlag, New York, 1996, pp. 223-238.
[8] F. Solomon, Random walk in a random environment, *Ann. Prob.*, Vol. **3**, 1975, pp. 1-31.

(Manuscript received January 7, 1996;
Revised March 4, 1996.)