ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 33, nº 2 (1997), p. 179-222

<http://www.numdam.org/item?id=AIHPB_1997__33_2_179_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 2, 1997, p. 179-222 *Probabilités et Statistiques*

Large deviation estimate of transition densities for jump processes

by

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ABSTRACT. – We give asymptotic upper and lower bounds of large deviation type for the transition density of a jump type processes on $\mathbb{R}^d$, which is composed of stable-like processes on the line and vector fields on $\mathbb{R}^d$. We use the theory of Malliavin calculus both for diffusion and for jump type processes. In the case where there is no drift, the upper and lower bounds coincide.

Key words: Jump process, large deviation, transition density.

RÉSUMÉ. – Dans cet article, nous démontrons un théorème de majoration et minoration de la densité pour une classe de processus avec sauts sur $\mathbb{R}^d$. Nous utilisons dans ce but un calcul de Malliavin pour processus avec sauts, et la théorie des grandes deviations.

INTRODUCTION

Consider $m + 1$ vector fields $X_0, X_j, j = 1, \cdots, m$, on $\mathbb{R}^d$ whose derivatives of all orders are bounded. Consider the SDE

$$(0.1) \qquad \begin{cases} dx_s(x) = \sum_{j=1}^{m} X_j(x_{s-}(x))dz_{j,s} + X_0(x_s(x))ds \\ x_0(x) = x \end{cases}$$

A.M.S. Classification : 60 J 25, 60 J 75.

where $z_{j,s}$ denote 1-dimensional compensated Lévy processes with the common smooth Lévy measure $g(\zeta)d\zeta$ and $\Delta z_{j,s} = z_{j,s} - z_{j,s-}$ for $j = 1, \cdots, m$. The Markov process $\{x_t(x)\}$ corresponds to a semigroup associated to the infinitesimal generator (integro-differential operator) of jump type

$$(0.2)\qquad Af(x) = \sum_{j=1}^{m} \int_{\mathbb{R}\setminus\{0\}} [f(x + X_j(x)\zeta) - f(x) - \zeta\langle X_j(x), \text{ grad } f(x)\rangle] \times g(\zeta)d\zeta + X_0 f(x),$$

$f \in C_0^\infty(\mathbb{R}^d)$.

It is known (*cf.* Léandre [24]) that, under a non-degeneracy condition on the Lie algebra $Lie(X_1, \cdots, X_m)$, the semigroup admits a regular density $p_t(x, dy) = p_t(x, y)dy$, $t > 0$, with respect to the d-dimensional Lebesgue measure dy.

We study the estimation concerning $p_t(x, y)$ of the above type, by using the large deviation theory. That is, consider, for each $\varepsilon > 0$, the semigroup associated with the generator

$$(0.3)\qquad A^\varepsilon f(x) = \sum_{j=1}^{m} \int_{\mathbb{R}\setminus\{0\}} [f(x + X_j(x)\varepsilon\zeta) - f(x) - \varepsilon\zeta\langle X_j(x), \text{ grad } f(x)\rangle] \times \frac{1}{\varepsilon} g(\zeta)d\zeta + X_0 f(x),$$

$f \in C_0^\infty(\mathbb{R}^d)$.

The law $p_t(x, dy, \varepsilon)$ corresponding to this semigroup possesses the density $p_t(x, y, \varepsilon) : p_t(x, dy, \varepsilon) = p_t(x, y, \varepsilon)dy$ for each $\varepsilon > 0$. We provide in the framework of large deviation theory the asymptotic estimate of $p_1(x, y, \varepsilon)$ as $\varepsilon \to 0$. This type of problem was studied by Freidlin and Ventcel [14] when the vector fields $X_j(x)$ are not degenerate (elliptic setting) (*see* also [13], [29]). Here we shall carry out our study in a hypoelliptic setting which is our feature. For this purpose we shall apply the theory of Malliavin calculus of jump type (*cf.* [5], [7], [24], [30], *see* also [20], [21] and [34]).

Intuitively, for small $\varepsilon > 0$ we can compare the jump process which corresponds to A^ε with a diffusion process corresponding to the infinitesimal generator B^ε given by

$$(0.4)\qquad B^\varepsilon f(x) = \frac{\varepsilon}{2} \sum_{j=1}^{m} X_j^2 f(x) + X_0 f(x), \quad f \in C_0^\infty(\mathbb{R}^d)$$

(*see* Section 1 for detail). The large deviation theory for diffusion processes has been extensively studied, and we can observe the diffusion trajectory above converges as $\varepsilon \to 0$ to a deterministic path exponentially quick (*cf.* e.g., [2], [14]). One may expect then that the density $p_1(x, y, \varepsilon)$ at y of the law deriving from A^ε will disappear exponentially quick as $\varepsilon \to 0$ with some rate functions depending on x and y.

A crucial idea in our proof is the notion of *skelton trajectories.* A skelton trajectory, denoted by $y_s(h)$, is a deterministic trajectory obtained from vector fields driven by a short path h in the Sobolev space. Those trajecories are supposed to approximate jump trajectories corresponding to A^ε. Two quantities given by "Lagrangian" attached to a skelton trajectory reaching y from x are expressed by functions $d(x, y), d_R(x, y)$. These functions play similar roles as control distances between x and y and provide the rate of convergence above.

In Section 1 we state our result as Theorem. The lower and the upper bounds will be proved in Sections 2 and 3, respectively. Sections 4 to 6 are devoted to the details of proof.

This study may be viewed as a continuation of Ishikawa [17], along the line in the introduction of Léandre [22]. It may be regarded as an extension to the jump case of various results on the large deviation theory in the diffusion case ([8], [14], [23] and [26]), since its way of argument relies on those in the diffusion case. The proof for the upper bound (Proposition 3.3) also depends on Léandre's method and results in [24].

1. NOTATION AND RESULTS

(1) *Basic processes.*

Given $\alpha \in (1, 2)$, let $z_{j,s}$ be a symmetric jump type process on $\mathbb{R}$ (*truncated stable process* (*cf.* [15])) corresponding to the generator

$$(1.1) \qquad \left\{ \begin{array}{l} L\varphi(x) = \displaystyle\int_{\mathbb{R}} [\varphi(x+\zeta) - \varphi(x) - \zeta\varphi'(x)]\eta(\zeta)|\zeta|^{-1-\alpha} d\zeta, \\ \qquad\qquad x \in \mathbb{R}, \quad \varphi \in C_0^\infty(\mathbb{R}) \end{array} \right.$$

with $z_{j,0} \equiv 0$, $j = 1, \cdots, m$. Here $\eta \in C_0^\infty(\mathbb{R}), 0 \leq \eta(\zeta) \leq 1$, η is symmetric, supp $\eta = \{\zeta; |\zeta| \leq c\}$ and $\eta(\zeta) \equiv 1$ in $\{\zeta; |\zeta| \leq c/2\}$ for some $0 < c < +\infty$. We put $g(\zeta)d\zeta \equiv \eta(\zeta)|\zeta|^{-1-\alpha} d\zeta$, that is, $g(\zeta)d\zeta$ is the Lévy measure of $z_{j,s}$ having a compact support.

By $z^{\varepsilon}_{j,s}, \varepsilon > 0$, we denote the perturbed process of $z_{j,s}$ corresponding to the generator

$$(1.2)\qquad \begin{aligned} L^{\varepsilon}\varphi(x) &= \frac{1}{\varepsilon}\int_{\mathbb{R}}[\varphi(x+\varepsilon\zeta)-\varphi(x)-\varepsilon\zeta\varphi'(x)]g(\zeta)d\zeta, \\ &= \int_{\mathbb{R}}[\varphi(x+\zeta)-\varphi(x)-\zeta\varphi'(x)]\frac{1}{\varepsilon^2}g\left(\frac{\zeta}{\varepsilon}\right)d\zeta, \\ &\qquad \varphi \in C_0^{\infty}(\mathbb{R}), \quad j=1,\cdots,m. \end{aligned}$$

with $z^{\varepsilon}_{j,0} \equiv 0, j = 1, \cdots, m$. We put $\tilde{z}^{\varepsilon}_{j,s} \equiv \frac{1}{\sqrt{\varepsilon}} z^{\varepsilon}_{j,s}$, $j = 1, \cdots, m$, and $z^{\varepsilon}_{s} \equiv (z^{\varepsilon}_{1,s}, \cdots, z^{\varepsilon}_{m,s})$, $\tilde{z}^{\varepsilon}_{s} \equiv (\tilde{z}^{\varepsilon}_{1,s}, \cdots, \tilde{z}^{\varepsilon}_{m,s}), \varepsilon > 0$. We assume $\{z^{\varepsilon}_{j,s}; j = 1, \cdots, m\}$ are mutually independent. The law of z^{ε}_{s} is denoted by Π_{ε}. We remark the process $\tilde{z}^{\varepsilon}_{j,s}$ has the generator $\tilde{L}^{\varepsilon}$ of the form

$$(1.3)\quad \tilde{L}^{\varepsilon}\varphi(x) = \int_{\mathbb{R}}[\varphi(x+\zeta)-\varphi(x)-\zeta\varphi'(x)]g_{\varepsilon}(\zeta)d\zeta, \quad \varphi \in C_0^{\infty}(\mathbb{R}),$$

where $g_{\varepsilon}(\zeta)d\zeta \equiv \frac{1}{\varepsilon^{\frac{3}{2}}} g(\frac{\zeta}{\sqrt{\varepsilon}})d\zeta$, in view of the expression (1.2). The Lévy measure $g_{\varepsilon}(\zeta)d\zeta$ of $\tilde{z}^{\varepsilon}_{j,s}$ is again a symmetric measure on $\mathbb{R} \setminus \{0\}$ having the compact support, $\varepsilon > 0$. Since $\tilde{L}^{\varepsilon}\varphi(x) \to c\varphi''(x)$ as $\varepsilon \to 0$ for $\varphi \in C_0^{\infty}(\mathbb{R})$, we observe $\tilde{z}^{\varepsilon}_{j,s} \to cw_{j,s}$ in law, where $w_{j,s}$ denotes 1-dimentional Wiener process (Brownian motion), $j = 1, \cdots, m$ (*cf.* [12], Theorem 4.2.5, [19], Theorem VII-5.4). We may assume $c = 1$ without loss of generality.

We denoted by $H(p_j)$ the "symbol" of the process $z_{j,s}$: $H(p_j) \equiv \log\ E[\exp(p_j \cdot z_{j,1})]$, $j = 1, \cdots, m$, and we put $\mathbf{H}(\mathbf{p}) = \sum_{j=1}^{m} H(p_j)$, $\mathbf{p} = (p_1, \cdots, p_m)$. Let $\mathbf{L}(\mathbf{q})$ be the Legendre transform (the Lagrangian) of $\mathbf{H}(\mathbf{p})$: $\mathbf{L}(\mathbf{q}) = \sup_{\mathbf{p}}[\langle \mathbf{p}, \mathbf{q}\rangle - \mathbf{H}(\mathbf{p})]$. $\mathbf{L}(\mathbf{q})$ is a smooth, convex, non-negative function which satisfies for each $R > 0$ large there exists $m_R > 0$, $M_R > 0$ such that $\mathbf{L}(\mathbf{q}) \le M_R$, $|\operatorname{grad} \mathbf{L}(\mathbf{q})| \le M_R$ for all $|\mathbf{q}| \le R$, and

$$(1.4)\qquad \frac{\partial^2 \mathbf{L}}{\partial q_i \partial q_j}(\mathbf{q}) \ge m_R I \qquad \text{for all} \quad |\mathbf{q}| \ge R.$$

Indeed, we can calculate $H(p_j)$ directly to have $H(p_j) \sim c|p_j|^{\alpha}$, $c > 0$, for $|p_j|$ large. Hence $\mathbf{H}(\mathbf{p}) = \sum_{j=1}^{m} H(p_j) \sim \sum_{j=1}^{m}(c|p_j|^{\alpha})$, and

(1.5)

$$\mathbf{L}(\mathbf{q}) \equiv \sup_{\mathbf{p}}[\langle \mathbf{p}, \mathbf{q}\rangle - \mathbf{H}(\mathbf{p})] \sim \sup_{\mathbf{p}}\left[\sum_{j=1}^{m}(p_j q_j - c|p_j|^{\alpha})\right] = \sum_{j=1}^{m} c'|q_j|^{\frac{\alpha}{\alpha-1}},$$

for $|q_j|$ large (*cf.* [1], Section 14). Since $1 < \alpha < 2$ we have $\frac{\alpha}{\alpha-1} > 2$, and we have (1.4).

(2) *Function spaces.*

We denote by $\mathcal{D}(I)$ the space of paths on $I = [0,1]$ to $\mathbb{R}^d$ such that all the components are right continuous and have left limits. For $x(\cdot), y(\cdot) \in \mathcal{D}(I)$, let us denote by $d(x(\cdot), y(\cdot))$ the Skorohod metric, and by $\rho(x(\cdot), y(\cdot))$ the sup-norm metric (*see* [10], Section 13.5). We remark that $(\mathcal{D}(I), d)$ is a Polish space.

Let $W^{1,p}(I)$, $p > 1$, be the Sobolev space

$$W^{1,p}(I) = \left\{ \varphi \in \mathcal{S}'(I); \|\varphi\|_{W^{1,p}} = (\|\dot{\varphi}\|_{L^p}^p + \|\varphi\|_{L^p}^p)^{1/p} < +\infty \right\}.$$

Then $(W^{1,p}(I), \|\cdot\|_{W^{1,p}})$ is a Banach space. On the set $W^{1,p}(I)$ we also put the sup-norm $\|\varphi\|_\infty = \sup_{t\in I} |\varphi(t)|$, $\varphi \in W^{1,p}(I)$. In what follows, we put $p = \frac{\alpha}{\alpha-1} > 2$, and denoted by $\mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$ the product space $\underbrace{W^{1,\frac{\alpha}{\alpha-1}}(I) \times \cdots \times W^{1,\frac{\alpha}{\alpha-1}}(I)}_{m}$ with the norm $\|\varphi\|_{\mathbf{W}^{1,\frac{\alpha}{\alpha-1}}} \equiv \left(\sum_{j=1}^m \|\varphi_j\|_{W^{1,\frac{\alpha}{\alpha-1}}}^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}$ for $\varphi = (\varphi_1, \cdots, \varphi_m)$.

For $h = (h_1, \cdots, h_m) \in \mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$, let $y_s(h)$ be the deterministic path in $\mathbb{R}^d$defined by

$$\text{(1.6)} \qquad \begin{cases} dy_s(h) = \sum_{j=1}^m X_j(y_s(h))\dot{h}_{j,s}ds + X_0(y_s(h))ds \\ y_0(h) = x. \end{cases}$$

Here $X_1, \cdots, X_m$ are C^∞-vector fields on $\mathbb{R}^d$ that are bounded including their derivatives. (Here and in the sequel we identify vector fields with the vector valued functions.) We put (*see* (1.17)) the restricted Hörmander condition on $X_1, \cdots, X_m$

$$\text{(1.7)} \qquad Lie(X_1, \cdots, X_m)(x) = T_x(\mathbb{R}^d) \ \ for \ \ all \ \ x \in \mathbb{R}^d.$$

We denote by Φ_x the C^∞ map $\mathbf{W}^{1,\frac{\alpha}{\alpha-1}} \to \mathbb{R}^d$, $h \mapsto y_1(h)$. The map Φ_x is said to be a submersion at $h_0 \in W^{1,\frac{\alpha}{\alpha-1}}$ if the Frechet derivative $D\Phi_x(h_0)(\cdot)$ at h_0 is onto from $\mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$ to $\mathbb{R}^d$.

Given $x, y \in \mathbb{R}^d$ we put quantities $d(x,y), d_R(x,y)$ as follows :

$$\text{(1.8)} \qquad d^{\frac{\alpha}{\alpha-1}}(x,y) \equiv \inf\left\{ \int_0^1 \mathbf{L}(\dot{h}_s)ds; \Phi_x(h) = y \right\}$$

(1.9)

$$d_R^{\frac{\alpha}{\alpha-1}}(x,y) \equiv \inf\left\{\int_0^1 \mathbf{L}(\dot{h}_s)ds; \Phi_x(h) = y, \Phi_x \text{ is a submersion at } h\right\}.$$

The function $d(x,y)$ is finite for $x, y \in \mathbb{R}^d (y \neq x)$ by (1.7) (*cf.* [8], Theorem 1.14). Further, if $X_0 \equiv 0$ then $d(x,y)$ and $d_R(x,y)$ coincide, since (1.7) implies the submersion condition (*cf.* the remark in Section II and the proof of Théorème 1.2 of [23], and Section II of 24). In this case the function $(x,y) \mapsto d(x,y)$ is continuous (*cf.* [8], Theorem 1.14). It follows from Theorem 5.2.1 of [14] that level set $\tilde{\Phi}(r) = \{h; \int_0^1 \mathbf{L}(\dot{h}_s)ds \leq r\}$ is compact in $(\mathcal{D}(I), \| \ \|_\infty)$. Further we have the following

LEMMA 1.1 (Freidlin and Ventcel [14], (5.2.5) and (5.2.6)). – *For any $\alpha > 0$, $\eta > 0$, $r_0 > 0$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, h satisfying $\int_0^1 \mathbf{L}(\dot{h}_s)ds \leq r_0$, we have*

$$(1.10) \qquad \Pi_\varepsilon\left\{\sup_{0\leq s\leq 1} |z_s^\varepsilon - h_s| < \alpha\right\} \geq \exp\left[-\frac{1}{\varepsilon}\left(\int_0^1 \mathbf{L}(\dot{h}_s)ds + \eta\right)\right]$$

and

$$(1.11) \quad \Pi_\varepsilon\left\{\inf_{\substack{h\,:\int_0^1 L(\dot{h}_s)ds\leq r \\ r\leq r_0}} \ \sup_{0\leq s\leq 1} |z_s^\varepsilon - h_s| \geq \alpha\right\} \leq \exp\left[-\frac{1}{\varepsilon}(r-\eta)\right].$$

(3) *Processes and semigroups.*

Let $x_s(\varepsilon)$ be the process given by the following SDE

$$(1.12) \qquad \begin{cases} dx_t(\varepsilon) = \displaystyle\sum_{j=1}^m X_j(x_{t-}(\varepsilon))dz_{j,t}^\varepsilon + X_0(x_{t-}(\varepsilon))dt \\ x_0(\varepsilon) \equiv x. \end{cases}$$

Here X_0 and $X_1, \cdots, X_m$ are as in (1.6), (1.7). We put $\varphi_s(\varepsilon) : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto x_s(\varepsilon)$. The law of $x_t(\varepsilon)$ (under Π_ε) is denoted by P. It is known that under (1.7) $\varphi_s(\varepsilon)$ is continuous for all s almost surely. Further we have

PROPOSITION 1.2 (Léandre [24], Proposition I1). – *Suppose that there exists $C > 0$ such that for all $j = 1, \cdots, m$,*

$$(1.13) \qquad \inf_{(\zeta/\varepsilon)\in \text{supp} g,\ x\in\mathbb{R}^d} \left|\det\left(I + \zeta\left(\frac{\partial}{\partial x}X_j(x)\right)\right)\right| > C.$$

Then, for all s and x, $(\frac{\partial}{\partial x}\varphi_s(\varepsilon))^{-1}(\omega)$ exists almost surely.

We remark that (1.13) is satisfied if the Jacobian matrices $(\frac{\partial}{\partial x}X_j(x))$ are anti-symmetric, or, $\varepsilon > 0$ is sufficiently small. Under (1.13), $\varphi_s(\varepsilon)$ defines a flow of C^∞-diffeomorphisms which is bounded in the sense that for all $p > 1$ and all multi-indices (α)

$$(1.14) \quad \begin{cases} E\left[\left(\sup_{u\leq s}\left|\left(\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\varphi_u(\varepsilon)\right)\right|\right)^p\right] \leq C_1((\alpha), s, p, \varepsilon) \\ \text{and} \\ E\left[\left(\sup_{u\leq s}\left|\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\left(\frac{\partial}{\partial x}\varphi_u(\varepsilon)\right)^{-1}\right|\right)^p\right] \leq C_1'((\alpha), s, p, \varepsilon) \end{cases}$$

for some constants $C_1((\alpha), s, p, \varepsilon)$ and $C_1'((\alpha), s, p, \varepsilon)$, which depend only on $(\alpha), s, p, \varepsilon$ and the uniform norm of derivatives of all orders of $X_0, X_1, \cdots, X_m$ (*cf.* Léandre [24], (1.9), (1.11)).

For $\varepsilon > 0$, let $\nu = \nu_\varepsilon$ be a C^∞-function : $\mathbb{R} \to [0, 1]$ with compact support such that it is equal to ζ^2/ε in a neighborhood of the origin. Let $t \mapsto \mathbf{K}_t(x, \varepsilon)$ be the *stochastic quadratic form associated to* $\{x_s(\varepsilon)\}$ defined by

$$(1.15) \quad \mathbf{K}_t(x,\varepsilon)(\cdot) = \sum_{j=1}^{m}\sum_{s\leq t}\nu(\Delta z^\varepsilon_{j,s})\left\langle\left(\frac{\partial}{\partial x}\varphi_s(\varepsilon)\right)^{-1}X_j(\varphi_{s-}(\varepsilon)),\ \cdot\right\rangle^2.$$

We remark

$$\left(\frac{\partial}{\partial x}\varphi_s(\varepsilon)\right)^{-1} = \left(\frac{\partial}{\partial x}\varphi_{s-}(\varepsilon)\right)^{-1}\left\{\sum_{j=1}^{m}\left(I + \frac{\partial}{\partial x}X_j(x_{s-}(\varepsilon)\Delta z^\varepsilon_{j,s})\right)^{-1}\right\}$$

(*cf.* [24] (1.13)). We have

PROPOSITION 1.3 (Léandre [24], Théorème I3). – *Suppose that the assumption* (1.13) *holds. In order that the law of* $x_{t_0}(x)$ $(t_0 > 0)$ *possesses a* C^∞ *density, it is sufficient that for all* $p > 1$,

$$(1.16) \quad E[|(\mathbf{K}_{t_0}(x,\varepsilon)(\cdot))^{-1}|^p] < +\infty.$$

We put $F_1(x) = (X_1, \cdots, X_m)(x), F_\ell(x) = [F_{\ell-1}, (X_1, \cdots, X_m)](x) \cup F_{\ell-1}(x), \ell = 2, 3, \cdots$. Here $[X, Y](x)$ denotes the Lie bracket of X and Y at x. We assume that there exists some integer N such that

$$(1.17) \quad \inf_{x\in\mathbb{R}^d, e\in S^{d-1}} \sum_{Y\in F_N(x)} \langle Y, e\rangle^2 > 0.$$

It follows from Théorème III.1 of [24] that (1.17) implies (1.16). We remark (1.17) is stronger than (1.7). We denote by $p_t(x, y, \varepsilon)$ the density of the law $p_t(x, dy, \varepsilon)$ of $x_t(\varepsilon)$. The semigroup corresponding to $(p_t(x, y, \varepsilon))_{t \geq 0}$ has the generator A^ε in (0.3). Our main result is the following

THEOREM. – *Assume* (1.17). *As* ε *tends to* 0,

$$\liminf_{\varepsilon \to 0} \varepsilon \log p_1(x, y, \varepsilon) \geq -d_R^{\frac{\alpha}{\alpha-1}}(x, y) \tag{1.18}$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log p_1(x, y, \varepsilon) \leq -d^{\frac{\alpha}{\alpha-1}}(x, y). \tag{1.19}$$

If in particular $X_0 \equiv 0$, *then we have*

$$\lim_{\varepsilon \to 0} \varepsilon \log p_1(x, y, \varepsilon) = -d_R^{\frac{\alpha}{\alpha-1}}(x, y). \tag{1.20}$$

The proof of (1.18), (1.19) will be given in Sections 2 and 3 respectively. The last statement follows by the remark just above Lemma 1.1.

The next lemma (*continuity lemma*) plays an important role in proving the upper bound (Lemma 3.5).

LEMMA 1.4. – *Let* $h \in \mathbf{W}^{1, \frac{\alpha}{\alpha-1}}$ *and let* $y_s(h)$ *be the solution of* (1.6). *For* $\varepsilon > 0$ *let* $x_s(\varepsilon)$ *be the flow defined by* (1.12). *Fix* $K > 0$ *and* $R > 0$. *Then there exist* $\varepsilon_0 > 0, r > 0$ *and* $C > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$

$$P\{\sup_{s \leq t} |z_s^\varepsilon - h_s| < r, \sup_{s \leq t} |x_s(\varepsilon) - y_s(h)| > R\} \leq C \exp(-K/\varepsilon). \tag{1.21}$$

Here $r > 0$ depends only on $\|h\|_{\mathbf{W}^{1, \frac{\alpha}{\alpha-1}}}$. The proof of this lemma will be given in Section 6.

2. LOWER BOUND

We denote by $y_s(h)$ the curve defined in Section 1, and by Φ_x the mapping $h \mapsto y_1(h)$. In this section, we prove the following

PROPOSITION 2.1. – *Assume that there exists* $h = (h_1, \cdots, h_m) \in \mathbf{W}^{1, \frac{\alpha}{\alpha-1}}$ *such that* $y_1(h) = y$ *and that* Φ_x *is a submersion at* h. *Then*

$$\liminf_{\varepsilon \to 0} \; \varepsilon \log p_1(x, y, \varepsilon) \geq -d_R^{\frac{\alpha}{\alpha-1}}(x, y). \tag{2.1}$$

Given $\eta > 0$. We have only to show

$$\varepsilon \log p_1(x, y, \varepsilon) \geq -d_R^{\frac{\alpha}{\alpha-1}}(x, y) - \eta \tag{2.1$'$}$$

for small $\varepsilon > 0$. Given $h \in \mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$ satisfying the assumption, we denote by $\tilde{x}_t(\varepsilon)$ the process defined by

$$\begin{cases} d\tilde{x}_t(\varepsilon) = \sum_{j=1}^{m} X_j(\tilde{x}_{t-}(\varepsilon))(dz^{\varepsilon}_{j,t} + dh_{j,t}) + X_0(\tilde{x}_t(\varepsilon))dt \\ x_0(\varepsilon) \equiv x. \end{cases}$$

By $\tilde{P}$ we denote the law of $\tilde{x}_t(\varepsilon)$. Then we have a transformation of measure P and $\tilde{P}$ (Girsanov transformation for jump processes) as follows (*cf.* [14], p. 149); Let $\alpha(s) \equiv \frac{d\mathbf{L}}{dq}(\dot{h}_s)$. Then

$$\tag{2.2} \begin{aligned} d\tilde{P} = \exp\Big\{ -\frac{1}{\varepsilon}\Big(\sum_{j=1}^{m}\int_0^1 \alpha_j(s)dz^{\varepsilon}_{j,s}\Big) \\ -\frac{1}{\varepsilon}\Big(\int_0^1 ds\langle\alpha(s),\dot{h}_s\rangle - \mathbf{H}(\alpha(s))\Big)\Big\}dP. \end{aligned}$$

Hence the law $\tilde{P}$ is uniquely defined up to h. Further, we have

$$\begin{aligned} E[f(x_1(\varepsilon))] = E\Big[f(\tilde{x}_1(\varepsilon))\exp\Big\{ -\frac{1}{\varepsilon}\Big(\sum_{j=1}^{m}\int_0^1 \alpha_j(s)dz^{\varepsilon}_{j,s}\Big) \\ -\frac{1}{\varepsilon}\Big(\int_0^1 ds\langle\alpha(s),\dot{h}_s\rangle - \mathbf{H}(\alpha(s))\Big)\Big\}\Big], \end{aligned}$$

for $f \in C_0^{\infty}(\mathbb{R}^d)$.

Let $(f_n; n \in \mathbf{N})$, $f_n \in C_0^{\infty}(\mathbb{R}^d)$ be a series of non-negative functions such that $f_n \to \delta_{\{0\}}$. Let $\psi \in C_0^{\infty}(\mathbb{R}^d)$ be a cut-off function such that $0 \le \psi \le 1$, $\psi \equiv 0$ in $\{|x| \ge \eta\}$ and $\psi \equiv 1$ in $\{|x| \le \frac{\eta}{2}\}$. By the Girsanov we have

$$\begin{aligned} E[f_n(x_1(\varepsilon) - y_1(h))] \ge \exp\Big[-\frac{1}{\varepsilon}\Big(\int_0^1 \mathbf{L}(\dot{h}_s)ds\Big)\Big] \\ \times E\Big[f_n(\tilde{x}_1(\varepsilon) - y_s(h))\psi\Big(\sum_{j=1}^{m}\int_0^1 \alpha_j(s)dz^{\varepsilon}_{j,s}\Big) \\ \times \exp\Big[-\frac{1}{\varepsilon}\Big(\sum_{j=1}^{m}\int_0^1 \alpha_j(s)dz^{\varepsilon}_{j,s}\Big)\Big]\Big] \end{aligned}$$

because $\mathbf{L}(\mathbf{q}) \equiv \sup_{\mathbf{p}}[\langle \mathbf{p}, \mathbf{q}\rangle - \mathbf{H}(\mathbf{p})]$. Since $y_1(h) = y$, we have

$$\begin{aligned} p_1(x, y, \varepsilon) &= \lim_{n\to 0} E[f_n(x_1(\varepsilon) - y_1(h))] \\ &\geq \exp\left[-\frac{1}{\varepsilon}(d_R^{\frac{\alpha}{\alpha-1}}(x, y) + 2\eta)\right] \\ &\quad \times \lim_{n\to 0} E\left[f_n(\tilde{x}_1(\varepsilon) - y_1(h))\psi\left(\sum_{j=1}^{m}\int_0^1 \alpha_j(s)dz_{j,s}^{\varepsilon}\right)\right]. \end{aligned}$$

We put $u_s(\varepsilon, h) \equiv x + \frac{1}{\sqrt{\varepsilon}}(\tilde{x}_s(\varepsilon) - y_s(h))$, $\varepsilon > 0$ and $\tilde{z}_{j,s}^{\varepsilon} \equiv \frac{1}{\sqrt{\varepsilon}} z_{j,s}^{\varepsilon}$, $\varepsilon > 0$. Then $u_s(\varepsilon, h)$ satisfies

$$(2.3) \quad \begin{cases} du_s(\varepsilon, h) = \displaystyle\sum_{j=1}^{m} X_j(u_{s-}(\varepsilon, h), s, \varepsilon)d\tilde{z}_{j,s}^{\varepsilon} + X_0(u_s(\varepsilon. h), s, \varepsilon)ds \\ u_0(\varepsilon, h) \equiv x \end{cases}$$

where

$$X_j(x', s, \varepsilon) \equiv X_j(\sqrt{\varepsilon}x' + y_s(h)),$$

$$\begin{aligned} X_0(x', s, \varepsilon) \equiv &\sum_{j=1}^{m} \frac{1}{\sqrt{\varepsilon}}(X_j(\sqrt{\varepsilon}x' + y_s(h)) - X_j(y_s(h)))\dot{h}_{j,s} \\ &+ \frac{1}{\sqrt{\varepsilon}}\{X_0(y_s(h) + \sqrt{\varepsilon}x') - X_0(y_s(h))\}. \end{aligned}$$

We write $\varphi_s(\varepsilon, h) : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto u_s(\varepsilon, h)$, which also defines a flow of C^∞- diffeomorphism. We have properties as (1.17) also for $\varphi_s(\varepsilon, h)$. That is, for all $p > 1$ and all multi-indices (α),

$$(2.4) \quad \begin{cases} E\left[\left(\sup_{u\leq s}\left|\left(\dfrac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\varphi_u(\varepsilon, h)\right)\right|\right)^p\right] \leq C_2((\alpha), s, p, \varepsilon), \\ E\left[\left(\sup_{u\leq s}\left|\dfrac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\left(\dfrac{\partial}{\partial x}\varphi_u(\varepsilon, h)\right)^{-1}\right|\right)^p\right] \leq C_2'((\alpha), s, p, \varepsilon) \end{cases}$$

for some constants $C_2((\alpha), s, p, \varepsilon)$ and $C_2'((\alpha), s, p, \varepsilon)$, which do not depend on x. Since $\tilde{z}_{j,s}^{\varepsilon} \equiv \frac{1}{\sqrt{\varepsilon}} z_{j,s}^{\varepsilon} \overset{\text{in law}}{\longrightarrow} w_{j,s}$ as $\varepsilon \to 0$ (*cf.* (1.4)), we have $u_s(\varepsilon, h) \overset{\text{in law}}{\longrightarrow} u_s(0, h)$ as $\varepsilon \to 0$ where $u_s(0, h)$ is the Gaussian (non-degenerate) diffusion process given by

$$(2.5)\qquad \begin{cases} du_s(0,h) = \sum_{j=1}^{m} X_j(y_s(h))\delta\omega_{s,j} \\ \qquad + \sum_{j=1}^{m} \Big(\frac{\partial X_j}{\partial x}(y_s(h)) \cdot u_s(0,h)\Big) dh_{j,s} \\ \qquad + \Big(\frac{\partial X_0}{\partial x}(y_s(h)) \cdot u_s(0,h)\Big) ds \\ u_0(0,h) \equiv x. \end{cases}$$

By the assumption (1.20) $u_s(0,h) - x$ possesses a C^∞−density $q_s(x,0,h)(z)$ such that $q_1(x,0,h)(0) > 0$, since it is Gaussian.

Then we observe

$$p_1(x,y,\varepsilon) \geq \exp\Big[-\frac{1}{\varepsilon}(d_R^{\frac{\alpha}{\alpha-1}}(x,y) + 2\eta)\Big] \times \varepsilon^{-d/2} \times \lim_{n\to 0} E\Big[f_n(u_1(\varepsilon,h) - x)\ \psi\Big(\sqrt{\varepsilon}\sum_{j=1}^{m}\int_0^1 \alpha_j(s)d\tilde{z}^{\varepsilon}_{j,s}\Big)\Big]$$

Let $\mu_t(x,\varepsilon)$ be the measure associated to $f \mapsto E[f(u_t(\varepsilon,h) - x)\ \psi(\sqrt{\varepsilon}\sum_{j=1}^{m}\int_0^t \alpha_j(s)d\tilde{z}^{\varepsilon}_{j,s})]$, $t > 0$. We can show the Malliavin quadratic form $K_t(x,\varepsilon,h)(\cdot)$ associated to $u_t(\varepsilon,h)$ satisfies

$$(2.6)\qquad \sup_{\substack{h\in F\\ \varepsilon\in(0,1]}} E[|K_t^{-1}(x,\varepsilon,h)(\cdot)|^p] \leq C(p,F) \quad \text{for all} \quad p>1, \qquad t>0$$

for any compact set $F \subset \mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$ (*cf.* [24] (1.16), [23] (1.17)), and hence $\mu_t(x,\varepsilon)$ posseses a C^∞ density $q_t(x,\varepsilon,h)(z)$ at $z \in \mathbb{R}^d$. Since $\psi\Big(\sqrt{\varepsilon}\sum_{j=1}^{m}\int_0^t \alpha_j(s)d\tilde{z}^{\varepsilon}_{j,s}\Big) \to 1$ as $\varepsilon \to 0$, we have only to show

$$(2.7)\qquad q_1(x,\varepsilon,h)(0) \longrightarrow q_1(x,0,h)(0) > 0$$

to get the lower bound.

To this end we have to show

$$(2.8)\qquad E[f^{(\alpha)}(u_1(\varepsilon,h))] \to E[f^{(\alpha)}(u_1(0,h))]$$

as $\varepsilon \to 0$, for all $f \in C_0^\infty(\mathbb{R}^d)$ and all multi-index α. Here we have

PROPOSITION 2.2. – *Integration-by-parts formulae hold:*

$$(2.9)\qquad E[f^{(\alpha)}(u_1(\varepsilon,h))] = E[\mathcal{A}_1^{(\alpha)}(\varepsilon,h) f(u_1(\varepsilon,h))]$$

(2.10) $$E[f^{(\alpha)}(u_1(0,h))] = E[\mathcal{A}_1^{(\alpha)}(0,h) f(u_1(0,h))],$$

and we have

$$\mathcal{A}_1^{(\alpha)}(\varepsilon,h) \to \mathcal{A}_1^{(\alpha)}(0,h)$$

in law as $\varepsilon \to 0$.

By this (2.7) follows. We prove this Proposition in Section 4. This completes the proof of Proposition 2.1.

Remark 2.3. – Our procedure of passing $\varepsilon \to 0$ in (2.3) may be regarded as "concentrating on small jumps" in view of (1.2), (1.3). Instead, there is another way to pass to the diffusion process from the jump process; namely $\alpha \to 2$ (*see* Bismut [9], p. 63, Remark 3 and [7], p. 187, Remarque 2).

3. UPPER BOUND

The object of this section is to prove the following

PROPOSITION 3.1.

(3.1) $$\limsup_{\varepsilon \to 0} \varepsilon \log p_1(x,y,\varepsilon) \le -d^{\frac{\alpha}{\alpha-1}}(x,y).$$

Let $\rho : \mathbb{R}^d \to [0,1]$ be a C^∞ function, and let $\mu(\rho,\varepsilon)$ be the measure associated to

(3.2) $$f \mapsto E[\rho(x_1(\varepsilon)) f(x_1(\varepsilon))].$$

As in Section 2, we can show the measure $\mu(\rho,\varepsilon)$ possesses a C^∞ density which we denote by $\tilde{p}_\rho(x,y,\varepsilon)$. Then $p_1(x,y,\varepsilon) = \tilde{p}_\rho(x,y,\varepsilon)$ if $\rho(y) = 1$.

Let $\eta > 0$ and $q > 1$. To obtain the point-wise upper bound, we may assume supp ρ is compact by the argument above : $F =$ supp ρ. Here we have

PROPOSITION 3.2. – *Let* $F \subset \mathbb{R}^d$ *be a closed set. For any* $\eta > 0$ *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

(3.3) $$\varepsilon \log P\{x_1(\varepsilon) \in F\} \le -\inf_{y \in F} d^{\frac{\alpha}{\alpha-1}}(x,y) + \eta.$$

The proof of this proposition will be given just below. By this proposition we have immediately, for $0 < \varepsilon < \varepsilon_0$,

(3.4)
$$\mu(\rho,\varepsilon)(f) \le \|f\|_\infty \exp\left(\frac{1}{\varepsilon}\left\{-\inf_{y \in \operatorname{supp}\rho} d^{\frac{\alpha}{\alpha-1}}(x,y) + \eta\right\}\right), \quad f \in C_0^\infty(\mathbb{R}^d).$$

However, to obtain the upper bound for the density $\tilde{p}_\rho(x,y,\varepsilon)$, we have to show further that

PROPOSITION 3.3. – *For* $0<\varepsilon<\varepsilon_0$, *for all multi-index* α *and* $q>1$, *there exist* $C(\alpha,q)$ *and* $M(\alpha,q)$ *such that*

$$(3.5)\qquad \begin{cases} \mu(\rho,\varepsilon)(D^\alpha f) \le C(\alpha,q)\varepsilon^{-M(\alpha,q)}\|f\|_\infty \\ \quad \times \exp\left(\dfrac{1}{\varepsilon q}\left\{-\inf_{y\in \operatorname{supp}\rho} d^{\frac{\alpha}{\alpha-1}}(x,y)+\eta\right\}\right), \\ \qquad f\in C_0^\infty(\mathbb{R}^d). \end{cases}$$

Indeed, Proposition 3.3 easily leads

$$(3.6)\quad \tilde{p}_\rho(x,y,\varepsilon) \le C(q)\varepsilon^{-M(q)}\exp\left(\frac{1}{\varepsilon q}\left\{-\inf_{y\in\operatorname{supp}\rho} d^{\frac{\alpha}{\alpha-1}}(x,y)+\eta\right\}\right),$$

for all $q>1$, and we have the conclusion of Proposition 3.1.

First we give the proof of Proposition 3.2, and after that of Proposition 3.3.

Proof of Proposition 3.2. – We first show

LEMMA 3.4. – *Given any closed set* E *in* $(\mathcal{D}(I),\rho)$, *for any* $\eta>0$ *there exists* $\varepsilon_0>0$ *such that*

$$(3.7)\quad \varepsilon\log P\{x_\cdot(\varepsilon)\in E\} \le -\inf_{y_\cdot(h)\in E}\left\{\int_0^1 \mathbf{L}(\dot{h}_s)ds\right\}+\eta,\qquad 0<\varepsilon<\varepsilon_0.$$

Then the conclusion of Proposition 3.2 follows if we put

$$(3.8)\qquad E=\{y_s(h)\in\mathcal{D}(I); y_1(h)\in F\},$$

which is a closed in $(\mathcal{D}(I),\rho)$.

Proof of Lemma 3.4. – We may assume

$$\inf_{y_\cdot(h)\in E}\left\{\int_0^1 \mathbf{L}(\dot{h}_s)ds\right\}>0$$

(otherwise the assertion is trivial). Choose

$$0<r<\inf_{y_\cdot(h)\in E}\left\{\int_0^1 \mathbf{L}(\dot{h}_s)ds\right\}.$$

Then E is disjoint from

$$\Phi(r)\equiv\left\{y_\cdot(h);\ \int_0^1 \mathbf{L}(\dot{h}_s)ds\le r\right\}.$$

Since $\Phi(r)$ is compact (*cf.* Section 1, [11], Proposition 3.1), there exists $c > 0$ such that $\rho(x.(\varepsilon), \Phi(r)) \geq c$ for $x.(\varepsilon) \in E$. Since

$$r < \inf_{y.(h) \in E} \left\{ \int_0^1 \mathbf{L}(\dot{h}_s) ds \right\}$$

is arbitrary, we have the assertion by the next lemma.

LEMMA 3.5. – *Given any* $c > 0, r > 0$ *and* $\eta > 0$, *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$,

$$\varepsilon \log P\{\rho(x.(\varepsilon), \Phi(r)) \geq c\} \leq -r + 2\eta. \tag{3.9}$$

Proof. – Let $M(r,c) = \{y.(h); \rho(y.(h), y.(h')) \geq c$ for all $y.(h')$ such that $\int_0^1 \mathbf{L}(\dot{h}'_s) ds \leq r\}$. Then $[\rho(y.(h), \Phi(r)) \geq c \iff y.(h) \in M(r,c)]$. Since $\tilde{\Phi}(r) \equiv \{h; \int_0^1 \mathbf{L}(\dot{h}_s) ds \leq r\}$ is compact (in $(\mathcal{D}(I), \|\cdot\|_\infty)$), there exist $h_1, \cdots, h_N \in \tilde{\Phi}(r)$ such that $\tilde{\Phi}(r) \subset \bigcup_{i=1}^N B(h_i, \alpha) \equiv U; B(h,\alpha) \equiv \{h'; \|h - h'\|_\infty < \alpha\}$. Then $y.(h_i) \in \Phi(r)$ and

$$\begin{aligned} \{z^\varepsilon_\cdot \in U\} \cap \{x.(\varepsilon) \in M(r,c)\} \\ \subset \bigcup_{i=1}^N \{\|z^\varepsilon_\cdot - h_i\|_\infty < \alpha, \rho(x.(\varepsilon), y.(h_i) \geq c\}. \end{aligned} \tag{3.10}$$

By Lemma 1.4 (continuity lemma) with $t = 1, K = r - \eta + 1$, we have, for some $\varepsilon_1 > 0, \alpha > 0$ and $c > 0$,

$$P\{z^\varepsilon_\cdot \in U, x.(\varepsilon) \in M(r,c)\} \leq NC \exp\left(-\frac{K}{\varepsilon}\right) \tag{3.11}$$

for $0 < \varepsilon < \varepsilon_1$.

On the other hand, it follows from (1.11) that, for some $\varepsilon_2 > 0$,

$$\varepsilon \log P\{z^\varepsilon_\cdot \notin U\} \leq -r + \frac{\eta}{2} \quad \text{for} \quad 0 < \varepsilon < \varepsilon_2. \tag{3.12}$$

Choose $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$. Then by (3.11), (3.12),

$$\begin{aligned} P\{x.(\varepsilon) \in M(r,c)\} &\leq P\{z^\varepsilon_\cdot \notin U\} + NC \exp\left(-\frac{K}{\varepsilon}\right) \\ &\leq \exp\left(-\frac{r}{\varepsilon} + \frac{\eta}{2\varepsilon}\right) + NC \exp\left(-\frac{K}{\varepsilon}\right) \\ &\leq \exp\left(-\frac{r}{\varepsilon} + \frac{2\eta}{\varepsilon}\right) \cdot \left\{\exp\left(-\frac{\eta}{2\varepsilon}\right) + NC \exp\left(-\frac{1}{\varepsilon}\right)\right\}, \end{aligned} \tag{3.13}$$

since $K = r - \eta + 1$. Choose $\varepsilon > 0$ small, and we have the assertion.

Proof of Proposition 3.3. – The proof is rather delicate and is carried out in a similar way as in [24], but it is more tedious in our case. We shall devide it into four steps.

(*Step* 0). – Let $\mathbf{K}_1(\cdot) = \mathbf{K}_1(x,\varepsilon)(\cdot)$ denote the stochastic quadratic form at $t = 1$ associated to $x_s(\varepsilon)$ ((1.14)). Let $(f_i, i \in I)$ be a family of functions : $\mathbb{R} \to \mathbb{R}$ with some index set I. For $\eta > 0$ small we write $f_i(\eta) = o_i(1)$ if $\lim_{\eta\to 0} f_i(\eta) = 0$ uniformly in i, and $f_i(\eta) = o_i(\eta^\infty)$ if $(f_i(\eta)/\eta^p) = o_i(1)$ for all $p > 1$. In case that $f_i(\eta)$ is a random variable $f_i(\omega,\eta), \omega \in \Omega$, we say as above if there exists a subset Ω_1 of probability 1 such that $\sup_{\omega\in\Omega_1, i\in I} f_i(\omega,\eta) = o_i(1)$.

In view of the integration-by-parts formula ([5], Section 4), we have

(3.14)
$$\mu(\rho,\varepsilon)(D^\alpha f) = E[\rho(x_1(\varepsilon))D^\alpha f(x_1(\varepsilon))] = E[J_1^{(\alpha)}(\varepsilon,\rho)f(x_1(\varepsilon))].$$

Hence

$$|\mu(\rho,\varepsilon)(D^\alpha f)| \le \|f\|_\infty (E[|J_1^{(\alpha)}(\varepsilon,\rho)|^p])^{1/p}(P\{x_1(\varepsilon) \in \text{supp } \rho\})^{1/q},$$

where

$$|J_1^{(\alpha)}(\varepsilon,\rho)| \le C|\mathbf{K}_1^{-1}(x,\varepsilon)|^{2|\alpha|}|Q^\alpha(\varepsilon,\rho)|$$

with

$$\sup_{\varepsilon\in(0,1]} E[|Q^\alpha(\varepsilon,\rho)|^p] < +\infty, \qquad p \ge 1$$

(*cf.* [7], (4.50), (4.91)). Hence it is sufficient to show, for all $p > 1$ and $e \in \mathcal{S}^{d-1}$,

$$E[|\mathbf{K}_1^{-1}(e)|^p] \le \frac{C'(\alpha,q)}{\varepsilon^{M(\alpha,q)}}, \qquad \varepsilon \in (0,1] \tag{3.15}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

To get (3.15), fix $p > 1$, $e \in \mathcal{S}^{d-1}$. Here we introduce a new parameter $\gamma = \gamma(\varepsilon) \equiv \frac{1}{\sqrt{\varepsilon}}$. Note that $\gamma(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We shall show

$$P\{\mathbf{K}_1^{-1}(e) > \gamma^N\eta^{-1}\} = P\{\mathbf{K}_1(e) < \gamma^{-N}\eta\} = o_{e,\varepsilon}(\eta^\infty) \tag{3.16}$$

for some integer N.

(*Step* 1). – Choose arbitrary integer $k \le \frac{1}{\gamma^N\eta}$. Since

$$P\left\{\sup_{k\le\frac{1}{\gamma^N\eta}}\left|\left(\frac{\partial\varphi_{k\gamma^N\eta}}{\partial x}\right)^{-1}\right| > \frac{1}{\eta}\right\} = o_{k,\varepsilon}(\eta^\infty) \tag{3.18}$$

(*cf.* [24], (1.11)), in order to get (3.16) it is sufficient to show, for some $r > 0$,

(3.19) $$P\{\mathbf{K}_{(k+1)\gamma^N\eta}(\bar{e}) - \mathbf{K}_{k\gamma^N\eta}(\bar{e}) < (\gamma^{-N}\eta)^r / \mathcal{F}_{k\gamma^N\eta}\} = o_{e,k,\varepsilon}(1)$$

where $\bar{e} = |(\frac{\partial\varphi_{k\gamma^N\eta}}{\partial x})^{-1}|^{-1}e$.

To get (3.19) it is enough to show

(3.20) $$P\Big\{\Big[\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta} ds 1_{((\gamma^{-N}\eta)^{-r_1},\infty)} \times \Big(\int_{|u|>(\gamma^{-N}\eta)^r} d\nu_s(e,k,\gamma^N\eta)(u)\Big)\Big] \leq (\gamma^N\eta)^{r_2} / \mathcal{F}_{k\gamma^N\eta}\Big\} = o_{e,k,\varepsilon}(1)$$

for some $r_1 > 0, r_2 > 0, r_1 > r_2$ where $d\nu_s(u) = d\nu_s(e,k,\gamma^N\eta)(u)$ is the Lévy measure of $\Delta\mathbf{K}_t(\bar{e})$ given $\mathcal{F}_{k\gamma^N\eta}$ (*i.e.* $ds \times d\nu_s(u)$ is the compensator of $\Delta\mathbf{K}_s(\bar{e})$ with respect to dP). Indeed, since

$$Y_t = \exp\Big[-\sum_{k\gamma^N\eta\leq s\leq t} 1_{((\gamma^{-N}\eta)^r,\infty)}(\Delta\mathbf{K}_s(\bar{e})) - \Big[\int_{k\gamma^N\eta}^t ds \int_{|u|>(\gamma^{-N}\eta)^r} (e^{-1} - 1) d\nu_s(u)\Big]\Big]$$

is a martingale (given $\mathcal{F}_{k\gamma^N\eta}$), we have

$$\begin{aligned} &P\Big\{\sum_{k\gamma^N\eta\leq s\leq(k+1)\gamma^N\eta} 1_{((\gamma^{-N}\eta)^r,\infty)}(\Delta\mathbf{K}_s(\bar{e})) = 0/\mathcal{F}_{k\gamma^N\eta}\Big\} \\ &\leq E\Big[\exp\Big[\int_{k\gamma^N\eta}^t ds \int_{|u|>(\gamma^{-N}\eta)^r} (e^{-1}-1) d\nu_s(u)\Big] \\ &: \Big\{\omega; \Big[\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta} ds 1_{((\gamma^{-N}\eta)^{-r_1},\infty)} \\ &\times \Big(\int_{|u|>(\gamma^{-N}\eta)^r} d\nu_s(u)\Big)\Big] > (\eta\gamma^N)^{r_2}\Big\}/\mathcal{F}_{k\gamma^N\eta}\Big] \\ &+ E\Big[\exp\Big[\int_{k\gamma^N\eta}^t ds \int_{|u|>(\gamma^{-N}\eta)^r} (e^{-1}-1) d\nu_s(u)\Big] \\ &: \Big\{\omega; \Big[\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta} ds 1_{((\gamma^{-N}\eta)^{-r_1},\infty)} \\ &\times \Big(\int_{|u|>(\gamma^{-N}\eta)^r} d\nu_s(u)\Big)\Big] \leq (\eta\gamma^N)^{r_2}\Big\}/\mathcal{F}_{k\gamma^N\eta}\Big]. \end{aligned}$$

By (3.20), R.H.S. is inferior to

$$\begin{aligned}&o_{e,k,\varepsilon}(1)+E\Big[\exp\Big[\int_{k\gamma^N\eta}^{t}ds\int_{|u|>(\gamma^{-N}\eta)^r}(e^{-1}-1)d\nu_s(u)\Big]\\&:\Big\{\omega;\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta}ds1_{((\gamma^{-N}\eta)^{-r_1},\infty)}\\&\times\Big(\int_{|u|>(\gamma^{-N}\eta)^r}d\nu_s(u)\Big)>(\eta\gamma^N)^{r_2}\Big\}/\mathcal{F}_{k\gamma^N\eta}\Big]\\&\le o_{e,k,\varepsilon}(1)+\exp[(e^{-1}-1)((\eta\gamma^N)^{r_2}\times(\gamma^{-N}\eta)^{-r_1})]=o_{e,k,\varepsilon}(1),\end{aligned}$$

since $\gamma^{N(r_2+r_1)}\times\eta^{r_2-r_1}\to\infty$ as $\eta\to0$. Hence we have (3.19).

(*Step* 2). – Next we show (3.20). For each vector field X we put the criterion processes

$$\text{(3.21)}\qquad\begin{cases}Cr(s,X,e,k,\varepsilon,\eta)\equiv\Big\langle\Big(\dfrac{\partial\varphi_s}{\partial x}(\varepsilon)\Big)^{-1}X(x_{s-}(\varepsilon)),\bar e\Big\rangle,\\[2ex] Cr(s,e,k,\varepsilon,\eta)\equiv\displaystyle\sum_{j=1}^{m}|Cr(s,X_j,e,k,\varepsilon,\eta)|\end{cases}$$

for $s\in[k\gamma^N\eta,(k+1)\gamma^N\eta]$. By $d\nu_s'(Y,e,k,\varepsilon,\eta)$ we denote the Lévy measure of $Cr(s,Y,e,k,\varepsilon,\eta)$. To get (3.20) it is sufficient to show that, for given $\eta>0$ there exist integers $n=n(\eta),n_1=n_1(\eta)$, such that

$$\text{(3.22)}\qquad\begin{aligned}P\Big\{&\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta}ds1_{((\gamma^{-N}\eta)^{n},\infty)}(Cr(s,e,k,\varepsilon,\eta))\\&<(\eta\gamma^N)^{n_1}/\mathcal{F}_{k\gamma^N\eta}\Big\}=o_{e,k,\varepsilon}(1).\end{aligned}$$

Indeed, consider the event $\{Cr(s,e,k,\varepsilon,\eta)\ge c>0,s\in[k\gamma^N\eta,(k+1)\gamma^N\eta]\}$. Then we can show

$$\text{(3.23)}\qquad\int_{|u|>(\gamma^{-N}\eta)^r}d\nu_s(e,k,\varepsilon,\eta)(u)\ge C_\varepsilon\eta^{-\alpha r/2},$$

for $\eta\le(c\gamma^N)^{\tilde\gamma_1}$ for some $\tilde\gamma_1>0$ (*cf.* [24], (3.16), (3.18)). Here we can choose $r>0$ such that $C_\varepsilon\eta^{-\alpha r/2}\ge(\gamma^{-N}\eta)^{-r_1}$ for η small, so that

(3.24)
$$\int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta}ds1_{((\gamma^{-N}\eta)^{-r_1},\infty)}\Big(\int_{|u|>(\gamma^{-N}\eta)^r}d\nu_s(e,k,\varepsilon,\eta)(u)\Big)\ge(\gamma^N\eta)^{r_2}$$

for some $r_1 > 0, r_2 > r_1$. Following (3.22), the probability of the complement of this event is small ($= o_{e,k,\varepsilon}(1)$), hence (3.20) follows.

(*Step* 3). – To show (3.22), note that it is equivalent to

$$(3.22)' \qquad P\left\{ \int_{k\gamma^N\eta}^{(k+1)\gamma^N\eta} ds 1_{((\gamma^{-N}\eta)^n,\infty)}(Cr(s,e,k,\varepsilon,\eta)) \geq (\eta\gamma^N)^{n_1} / \mathcal{F}_{k\gamma^N\eta} \right\} = 1 - o_{e,k,\varepsilon}(1).$$

The process $Cr(s, Y, e, k, \varepsilon, \eta)$, where Y is a vector field, has the following decomposition

$$(3.25) \quad Cr(t, Y, e, k, \varepsilon, \eta) = Cr(k\gamma^N\eta, Y, e, k, \varepsilon, \eta) + \sum_{k\gamma^N\eta \leq s \leq t}^{c} \Delta Cr(s, Y, e, k, \varepsilon, \eta) + \int_{k\gamma^N\eta}^{t} \left\{ A(s, Y, e, k, \varepsilon, \eta) + \left\langle \bar{e}, \left(\frac{\partial \varphi_s}{\partial x}(\varepsilon) \right)^{-1} [X_0, Y](x_s(\varepsilon)) \right\rangle \right\} ds$$

for $t \in (k\gamma^N\eta, (k+1)\gamma^N\eta)$. Here

$$(3.26) \quad A(s, Y, e, k, \varepsilon, \eta) = \sum_{j=1}^{m} \int \left\langle \bar{e}, \left(\frac{\partial \varphi_{s-}}{\partial x}(\varepsilon) \right)^{-1} \times \left\{ \left(I + \frac{\partial X_j}{\partial x}(x_{s-}(\varepsilon)) \sqrt{\varepsilon}\zeta \right)^{-1} \times \left(Y(x_{s-}(\varepsilon) + X_j(x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta \right) - Y(x_{s-}(\varepsilon)) - [X_j, Y](x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta \right\} \right\rangle g_\varepsilon(\zeta) d\zeta,$$

$\sum_{k\gamma^N\eta \leq s \leq t}^{c}$ denotes the compensated sum so that this term is a martingale, and $g_\varepsilon(\zeta)$ is the Lévy measure of $\tilde{z}_{j,s}^{\varepsilon}$ (*cf.* Section 1 and the beginning of

Section 4 below). If we denote by $F_{j,s}$ the mapping

$$\begin{aligned}(3.27)\qquad \zeta \mapsto \Big\langle \bar{e}, \Big(\frac{\partial \varphi_{s-}}{\partial x}(\varepsilon)\Big)^{-1} \\ \times \Big\{ \Big(I + \frac{\partial X_j}{\partial x}(x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta\Big)^{-1} \\ \times (Y(x_{s-}(\varepsilon) + X_j(x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta) \\ - Y(x_{s-}(\varepsilon)) - [X_j, Y](x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta) \Big\} \Big\rangle,\end{aligned}$$

then $\frac{\partial F_{j,s}}{\partial \zeta}(0) = \gamma^{-1}\Big\langle \bar{e}, \Big(\frac{\partial \varphi_{s-}}{\partial x}(\varepsilon)\Big)^{-1} [X_j, Y](x_{s-}(\varepsilon))\Big\rangle$. Hence we have to estimate the term $Cr(s, [Y, X_j], e, k, \varepsilon, \eta)$ to estimate $dCr(s, Y, e, k, \varepsilon, \eta)$. Note that if we assume

$$(3.28)\qquad \sum_{j=1}^{m} |Cr(s, [Y, X_j], e, k, \varepsilon, \eta)| \geq c \quad (> 0),$$

then we can show there exist $\gamma' > 0, \gamma'_1 > 0$ such that

$$(3.29)\qquad \int_{|u|\geq \eta} d\nu'_s(Y, e, k, \varepsilon, \eta) > C_\gamma \eta^{-\alpha/2}$$

for $\eta \leq (c\gamma^{-1})^{\gamma'_1}$ and

$$(3.30)\qquad \Big|\Big(\frac{\partial \varphi_{s-}}{\partial x}(\varepsilon)\Big)^{-1}\Big| \cdot \Big|\Big(\frac{\partial \varphi_{k\gamma^N\eta}}{\partial x}(\varepsilon)\Big)^{-1}\Big|^{-1} \leq (c\varepsilon')^{-\gamma'}$$

for $s \in [k\gamma^N\eta, (k+1)\gamma^N\eta]$ (*cf.* [24], (3.28), (3.29)).

Let N be the maximal degree of degeneracy of $Lie(X_1, \cdots, X_m)$ on $\mathbb{R}^d$ (*i.e.* the subalgebra consiting of the Lie brackets up to order N spans the whole space at each point, *cf.* (1.17)). Let Y be an arbitrary vector field in this subalgebra. We have the following

LEMMA 3.6. – *If there exist integers* $n = n(\eta), n_1 = n_1(\eta)$ *and a stopping time* $T = T(k, e, \varepsilon, \eta) \in [k\gamma^N\eta, (k+1)\gamma^N\eta]$ *such that*

$$\begin{aligned}(3.31)\qquad P\Big\{ & \textit{for all } s \in [T, T + (\gamma^N\eta)^{n_1}], \\ & \gamma^{-1}\sum_{j=1}^{m}\Big|\Big\langle\Big(\frac{\partial \varphi_{(k+1)\varepsilon'^N}}{\partial x}(\varepsilon)\Big)^{-1}[X_j, Y](x_{s-}(\varepsilon)), e_T\Big\rangle\Big| \\ & \geq (\gamma^{-N}\eta)^n,\ T + (\gamma^N\eta)^{n_1} \leq (k+1)(\gamma^N\eta)/\mathcal{F}_T\Big\} \\ & = 1 - o_{e,k,\varepsilon}(\eta),\end{aligned}$$

then there exist another integers $n' = n'(\eta), n'_1 = n'_1(\eta)$ *and another stopping time* $T' = T'(k, e, \varepsilon, \eta)$ *such that*

$$(3.32)\quad P\Big\{\textit{for all } s \in [T', T' + (\gamma^N \eta)^{n'_1}], \\ \left|\left\langle \left(\frac{\partial \varphi_s}{\partial x}(\varepsilon)\right)^{-1} Y(x_{s-}(\varepsilon)), e_{T'} \right\rangle\right| \geq (\gamma^{-N}\eta)^{n'} / \mathcal{F}_{T'}\Big\} \\ = 1 - o_{e,k,\varepsilon}(\eta),$$

and

$$(3.33)\qquad P\{[T', T' + (\gamma^N \eta)^{n'_1}] \subset [T, T + (\gamma^N \eta)^{n_1}]\} = 1 - o_{e,k,\varepsilon}(\eta).$$

Here we put $e_T \equiv |(\frac{\partial \varphi_T}{\partial x}(\varepsilon))^{-1}|^{-1} e$.

Granting this lemma for a while (the proof will be given in Section 5), we observe that, for $r = 1, \cdots, N$,

$$(3.31)'\quad P\Big\{ \text{for all } s \in [T, T + (\gamma^N \eta)^{n_1}], \\ \gamma^{-r} \sum_{j=1}^{m} \left|\left\langle \left(\frac{\partial \varphi_s}{\partial x}(\varepsilon)\right)^{-1} [X_j, Y](x_{s-}(\varepsilon)), e_T \right\rangle\right| \\ \geq (\gamma^{-N}\eta)^n,\ T + (\gamma^N \eta)^{n_1} \leq (k+1)(\gamma^N \eta) / \mathcal{F}_T\Big\} \\ = 1 - o_{e,k,\varepsilon}(\eta)$$

implies

there exist $n' = n'(\eta)$, $n'_1 = n'_1(\eta)$ and T' such that

$$(3.32)'\quad P\Big\{ \text{for all } s \in [T', T' + (\gamma^N \eta)^{n'_1}], \\ \gamma^{-r+1} \left|\left\langle \left(\frac{\partial \varphi_s}{\partial x}(\varepsilon)\right)^{-1} Y(x_{s-}(\varepsilon)), e_{T'} \right\rangle\right| \\ \geq (\gamma^{-N}\eta)^{n'} / \mathcal{F}_{T'}\Big\} = 1 - o_{e,k,\varepsilon}(\eta),$$

and satisfy (3.33). Iterating $(3.31)' \Longrightarrow (3.32)'$, we have, for a vector field Y which has the order N on its Lie brackets,

$$(3.34)\qquad P\Big\{ \text{for all } s \in [T, T + (\gamma^N \eta)^{n_1}],$$

$$\gamma^{-N}\sum_{j=1}^{m}\left|\left\langle\left(\frac{\partial\varphi_s}{\partial x}(\varepsilon)\right)^{-1}[X_j,Y](x_{s-}(\varepsilon)),e_T\right\rangle\right|$$
$$\left.\geq(\gamma^{-N}\eta)^{n},T+(\gamma^{N}\eta)^{n_1}\leq(k+1)(\gamma^{N}\eta)/\mathcal{F}_T\right\}$$
$$=1-o_{e,k,\varepsilon}(\eta),$$

implies

there exist integers $n''=n''(\eta), n_1''=n_1''(\eta)$ and T'' such that

$$P\left\{\text{ for all } s\in[T'',T''+(\gamma^N\eta)^{n_1''}],\right. \tag{3.35}$$
$$\sum_{j=1}^{m}\left|\left\langle\left(\frac{\partial\varphi_s}{\partial x}(\varepsilon)\right)^{-1}X_j(x_{s-}(\varepsilon)),e_{T''}\right\rangle\right|$$
$$\left.\geq(\gamma^{-N}\eta)^{n''}/\mathcal{F}_{T''}\right\}=1-o_{e,k,\varepsilon}(\eta),$$

and

$$P\{[T'',T''+(\gamma^N\eta)^{n_1''}]\subset[T,T+(\gamma^N\eta)^{n_1}]\}=1-o_{e,k,\varepsilon}(\eta).$$

We remark that the assumption (3.34) is verified for some $Y, n=n(\eta)$ and $n_1=n_1(\eta)$ by the assumption (1.17) in view of (3.27), and that (3.34) implies (3.28). Hence we have (3.35) which implies (3.22)′–granting Lemma 3.6, and we finish the proof of Proposition 3.3.

We will give the proof of Lemma 3.6 in Section 5. This completes the proof of Proposition 3.1.

4. PROOF OF PROPOSITION 2.2

The proof of this proposition is also a bit long.

[A] *Integration by parts of order* 1.

Let $\nu(\zeta)$ be the function appearing in (1.18), that is, $\nu(\zeta)\sim\zeta^2/\varepsilon$ for $|\zeta|$ small. The symmetric Lévy process $\tilde{z}_{j,s}^{\varepsilon}=\frac{1}{\sqrt{\varepsilon}}z_{j,s}^{\varepsilon}$ (*cf.* Section 1) has a Poisson-point-process representation $\tilde{z}_{j,s}^{\varepsilon}=\int_0^s\int\zeta N_j^{\varepsilon}(dsd\zeta)$, where

$N_j^\varepsilon(dsd\zeta)$ is a Poisson counting measure on $\mathbb{R}$ associated to $\tilde{z}_{j,s}^\varepsilon$, with the mean measure

$$(4.1) \qquad ds \times g_\varepsilon(\zeta)d\zeta \equiv ds \times \frac{1}{\varepsilon^{\frac{3}{2}}} g(\frac{\zeta}{\sqrt{\varepsilon}})d\zeta$$

(*cf.* (1.3)). We denote by P^ε the law of $\tilde{z}_{j,s}^\varepsilon, j = 1, \cdots, m$.

Recall that (*cf.* (2.3)) the process $u_s(\varepsilon, h)$ is given by

$$(2.3)' \quad \begin{cases} du_s(\varepsilon, h) = \sum_{j=1}^{m} X_j(u_{s-}(\varepsilon, h), s, \varepsilon)d\tilde{z}_{j,s}^\varepsilon + X_0(u_s(\varepsilon, h), s, \varepsilon)ds \\ u_0(\varepsilon, h) \equiv x. \end{cases}$$

We follow [5], Section 6. Let $v = (v_1, \cdots, v_m)$ be a bounded predictable process on $[0, +\infty)$ to $\mathbb{R}^m$. We consider the perturbation

$$\theta_j^\lambda : \zeta_j \mapsto \zeta_j + \lambda\nu(\sqrt{\varepsilon}\zeta_j)v_j, \qquad \lambda \in \mathbb{R}, \quad j = 1, \cdots, m.$$

Let $N_j^{\lambda,\varepsilon}(dsd\zeta)$ be the Poisson random measure defined by

$$(4.2) \quad \int_0^t \int \phi(\zeta)N_j^{\lambda,\varepsilon}(dsd\zeta) = \int_0^t \int \phi(\theta_j^\lambda(\zeta))N_j^\varepsilon(dsd\zeta), \quad \phi \in C_0^\infty(\mathbb{R}).$$

We put $\tilde{z}_{j,s}^{\lambda,\varepsilon} = \int_0^t \int \zeta N_j^{\lambda,\varepsilon}(dud\zeta)$, and denote by $P^{\lambda,\varepsilon}$ its law, $j = 1, \cdots, m$. Set $\Lambda_j^\lambda(\zeta) = \{1 + \lambda\sqrt{\varepsilon}\nu'(\sqrt{\varepsilon}\zeta)v_j\}\frac{g_\varepsilon(\theta_j^\lambda(\zeta))}{g_\varepsilon(\zeta)}$, and

$$(4.3) \quad Z_t^\lambda(\varepsilon, h) = \exp\left[\sum_{j=1}^{m}\left\{\int_0^t \int \log\Lambda_j^\lambda(\zeta_j)N_j^\varepsilon(dsd\zeta_j) - \int_0^t ds \int (\Lambda_j^\lambda(\zeta_j) - 1)g_\varepsilon(\zeta_j)d\zeta_j\right]\right\}.$$

Then $Z_t^\lambda(\varepsilon, h)$ is a martingale, and $P^{\lambda,\varepsilon}$ has the derivative

$$(4.4) \qquad \frac{dP^{\lambda,\varepsilon}}{dP^\varepsilon} = Z_t^\lambda(\varepsilon, h) \text{ on } \mathcal{F}_{t,\varepsilon}.$$

where $\mathcal{F}_{t,\varepsilon}$ denotes the $\sigma-$field generated by $\tilde{z}_{j,s}^\varepsilon, j = 1, \cdots, m$ (*cf.* [5], Theorem 6-16, Bismut [7], (2.34)).

Consider the perturbed process $u_s^\lambda(\varepsilon, h)$ defined by

$$(4.5)\quad \begin{cases} du_s^\lambda(\varepsilon,h) = \sum_{j=1}^m X_j(u_{s-}^\lambda(\varepsilon,h), s, \varepsilon) d\tilde{z}_{j,s}^{\lambda,\varepsilon} + X_0(u_s^\lambda(\varepsilon,h), s, \varepsilon) ds \\ u_0^\lambda(\varepsilon,h) \equiv x. \end{cases}$$

Then $E^{P^\varepsilon}[f(u_t(\varepsilon,h))] = E^{P^{\lambda,\varepsilon}}[f(u_t^\lambda(\varepsilon,h))] = E^{P^\varepsilon}[f(u_t^\lambda(\varepsilon,h))Z_t^\lambda]$, and we have $0 = \frac{\partial}{\partial\lambda} E^{P^\varepsilon}[f(u_t^\lambda(\varepsilon,h))Z_t^\lambda]$, $f \in C_0^\infty(\mathbb{R}^d)$. By the chain rule, for $|\lambda|$ small, we have

$$\frac{\partial f}{\partial\lambda}(u_t^\lambda(\varepsilon,h)) = D_x f(u_t^\lambda(\varepsilon,h)) \cdot \frac{\partial u_t^\lambda(\varepsilon,h)}{\partial\lambda}, \quad f \in C_0^\infty(\mathbb{R}^d).$$

We have for $\lambda = 0$

(4.6)
$$E^{P^\varepsilon}\left[D_x f(u_t(\varepsilon,h)) \cdot \frac{\partial u_t^\lambda}{\partial\lambda}\bigg|_{\lambda=0}(\varepsilon,h)\right] = -E^{P^\varepsilon}\left[f(u_t(\varepsilon,h))\frac{\partial}{\partial\lambda}Z_t^\lambda\bigg|_{\lambda=0}\right].$$

First, by Corollary 6-17 of [5], we may differentiate Z_t^λ with respect to λ to obtain

$$(4.7)\quad \begin{aligned} R_t(\varepsilon,h) &\equiv \frac{\partial}{\partial\lambda} Z_t^\lambda|_{\lambda=0} \\ &= \sum_{j=1}^m \int_0^t \int \frac{\operatorname{div}\{g_\varepsilon(\cdot)v_j\nu(\sqrt{\varepsilon}\cdot)\}(\zeta_j)}{g_\varepsilon(\zeta_j)} \\ &\quad \times \{N_j^\varepsilon(ds d\zeta_j) - ds g_\varepsilon(\zeta_j) d\zeta_j\}. \end{aligned}$$

Here

$$\begin{aligned} &\int_0^t \int \frac{\operatorname{div}\{g_\varepsilon(\cdot)v_j\nu(\sqrt{\varepsilon}\cdot)\}(\zeta_j)}{g_\varepsilon(\zeta_j)} \{N_j^\varepsilon(ds d\zeta_j) - ds g_\varepsilon(\zeta_j) d\zeta_j\} \\ &= \sum_{s\le t}^c \left\{\sqrt{\varepsilon}v_j \cdot \nu'(\sqrt{\varepsilon}\Delta\tilde{z}_{j,s}^\varepsilon) + v_j \cdot \nu(\sqrt{\varepsilon}\Delta\tilde{z}_{j,s}^\varepsilon)\frac{g'_\varepsilon(\Delta\tilde{z}_{j,s}^\varepsilon)}{g_\varepsilon(\Delta\tilde{z}_{j,s}^\varepsilon)}\right\}, \end{aligned}$$

and $\{\cdots\}$ in R.H.S. $\sim (2 + (-1 - \alpha))(v_j\Delta\tilde{z}_{j,s}^\varepsilon)$ as $\varepsilon \to 0$, since $g_\varepsilon(\zeta) \sim \frac{1}{\varepsilon^{\frac{3}{2}}}(\frac{|\zeta|}{\sqrt{\varepsilon}})^{-1-\alpha}$ for $|\zeta|$ small (*cf.* Section 1). Since $\tilde{z}_{j,s}^\varepsilon \to w_{j,s}$ in law, $R_t(\varepsilon,h) \to (1-\alpha)\sum_{j=1}^m \int_0^t v_j \delta w_{j,s}$ in law.

Next we compute $H_t^\lambda = H_t^\lambda(\varepsilon,h) \equiv \frac{\partial u_t^\lambda}{\partial\lambda}$. u_t^λ is differentiable a.s. for $|\lambda|$ small, and its derivative at $\lambda = 0, H_t = H_t^0(\varepsilon,h)$, is obtained as the solution of the following equation

$$H_t = \sum_{j=1}^{m} \Big\{ \sum_{s\le t}^{c} \Big(\frac{\partial X_j}{\partial x}(u_{s-}, s, h) \Big) H_{s-} \Delta \tilde{z}^{\varepsilon}_{j,s} + \sum_{s\le t} X_j(u_{s-}) v_j(s) \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}) \Big\} + \int_0^t \frac{\partial X_0}{\partial x}(u_{s-}, s, h) H_{s-} ds$$

(*cf.* [5], Theorem 6-24). Namely, H_t is given by

$$\varphi_t^* \sum_{j=1}^{m} \sum_{s\le t} \nu(\sqrt{\varepsilon}\Delta \tilde{z}^{\varepsilon}_{j,s}) \times \Big\{ (\frac{\partial}{\partial x}\varphi_{s-}(\varepsilon, h))^{-1} \times \Big\{ \sum_{j=1}^{m} \Big(I + \frac{\partial}{\partial x} X_j(u_{s-}(\varepsilon, h)) \Delta \tilde{z}^{\varepsilon}_{j,s} \Big)^{-1} \Big\} \times X_j(u_{s-}(\varepsilon, h)) \Big\} v_j(s) \tag{4.8}$$

(*cf.* Bismut [7], (2.47), (4.62), [5] (6-37)). Here $\sum_{s\le t}^{c}$ denotes the compensated sum (*cf.* [7]), and φ_t^* denotes the push-forward $\varphi_t^* Y(x) = (\frac{\partial \varphi_t}{\partial x}(\varepsilon, h))Y(x)$, that is, $\varphi_t^* = (\frac{\partial \varphi_t}{\partial x}(\varepsilon, h))$. We will also use the following notation; φ_t^{*-1} denotes the pull back

$$\varphi_t^{*-1} Y(x) = \Big(\frac{\partial}{\partial x} \varphi_t(\varepsilon, h) \Big)^{-1} Y(u_{t-}(\varepsilon, h)),$$

where

$$\Big(\frac{\partial}{\partial x} \varphi_t(\varepsilon, h) \Big)^{-1} = \Big(\frac{\partial}{\partial x} \varphi_{t-}(\varepsilon, h) \Big)^{-1} \Big\{ \sum_{j=1}^{m} \Big(I + \frac{\partial}{\partial x} X_j(u_{t-}(\varepsilon. h)) \Delta \tilde{z}^{\varepsilon}_{j,t} \Big)^{-1} \Big\}$$

(*cf.* remark just below (1.15)). We remark H_t is the Fréchet derivative of $u_t(\varepsilon, h)$ to the direction of v_j. Observe that as $\varepsilon \to 0$ $H_t(\varepsilon. h)$ tends in law to

$$(\varphi_t^0)^* \sum_{j=1}^{m} \int_0^t ((\varphi_s^0)^{*-1} X_j)(x) v_j(s) ds \tag{4.9}$$

where $\varphi_s^0 : x \mapsto y_s(h)$. Indeed, since ν has a compact support and $\nu(\zeta) = \zeta^2/\varepsilon$ if ζ is in a neighbourhood of 0, we may assume $\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) = \frac{1}{\varepsilon}(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s})^2 = (\Delta\tilde{z}^{\varepsilon}_{j,s})^2$. In view of that $\Delta\tilde{z}^{\varepsilon}_{j,s} \to \delta w_{j,s}$ in law, and that $(\Delta w_{j,s})^2 = \Delta[w_j]_s = \Delta s$, we have (4.9) (*cf.* [6] (1.13), (2.14)). Remark that we can regard H_t as a linear functional : $\mathbb{R}^d \to \mathbb{R}$ by (using the above notation)

(4.10)
$$\langle H_t, p\rangle = \left\langle \varphi_t^* \sum_{j=1}^{m} \sum_{s\le t} \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s})(\varphi_s^{*-1}X_j)(x)v_j(s), p\right\rangle, \quad p \in \mathbb{R}^d.$$

Further the process v_j may be replaced by the process $\tilde{v}_j$ with values in $T_x(\mathbb{R}^d)$, which can be identified with the former one by the expression $v_j = \langle \tilde{v}_j, q\rangle, q \in \mathbb{R}^d$. That is, $v_j = \tilde{v}_j(q)$.

We put

$$\tilde{v}_j \equiv (\alpha - 1)\left(\frac{\partial}{\partial x}\varphi_{s-}(\varepsilon, h)\right)^{-1} X_j(u_{s-}(\varepsilon, h))$$

in what follows, so that v_j is predictable. Using this expression, H_t defines a liner mapping from $T_x^*(\mathbb{R}^d)$ to $T_x(\mathbb{R}^d)$ defined by

$$q \mapsto H_t(q) = \varphi_t^* \sum_{j=1}^{m} \sum_{s\le t} \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s})(\varphi_s^{*-1}X_j)(x)\tilde{v}_j(q). \tag{4.11}$$

We shall identify $H_t(\varepsilon, h)$ with this linear mapping. (In the sequel we shall use the notation $(\tilde{\varphi}_s^{*-1}X_j)(x) = (\frac{\partial}{\partial x}\varphi_{s-}(\varepsilon, h))^{-1}X_j(u_{s-}(\varepsilon, h))$ for simplicity.)

Let $K_t = K_t(x, \varepsilon, h)$ be the stochastic quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$ which is essentially the same as what appeared in Section 2 :

$$K_t(p, q) = \sum_{j=1}^{m} \sum_{s\le t} \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s})\langle(\varphi_s^{*-1}X_j)(x), p\rangle\langle q, (\varphi_s^{*-1}X_j)(x)\rangle.$$

We put, for $0 < s < t$, $K_{s,t} = K_{s,t}(\cdot,\cdot)$ be the quadratic form

$$K_{s,t}(p, q) = \sum_{j=1}^{m} \sum_{s<u\le t} \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s})\langle(\varphi_s^{*-1}X_j)(x), p\rangle\langle q, (\varphi_s^{*-1}X_j)(x)\rangle.$$

That is, $K_t = K_{0,t}$. We remark that by the similar reason as above, $K_{s,t}$ tends in law as $\varepsilon \to 0$ to $K^0_{s,t}$ defined by

$$K^0_{s,t}(p, q) \equiv \sum_{j=1}^{m} \int_s^t du\langle((\varphi_u^0)^{*-1}X_j)(x), p\rangle\langle q, ((\varphi_u^0)^{*-1}X_j)(x)\rangle. \tag{4.12}$$

It follows from (2.6) that

$$\sup_{h\in F,\varepsilon\in(0,1]} \|H_t^{-1}(\varepsilon,h)\| \in L^p, \quad p\geq 1, \quad t>0 \tag{4.13}$$

for any compact $F \subset \mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$. By the inverse mapping theorem we can guarantee the existence and the differentiability of the inverse of $H_t^\lambda(\varepsilon,h)$ for $|\lambda|$ small, which we denote by $H_t^{\lambda,-1}$: $H_t^{\lambda,-1} = [H_t^\lambda(\varepsilon,h)]^{-1}$.

Using the identification above, we can carry out the integration-by-parts procedure for $F_t^\lambda(\varepsilon,h) = f(u_t^\lambda(\varepsilon,h))H_t^{\lambda,-1}$. Recall we have $E[F_t^0(\varepsilon,h)] = E[F_t^\lambda(\varepsilon,h)\cdot Z_t^\lambda]$. Taking the Fréchet derivation $\frac{\partial}{\partial\lambda}|_{\lambda=0}$ for both sides yields

$$\begin{aligned} 0 =& E[D_x f(u_t(\varepsilon,h))H_t^{-1}H_t] \\ &+ E\left[f(u_t(\varepsilon,h))\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}\Big|_{\lambda=0}\right] + E[f(u_t(\varepsilon,h))H_t^{-1}\cdot R_t]. \end{aligned} \tag{4.14}$$

Here $\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}$ is defined by

$$\left\langle \frac{\partial}{\partial\lambda}H_t^{\lambda,-1}, e\right\rangle = \text{trace } \left[e' \mapsto \left\langle -H_t^{\lambda,-1}\Big(\frac{\partial}{\partial\lambda}H_t^\lambda\cdot e'\Big)H_t^{\lambda,-1}, e\right\rangle\right],$$

$e \in \mathbb{R}^d$, where $\frac{\partial}{\partial\lambda}H_t^\lambda$ is the second Fréchet derivative of $u_t^\lambda(\varepsilon,h)$ defined as in [5], Theorem 6-44. We put $DH_t = \frac{\partial}{\partial\lambda}H_t^\lambda|_{\lambda=0}$, then $\frac{\partial}{\partial\lambda}H_t^{\lambda,-1}|_{\lambda=0} = -H_t^{-1}DH_tH_t^{-1}$. This yields

$$E[D_x f(u_t(\varepsilon,h))] = E[f(u_t(\varepsilon,h))\mathcal{A}_t^{(1)}(\varepsilon,h)] \tag{4.15}$$

where

$$\mathcal{A}_t^{(1)}(\varepsilon,h) = \{H_t^{-1}DH_tH_t^{-1} - H_t^{-1}(\varepsilon,h)R_t(\varepsilon,h)\}.$$

Since $H_t(\varepsilon,h)$ tends in law to

$$(\alpha-1)(\varphi_t^0)^*\sum_{j=1}^m\int_0^t ds\langle((\varphi_s^0)^{*-1}X_j)(x)\rangle\langle((\varphi_s^0)^{*-1}X_j)(x) = (\alpha-1)(\varphi_t^0)^*K_t^0\rangle$$

(*cf.* (4.9)), $H_t^{-1}(\varepsilon,h) \to \frac{1}{(\alpha-1)}K_t^{0,-1}(\varphi_t^0)^{*-1}$ in law as $\varepsilon \to 0$. This implies

$$H_t^{-1}(\varepsilon,h)R_t(\varepsilon,h) \to -K_t^{0,-1}(\varphi_t^0)^{*-1}\sum_{j=1}^m\int_0^t((\varphi_s^0)^{*-1}X_j)(x)\delta w_{j,s} \tag{4.16}$$

in view of (4.7) (*cf.* [6], (4.30)).

Lastly, we compute $H_t^{-1} DH_t H_t^{-1}$. Observe that DH_t satisfies the following SDE

$$
\begin{aligned}
(4.17) \qquad DH_t = \sum_{j=1}^{m} \Bigg[\Bigg\{ \sum_{s\le t}^{c} \bigg(& \frac{\partial^2 X_j}{\partial x^2}(u_{s-}, s, h) H_{s-} \\
& + \frac{\partial X_j}{\partial x}(u_{s-}, s, h) \frac{\partial}{\partial x} H_{s-} \bigg) \Delta \tilde{z}^{\varepsilon}_{j,s} \\
& + \frac{\partial X_j}{\partial x}(u_{s-}, s, h) \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}) v_j(s) \\
& + \frac{\partial X_j}{\partial x}(u_{s-}, s, h) \Delta \tilde{z}^{\varepsilon}_{j,s} \Bigg\} DH_{s-} \\
& + \sum_{s\le t} \bigg\{ \frac{\partial X_j}{\partial x}(u_{s-}, s, h) H_{s-} \\
& + X_j(u_{s-}, s, h) \sqrt{\varepsilon} \nu'(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}) v_j(s) \bigg\} \\
& \times \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}) v_j(s) \Bigg] \\
& + \int_0^t \bigg\{ \frac{\partial^2 X_0}{\partial x^2}(u_s, s, h) H_s \\
& - \sum_{j=1}^{m} X_j(u_{s-}, s, h) \\
& \times \int \nu(\sqrt{\varepsilon} \zeta_j) v_j(s) d\zeta_j \bigg\} DH_s ds \\
& DH_0 = 0
\end{aligned}
$$

(*cf.* [5] (6-25), the proof of Theorem 6-44). By the argument similar to that in [6], p. 477, we have

$$
\begin{aligned}
(4.18) \; DH_t = \sum_{j=1}^{m} \varphi_t^* \bigg\{ & \sum_{s\le t} \frac{\partial}{\partial x} (\varphi_s^{*-1} X_j)(x) H_{s-} \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}) v_j(s) \\
& + \sum_{s\le t} \sum_{i=1}^{m} \bigg\langle (\varphi_s^{*-1} X_j)(x) \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,s}), \\
& \times \sum_{u\le s} \frac{\partial}{\partial x} [(\varphi_u^{*-1} X_i)(x) v_i(u), v_j(s)] \bigg\rangle \nu(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{i,u}) \\
& + \sqrt{\varepsilon} \sum_{s\le t} (\varphi_s^{*-1} X_j)(x) \nu'(\sqrt{\varepsilon} \Delta \tilde{z}^{\varepsilon}_{j,u})
\end{aligned}
$$

$$
\begin{aligned}
&\times v_j(s)\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,u})v_j(s)\Big\} \\
&= \sum_{j=1}^{m} \varphi_t^* \Big\{ \sum_{s\le t} (\alpha-1) \Big(\frac{\partial}{\partial x} (\varphi_s^{*-1} X_j)(x) \\
&\times \sum_{u\le s} \sum_{i=1}^{m} \langle [(\tilde{\varphi}_u^{*-1} X_i)(x), (\tilde{\varphi}_s^{*-1} X_j)(x)], \\
&(\tilde{\varphi}_u^{*-1} X_i)(x)\rangle \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{i,u}) \Big) \\
&+ \sum_{s\le t} \sum_{i=1}^{m} (\alpha-1)^2 \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) \big\langle (\varphi_s^{*-1} X_j)(x), \\
&\times \sum_{u\le s} \frac{\partial}{\partial x} [(\varphi_u^{*-1} X_i)(x), (\tilde{\varphi}_s^{*-1} X_j)(x)] \\
&\times (\tilde{\varphi}_u^{*-1} X_i)(x) \big\rangle \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{i,u}) \\
&+ \sqrt{\varepsilon} \sum_{s\le t} (\varphi_s^{*-1} X_j)(x) \nu'(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,u}) \\
&\times v_j(s)\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,u})v_j(s)\Big\} \\
&= (\alpha-1)^2 \sum_{j=1}^{m} \varphi_t^* \Big\{ \sum_{s\le t} M_j^{1,(1)}(t,s,\varepsilon,h)\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) \\
&+ \sum_{s\le t} M_j^{2,(1)}(t,s,\varepsilon,h)\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) \\
&+ \sum_{s\le t} M_j^{3,(1)}(t,s,\varepsilon,h)\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) \Big\} \ (say).
\end{aligned}
$$

Here

(4.19)

$$
\sup_{h\in F, \varepsilon\in(0,1]} \| \sum_{s\le t} \nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) M_j^{i,(1)}(t,s,\varepsilon,h) \| \in L^p, \quad p\ge 1, \quad i=1,2,3
$$

for any compact set $F \subset \mathbf{W}^{1,\frac{\alpha}{\alpha-1}}, j = 1,\cdots,m$. Indeed, we have $\sup_{s\in(0,t],h\in F,\varepsilon\in(0,1]} \|M_j^{1,(1)}(t,s,\varepsilon,h)\| \in L^p$, $p\ge 1$ by (4.13), the uniform L^p-boundedness of $(\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\varphi_u(\varepsilon,h))$ and $\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}(\frac{\partial}{\partial x}\varphi_u(\varepsilon,h))^{-1}$ (2.4). Further by the assumption on $\nu(\zeta)$ ((1.18)), we have $\sum_{s\le t}\nu(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,s}) \in L^p$, $p\ge 1$ (*cf.* [7] (4.22)), which implies (4.19) for cases $i=1,2$. The case for $i=3$ follows since the factor $\sqrt{\varepsilon}\nu'(\sqrt{\varepsilon}\Delta\tilde{z}^{\varepsilon}_{j,u})$ tends to 0 in law while others remain bounded.

Hence $H_t^{-1} D H_t H_t^{-1}$ tends in law as $\varepsilon \to 0$ to

$$(4.20)\quad K_t^{0,-1} (\varphi_t^0)^{*-1} \sum_{j=1}^m \Big\{ (\varphi_t^0)^* \int_0^t ds \Big(\frac{\partial}{\partial x} (\varphi_s^{0*-1} X_j)(x) \times \sum_{i=1}^m \int_0^s dv \langle [(\varphi_v^{0*-1} X_i)(x), \times (\varphi_s^{0*-1} X_j)(x)], (\varphi_v^{0*-1} X_i)(x) \rangle \Big) + (\varphi_t^0)^* \int_0^t ds \Big(\langle (\varphi_s^{0*-1} X_j)(x), \times \int_0^s dv \frac{\partial}{\partial x} [(\varphi_v^{0*-1} X_i)(x), \times (\varphi_s^{0*-1} X_j)(x)] (\varphi_v^{0*-1} X_i)(x) \rangle \Big) \Big\} \times K_t^{0,-1} (\varphi_t^0)^{*-1}.$$

We put

$$M_j^{1,(1)}(t,s,0,h) \equiv \frac{\partial}{\partial x} (\varphi_s^{0*-1} X_j)(x) \sum_{i=1}^m \int_0^s dv \times \langle [(\varphi_v^{0*-1} X_i)(x),\ (\varphi_s^{0*-1} X_j)(x)], (\varphi_v^{0*-1} X_i)(x) \rangle,$$

and

$$M_j^{2,(1)}(t,s,0,h) \equiv \Big\langle (\varphi_s^{0*-1} X_j)(x), \int_0^s dv \times \frac{\partial}{\partial x} [(\varphi_v^{0*-1} X_i)(x), (\varphi_s^{0*-1} X_j)(x)] (\varphi_v^{0*-1} X_i)(x) \Big\rangle,$$

$$j = 1, \cdots, m,$$

which are in L^p $p \geq 1$ (*cf.* [6] (1.4)). Here the term for $M_j^{3,(1)}(t,s,\varepsilon,h)$ tends to 0 as $\varepsilon \to 0$ by Fatou's lemma.

Combining (4.16), (4.20) ($t = 1$)

$$(4.21)\quad E[D_x f(u_1(\varepsilon,h))] = E[f(u_1(\varepsilon,h)) \mathcal{A}_1^{(1)}(\varepsilon,h)]$$

where

$$\begin{aligned}(4.22)\qquad \mathcal{A}_1^{(1)}(\varepsilon,h) &\overset{\text{in law}}{\longrightarrow} K_1^{0,-1}(\varphi_1^0)^{*-1}\\ &\times \sum_{j=1}^m \int_0^1 ((\varphi_s^0)^{*-1}X_j)(x)\delta w_{j,s}\\ &+ K_1^{0,-1}(\varphi_1^0)^{*-1}\sum_{j=1}^m(\varphi_1^0)^*\\ &\times \int_0^1 \{M_j^{1,(1)}(1,s,0,h)+M_j^{2,(1)}(1,s,0,h)\}\\ &\times dsK_1^{0,-1}(\varphi_1^0)^{*-1}.\end{aligned}$$

On the other hand, recalling the definition of $u_s(0,h)$ (*cf.* (2.5)), we have the integration-by-parts setting for diffusion processes, by Bismut [6] (4.14) with $Y = D_x$, that

$$(4.23)\qquad E[D_x f(u_1(0,h))] = E[f(u_1(0,h))\mathcal{A}_1^{(1)}(0,h)]$$

where

$$\begin{aligned}(4.24)\qquad \mathcal{A}_1^{(1)}(0,h) =&\sum_{j=1}^m K_1^{0,-1}(\varphi_1^0)^{*-1}\int_0^1 (\varphi_s^{0*-1}X_j)(x)\delta w_{j,s}\\ &+\sum_{j=1}^m K_1^{0,-1}(\varphi_1^0)^{*-1}(\varphi_1^0)^*\\ &\times\int_0^1 \{M_j^{1,(1)}(1,s,0,h)\\ &\qquad + M_j^{2,(1)}(1,s,0,h)\}dsK_1^{0,-1}(\varphi_1^0)^{*-1}.\end{aligned}$$

This leads our assertion of order 1.

[B] *Integration by parts of order* 2.

We identify the second derivative $D_x^2 f(x)$ with the tensor (matrix) $(p,q)\mapsto \langle D_x^2 f(x)p,q\rangle$, p, $q\in\mathbb{R}^d$.

We devide the interval $[0,1]$ into $[0,\frac{1}{2})$ and $[\frac{1}{2},1]$, to avoid iteration of integration by parts on the same interval (*integration by parts by blocks*, *cf.* [7], Section 4-g)). By the result in [A] and by the strong Markov property of $u_s(\varepsilon, h)$, we have

$$(4.25)\quad \begin{aligned} &E[D_x^2 f(u_1(\varepsilon,h))] \\ &= E[E^{u_{1/2}}[D_x f(u_{1/2}(\varepsilon,h))] \\ &\quad \times \{H_{1/2}^{-1}(\varepsilon,h) D H_{1/2}(\varepsilon,h) H_{1/2}^{-1} - H_{1/2}^{-1}(\varepsilon,h) R_{1/2}(\varepsilon,h)\}] \end{aligned}$$

where E^x denotes the conditional expectation with respect to trajectories starting from x. Here

$$(4.26)\quad \begin{aligned} &E^{u_{1/2}}[D_x f(u_{1/2}(\varepsilon,h))] \\ &= E\Bigg[f(u_1(\varepsilon,h)) \\ &\quad \times \Bigg\{ [(\varphi_1 \circ \varphi_{1/2}^{-1})^* K_{1/2,1}]^{-1} (\varphi_1 \circ \varphi_{1/2}^{-1})^* \\ &\quad \times \sum_{j=1}^m \sum_{1/2 < s \le 1} \{\nu(\sqrt{\varepsilon}\Delta \tilde{z}_{j,s}^\varepsilon) M_j^{1,(2)}(1, 1/2, s, \varepsilon, h) \\ &\quad + \nu(\sqrt{\varepsilon}\Delta \tilde{z}_{j,s}^\varepsilon) M_j^{2,(2)}(1, 1/2, s, \varepsilon, h) \\ &\quad + \nu(\sqrt{\varepsilon}\Delta \tilde{z}_{j,s}^\varepsilon) M_j^{3,(2)}(1, 1/2, s, \varepsilon, h)\} \\ &\quad \times [(\varphi_1 \circ \varphi_{1/2}^{-1})^* K_{1/2,1}]^{-1} \\ &\quad - [(\alpha - 1)(\varphi_1 \circ \varphi_{1/2}^{-1})^* K_{1/2,1}]^{-1} \\ &\quad \times \sum_{j=1}^m \int_{1/2}^t \int \frac{\text{div }\{g_\varepsilon(\cdot) v_j \nu(\sqrt{\varepsilon}\cdot)\}(\zeta_j)}{g_\varepsilon(\zeta_j)} \\ &\quad \times \{N_j^\varepsilon(ds d\zeta_j) - ds g_\varepsilon(\zeta_j) d\zeta_j\} \Bigg\} \Bigg], \end{aligned}$$

where we put

$$(4.27)\quad \begin{aligned} &M_j^{1,(2)}(t_2, t_1, s, \varepsilon, h) \\ &= \frac{\partial}{\partial x} (\varphi_s^{*-1} X_j)(x) \\ &\quad \times \sum_{i=1}^m \sum_{u \le s} \langle [(\varphi_u^{*-1} X_i)(x), (\tilde{\varphi}_s^{*-1} X_j)(x)], \\ &\qquad\qquad (\tilde{\varphi}_u^{*-1} X_i)(x) \rangle \nu(\sqrt{\varepsilon}\Delta \tilde{z}_{i,u}^\varepsilon), \\ &M_j^{2,(2)}(t_2, t_1, s, \varepsilon, h) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \Big\langle (\varphi_s^{*-1} X_j)(x), \\
&\qquad \sum_{u \le s} \frac{\partial}{\partial x}[(\varphi_u^{*-1} X_i)(x), (\tilde{\varphi}_u^{*-1} X_i)(x) \Big\rangle \\
&\quad \times \nu(\sqrt{\varepsilon}\Delta \tilde{z}_{i,u}^{\varepsilon}), \\
&M_j^{3,(2)}(t_2, t_1, s, \varepsilon, h) \\
&\quad = \sqrt{\varepsilon}(\varphi_s^{*-1} X_j)(x) \nu'(\sqrt{\varepsilon}\Delta \tilde{z}_{i,u}^{\varepsilon}) \\
&\quad \times \langle (\tilde{\varphi}_s^{*-1} X_j)(x), (\tilde{\varphi}_s^{*-1} X_j)(x) \rangle
\end{aligned}
$$

for $0 \le t_1 < t_2 \le 1$. Here we have similarly as in (4.19)

$$
(4.28) \quad \begin{cases} \sup_{h \in F, \varepsilon \in (0,1]} \| \sum_{t_1 < s \le t_2} \nu(\sqrt{\varepsilon}\Delta \tilde{z}_{j,s}^{\varepsilon}) M_j^{i,(2)}(t_2, t_1, s, \varepsilon, h) \| \in L^p, \\ \qquad p \ge 1, \quad i = 1, 2, 3 \end{cases}
$$

for any compact set $F \subset \mathbf{W}^{1, \frac{\alpha}{\alpha - 1}}, j = 1, \cdots, m$. We let $\varepsilon \to 0$, then R.H.S. of (4.26) $\to$

$$
\begin{aligned}
(4.29) \quad E\Bigg[f(u_1(0,h)) \Bigg\{ & K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
& \times \sum_{j=1}^{m} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*} \\
& \times \int_{1/2}^{1} \{ M_j^{1,(2)}(1, 1/2, s, 0, h) \\
& + M_j^{2,(2)}(1, 1/2, s, 0, h) \} \\
& \times ds K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
& + K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
& \times \sum_{j=1}^{m} \int_{1/2}^{1} (\varphi_s^{0*-1} X_j)(x) \delta w_{j,s} \Bigg\} \Bigg].
\end{aligned}
$$

Here

$$
\begin{aligned}
&M_j^{1,(2)}(1, 1/2, s, 0, h) \\
&\quad = \frac{\partial}{\partial x} (\varphi_s^{0*-1} X_j)(x) \\
&\quad \times \sum_{i=1}^{m} \int_0^s dv \langle [(\varphi_v^{0*-1} X_i)(x), (\varphi_s^{0*-1} X_j)(x)], (\varphi_v^{0*-1} X_i)(x) \rangle,
\end{aligned}
$$

and

$$\begin{aligned}
&M_j^{2,(2)}(1, 1/2, s, 0, h) \\
&\quad = \sum_{i=1}^{m} \Big\langle (\varphi_s^{0*-1} X_j)(x), \\
&\qquad\qquad \times \int_0^s dv \frac{\partial}{\partial x} [(\varphi_v^{0*-1} X_i)(x),\ (\varphi_s^{0*-1} X_j)(x)] \\
&\qquad\qquad \times (\varphi_v^{0*-1} X_i)(x) \Big\rangle,
\end{aligned}$$

which are in $L^p,\ p \geq 1$.

Combining these yields that $E[D_x^2 f(u_1(\varepsilon, h))]$ tends as $\varepsilon \to 0$ to

$$\begin{aligned}
(4.30) \qquad E\Bigg[& f(u_1(0,h)) \Big\{ K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
&\quad \times \sum_{j=1}^{m} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^* \\
&\quad \times \int_{1/2}^{1} \{ M_j^{1,(2)}(1, 1/2, s, 0, h) \\
&\quad + M_j^{2,(2)}(1, 1/2, s, 0, h) \} ds \\
&\quad \times K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
&\quad + K_{1/2,1}^{0,-1} (\varphi_1^0 \circ \varphi_{1/2}^{0,-1})^{*-1} \\
&\quad \times \sum_{j=1}^{m} \int_{1/2}^{1} (\varphi_s^{0*-1} X_j)(x) \delta w_{j,s} \Big\} \\
\times \Big\{ & K_{1/2}^{0,-1} (\varphi_{1/2}^0)^{*-1} \sum_{j=1}^{m} \int_0^{1/2} (\varphi_s^{0*-1} X_j)(x) \delta w_{j,s} \\
+ & K_{1/2}^{0,-1} (\varphi_{1/2}^0)^{*-1} \\
\times & \sum_{j=1}^{m} (\varphi_{1/2}^0)^* \int_0^{1/2} \Big\{ M_j^{1,(1)}(1/2, s, 0, h) \\
&\qquad + M_j^{2,(1)}(1/2, s, 0, h) \} ds \\
&\qquad \times K_{1/2}^{0,-1} (\varphi_{1/2}^0)^{*-1} \Big\} \Bigg].
\end{aligned}$$

On the other hand, in view of Remark 14 of [7], p. 227, we obtain

$$(4.31)\qquad E[D_x^2 f(u_1(0,h))] = E[f(u_1(0,h))\mathcal{A}_1^{(2)}(0,h)]$$

where

$$\begin{aligned}(4.32)\ \mathcal{A}_1^{(2)}(0,h) = \Bigg[& K_{1/2}^{0,-1}(\varphi_{1/2}^0)^{*-1}\sum_{j=1}^m \int_0^{1/2} (\varphi_s^{0*-1}X_j)(x)\delta w_{j,s} \\ & + K_{1/2}^{0,-1}(\varphi_{1/2}^0)^{*-1} \\ & \times \sum_{j=1}^m (\varphi_{1/2}^0)^* \\ & \times \int_0^{1/2} \{M_j^{1,(1)}(1/2,s,0,h) \\ & \qquad + M_j^{2,(1)}(1/2,s,0,h)\} \\ & \times ds K_{1/2}^{0,-1}(\varphi_{1/2}^0)^{*-1}\Bigg] \\ \times \Bigg[& K_{1/2,1}^{0,-1}(\varphi_1^0\circ\varphi_{1/2}^{0,-1})^{*-1}\sum_{j=1}^m \int_{1/2}^1 (\varphi_s^{0*-1}X_j)(x)\delta w_{j,s} \\ & + K_{1/2,1}^{0,-1}(\varphi_1^0\circ\varphi_{1/2}^{0,-1})^{*-1} \\ & \times \sum_{j=1}^m (\varphi_1^0\circ\varphi_{1/2}^{0,-1})^* \\ & \times \int_{1/2}^1 \{M_j^{1,(2)}(1,1/2,s,0,h) + M_j^{2,(2)}(1,1/2,s,0,h)\}ds \\ & \times K_{1/2,1}^{0,-1}(\varphi_1^0\circ\varphi_{1/2}^{0,-1})^{*-1}\Bigg],\end{aligned}$$

and we have the assertion of order 2.

[C] *Integration by parts of order n.*

Calculations for higher order of derivatives are similar. That is, to compute $E[D^n f(u_1(\varepsilon,h))]$ for the n-tensor $D^n f$, we devide $[0,1]$ into $[0,\frac{1}{2})\cup[\frac{1}{2},\frac{3}{4})\cup[\frac{3}{4},\frac{7}{8})\cup\cdots\cup[\frac{2^{n-1}-1}{2^{n-1}},1]$, and execute the integration by parts on each step. We put for $0 \leq t_1 < t_2 \leq 1$

$$
\begin{aligned}
&c_{i,1}(t_1, t_2; \varepsilon) \\
&\equiv E\Bigg[\|K^{-1}_{t_1,t_2}(\varphi_{t_2} \circ \varphi^{-1}_{t_1})^{*-1}\|^{(i+1)} \\
&\qquad \times \sum_{j=1}^{m} \Big\| (\varphi_{t_2} \circ \varphi^{-1}_{t_1})^* \sum_{t_1 < s \leq t_2} \nu(\sqrt{\varepsilon}\Delta \tilde{z}^{\varepsilon}_{j,s}) \\
&\qquad\qquad \times \{M_j^{1,(2)} + M_j^{2,(2)}\}(t_2, t_1, s, \varepsilon, h) \Big\| \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&c_{i,2}(t_1, t_2; \varepsilon) \\
&\equiv E\Bigg[\|K^{-1}_{t_1,t_2}(\varphi_{t_2} \circ \varphi^{-1}_{t_1})^{*-1}\|^{(i+1)} \\
&\qquad \times \sum_{j=1}^{m} \int_{t_1}^{t_2} \int \left| \frac{\mathrm{div}\ \{g_\varepsilon(\cdot) v_j \nu(\sqrt{\varepsilon}\cdot)\}(\zeta_j)}{g_\varepsilon(\zeta_j)} \right| \\
&\qquad \times \{N_j^{\varepsilon}(ds d\zeta_j) - ds g_\varepsilon(\zeta_j) d\zeta_j\} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&d_{i,1}(t_1, t_2; \varepsilon) \\
&\equiv E\Bigg[\|K^{-1}_{t_1,t_2}(\varphi_{t_2} \circ \varphi^{-1}_{t_1})^{*-1}\|^{2(i+1)} \\
&\qquad \times \Bigg(\sum_{j=1}^{m} \Big\| (\varphi_{t_2} \circ \varphi^{-1}_{t_1})^* \sum_{t_1 < s \leq t_2} \nu(\sqrt{\varepsilon}\Delta \tilde{z}^{\varepsilon}_{j,s}) \\
&\qquad\qquad \times \{M_j^{1,(2)} + M_j^{2,(2)}\}(t_2, t_1, s, \varepsilon, h) \Big\| \Bigg)^2 \Bigg]^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
&d_{i,2}(t_1, t_2; \varepsilon) \\
&\equiv E\Bigg[\|K^{-1}_{t_1,t_2}(\varphi_{t_2} \circ \varphi^{-1}_{t_1})^{*-1}\|^{2(i+1)} \\
&\qquad \times \Bigg(\sum_{j=1}^{m} \int_{t_1}^{t_2} \int \left| \frac{\mathrm{div}\ \{g_\varepsilon(\cdot) v_j \nu(\sqrt{\varepsilon}\cdot)\}(\zeta_j)}{g_\varepsilon(\zeta_j)} \right| \\
&\qquad \times \{N_j^{\varepsilon}(ds d\zeta_j) - ds g_\varepsilon(\zeta_j) d\zeta_j\} \Bigg)^2 \Bigg]^{1/2}, \quad i = 0, 1
\end{aligned}
$$

and let $m(t_1, t_2; \varepsilon) \equiv \max(c_{1,1}, c_{0,2}, d_{1,1}, d_{0,2}, 1)$. Then the procedure of estimation of of order 1 leads that

$$(4.33) \quad \|E[D^n_x f(u_1(\varepsilon, h))]\|$$

$$\leq C(n)m\left(0,\frac{1}{2};\varepsilon\right)\cdots m\left(\frac{2^{n-3}-1}{2^{n-3}},\frac{2^{n-3}-1}{2^{n-3}}+\frac{1}{2^{n-2}};\varepsilon\right)$$
$$\times m\left(\frac{2^{n-2}-1}{2^{n-2}},\frac{2^{n-2}-1}{2^{n-2}}+\frac{1}{2^{n-1}};\varepsilon\right)$$
$$\times m^2\left(\frac{2^{n-1}-1}{2^{n-1}},1;\varepsilon\right)I_n\|f\|_\infty,$$

where I_n is a linear combination of products of expectations (*cf.* [30], Section 2.e). Since

$$\sup_{h\in F,\varepsilon\in(0,1]}\|K_{t_1}^{-1}(\varepsilon,h)\|\in L^p,\quad \sup_{h\in F,\varepsilon\in(0,1]}\|K_{t_1,t_2}^{-1}(\varepsilon,h)\|\in L^p\quad(p\geq 1)$$

(*cf.* (2.5)), and since

$$(4.34)\qquad \begin{cases} \sup_{h\in F,\varepsilon\in(0,1]}\|\sum_{s\leq t_1}\nu(\sqrt{\varepsilon}\Delta\tilde{z}^\varepsilon_{j,s})M_j^{i,(1)}(t_1,s,\varepsilon,h)\|\in L^p,\\ \sup_{h\in F,\varepsilon\in(0,1]}\|\sum_{t_1<s\leq t_2}\nu(\sqrt{\varepsilon}\Delta\tilde{z}^\varepsilon_{j,s})M_j^{i,(2)}(t_2,t_1,s,\varepsilon,h)\|\in L^p,\\ \qquad i=1,2,3\quad(p\geq 1)\end{cases}$$

for $0\leq t_1<t_2\leq 1$ and for any compact set $F\subset\mathbf{W}^{1,\frac{\alpha}{\alpha-1}}$, the limiting procedures are justified, and we have the convergence of order n.

This completes the proof of Proposition 2.2.

5. PROOF OF LEMMA 3.6

Lemma 3.6 follows from the following two lemmas, which we cite from Léandre [24].

LEMMA 5.1 (cf. Léandre [24], Lemme III.2). – *Let $X_{e,\varepsilon,\eta}(t)$ be a process which has the following canonical (Doob-Meyer) decomposition*

$$(5.1)\qquad X_{e,\varepsilon,\eta}(t)=X_{e,\varepsilon,\eta}(0)+M_{e,\varepsilon,\eta}(t)+\int_0^t A_{e,\varepsilon,\eta}(s)ds,$$

with $d\langle M_{e,\varepsilon,\eta},M_{e,\varepsilon,\eta}\rangle_t=B_{e,\varepsilon,\eta}(t)dt$. Here we assume

$$(5.2)\qquad P\{\sup_{t\leq 1}|A_{e,\varepsilon,\eta}(t)|>(\gamma^{-N}\eta)^{-p}\}=o_{e,\varepsilon}(\eta^\infty)$$

$$\sup_{e \in S^{d-1}, \varepsilon \in (0,1]} E[\sup_{s \le 1} |B_{e,\varepsilon,\eta}(s)|^2] < +\infty. \tag{5.3}$$

If $|X_{e,\varepsilon,\eta}(0)| \ge (\gamma^{-N}\eta)^n$, *then*

$$P\left\{\exists t \le (\gamma^N\eta)^{\frac{5}{2}n}; |X_{e,\varepsilon,\eta}(t) - X_{e,\varepsilon,\eta}(0)| \ge \frac{(\gamma^{-N}\eta)^n}{2}\right\} = o_{e,\varepsilon}(1). \tag{5.4}$$

LEMMA 5.2 (cf. Léandre [24], Lemme III.3). – *Assume that there exists a process* $X_{e,\varepsilon,\eta}(t)$ *which has the decomposition* (5.1) *satisfing* (5.2), (5.3). *Assume further that the criterion processes* $C1_{e,\varepsilon,\eta}(t) \equiv A_{e,\varepsilon,\eta}(t), Cr_{e,\varepsilon,\eta}(t)$ *satisfy the following: there exist* $\alpha > 0, \gamma > 0, \gamma_1 > 0, n > 0$ *and* $c > 0$ *such that if*

$$\sup_{s \le 1} |C1_{e,\varepsilon,\eta}(s)| \le (\gamma^{-N}\eta)^{-\gamma}, \quad \sup_{s \le 1} |Cr_{e,\varepsilon,\eta}(s)| \ge (\gamma^{-N}\eta)^n, \tag{5.5}$$

then the Lévy measure $d\nu_{e,\varepsilon,\eta,t}(u)$ *of* $M(t)_{e,\varepsilon,\eta}$ *satisfies*

$$\int_{|u| \ge \eta} d\nu_{e,\varepsilon,\eta,t}(u) > C_\varepsilon \eta^{-\alpha} \tag{5.6}$$

for $\eta \le (c\gamma^{-1})^{\gamma_1}$.

If for all $p > 0$ *there exists some integer* n_1 *such that*

$$\begin{cases} P\{\sup_{s \le 1} |C1_{e,\varepsilon,\eta}(s)| > (\gamma^{-N}\eta)^{-p}\} = o_{e,\varepsilon}(\eta^\infty) \\ \text{and} \\ P\{\exists t \in [0, (\gamma^N\eta)^{n_1}], |Cr_{e,\varepsilon,\eta}(t)| \le (\gamma^{-N}\eta)^{n_1}\} = o_{e,\varepsilon}(\eta), \end{cases} \tag{5.7}$$

then, for all $p > 0$ *there exists* $n' = n'(\eta)$ *such that*

$$P\{T(n', \eta, e, \varepsilon) \ge (\gamma^{-N}\eta)^p\} = o_{e,\varepsilon}(1), \tag{5.8}$$

where $T(n, \eta, e, \varepsilon) \equiv \inf\{t > 0; |X_{e,\varepsilon,\eta}(t) - X_{e,\varepsilon,\eta}(0)| \ge (\gamma^{-N}\eta)^n\}$.

Let

$$X_{e,\varepsilon,\eta}(t) \equiv \sum_{j=1}^m \left\langle \left(\frac{\partial \varphi_{k\gamma^N\eta+t}}{\partial x}\right)^{-1} Z(x_{k\gamma^N\eta+t-}(\varepsilon)), \bar{e}\right\rangle,$$

where Z is a vector field in *Lie*$(X_1, \cdots, X_m)$, and we apply those lemmas for (Y, Z) such that $e^{t\sum_{j=1}^m X_j} e^{tY} = e^Z$ by using the Campbell-Hausdorff formula. We shall show that assumptions (5.2), (5.3), (5.6), (5.7) are satisfied later. Lemma 5.1 follows from the Burkholder-Davis-Gundy inequality, and we omit the detail. *See* [24] and [3.25], p. 240.

Lastly we check conditions (5.2), (5.3), (5.6), (5.7). By (3.25), we can put

$$\begin{aligned}A_{e,\varepsilon,\eta}(t) \equiv & A((k+t)\gamma^N\eta, Z, e, k, \varepsilon, \eta)\\ & + \left\langle \bar{e}, \left(\frac{\partial\varphi_{(k+t)\gamma^N\eta}}{\partial x}(\varepsilon)\right)^{-1} [X_0, Z](x_{((k+t)\gamma^N\eta)-}(\varepsilon))\right\rangle,\end{aligned}$$

$t \in (0,1]$. Observe that

$$\left|\left(\frac{\partial}{\partial x}\varphi_{k\gamma^N\eta}(\varepsilon)\right)^{-1}\right|^{-1}.$$

$$\left|\left(\frac{\partial\varphi_{(k+t)\gamma^N\eta}}{\partial x}\right)^{-1}\right| \leq c^{-\gamma'} \quad ((3.30)),$$

$$\left|\left(I + \sqrt{\varepsilon}\zeta\frac{\partial}{\partial x}X_j(x_{s-}(\varepsilon))\right)^{-1}\right| \leq C \quad ((1.13)),$$

and

$$|Z(x_{s-}(\varepsilon)+X_j(x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta)-Z(x_{s-}(\varepsilon))-[X_j,Z](x_{s-}(\varepsilon))\sqrt{\varepsilon}\zeta| \leq C\varepsilon\zeta^2,$$

where $\int C\varepsilon\zeta^2 g_\varepsilon(\zeta)d\zeta < +\infty$. Hence

$$\begin{aligned}(5.9) \qquad & P\{(\sup_{t\leq 1}|A(t,Z)|)^p > (\gamma^{-N}\eta)^{-p'}\}\\ & = P\{\sup_{t\leq 1}|A(t,Z)| > (\gamma^{-N}\eta)^{-p'/p}\} = o_{e,k,\varepsilon}(1)\end{aligned}$$

for all $p' > 0, p > 1$. This leads (5.2).

Next we put

$$M_{e,\varepsilon,\eta}(t) = \sum_{k\gamma^N\eta\leq s\leq t}^{c} \Delta Cr(s, Z, e, k, \varepsilon, \eta).$$

Then $dB_{e,\varepsilon,\eta}(t) = \langle M_{e,\varepsilon,\eta}, M_{e,\varepsilon,\eta}\rangle_t dt = \int u^2 d\nu'_t(u) \times dt$, where $d\nu'_t(u)$ is the sum of transformed measure of $g_\varepsilon(\zeta_j)d\zeta_j$ by

$$\begin{aligned}u_j(\zeta_j) = & \left\langle \left(\frac{\partial\varphi_{t-}}{\partial x}(\varepsilon)\right)^{-1}\right.\\ & \times\left\{\left(I + \frac{\partial X_j}{\partial x}(x_{t-}(\varepsilon))\sqrt{\varepsilon}\zeta_j\right)^{-1} Z(x_{t-}(\varepsilon)\right.\\ & \left.\left. + X_j(x_{t-}(\varepsilon))\sqrt{\varepsilon}\zeta_j) - Z(x_{t-}(\varepsilon))\right\}, \bar{e}\right\rangle.\end{aligned}$$

Hence

$$\int u^2 d\nu'_t(u) = \sum_{j=1}^m \int u_j^2(\zeta_j) g_\varepsilon(\zeta_j) d\zeta_j \tag{5.10}$$

$$\leq C \sum_{j=1}^m \left| \left(\frac{\partial}{\partial x} \varphi_{(k+1)\gamma^N \eta}(\varepsilon) \right) \left(\frac{\partial \varphi_{t-}}{\partial x}(\varepsilon) \right)^{-1} \right|^2 \varepsilon$$

$$\times \int \zeta_j^2 g_\varepsilon(\zeta_j) d\zeta_j < +\infty.$$

Hence $\sup_{e \in S^{d-1}, \varepsilon \in (0,1]} E[\sup_{t \leq 1} |B_{e,\varepsilon,\eta}(t)|^2] < +\infty$. This leads (5.3).

We put $C1_{e,\varepsilon,\eta}(t)$ as above, and

$$Cr_{e,\varepsilon,\eta}(t) = \sum_{j=1}^m \left| \left\langle \bar{e}, \left(\frac{\partial \varphi_{(k+t)\gamma^N \eta}}{\partial x}(\varepsilon) \right)^{-1} [X_j, Z](x_{((k+t)\gamma^N \eta)-}(\varepsilon) \right\rangle \right|.$$

Then (5.6), (5.7) follows from (3.29), and (5.9)-(3.34) respectively.

Q.E.D.

6. PROOF OF LEMMA 1.4

In this section we give the proof of Lemma 1.4 along the idea of Léandre [27] (*cf.* also Azencott [2]). Recall the definition of $\{x_t(\varepsilon)\}$ and $\{y_t(h)\}$:

$$\begin{cases} dx_t(\varepsilon) = \sum_{j=1}^m X_j(x_{t-}(\varepsilon)) dz_{j,t}^\varepsilon + X_0(x_{t-}(\varepsilon)) dt, \\ x_0(\varepsilon) = x, \end{cases} \tag{6.1}$$

$$\begin{cases} dy_t(h) = \sum_{j=1}^m X_j(y_t(h)) \dot{h}_{j,s} dt + X_0(y_t(h)) dt, \\ y_0(h) = x. \end{cases} \tag{6.2}$$

Now rewrite

$$x_t(\varepsilon) - y_t(h) = \sum_{j=1}^m \int_0^t X_j(x_{s-}(\varepsilon))(dz_{j,s}^\varepsilon - dh_{j,s}) \tag{6.3}$$

$$+\sum_{j=1}^{m}\int_0^t (X_j(x_{s-}(\varepsilon)) - X_j(y_s(h)))dh_{j,s})$$
$$+\int_0^t (X_0(x_s(\varepsilon)) - X_0(y_s(h)))ds.$$

Since the vector fields $X_0, X_1, \cdots, X_m$ are C^∞ and bounded, we have only to estimate the first term, due to the Gronwall lemma (*cf.* [12], Appendixes 5). We can rewrite the first term as

$$(6.4)\quad -\sum_{j=1}^{m}\left\{\int_0^t (z^\varepsilon_{j,s-} - h_{j,s})\left(\frac{\partial}{\partial x}X_j(x_{s-}(\varepsilon))\right)\right.$$
$$\left.\times\left(\sum_{k=1}^{m} X_k(x_{s-}(\varepsilon))dz^\varepsilon_{j,s} + X_0(x_{s-}(\varepsilon))ds\right)\right\}$$
$$-\sum_{j=1}^{m}[z^\varepsilon_{j,\cdot}, X_j(x_\cdot(\varepsilon))]_t$$

by the integration by parts formula. We write here $[z^\varepsilon_{j,\cdot}, X_j(x_\cdot(\varepsilon))]_t = ([z^\varepsilon_{j,\cdot}, X_j^{(i)}(x_\cdot(\varepsilon))]_t)$ as a vector. We write

$$(6.5)\quad w_t \equiv -\sum_{j=1}^{m}\sum_{k=1}^{m}\left\{\int_0^t (z^\varepsilon_{j,s-} - h_{j,s})\left(\frac{\partial}{\partial x}X_j(x_{s-}(\varepsilon))\right)\right.$$
$$\left.\times X_k(x_{s-}(\varepsilon))dz^\varepsilon_{j,s}\right\}$$
$$-\sum_{j=1}^{m}[z^\varepsilon_{j,\cdot}, X_j(x_\cdot(\varepsilon))]_t$$
$$= -w_{(1),t} - w_{(2),t}\quad (say).$$

In what follws we write $w_{(1),t} = (w^{(1)}_{(1),t}, \cdots, w^{(d)}_{(1),t})$ by coordinates.

Consider the ball $B(R) = \{x \in \mathbb{R}^d; |x| \leq R\}$, and let T be the exit time of $w_{(1),t}$ from $B(R)$. Let $T_i = T_i(R)$ be the exit time of $w^{(i)}_{(1),t}$ from the interval $[-R/\sqrt{d}, R/\sqrt{d}]$. Let O be the event $\{\sup_{s\leq 1}|z^\varepsilon_s - h_s| < r\}$. The process

(6.6)

$$W_t^{(i)} = \exp\left[\lambda w^{(i)}_{(1),t} - \int_0^t ds\sum_{j=1}^{m}\int (e^{\lambda F_{j,s,i}(\sqrt{\varepsilon}\zeta)} - 1 - \lambda F_{j,s,i}(\sqrt{\varepsilon}\zeta))g_\varepsilon(\zeta)d\zeta\right]$$

is an exponential martingale. Here $F_{j,s,i}$ denotes the mapping

$$(6.7)\quad F_{j,s,i} : \zeta \mapsto \left\{ \sum_{k=1}^{m} (z^{\varepsilon}_{j,s-} - h_{j,s}) \left(\frac{\partial}{\partial x} X_j(x_{s-}(\varepsilon)) \right) X_k(x_{s-}(\varepsilon)) \right\}^{(i)} \zeta.$$

Fix s, i, and we put $\zeta'_j = F_{j,s,i}(\sqrt{\varepsilon}\zeta_j)$. If $|\zeta_j| \leq K_1$ then $|\zeta'_j| \leq Cr\sqrt{\varepsilon}$ on "$\{\sup_{s\leq 1} |z^{\varepsilon}_s - h_s| < r\}$" where $C = C_{K_1}$ does not depend on s, i. Then $|e^{\lambda\zeta'_j} - 1 - \lambda\zeta'_j| \leq C'\lambda^2\zeta'^2_j$ for $|\lambda\zeta'_j| \leq K_2$. Hence, for $\varepsilon > 0$ small and $r > 0$ small,

$$(6.8)\quad \begin{aligned} \langle W^{(i)}, W^{(i)} \rangle_t &\leq \sum_{j=1}^{m} \int_0^t ds C'' r^2 \varepsilon \lambda^2 \int_{\mathbf{R}} (\sqrt{\varepsilon}\zeta_j)^4 g_\varepsilon(\zeta_j) d\zeta_j \\ &\leq C_1 r^2 \varepsilon^3 \lambda^2 \varepsilon^{\frac{1}{2}\alpha - 1} t, \end{aligned}$$

since $g_\varepsilon(\zeta) = (1/\varepsilon^{\frac{3}{2}}) g(\zeta/\sqrt{\varepsilon})$ with $g(z) \sim c_\alpha |z|^{-1-\alpha}$ for $|z|$ small, and since supp g_ε is bounded.

We apply the martingale property and the upper bound of exponential type (*cf.* [28], Lemme 17) to $W^{(i)}_T$ with $T \leq 1$, we have in view of (6.6) and that supp g_ε is bounded

$$(6.9)\quad \begin{aligned} & P\{O; T \leq 1\} \exp(\lambda^2 R/\sqrt{d}) \\ &\leq P\{O; \exists i \in \{1, \cdots, d\}, T_i \leq 1\} \exp(\lambda^2 R/\sqrt{d}) \\ &\leq E[\exp(\lambda^2 w^{(i)}_{(1),T_i}); \\ &\qquad O \cap \{\exists i \in \{1, \cdots, d\}, \sup_{0\leq s\leq 1} |w^{(i)}_{(1),s}| \geq (R/\sqrt{d})\}] \\ &\leq 2d \exp[-(R/\sqrt{d})\frac{\lambda'}{d} + \lambda^2 R/\sqrt{d} \\ &\qquad + (1/2)\lambda'^2 C_1 r^2 \lambda^2 \varepsilon^{\frac{1}{2}\alpha+2}(1 + \exp[\lambda'])] \end{aligned}$$

for $\lambda' > 0$. Choose $\lambda' = 2\log|\lambda|$ and $\lambda = CR^2/r\sqrt{\varepsilon}$, then

$$\begin{aligned} R.H.S. \leq 2d \exp\Big[& \frac{R}{\sqrt{d}} \Big\{ (CR^2/r\sqrt{\varepsilon})^2 - \frac{1}{d}\log(CR^2/r\sqrt{\varepsilon}) \Big\} \\ & + 2(\log(CR^2/r\sqrt{\varepsilon})^2 \varepsilon^{\frac{\alpha}{2}} (C^2R^4\varepsilon + C^4R^8/r^2) \Big]. \end{aligned}$$

This implies, since $1 < \alpha < 2$,

$$(6.10)\quad P\{O; T \leq 1\} \leq C_5 \exp[-C_6 R^5/r^2\varepsilon].$$

Next we study the term $w_{(2),t}$. We remark that $\langle w_{(2),\cdot}, w_{(2),\cdot}\rangle_t \leq C\varepsilon t$ for some $C > 0$. We can apply again the upper bound for the exponential type, so that

$$P\{\sup_{s\leq t} |w_{(2),s}| \geq R\} \leq 2d \exp\left\{-\frac{\lambda}{d}R + \frac{\lambda^2}{2}C\varepsilon t(1+e^{|\lambda|})\right\} \tag{6.11}$$

for $\lambda \in \mathbb{R}$. We put $\lambda = \frac{1}{\varepsilon}\log(\frac{1}{K})$. Then

$$\begin{aligned} R.H.S. &\leq 2d \exp\left\{\frac{1}{\varepsilon}\left(\log\left(\frac{1}{K}\right)\right)\left(\frac{1}{2}Ct\left(\log\left(\frac{1}{K}\right)\right)\right.\right. \\ &\qquad \left.\left.\times\left(1+\left(\frac{1}{K}\right)^{\frac{1}{\varepsilon}}\right) - R\right)\right\} \\ &\leq C' \exp\{-K'/\varepsilon\} \end{aligned} \tag{6.12}$$

for $C > 0$ small, $\varepsilon > 0$ small and for some $K' > 0$.

Since the vector fields $X_0, X_1, \cdots, X_m$ are smooth and bounded including their derivatives, the remaining term

$$\sum_{j=1}^{m} \int_0^t (z_{j,s-}^{\varepsilon} - h_{j,s})\left(\frac{\partial}{\partial x}X_j(x_{s-}(\varepsilon))\right)X_0(x_{s-}(\varepsilon))ds \tag{6.13}$$

remain small on O if we choose r small.

Combining this with (6.10), (6.12) implies (1.24). This completes the proof of Lemma 1.4.

7. CONCLUDING REMARK

Our aim was to provide a framework of large deviation in case of jump processes. Unfortunately, we can not proceed (contrary to the diffusion case) from estimates (1.18)-(1.20) to the short time asymptotic of $p_t(x, y, 1)$ as $t \to 0$. This is because the driving processes $z_{j,s}^{\varepsilon}$ do not have the scaling property, and hence $p_1(x, y, \varepsilon) \neq p_{\varepsilon^{\alpha/2}}(x, y, 1)$. (As for the short time asymptotic for the type of jump processes treated in this article, *see* [16], [18].)

The reason for this is that we have confined ourselves in Section 1 to the simple case where the support of the Lévy measure of the driving process is compact. Although this condition may not be indispensable, we have not been able to confirm Lemma 1.4 when the support is not compact.

ACKNOWLEDGEMENT

A major part of this work was done at Université de Strasbourg (Strasbourg, France) in 1993 while the author was on leave from Hirosaki University. The author would like to thank Professor Rémi Léandre for valuable discussions. Thanks are also to IRMA for its hospitality. The author is deeply indebted also to the referee for his careful reading and for valuable comments.

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(Manuscript received April 28, 1995;
Revised March 6, 1996.)