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One-dimensional random walks, decreasing rearrangements and discrete Steiner symmetrization

by

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ABSTRACT. – Take a simple random walk in the "blind alley" $\{1, 2, \ldots, N+1\}$, starting at 1, with the boundary condition that movement to the left of 1 results in staying put at 1. Each time the random walk visits a point $n \in \{1, 2, \ldots, N\}$, it is subject to a danger and has a probability d_n of being consumed by it. We prove that the probability of safe arrival at $N+1$ is increased if the d_n are replaced by their non-decreasing rearrangement $d_n^{\#}$. Next, we consider the same random walk but now on *all* of $\mathbb{Z}^+$, again with a danger d_n at each point $n \in \mathbb{Z}^+$. Let T_d be the time of first absorption by one of the dangers d_n. We prove that $P(T_d \geq \lambda) \leq P(T_{d^\#} \geq \lambda)$ for all $\lambda \in \mathbb{Z}^+$. Finally, we obtain a theorem on Steiner rearrangement and generalized discrete harmonic measure for discrete cases which are *a priori* symmetric under a reflection in an appropriate axis. Our methods are completely elementary.

RÉSUMÉ. – On considère une marche aléatoire dans le "cul-de-sac" $\{1, 2, \ldots, N+1\}$ avec 1 comme point de départ et qui doit rester sur place dès qu'elle est tentée d'aller à gauche de 1. En chaque point n de $\{1, \ldots, N\}$ il y a une probabilitée d_n que la marche soit absorbée par un danger dès qu'elle arrive à ce point. Nous démontrons que la probabilité d'arriver sain et sauf au point $N+1$ croît si on remplace les d_n par leurs réarrangements non-decroissants $d_n^{\#}$. Ensuite, nous considérons la même marche mais cette fois sur *tout* l'ensemble $\mathbb{Z}^+$, avec encore un danger d_n sur chaque point $n \in \mathbb{Z}^+$. Si T_d est le temps de première absorption par l'un de dangers d_n, nous démontrons que $P(T_d \geq \lambda) \leq P(T_{d^\#} \geq \lambda)$ pour

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chaque $\lambda \in \mathbb{Z}^+$. Enfin, nous établissons un théorème sur la symétrization de Steiner et la mesure harmonique généralisée dans les cas discrets qui sont *a priori* symétriques par rapport à la réflexion dans l'axe approprié. Les méthodes sont élémentaires.

Key words and phrases: Non-increasing rearrangement, Steiner symmetrization, random walks with dangers, second order difference equations, ordinary differential equations, Baernstein $*$-functions. The author would like to thank Albert Baernstein II, Arie Harel and Ravi Vakil for interesting discussions on these topics. In particular, he would like to thank Professor Baernstein for having suggested that the author also consider the case of $p \neq \frac{1}{2}$. The author would also like to thank the referees for their suggestions and their careful reading. The research was partially supported by Professor J. J. F. Fournier's NSERC grant #4822 and was done while the author was at the University of British Columbia. The present paper largely coincides with Section IV.9 of the author's doctoral dissertation.

1. STATEMENT OF RESULTS

We write $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and $\mathbb{Z}_0^+ = \{0\} \cup \mathbb{Z}^+$. Fix $p \in [0, 1]$. Let $\{r_i^p : i \in \mathbb{Z}_0^+\}$ be a random walk on $\{1, 2, \ldots, N+1\}$, with $r_0^p = 1$,

$$P(r_{i+1}^p = r_i^p + 1 \mid r_i^p) = p,$$

$$P(r_{i+1}^p = n - 1 \mid r_i^p = n) = 1 - p, \qquad \text{if } n > 1,$$

and

$$P(r_{i+1}^p = 1 \mid r_i^p = 1) = 1 - p.$$

Thus, we have a simple random walk on a "blind alley," with the boundary condition that at the "wall" (*i.e.*, at 1) when we try to go to the left then we stay put. The open end of the blind alley is at $N + 1$.

Let $s_1, s_2, \ldots, s_N \in [0, 1]$ be given. Every time the random walk r_i^p is at a point $n \in \{1, 2, \ldots, N\}$, let there be a new danger (independent of anything that had happened until that time, and in particular independent of the outcomes of any previous visits to the point n) and let the probability of surviving it be s_n. Let $P_N^p(s_1, \ldots, s_N)$ be the probability that the random walk has survived all the time up to its arrival at the point $N + 1$. More precisely, let $X_0, X_1, \ldots$ be random variables which are independent and identically uniformly distributed on $[0, 1]$. Let

$$T_N = \inf \{i \geq 0 : r_i^p = N + 1\}.$$

Of course $P(T_N < \infty) = 1$ if $p > 0$. Then we have

$$P_N^p(s_1, \dots, s_N) = P\left(\bigcap_{i=0}^{T_N - 1} \{X_i \leq s_{r_i^p}\}\right).$$

Note that

$$P_N^1(s_1, \dots, s_N) = s_1 s_2 \cdots s_N, \tag{1}$$

$$P_N^0(s_1, \dots, s_N) \equiv 0$$

and

$$P_N^p(1, \dots, 1) = 1,$$

for every $p > 0$.

Throughout, the terms "increasing" and "decreasing" shall be of the weaker variety, *i.e.*, they shall mean "non-decreasing" and "non-increasing," respectively.

THEOREM 1. – *Let* $s_1, \dots, s_N \in [0, 1]$, *and let* $s_1^*, \dots, s_N^*$ *be* $s_1, \dots, s_N$ *rewritten in decreasing order. Then for* $p \in [0, 1]$ *we have*

$$P_N^p(s_1, \dots, s_N) \leq P_N^p(s_1^*, \dots, s_N^*),$$

with equality if and only if at least one of the following conditions holds:

(*a*) $(s_1, \dots, s_N) = (s_1^*, \dots, s_N^*)$;

(*b*) $s_k = 0$ *for some* $k \in \{1, \dots, N\}$;

(*c*) $p = 1$;

(*d*) $p = 0$.

This result is analogous to an inequality of Essén [4, Thm. 2] concerning rearrangement in a certain second order difference equation. His difference equation is very similar to that which must be solved to compute $P_N^{1/2}(s_1, \dots, s_N)$, but there are still some essential differences. We will say more regarding the work of Essén in the proof of Theorem 5, below, where we shall state the actual difference equations whose solution gives $P_N^{1/2}$, and in §4 of the paper where we shall discuss the connection with Essén's analogous continuous case [5, Thm. 5.2].

It is quite possible that Essén's methods [4] could be adapted to prove Theorem 1, at least in the case $p = \frac{1}{2}$, even though his results do not appear to apply directly. However, we prefer to use different tactics (keeping the same overall strategy) which, in an elementary way, exploit the linearity properties of a function appearing in the explicit formula for P_N^p. Our proof will be given in §2. Finally, it should be noted that it does not seem that the methods of Baernstein [1] can be used to prove results like Theorem 1.

The heuristics behind Theorem 1 say that if we consider the random walk only until such time as it hits the point $N+1$, then it will spend more time further away from this point than it does nearer to it, so we will improve safety if we reorder the dangers so the more dangerous ones are near $N+1$ where the random walk spends less time. The author has not found a way of making this intuition into a rigorous proof. One might hope to find a probabilistic proof along these lines, but no such proof appears to be available right now, and it does not appear at all easy to produce such a proof.

If $p = \frac{1}{2}$ then Theorem 1 may be thought of as a discrete one-dimensional analogue of a conjecture concerning harmonic measure and radial rearrangement of circularly symmetric domains in the plane; *see* [9, Appendix B].

THEOREM 2. – *Let* $s_1, \ldots, s_N \in [0,1]$ *be given. Fix* $j \in \{1, \ldots, N\}$. *Then for* $p \in [0,1]$ *we have*

$$P_N^p(s_1, \ldots, s_N) \leq P_{N-1}^p(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N), \tag{2}$$

with the obvious conventions if j *is* 1 *or* N. *Moreover, equality holds if and only if at least one of the following conditions holds:*

(*a*) *there is a* $k \in \{1, \ldots, j-1, j+1, \ldots, N\}$ *with* $s_k = 0$;

(*b*) $s_k = 1$ *for every* $k \in \{1, \ldots, j\}$;

(*c*) $p = 1$ *and* $s_j = 1$;

(*d*) $p = 0$.

Intuitively Theorem 2 says that if we make a dangerous road shorter by removing a segment then the road becomes safer for a random walk. We will give a proof of Theorem 2 in §2 as a by-product of our proof of Theorem 1.

Now, let $s_1, s_2, \ldots \in [0,1]$ be an infinite sequence. Define the random walk r_i^p on $\mathbb{Z}^+$ with the same transition probabilities as the previous walk on $\{1, \ldots, N+1\}$. Let L_s be the first time that the random walk fails to survive a step. More precisely, we define

$$L_s = \inf \{i \geq 0 : X_i > s_{r_i}\}.$$

THEOREM 3. – *Let* $s_1, s_2, \ldots \in [0,1]$ *and let* $s_1^*, s_2^*, \ldots$ *be the decreasing rearrangement of the* s_i. *Let* $p \in [0, \frac{1}{2}]$. *Then*

$$P(L_s > n) \leq P(L_{s^*} > n),$$

for every $n \geq 0$.

We shall give a proof of Theorem 3 in §5, where we shall also state some closely related results, including one on discrete Steiner rearrangement. It is not known whether the condition $p \in [0, \frac{1}{2}]$ can be relaxed to $p \in [0, 1]$, although it is easy to *see* that Theorem 3 does hold for $p = 1$.

OPEN PROBLEM 1. – Prove or disprove that Theorem 3 also holds for $p \in (\frac{1}{2}, 1)$.

We now make a tangential remark in response to a question posed by a referee.

Remark. – Can we say anything about the question of when one has $E[L_s] < \infty$? Suppose that $p \leq \frac{1}{2}$ and that there exists a $k \in \mathbb{Z}^+$ such that $s_k < 1$. Since $p \leq \frac{1}{2}$, it is easy to see that almost surely the random walk r_i visits the point k infinitely often. Let T_n be the time of the nth visit of the random walk to the point k. It is easy to see that $E[T_1] < \infty$ and that $E[T_{n+1} - T_n] < \infty$ for all $n \in \mathbb{Z}^+$. Let $A = E[T_1]$ and $B = E[T_2 - T_1]$. Note that $E[T_{n+1} - T_n] = B$ for all n by the Markov property. Then, using the Markov property, we can see that:

$$
\begin{aligned}
E[L_s] &\leq E[T_1] + s_k E[T_2] + s_k^2 E[T_3] + s_k^3 E[T_4] + \cdots \\
&= A + s_k(A + B) + s_k^2(A + 2B) + s_k^3(A + 3B) + \cdots < \infty,
\end{aligned}
$$

since $0 \leq s_k < 1$. Conversely, it is clear that if $p \leq \frac{1}{2}$ and $s_k = 1$ for all k then $L_s = \infty$ almost surely. Hence, we have seen that for $p \leq \frac{1}{2}$ we have $E[L_s] < \infty$ if and only if there is a k with $s_k < 1$. For $p > \frac{1}{2}$ we only note the easy result that if $\sup_k s_k < 1$ then $E[L_s] < \infty$.

We now proceed to give a second open problem. Fix $p \in [0, 1]$. Let Φ be a real valued function on $\mathbb{Z}^+$ satisfying the "convexity" (one might also use the term "subharmonicity") condition

$$
\Phi(n) \leq (1 - p)\Phi(n - 1) + p\Phi(n + 1), \tag{3}
$$

for $n \in \mathbb{Z}^+$, where $\Phi(0) \stackrel{\text{def}}{=} \Phi(1)$. Condition (3) is equivalent to assuming that $\Phi(r_i)$ is a submartingale. It is easy to inductively see (starting with the fact that $\Phi(1) = \Phi(0)$ so that $\Phi(1) \geq \Phi(0)$) that if $p > 0$ then (3) implies that Φ is increasing.

OPEN PROBLEM 2. – Does it follow that

$$
E[\Phi(r_{L_s})] \leq E[\Phi(r_{L_{s^*}})]?
$$

If $p = \frac{1}{2}$ then this is a one-dimensional discrete analogue of a conjecture of Pruss concerning least harmonic majorants and radial rearrangement of circularly symmetric domains; *see* [9]. Here, we just wish to note that some sort of convexity condition like (3) on Φ in addition to requiring Φ to be increasing is necessary if $p \in (0,1)$. For, if we do not have this condition, then we may adapt a counterexample given in [9] to [9, Conjecture 3]. In fact, we can set $s_1 = \frac{1}{2}$, $s_2 = 0$, $s_3 = \frac{1}{2}$ and $s_4 = s_5 = \cdots = 0$, and let $\Phi(n)$ be 0 for $n \leq 1$ and 1 otherwise; a simple computation shows that then the answer to Problem 2 would be negative. Note also that if we let $s_{N+1} = s_{N+2} = \cdots = 0$ and set $\Phi(n) = \max(n - N, 0)$ then $E[\Phi(r_{L_s})] = P_N^p(s_1, \ldots, s_N)$ so that Theorem 1 is a special case of Problem 2.

Finally, the following result should surprise no one, but we state it for completeness. If we increase the probability of going towards our goal then certainly the probability of arriving at it should increase.

THEOREM 4. – *Let* $0 \leq p < r \leq 1$ *and let* $s_1, \ldots, s_N \in [0,1]$. *Then*

$$P_N^p(s_1, \ldots, s_N) \leq P_N^r(s_1, \ldots, s_N),$$

with equality if and only if one of the following conditions holds:

(*a*) $s_k = 0$ *for some* $k \in \{1, \ldots, N\}$

(*b*) $s_1 = \cdots = s_N = 1$ *and* $p > 0$.

We now outline a proof of Theorem 4, leaving the details as an exercise to the reader. Consider a more general case of a random walk defined as above, but instead of having a constant probability p of going to the right and $1 - p$ of going to the left, allow this probability to vary with position, so that the probability of moving to the right from $n \in \{1, \ldots, N\}$ is $t_n \in [0,1]$ and the probability of moving to the left is $1 - t_n$. As before, moving to the left from 1 results in standing still. Just as before, we can define the probability of the random walk getting from 1 to $N + 1$ without having fallen into any of the dangers. I claim that this probability will increase if any one of the t_n is increased; clearly this would be a more general result than Theorem 4 (though of course we would have to ensure that appropriate conditions of equality hold, the verification of which we leave as an exercise for the reader).

To prove the claim, fix n. Assume $n > 1$; the case $n = 1$ is handled similarly. We want to see the dependence on t_n. So, let A be the probability that a random walk (with movement probabilities defined by the t_j) starting from $n - 1$ will eventually arrive at n without having fallen into any of the dangers. Let B be the probability that such a random walk starting from $n + 1$ eventually arrives at n without having fallen into any of the dangers and without having first arrived at $N + 1$. Let C be the probability that such a random walk when started from $n + 1$ eventually arrives at

$N+1$ without having fallen into any of the dangers and without having first arrived at n. Finally, let P be the probability that a random walk starting at n eventually arrives at $N+1$ without having fallen into any of the dangers. The probability of a random walk from 1 arriving safely at $N+1$ is proportional to P, so we need only compute how P depends on t_n. Also, A, B and C are independent of t_n and satisfy the equation

$$P = s_n(1-t_n)AP + s_n t_n(BP+C).$$

From this point on it is an elementary exercise to verify that P increases with t_n, and to determine the conditions under which the increase fails to be strict.

2. VARIOUS USEFUL IDENTITIES, FORMULAE AND SOME PROOFS

In this section we shall prove Theorems 1 and 2, assuming an explicit formula (Theorem 5, below) for $P_N^p(s_1,\ldots,s_N)$. The proof of this formula will be given in §3.

First we note a simple probabilistic identity which will later prove to be of use. Suppose $p \in (0,1)$, $N \geq 2$ and $s_1 = 1$. Then it does not matter how long the random walk spends at the point 1, since it will survive to eventually leave 1 and go to 2. Whenever it subsequently goes left from 2, it will survive until its eventual return to 2. Hence, we may form a certain correspondence between random walks on $\{1,2,\ldots,N\}$ and those on $\{2,\ldots,N\}$ in such a way as to prove that

$$P_N^p(1,s_2,\ldots,s_N) = P_{N-1}^p(s_2,\ldots,s_N). \tag{4}$$

It is trivial to also verify that this continues to hold if $p \in \{0,1\}$.

Now, for positive n, let $\psi_{N,n}(a_1,\ldots,a_N)$ be the sum of all terms of the form

$$a_{i_1}a_{i_1+1}a_{i_2}a_{i_2+1}\cdots a_{i_n}a_{i_n+1} \tag{5}$$

with

$$1 \leq i_1 < i_1+1 < i_2 < i_2+1 < \cdots < i_n < i_n+1 \leq N.$$

Explicitly we have

$$\psi_{N,n}(a_1,\ldots,a_N) = \sum_{i_1=1}^{N-2n+1}\ \sum_{i_2=i_1+2}^{N-2n+3} \cdots \sum_{i_n=i_{n-1}+2}^{N-1} a_{i_1}a_{i_1+1}a_{i_2}a_{i_2+1}\cdots a_{i_n}a_{i_n+1},$$

with the convention that empty sums are equal to zero. Clearly $\psi_{N,n}$ is a function of N variables, is linear in each variable if the others are fixed, and vanishes identically for $2n > N$. Let

$$\Psi_N(a_1, \dots, a_N) = 1 + \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n \psi_{N,n}(a_1, \dots, a_N),$$

for $N \in \mathbb{Z}_0^+$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Note that $\Psi_N \equiv 1$ for $N \in \{0, 1\}$.

Now, I claim that

$$\Psi_{N+1}(a_1, \dots, a_{N+1}) = \Psi_N(a_2, a_3, \dots, a_{N+1}) - a_1 a_2 \Psi_{N-1}(a_3, \dots, a_{N+1}), \tag{6}$$

for $N \geq 1$. This identity is central to our work.

The proof of the identity is not very difficult. For, take one of the terms in $\Psi_{N+1}(a_1, \dots, a_{N+1})$. It will be either of the form

$$(-1)^n a_{i_1} a_{i_1+1} a_{i_2} a_{i_2+1} \cdots a_{i_n} a_{i_n+1}$$

with $1 \leq n \leq \lfloor \frac{N+1}{2} \rfloor$ and

$$1 \leq i_1 < i_1 + 1 < i_2 < i_2 + 1 < \cdots < i_n < i_n + 1 \leq N + 1,$$

or else it will be identically 1. If a_1 occurs in this term then $i_1 = 1$ so that a_2 must also occur in it. It is easy to see by the definitions that it must then also be a term of $-a_1 a_2 \Psi_{N-1}(a_3, \dots, a_{N+1})$. On the other hand, if a_1 fails to occur in the term, then this term must be a term of $\Psi_N(a_2, \dots, a_{N+1})$. Conversely, it is easy to verify that any term of the right hand side of (6) is also a term of the left hand side, and the proof of the claim is complete.

As a corollary of (6), we can see that

$$\Psi_{N+1}(0, a_1, \dots, a_N) = \Psi_N(a_1, \dots, a_N), \tag{7}$$

for $N \geq 1$. For $N = 0$ this also holds trivially, and hence (7) is valid for all $N \geq 0$. Also, by (6) and (7) we obtain

$$\Psi_{N+1}(a_1, 0, a_3, \dots, a_{N+1}) = \Psi_N(0, a_3, \dots, a_{N+1}) = \Psi_{N-1}(a_3, \dots, a_{N+1}), \tag{8}$$

for $N \geq 1$.

Note that $\Psi_N(a_1, \dots, a_N) = \Psi_N(a_N, \dots, a_1)$, so that

$$\Psi_N(a_1, \dots, a_N) = \Psi_{N-1}(a_1, \dots, a_{N-1}) - a_N a_{N-1} \Psi_{N-2}(a_1, \dots, a_{N-2}), \tag{9}$$

whenever $N \geq 2$, by (6).

Now, define

$$\phi_n(p) = \begin{cases} p, & \text{if } n \text{ is even} \\ 1-p, & \text{if } n \text{ is odd.} \end{cases}$$

Note that $\phi_{n+1}(p) = \phi_n(1-p) = 1 - \phi_n(p)$ for every n and p. Because the expressions that will be involved would be unmanageable otherwise, it will be useful to have two more abbreviations. Let

$$\overline{\Psi}_N^p(a_1, \dots, a_N) = \Psi_{N+1}(1, \phi_1(p)a_1, \dots, \phi_N(p)a_N)$$

and

$$\Psi_N^p(a_1, \dots, a_N) = \Psi_N(\phi_1(p)a_1, \dots, \phi_N(p)a_N).$$

At times the reader will be implicitly expected to be able to use the definitions to mentally rewrite the $\overline{\Psi}_N^p$ and Ψ_N^p in terms of the Ψ_N.

The following result then gives a formula for the probability of traversal; a proof will be given in §3.

THEOREM 5. – *For $p \in (0,1]$ and $s_1, \dots, s_N \in [0,1]$ we have*

$$P_N^p(s_1, \dots, s_N) = \frac{p^N s_1 s_2 \cdots s_N}{\overline{\Psi}_N^p(s_1, s_2, \dots, s_N)}. \tag{10}$$

Moreover, the denominator is always strictly positive under the above conditions.

Assuming Theorem 5, I claim that

$$\overline{\Psi}_N^p(1, a_2, \dots, a_N) = p\overline{\Psi}_{N-1}^p(a_2, \dots, a_N) \tag{11}$$

For, if p is fixed then both sides are linear in any one variable when the others are fixed, so that it is enough to verify (11) for $a_2, \dots, a_N \in (0,1]$. Moreover, both sides of (11) are continuous in p and hence it suffices to consider $p \in (0,1]$. But under such circumstances (11) follows from (4) and Theorem 5. Note that if $p = \frac{1}{2}$ then (11) takes the particularly simple form

$$\Psi_{N+1}\left(1, \frac{1}{2}, x_2, \dots, x_N\right) = \frac{1}{2}\Psi_N(1, x_2, \dots, x_N).$$

LEMMA 1. – *Let* $N \geq 1$ *and fix* $a_1, \ldots, a_N \in [0,1]$. *Suppose* $p \in (0,1]$. *Then* $\Psi_{N+1}(x, \phi_1(p)a_1, \ldots, \phi_N(p)a_N)$ *is strictly positive for every* $x \in [0,1]$.

Proof. – Fix $a_1, \ldots, a_N$. Now,

$$x \mapsto \Psi_{N+1}(x, \phi_1(p)a_1, \ldots, \phi_N(p)a_N)$$

is a linear function and hence it suffices to verify its strict positivity for $x \in \{0,1\}$. If $x = 1$, then the strict positivity immediately follows from the "moreover" in Theorem 5. Now, for $x = 0$, by (8) we may write

$$\begin{aligned}
&\Psi_{N+1}(0, \phi_1(p)a_1, \ldots, \phi_N(p)a_N) \\
&\quad = \Psi_{N+2}(1, 0, \phi_1(p)a_1, \ldots, \phi_N(p)a_N) \\
&\quad = \Psi_{N+2}(1, \phi_1(1-p) \cdot 0, \phi_2(1-p)a_1, \ldots \phi_{N+1}(1-p)a_N).
\end{aligned}$$

The strict positivity of this again immediately follows from the "moreover" of Theorem 5. □

We also note that

$$\Psi_{M+N+1}(a_1, \ldots, a_M, 0, b_1, \ldots, b_N) = \Psi_M(a_1, \ldots, a_M)\Psi_N(b_1, \ldots, b_N). \tag{12}$$

The easiest way to prove this is to note that every term of the right hand side is a term of the left hand side and vice versa, much as in the proof of (6).

Finally, it is easy to use the fact that $\phi_n(p)\phi_{n+1}(p) = p(1-p) = \phi_n(1-p)\phi_{n+1}(1-p)$ for every n together with the way that Ψ_M is defined to show that

$$\Psi_M(\phi_1(p)a_1, \ldots, \phi_M(p)a_M) = \Psi_M(\phi_1(1-p)a_1, \ldots, \phi_M(1-p)a_M). \tag{13}$$

We can write this concisely as $\Psi_M^p = \Psi_M^{1-p}$. Now, recalling that $1 - \phi_n(p)$ is either p or $1-p$ for any n, and applying (12), followed by (13) if necessary, we see that

$$\begin{aligned}
&\overline{\Psi}^p_{M+N+1}(a_1, \ldots, a_M, 0, b_1, \ldots, b_N) \\
&\quad = \overline{\Psi}^p_M(a_1, \ldots, a_M)\Psi^r_N(b_1, \ldots, b_N) \\
&\quad = \overline{\Psi}^p_M(a_1, \ldots, a_M)\Psi^p_N(b_1, \ldots, b_N),
\end{aligned} \tag{14}$$

where $r = 1 - \phi_{M+2}(p)$.

LEMMA 2. – *Let* $a_1, \ldots, a_m$ *and* $b_1, \ldots, b_n$ *be in* $[0, 1]$. *Let* $p \in (0, 1)$. *Suppose that*

$$\min\,(a_1, \ldots, a_m) \geq \max\,(b_1, \ldots, b_n). \tag{15}$$

Then

$$\overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\Psi^p_n(b_1, \ldots, b_n) \geq \overline{\Psi}^p_m(a_1, \ldots, a_m)\Psi^p_{n-1}(b_2, \ldots, b_n). \tag{16}$$

Moreover if equality holds then at least one of the a_j *vanishes.*

Proof. – We proceed by induction on $\max\,(m, n)$. If $\max\,(m, n) = 1$ then (16) becomes

$$1 \geq \overline{\Psi}^p_1(a_1) = 1 - \phi_1(p)a_1.$$

This is clearly true, and strict inequality holds unless $a_1 = 0$.

Now suppose Lemma 2 has been proved when $\max\,(m, n) = N - 1$ and that we have $\max\,(m, n) = N > 1$. By (6) and (9), we see that (16) is equivalent to the inequality

$$\begin{aligned}
&\overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\Psi^p_{n-1}(b_2, \ldots, b_n) \\
&\qquad - \overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\phi_1(p)\phi_2(p)b_1b_2\Psi^p_{n-2}(b_3, \ldots, b_n) \\
&\quad \geq \overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\Psi^p_{n-1}(b_2, \ldots, b_n) \\
&\qquad - \phi_{m-1}(p)\phi_m(p)a_{m-1}a_m\overline{\Psi}^p_{m-2}(a_1, \ldots, a_{m-2})\Psi^p_{n-1}(b_2, \ldots, b_n).
\end{aligned}$$

Note that we have implicitly used (13) after applying (6). Clearly our last inequality is equivalent to

$$\begin{aligned}
&p(1-p)a_{m-1}a_m\overline{\Psi}^p_{m-2}(a_1, \ldots, a_{m-2})\Psi^p_{n-1}(b_2, \ldots, b_n) \\
&\quad \geq p(1-p)b_1b_2\overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\Psi^p_{n-2}(b_3, \ldots, b_n).
\end{aligned}$$

But this is true by the induction hypothesis since $\max\,(m-1, n-1) = N-1$ and since $a_{m-1}a_m \geq b_1b_2$ because of (15). Moreover, if equality holds then either $a_{m-1}a_m$ vanishes or, again by the induction hypothesis, one of $a_1, \ldots, a_{m-1}$ vanishes. $\square$

The following lemma is an exact equivalent of Essén's [4, Lemma 1], and indeed our strategy for the proof of Theorem 1 is quite similar to that of Essén. Of course we use the convention that the infimum of an empty set is equal to $+\infty$.

LEMMA 3. – *Fix $p \in (0,1)$. Suppose that $a_1, \ldots, a_N \in [0,1]$ and assume that $i \in \{1, \ldots, N-1\}$ has the property that*

$$\inf \{a_1, \ldots, a_{i-1}\} \geq \max (a_i, \ldots, a_N) \tag{17}$$

(*this condition on i is trivially satisfied if $i = 1$*). *Finally suppose that*

$$a_i < \max (a_i, \ldots, a_N) \tag{18}$$

and that $j \in \{i+1, \ldots, N\}$ is such that $a_j = \max (a_i, \ldots, a_N)$. Then

$$\overline{\Psi}_N^p(a_1, \ldots, a_N) > \overline{\Psi}_N^p(a_1, \ldots, a_{i-1}, a_j, a_i, a_{i+1} \ldots, a_{j-1}, a_{j+1}, \ldots, a_N). \tag{19}$$

For the rest of this section, in interpreting (19) and similar expressions we use the convention that a sequence of the form $a_k, \ldots, a_n$ is empty and omitted if $n < k$. We shall assume Lemma 3 for now and show how it implies Theorems 1 and 2. A more elementary method of proof of Theorem 2 was kindly communicated to the author by Mr. Ravi Vakil. His approach in effect reduces the question to consideration of the movement of the system between the points $j-1$, j, $j+1$, N and ∞, where ∞ indicates that the random walk has been terminated by having fallen into one of the dangers. This new system is sufficiently small that explicit computation can be used to prove the desired result (*cf.* the outline of proof of Theorem 4, above). However, since we have Lemma 3 available (and we will definitely need it for Theorem 1), we proceed as follows.

Proof of Theorem 2. – Assume that $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N \in (0,1]$. (If one of these vanishes then the result is trivial.) The result is easy if $p \in \{0,1\}$ so assume $0 < p < 1$. It is clear on probabilistic grounds that we may assume that $s_j = 1$ since changing $s_j = 1$ to $s_j < 1$ would strictly decrease the left side of (2) and leave the right side unchanged. By Theorem 5, we need only show that

$$p^{-1}\overline{\Psi}_N^p(s_1, \ldots, s_N) \geq \overline{\Psi}_{N-1}^p(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, a_N) \tag{20}$$

and that equality holds if and only if $s_1 = \cdots = s_j = 1$. We shall prove this by induction on N. If $N = 1$ then the result follows immediately from the definition of the Ψ_N. Suppose that $N > 1$ and the desired result has been proved for $N-1$. Assume first that $s_1 = 1$. If $j = 1$ then by (11) we do have equality in (20) as desired. If $j > 1$, on the other hand, then we may apply (11) to both sides of (20) and the desired result will then follow by the induction hypothesis. Hence we may assume that $s_1 < 1$. Then, the hypotheses of Lemma 3 are satisfied with $i = 1$ and j as above, so that

$$\overline{\Psi}_N^p(s_1, \ldots, s_N) > \overline{\Psi}_N^p(s_j, s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N).$$

Now since $s_j = 1$, an application of (11) to the right hand side of the above inequality proves that strict inequality holds in (20) as desired. □

Proof of Theorem 1. – Again, we may assume that $0 < p < 1$ and that the s_k are all strictly positive. Then, assuming Lemma 3 and given $s_1, \ldots, s_N \in (0, 1]$, I claim that

$$\overline{\Psi}_N^p(s_1, \ldots, s_N) \geq \overline{\Psi}_N^p(s_1^*, \ldots, s_N^*) \tag{21}$$

with equality if and only if $s_1 \geq s_2 \geq \cdots \geq s_N$. For, if it is not true that $s_1 \geq s_2 \geq \ldots \geq s_N$, then we may let i be the maximum of the numbers $i_1 \in \{1, \ldots, N\}$ which have the properties that $s_1, \ldots, s_{i_1 - 1}$ are in decreasing order and that whenever $1 \leq k < i_1$ then $s_k \geq \max\,(s_{i_1}, \ldots, s_N)$ (note that the conditions on i_1 are automatically satisfied for $i_1 = 1$). Because $s_1, \ldots, s_N$ are *not* all in decreasing order, it follows that $i < N$ and the maximality of i implies that $s_i < \max\,(s_i, \ldots, s_N)$. We may then apply Lemma 3, and let

$$(s'_1, \ldots, s'_N) = (s_1, \ldots, s_{i-1}, s_j, s_i, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N).$$

Note that $s'_1, \ldots, s'_N$ are a permutation of $s_1, \ldots, s_N$. Hence, if $s'_1, \ldots, s'_N$ are in decreasing order then we are done by (19). Otherwise, proceed just as before and define i' in terms of the s'_k just as i was defined in terms of the s_k. Then it is easy to see that $i' > i$. We may iterate this procedure at most $N - 1$ times until we have sorted the s_k into decreasing order, and so the claim is proved. Theorem 1 then follows from Theorem 5 and this claim. □

We now prove Lemma 3 by exploiting the linearity properties of the Ψ_N, using a reduction reminiscent of Hardy and Littlewood's [7] reduction of a certain rearrangement inequality to the case where all the variables were in $\{0, 1\}$ (*see* also Lemma 4, below).

Proof of Lemma 3. – Throughout $p \in (0, 1)$ shall be fixed. Let j be as in the statement of the Lemma and set $\lambda = a_j$. Note that by (18) we have $\lambda > 0$. Fix a_j as well as $a_1, \ldots, a_{i-1}$. What we must prove is that

$$\overline{\Psi}_N^p(a_1, \ldots, a_N) - \overline{\Psi}_N^p(a_1, \ldots, a_{i-1}, a_j, a_i, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N) > 0 \tag{22}$$

whenever $a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N \in [0, \lambda]$ and $0 \le a_i < \lambda$. We first consider the case when $j = i + 1$. In that case, the two N-tuples serving as arguments to the $\overline{\Psi}^p_N$ in (22) will only differ by an exchange of their ith and jth elements. Moreover, if all variables other than a_i are fixed, then the left hand side of (22) will be a linear function of a_i. If we had $a_i = \lambda$ then the left side of (22) would vanish since $a_j = \lambda$ too. On the other hand if we had $a_i = 0$ then this left hand side would become

$$\overline{\Psi}^p_N(a_1, \ldots, a_{i-1}, 0, a_j, \ldots, a_N) - \overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, a_j, 0, a_{j+1}, \ldots, a_N).$$

But applying (14) to both terms and then using Lemma 2, we see that this is strictly positive. Note that Lemma 2 is applicable since by choice of j and by (17), we have

$$\min (a_1, \ldots, a_{i-1}, a_j) = \lambda \ge \lambda = \max (a_j, \ldots, a_N),$$

and moreover $\lambda > 0$ so that strict inequality must hold. Hence, the left side of (22) is strictly positive if $a_i = 0$, vanishes if $a_i = \lambda$ and hence by linearity is strictly positive if $a_i \in [0, \lambda)$. This completes the proof if $j = i + 1$.

Now suppose $j > i + 1$. By linearity considerations we need only verify (22) when $a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N \in \{0, \lambda\}$ and the conclusion for them lying in $[0, \lambda]$ will immediately follow. Of course we always have $a_j = \lambda$. Thus from now on we assume that $a_{i+1}, \ldots, a_N \in \{0, \lambda\}$. Now, take the least integer $j_1 \in \{i + 1, \ldots, j\}$ with the property that $\lambda = a_{j_1} = a_{j_1+1} = \cdots = a_j$. Then, the N-tuple

$$(a_1, \ldots, a_{i-1}, a_j, a_i, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N) \tag{23}$$

will not at all change if we replace j by j_1 throughout its definition, since when we are moving one of the λ's from the string $a_{j_1}, \ldots, a_j$, then it clearly does not matter which one we move (see *Fig.* 1). Thus, we may replace j by j_1 and by minimality of j_1 assume that either $j = i + 1$ or that $a_{j-1} \neq \lambda$ (or both). We have already handled the case $j = i + 1$.

			i ↓			$i_1{+}1$ ↓									j_1 ↓		j ↓		
a_1	a_2	a_3	λ	λ	λ	0	λ	0	λ	λ	0	λ	λ	0	λ	λ	λ	a_{19}	a_{20}

Figure 1. – An example of the original N-tuple $(a_1, \ldots, a_N)$ for $N = 20$, $i = 4$ and $j = 18$. The new N-tuple (23) will be formed from this N-tuple by cutting out the jth element and pasting it to the left of the ith. Clearly the result of this operation will be the same whether it is the element in position j or the element in position j_1 that we cut out. The result will also be the same whether it is to the left of position i or to the left of position $i_1 + 1$ that we paste this element.

Hence, we have $a_{j-1} \neq \lambda$ and $j > i+1$. Moreover $a_{j-1} \in \{0, \lambda\}$ so that $a_{j-1} = 0$. Now keep $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N$ fixed. We shall show that in our present case (22) holds whenever $a_i \in [0, \lambda]$. By linearity it suffices to consider $a_i \in \{0, \lambda\}$. We first note that we can reduce the case $a_i = \lambda$ to the case $a_i = 0$ as follows. Suppose $a_i = \lambda$. Then, let i_1 be the greatest integer $i_1 \in \{i, \ldots, N\}$ with the property that $a_i = a_{i+1} = \cdots = a_{i_1} = \lambda$. Since $a_{j-1} = 0$, we have $i_1 < j-1$. Just as in our work with j_1 we can see that the N-tuple (23) will not change if i is replaced by $i_1 + 1$ (this is so because $a_i, \ldots, a_{i_1}$ is a string of λ's and it does not matter on which side of this string we insert $a_j = \lambda$; *see* Figure 1, except that now j should be in the same place as j_1 was). But the maximality of i_1 implies that $a_{i_1+1} \neq \lambda$, hence $a_{i_1+1} = 0$. Hence, indeed, replacing i by $i_1 + 1$ if necessary, we may assume that $a_i = 0$.

We now thus need only consider the case where $a_i = a_{j-1} = 0$ and $a_j = \lambda$. The case $j = i+1$ was already handled, so we may still assume that $j > i+1$. Then, we may rewrite the left hand side of (22) as

$$\begin{aligned}&\overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{j-2}, 0, a_j, \ldots, a_N)\\ &\quad - \overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, a_j, 0, a_{i+1}, \ldots, a_{j-2}, 0, a_{j+1}, \ldots, a_N).\end{aligned}$$

Applying (14) twice in each of the two terms, we see that this equals

$$\begin{aligned}&\overline{\Psi}^p_{i-1}(a_1, \ldots, a_{i-1})\Psi^p_{j-2-i}(a_{i+1}, \ldots, a_{j-2})\Psi^p_{N-j+1}(a_j, \ldots, a_N)\\ &- \overline{\Psi}^p_i(a_1, \ldots, a_{i-1}, a_j)\Psi^p_{j-2-i}(a_{i+1}, \ldots, a_{j-2})\Psi^p_{N-j}(a_{j+1}, \ldots, a_N). \quad (24)\end{aligned}$$

But the middle factor in both terms is the same, and by Lemma 1 it is strictly positive. Moreover,

$$\min\,(a_1, \ldots, a_{i-1}, a_j) = \lambda \geq \lambda = \max\,(a_j, \ldots, a_N)$$

and $\lambda > 0$ so that the left hand side of (24) is strictly positive by Lemma 2. □

3. PROOF OF THE FORMULA FOR THE PROBABILITY OF SAFE TRAVERSAL

Instead of giving a probabilistic proof, we give one coming from a solution of an associated system of simultaneous equations.

Proof of Theorem 5. – If $p = 1$ then $\Psi^1_N \equiv 1$ for all $N \geq 1$ by a repeated application of (8), so that the content of the Theorem for $p = 1$ follows from (1). From now on we assume that $p \in (0, 1)$. Let $q = 1 - p$.

Consider a random walk with the same transition probabilities as $\{r_i^p\}$, with the same boundary condition at 1, but not necessarily starting at the point 1. Let p_n be the probability that when started at n, it arrives at N without having fallen into any of the dangers along the route. Then,

$$p_1 = P_N^p(s_1, \ldots, s_N).$$

The following equations are easy to verify:

$$\begin{aligned} p_1 &= s_1(qp_1 + pp_2) \\ p_2 &= s_2(qp_1 + pp_3) \\ p_3 &= s_3(qp_2 + pp_4) \\ &\ldots \\ p_{N-1} &= s_{N-1}(qp_{N-2} + pp_N) \\ p_N &= s_N(qp_{N-1} + p). \end{aligned}$$

This is a tridiagonal system of N equations in the N unknowns $p_1, \ldots, p_N$. If $p = q = \frac{1}{2}$ then all but the first and last equations can be rewritten as

$$D^2 p_j - \delta_j p_j = 0$$

where $2 \le j \le N-1$, $D^2 p_j = \frac{1}{2}(p_{j-1} + p_{j+1}) - p_j$ and $\delta_j = s_j^{-1} - 1$. This shows the similarity with the work of Essén [4] who considers a similar question but with different boundary conditions and with D^2 replaced by Δ^2, where $\Delta^2 p_j = 2D^2 p_{j-1}$ so that $\Delta^2 p_j = \Delta(\Delta p_j)$ where $\Delta p_j = p_j - p_{j-1}$.

In fact, for general $p \in (0,1)$, our system can be solved by a simple and standard elimination scheme. First we transform it into the upper triangular system of equations

$$\begin{pmatrix} A_1 & ps_1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & A_2 & ps_2 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & A_{N-1} & ps_{N-1} \\ 0 & 0 & 0 & \ldots & 0 & 0 & A_N \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N-1} \\ p_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -ps_N \end{pmatrix},$$

where the A_i are inductively defined by

$$A_1 = qs_1 - 1$$

and by

$$A_{n+1} = -1 - \frac{pqs_n s_{n+1}}{A_n},$$

for $n = 2, \ldots, N$. It is easy to inductively verify that we will have $A_n \leq -\min(p, q) < 0$ for $n = 1, \ldots, N$ so that everything is well defined.

Then, a further reduction (starting from the bottom and working our way up) transforms the system into a diagonal system and shows that

$$p_1 = (-1)^N \frac{(ps_1)(ps_2)\cdots(ps_N)}{A_1 A_2 \cdots A_N}.$$

Comparing this with (10), we see that we will be done as soon as we show that

$$(-A_1)(-A_2)\cdots(-A_N) = \Psi_{N+1}(1, \phi_1(p)s_1, \ldots, \phi_N(p)s_N). \tag{25}$$

Since we have seen that $A_n < 0$ for $n = 1, \ldots, N$, the positivity of the denominator in (10) will also follow from (25).

Let

$$B_n = -A_{N-n+1},$$

for $n = 1, \ldots, N$ and set

$$t_n = pqs_{N-n}s_{N-n+1},$$

for $n = 1, \ldots, N-1$. Define $t_N = qs_1$. Then from the inductive definition of the A_n we find that

$$B_N = 1 - t_N$$

while

$$B_n = 1 - t_n B_{n+1}^{-1}, \tag{26}$$

for $n = 1, \ldots, N-1$. Let

$$B_{N+1} = 1.$$

Then (26) also holds for $n = N$. We then have

$$B_n B_{n+1} = B_{n+1} - t_n, \tag{27}$$

for $n = 1, \ldots, N$. Let $\Gamma_n = B_1 B_2 \cdots B_n$ for $n \leq N+1$. Then since $B_{N+1} = 1$, and since $\Psi_{N+1}(a_1, \ldots, a_{N+1}) = \Psi_{N+1}(a_{N+1}, \ldots, a_1)$, we see that (25) is equivalent to the assertion that

$$\Gamma_{N+1} = \Psi_{N+1}(a_1, a_2, \ldots, a_{N+1}), \tag{28}$$

where $a_n = \phi_{N-n+1}(p)s_{N-n+1}$ for $n = 1, \ldots, N$ and $a_{N+1} = 1$. Recall that

$$\phi_n(p)\phi_{n+1}(p) = pq$$

for every n and that $\phi_1(p) = q$ so that $t_n = a_n a_{n+1}$ for $n = 1, \ldots, N$. We shall now work exclusively in terms of the a_n, t_n, B_n and Γ_n.

To compute Γ_n, note that

$$\Gamma_1 = B_1.$$

Suppose that

$$\Gamma_n = \alpha_n B_n + \beta_n.$$

Then

$$\begin{aligned}\Gamma_{n+1} &= \Gamma_n B_{n+1}\\ &= \alpha_n B_n B_{n+1} + \beta_n B_{n+1}\\ &= \alpha_n(B_{n+1} - t_n) + \beta_n B_{n+1}\\ &= (\alpha_n + \beta_n)B_{n+1} - t_n\alpha_n,\end{aligned}$$

where we have used (27) to obtain the second-last equality. Thus, if we define α_n and β_n inductively by

$$\begin{aligned}\alpha_1 &= 1\\ \beta_1 &= 0\end{aligned}$$

and

$$\alpha_{n+1} = \alpha_n + \beta_n \tag{29a}$$

$$\beta_{n+1} = -t_n\alpha_n, \tag{29b}$$

for $n = 1, \ldots, N$, then it follows by induction that we will always have

$$\Gamma_n = \alpha_n B_n + \beta_n.$$

Since $B_{N+1} = 1$, it follows that

$$\Gamma_{N+1} = \alpha_{N+1} + \beta_{N+1}. \tag{30}$$

I claim that

$$\alpha_n = \Psi_{n-1}(a_1, \ldots, a_{n-1}) \tag{31a}$$

$$\beta_n = \Psi_n(a_1, \ldots, a_n) - \Psi_{n-1}(a_1, \ldots, a_{n-1}), \tag{31b}$$

for $n = 1, \ldots, N+1$. If this were true then (28) would immediately follow from (30). We prove (31a) and (31b) by induction. For $n = 1$ they are true since Ψ_1 and Ψ_0 are both identically 1. Suppose that they hold for n. Then by applying (29a) to (31a) and (31b), we see that

$$\alpha_{n+1} = \Psi_n(a_1, \ldots, a_n)$$

as desired. Now applying (29b) to (31a) we find that

$$\beta_{n+1} = -t_n \Psi_{n-1}(a_1, \ldots, a_{n-1}).$$

Thus, to obtain (31b) for $n+1$ we must show that

$$\Psi_{n+1}(a_1, \ldots, a_{n+1}) - \Psi_n(a_1, \ldots, a_n) = -t_n \Psi_{n-1}(a_1, \ldots, a_{n-1}). \quad (32)$$

But $t_n = a_n a_{n+1}$ so that (32) follows from (9). $\square$

4. THE ONE-DIMENSIONAL CONTINUOUS CASE

We now show how our result is connected with a one-dimensional continuous rearrangement inequality of Essén [5].

Suppose that $p = \frac{1}{2}$. For a sequence p_j, let $D^2 p_j = \frac{1}{2}(p_{j-1} + p_{j+1}) - p_j$. Then, it is not difficult to verify (*cf.* the proof of Theorem 5, above) that to find $P_N^{1/2}(s_1, \ldots, s_N)$ one needs to solve

$$D^2 p_j - \delta_j p_j = 0,$$

for $j \in \{-N+1, \ldots, N\}$, subject to the conditions

$$p_{N+1} = p_{-N} = 1$$

and

$$p_0 = p_1,$$

where $\delta_j = \delta_{1-j} = s_j^{-1} - 1$ if $j \in \{1, \ldots, N\}$. Then one will have

$$p_0 = P_N^{1/2}(s_1, \ldots, s_N).$$

The symmetry of the problem easily shows that the solution will have the property that if $j \in \{1, \ldots, N\}$ then $p_j = p_{1-j}$, and this symmetry easily shows why this system is equivalent to the one exhibited at the beginning of the proof of Theorem 5. (Note that we are in effect now considering a random walk on $\{-N+1,\ldots,N\}$ in place of our reflecting walk on $\{1,\ldots,N\}$.)

The reason for writing the system as above is that it suggests as a continuous analogue the differential equation

$$p''(x) - \delta(x)p(x) = 0 \tag{33}$$

on $[-L, L]$, where δ is even, and where p is subject to the conditions that

$$p(L) = p(-L) = 1$$

and

$$p'(0) = 0. \tag{34}$$

To solve this, by symmetry we need only solve (33) on $[0, L]$ subject to (34) and to the condition that

$$p(L) = 1. \tag{35}$$

We now define the function $\delta^{\#}$ on $[0, L]$ to be the equimeasurable increasing rearrangement of the restriction of δ to $[0, L]$ and put $\delta^{\#}(x) = \delta^{\#}(-x)$ for $-L \leq x < 0$. (Note that we are rearranging in the opposite order from the way we rearranged the p_j because $\delta(x)$ corresponds to $p_j^{-1} - 1$.)

The following result is then an exact continuous equivalent of the $p = \frac{1}{2}$ case of Theorem 1.

THEOREM A (special case of Essén [5, Thm. 5.2]). – *Let δ be a nonnegative lower semicontinuous piecewise constant function on* $[0, L]$, *and let p be the solution of* (33), (34) *and* (35). *Let $p^{\#}$ be the solution of* (33), (34) *and* (35) *after replacing δ with $\delta^{\#}$. Then $p^{\#}(0) \geq p(0)$.*

It is not unlikely that Theorem A can be given some probabilistic interpretation in terms of Brownian motion, but such an interpretation is not as interesting as the probabilistic interpretation of our discrete results.

5. SURVIVAL TIMES AND DISCRETE STEINER REARRANGEMENT

To prove Theorem 3 and related results, we first consider a more general situation. Let $p^{(0)}, p^{(1)}, \ldots$ be a sequence in $[0,1] \cup \{-\infty\}$. For each fixed $i \in \mathbb{Z}_0^+$, let $s_1^{(i)}, s_2^{(i)}, \ldots$ be a sequence of numbers in $[0, 1]$. Now, as i

runs over $\mathbb{Z}_0^+$ let R_i be a random walk on $\mathbb{Z}^+$, which, if $p^{(i)} > -\infty$, has probability $p^{(i)}$ of moving to the right at time i and probability $1 - p^{(i)}$ of moving to the left at that time, and if $p^{(i)} = -\infty$ then it satisfies $R_i = R_{i+1}$. Again, if the walk moves to the left of 1 then we constrain it to remain at 1 for the time step. More generally than before, let

$$L_s = \inf\{i \geq 0 : X_i > s_{R_i}^{(i)}\},$$

where as before the X_i are i.i.d. and uniformly distributed on $[0, 1]$. Then, $L_s - 1$ represents the survival time of the random walk. Moreover, $s_k^{(i)}$ is the survival probability of the random walk at time i if this random walk happens to be at point k at this time.

For each fixed $i \in \mathbb{Z}_0^+$, let $(s^*)_1^{(i)}, (s^*)_2^{(i)}, \ldots$ be the decreasing rearrangement of $s_1^{(i)}, s_2^{(i)}, \ldots$.

Note that we have not defined where our random walk is to start. Because of this, we shall write $P^j(\cdot)$ for probabilities where the random walk is conditioned to start at j.

THEOREM 6. – *Suppose $p^{(i)} \in [0, \frac{1}{2}] \cup \{-\infty\}$ for $i \in \mathbb{Z}_0^+$. Let J be any set of precisely m positive integers. Then*

$$\sum_{j \in J} P^j(L_s > n) \leq \sum_{j=1}^{m} P^j(L_{s^*} > n),$$

for every nonnegative n.

It is easy to verify that this need not hold if the condition $p^{(i)} \leq \frac{1}{2}$ is dropped (counterexamples can be found even with $n = 1$ and $p^{(0)} = 1$); nevertheless, we do conjecture that the condition $p \leq \frac{1}{2}$ can be omitted in Theorem 3. Note that Theorem 3 does hold for $p = 1$.

Clearly, Theorem 3 will follow from Theorem 6 if we let $J = \{1\}$, and define $p^{(i)} = p$ for each i and $s_k^{(i)} = s_k$ for each i and k.

In order to set things up for the proof of Theorem 6, we now define $a_i: \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, 1]$ as follows. If $p^{(i)} > -\infty$ then let

$$a_i(j, k) = \begin{cases} p^{(i)}, & \text{if } k = j + 1 \\ 1 - p^{(i)}, & \text{if } k = j - 1 \text{ and } j > 1 \\ 1 - p^{(i)}, & \text{if } j = k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $p^{(i)} = -\infty$ then let $a_i(j, k) = \delta_{jk}$, where δ_{jk} is 1 when $j = k$ and 0 otherwise. Then, the a_i are the transition matrices corresponding to the random walk R_i, *i.e.*,

$$a_i(j, k) = P(R_{i+1} = k \mid R_i = j).$$

Let $\nu \mapsto F_\nu$ be the indicator function of J. Then

$$\sum_{j\in J} P^j(L_s > n)$$
$$= \sum_{\nu_0=1}^{\infty}\sum_{\nu_1=1}^{\infty}\cdots\sum_{\nu_n=1}^{\infty} F_{\nu_0} s^{(0)}_{\nu_0} a_0(\nu_0,\nu_1) s^{(1)}_{\nu_1}\cdots a_{n-1}(\nu_{n-1},\nu_n) s^{(n)}_{\nu_n}. \quad (36)$$

Moreover, the sums only appear to be infinite since all but finitely many summands vanish as J is finite.

We shall prove that if we have $p^{(i)} \leq \frac{1}{2}$ for every $i \geq 0$ then a simultaneous replacement of s with s^* and F with F^* in (36) cannot decrease (36). Since replacing F with F^* is equivalent to replacing J with $\{1,\ldots,\text{ Card } J\}$, this will immediately yield Theorem 6. It is to be noted that the above replacement inequality is very similar to a result of Haliste [6, Lemma 8.1] and the structure of our proof will be very similar, too.

Throughout we shall use $*$'s to denote decreasing rearrangements.

LEMMA 4. – *Let $t_1,\ldots,t_n$ and $t'_1,\ldots,t'_n$ be any finite real numbers. Suppose that whenever $x_1,\ldots,x_n \in \{0,1\}$ then we have*

$$t_1x_1 + \cdots + t_nx_n \leq t'_1x^*_1 + \cdots + t'_nx^*_n. \quad (37)$$

Then (37) *holds for any choice of* $x_1,\ldots,x_n \in [0,\infty)$.

This says that we may proceed from linear rearrangement results valid on the corners of an n-cube to ones valid on a whole octant.

Proof of Lemma 4. – Fix any $x_1,\ldots,x_n \in [0,\infty)$. By the decomposition result of Hardy, Littlewood and Pólya [8, §10.3(2)] we may find sequences $x^i_1,\ldots,x^i_n \in \{0,1\}$ for $i = 1,\ldots,n$ and coefficients $\alpha_1,\ldots,\alpha_n \in [0,\infty)$ such that

$$x_i = \alpha_1 x^i_1 + \cdots + \alpha_n x^i_n$$

and

$$x^*_i = \alpha_1 (x^i)^*_1 + \cdots + \alpha_n (x^i)^*_n,$$

both for every $i \in \{1,\ldots,n\}$. Since, for each fixed i, we have (37) holding for $x^i_1,\ldots,x^i_n$, we may then use positive linear combinations (with coefficients α_i) of (37) for these sequences to prove that (37) also holds for $x_1,\ldots,x_n$. □

Proof of Theorem 6. – As a first step in reducing the problem to a more manageable one, clearly we may assume that, for each fixed i, we have

$s_k^{(i)}$ vanishing if k is sufficiently large. Note that then for any fixed i, (36) may be rewritten in the form

$$t_1^{(i)} s_1^{(i)} + \cdots + t_N^{(i)} s_N^{(i)},$$

for some large N, where $t_1^{(i)}, \ldots, t_N^{(i)}$ do not depend on $s_1^{(i)}, \ldots, s_N^{(i)}$. Then by $n+1$ applications of Lemma 4 we see that we need only consider the case where all of the $s_k^{(i)}$ lie in $\{0,1\}$. In that case, let

$$\mu_i = \text{Card } \{k : s_k^{(i)} = 1\}.$$

Set $A_{\nu_0} = F_{\nu_0} s_{\nu_0}^{(0)}$. Note that A_{ν_0} vanishes for all but at most min (μ_0, m) values of ν_0, where $m = \text{Card } J$, and that $F_{\nu_0}^*(s^{(0)})_{\nu_0}^*$ is 1 for $\nu_0 = 1, \ldots, \min(\mu_0, m)$. Hence $A_{\nu_0}^* \leq F_{\nu_0}^*(s^{(0)})_{\nu_0}^*$ for each ν_0.

Thus, by (36), we will be done as soon as we can show that in general if each a_i has the form given above with $p^{(i)} \leq \frac{1}{2}$, if A_ν is a nonnegative sequence, and if the $s_k^{(i)}$ are arbitrary $\{0,1\}$ sequences with the number of nonzero entries for a fixed i equaling μ_i, then

$$\sum_{\nu_0=1}^{\infty} \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_n=1}^{\infty} A_{\nu_0} a_0(\nu_0, \nu_1) s_{\nu_1}^{(1)} \cdots a_{n-1}(\nu_{n-1}, \nu_n) s_{\nu_n}^{(n)}$$
$$\leq \sum_{\nu_0=1}^{\infty} \sum_{\nu_1=1}^{\mu_1} \cdots \sum_{\nu_n=1}^{\mu_n} A_{\nu_0}^* a_0(\nu_0, \nu_1) \cdots a_{n-1}(\nu_{n-1}, \nu_n). \tag{38}$$

We proceed by induction. If $n = 0$ then (38) is trivial. Suppose now that (38) holds for $n-1$, and that we are to prove it for n. Exactly as in [6, proof of Lemma 8.1], let

$$B_{\nu_1} = \sum_{\nu_0=1}^{\infty} A_{\nu_0} a_0(\nu_0, \nu_1) s_{\nu_1}^{(1)}$$

and

$$c_{\nu_1} = \sum_{\nu_2=1}^{\mu_2} \cdots \sum_{\nu_n=1}^{\mu_n} a_1(\nu_1, \nu_2) \cdots a_{n-1}(\nu_{n-1}, \nu_n).$$

Since (38) holds for $n-1$, we must have

$$\sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} \cdots \sum_{\nu_n=1}^{\infty} B_{\nu_1} a_1(\nu_1, \nu_2) s_{\nu_2}^{(2)} \cdots a_{n-1}(\nu_{n-1}, \nu_n) s_{\nu_n}^{(n)} \leq \sum_{\nu_1=1}^{\mu_1} B_{\nu_1}^* c_{\nu_1}. \tag{39}$$

Again following [6], let φ be a permutation of $\mathbb{Z}^+$ such that $B^*_{\varphi(\nu)} = B_\nu$ for each ν, and define $C_\nu = c_{\varphi(\nu)}$ as well as $d_\nu = s^{(1)}_\nu C_\nu$. Then,

$$\sum_{\nu_1=1}^{\mu_1} B^*_{\nu_1} c_{\nu_1} = \sum_{\nu_1=1}^{\infty} B_{\nu_1} C_{\nu_1} = \sum_{\nu_0=1}^{\infty}\sum_{\nu_1=1}^{\infty} A_{\nu_0} a_0(\nu_0,\nu_1) d_{\nu_1}. \tag{40}$$

I now claim that (40) cannot exceed

$$\sum_{\nu_0=1}^{\infty}\sum_{\nu_1=1}^{\infty} A^*_{\nu_0} a_0(\nu_0,\nu_1) d^*_{\nu_1}.$$

This claim will follow from the general inequality that for our a_0

$$\sum_{\nu_0=1}^{\infty}\sum_{\nu_1=1}^{\infty} \alpha_{\nu_0} a_0(\nu_0,\nu_1)\beta_{\nu_1} \le \sum_{\nu_0=1}^{\infty}\sum_{\nu_1=1}^{\infty} \alpha^*_{\nu_0} a_0(\nu_0,\nu_1)\beta^*_{\nu_1}, \tag{41}$$

whenever the α_ν and β_ν are nonnegative numbers which vanish for all but finitely many ν. If $p^{(0)} = -\infty$ then (41) follows from the Hardy-Littlewood inequality

$$\alpha_1\beta_1 + \cdots + \alpha_N\beta_N \le \alpha^*_1\beta^*_1 + \cdots + \alpha^*_N\beta^*_N.$$

Hence assume that $p \overset{\text{def}}{=} p^{(0)}$ lies in $[0, \frac{1}{2}]$. By applying Lemma 4 twice, we may assume that α_ν and β_ν both take values only in $\{0,1\}$. In that case, let $\gamma = \text{Card}\,\{\nu : \alpha_\nu = 1\}$ and $\delta = \text{Card}\,\{\nu : \beta_\nu = 1\}$. Assume that γ and δ are both at least 1 (for if one of them vanishes then (41) is trivial). Then, (41) is equivalent to the assertion that

$$\begin{aligned} &p\sum_{\nu=1}^{\infty} \alpha_\nu\beta_{\nu+1} + (1-p)\sum_{\nu=2}^{\infty}\alpha_\nu\beta_{\nu-1} + (1-p)\alpha_1\beta_1 \\ &\qquad \le p\min\,(\gamma,\delta-1) + (1-p)\min\,(\gamma-1,\delta) + (1-p). \end{aligned} \tag{42}$$

Suppose now that $\alpha_1 = \beta_1 = 1$. Then,

$$\sum_{\nu=1}^{\infty} \alpha_\nu\beta_{\nu+1} \le \min\,(\gamma,\delta-1) \tag{43a}$$

and

$$\sum_{\nu=2}^{\infty} \alpha_\nu\beta_{\nu-1} \le \min\,(\gamma-1,\delta), \tag{43b}$$

so that (42) holds. Now suppose that precisely one of α_1 and β_1 is 1. Then, it is easy to see that at least one of (43a) and (43b) must hold;

moreover, the other one will also hold provided we delete the "-1" in its right hand side. Using the fact that $(1-p)\alpha_1\beta_1 = 0$ and that $p \le 1-p$ (since $p \le \frac{1}{2}$), it follows that (42) must hold. The remaining case is when $\alpha_1 = \beta_1 = 0$. Considerations as above show that if (43a) or (43b) holds, then (42) will follow. Hence, we may assume that neither (43a) nor (43b) holds, and it follows that

$$\sum_{\nu=1}^{\infty} \alpha_\nu \beta_{\nu+1} = \min(\gamma, \delta) = \sum_{\nu=2}^{\infty} \alpha_\nu \beta_{\nu-1}.$$

Assume now that $\gamma \le \delta$. It follows that whenever $\alpha_\nu = 1$ then both $\beta_{\nu-1}$ and $\beta_{\nu+1}$ must be 1. If this is to be the case, then we must in fact have $\gamma + 1 \le \delta$. In that case, the right hand side of (42) becomes

$$p\gamma + (1-p)(\gamma - 1) + (1-p) = \gamma.$$

Clearly, the left hand side of (42) is also γ, and so in this case we are done. Now, if $\gamma > \delta$, then the left side of (42) is δ, while the right hand side is

$$p(\delta - 1) + (1-p)\delta + (1-p) = \delta + 1 - 2p \ge \delta,$$

if $p \le \frac{1}{2}$. This completes the proof of (42) and hence also of (41).

Returning to (40), we now have

$$\sum_{\nu_1=1}^{\mu_1} B^*_{\nu_1} c_{\nu_1} \le \sum_{\nu_0=1}^{\infty} \sum_{\nu_1=1}^{\infty} A^*_{\nu_0} a_0(\nu_0, \nu_1) d^*_{\nu_1}.$$

Note that $s^{(1)}_{\nu_1}$ vanishes for all but at most μ_1 values of ν_1, and hence the range of summation of ν_1 on the right hand side of the above can be restricted to $\{1, \ldots, \mu_1\}$. I claim that $c_1 \ge c_2 \ge c_3 \ge \cdots$. For now, suppose that this claim is just. Then, since we always have $s^{(1)}_{\nu_1} \le 1$, it would necessarily follow that $d^*_{\nu_1} \le c_{\nu_1}$. Thus, we would have

$$\sum_{\nu_1=1}^{\mu_1} B^*_{\nu_1} c_{\nu_1} \le \sum_{\nu_0=1}^{\infty} \sum_{\nu_1=1}^{\mu_1} A^*_{\nu_0} a_1(\nu_0, \nu_1) c_{\nu_1}.$$

Then, (38) for n would follow by (39) and the definition of the c_{ν_1}.

It remains to prove the right monotonicity property of the c_{ν_1}. This will follow as soon as we show that whenever the a_i are defined as above and the ν_i are nonnegative, then, for every $m \ge 1$, the expression

$$\sum_{\nu_1=1}^{\mu_1} \cdots \sum_{\nu_m=1}^{\mu_m} a_0(\nu_0, \nu_1) \cdots a_{m-1}(\nu_{m-1}, \nu_m)$$

is decreasing in ν_0. We proceed by induction on m, and we shall prove the induction hypothesis and the induction step simultaneously. Suppose either that $m > 1$ and the result holds for $m-1$, or that $m = 1$. If $m > 1$, then let

$$\gamma_{\nu_1} = \sum_{\nu_2=1}^{\mu_2} \cdots \sum_{\nu_m=1}^{\mu_m} a_1(\nu_1, \nu_2) \cdots a_{m-1}(\nu_{m-1}, \nu_m).$$

The assumption that the result holds for $m - 1$ then shows that γ_{ν_1} is decreasing. If $m = 1$, then let $\gamma_{\nu_1} \equiv 1$, which is trivially decreasing.

What we must now show is that

$$\varepsilon_{\nu_0} \overset{\text{def}}{=} \sum_{\nu_1=1}^{\mu_1} a_0(\nu_0, \nu_1)\gamma_{\nu_1}$$

is decreasing in ν_0. If $p^{(0)} = -\infty$, then this is obvious. Otherwise, let $p = p^{(0)}$ and fix $\nu \geq 1$. We must show that $\varepsilon_\nu \geq \varepsilon_{\nu+1}$. Suppose first that $\nu = 1$. Then $\varepsilon_1 = (1-p)\gamma_1 + p\gamma_2$ and $\varepsilon_2 = (1-p)\gamma_1 + p\gamma_3$, so that $\varepsilon_1 \geq \varepsilon_2$ since $\gamma_2 \geq \gamma_3$. Now, suppose $\nu > 1$. Then $\varepsilon_\nu = (1-p)\gamma_{\nu-1} + p\gamma_{\nu+1}$ while $\varepsilon_{\nu+1} = (1-p)\gamma_\nu + p\gamma_{\nu+2}$, so that again the desired inequality holds because of the decreasing character of the γ_ν. This completes the simultaneous proof of both the induction hypothesis and the induction step, and hence gives a proof of the claim, so that we have finished proving the Theorem. □

As a corollary, we obtain a discrete Steiner rearrangement result in the case of a certain reflection symmetry. We work on the half lattice $\mathfrak{H} = \mathbb{Z} \times \mathbb{Z}^+ \subset \mathbb{C}$. Fix $p \in [0, \frac{1}{2}]$. Let $\mathfrak{r}_i$ be a random walk on $\mathfrak{H}$ with transition probabilities

$$P(\mathfrak{r}_{i+1} = \mathfrak{r}_i + (0,1) \mid \mathfrak{r}_i) = \frac{p}{2},$$

$$P(\mathfrak{r}_{i+1} = \mathfrak{r}_i + (1,0) \mid \mathfrak{r}_i) = P(\mathfrak{r}_{i+1} = \mathfrak{r}_i - (1,0) \mid \mathfrak{r}_i) = \frac{1}{4},$$

$$P(\mathfrak{r}_{i+1} = \mathfrak{r}_i - (0,1) \mid \mathfrak{r}_i = (x,y)) = \frac{1-p}{2}, \qquad \text{if } y > 1,$$

and

$$P(\mathfrak{r}_{i+1} = \mathfrak{r}_i \mid \mathfrak{r}_i = (x,1)) = \frac{1-p}{2}.$$

If $p = \frac{1}{2}$, then our random walk has equal probability of moving in any one of 4 directions at any time step, except that moving down from the line $\{(x,1) : x \in \mathbb{Z}\}$ is interpreted as staying put. If $p < \frac{1}{2}$, then we have much the same situation, except that the walk is biased to move towards the line $\{(x,1) : x \in \mathbb{Z}\}$.

Remark. – All the results below hold for some more general walks. In fact, we may choose any α_1, α_2, α_3 and α_4 in $[0,1]$ whose sum is 1 and which satisfy $\alpha_1 \leq \alpha_2$, and say that the probability of the random walk moving up, down, left or right in a time step is α_1, α_2, α_3 and α_4, respectively, with the usual condition that moving down from the line $\{(x,1) : x \in \mathbb{Z}\}$ results in staying put. In fact, we can even make α_1, α_2, α_3 and α_4 depend on the time i and even on the first coordinate Re $\mathfrak{r}_i$ of $\mathfrak{r}_i$. It is a very simple matter to adapt the work below to this situation and we shall say a few words about this after the proof of our main Steiner rearrangement result, Theorem 7, below.

For each $z \in \mathfrak{H}$, let $\mathfrak{s}_z$ represent a survival probability and be chosen in $[0,1]$. Fix $M \in \mathbb{Z}$. Let

$$\mathfrak{T}_M = \inf\,\{i \geq 0 : \text{Re}\ \mathfrak{r}_i > M\},$$

and set

$$\mathfrak{L}_\mathfrak{s} = \inf\,\{i \geq 0 : X_i > \mathfrak{s}_{\mathfrak{r}_i}\}.$$

We now write $P^z(\cdot)$ for probabilities under the conditioning that the random walk start at z, *i.e.*, that $\mathfrak{r}_0 = z$.

We define the (discrete and one-sided) Steiner rearrangement $\mathfrak{s}^*$ of $\mathfrak{s}$ by letting $\mathfrak{s}^*_{(x,1)}, \mathfrak{s}^*_{(x,2)}, \ldots$ be the decreasing rearrangement of $\mathfrak{s}_{(x,1)}, \mathfrak{s}_{(x,2)}, \ldots$ for each fixed $x \in \mathbb{Z}$.

THEOREM 7. – *Let $x \leq M$ be an integer, and let $\mathfrak{J}$ be any m-element subset of $\{x\} \times \mathbb{Z}^+$. For any $n \geq 0$ we then have*

$$\sum_{z \in \mathfrak{J}} P^z(\mathfrak{L}_\mathfrak{s} > n) \leq \sum_{y=1}^{m} P^{(x,y)}(\mathfrak{L}_{\mathfrak{s}^*} > n)$$

and

$$\sum_{z \in \mathfrak{J}} P^z(\mathfrak{L}_\mathfrak{s} \geq \mathfrak{T}_M) \leq \sum_{y=1}^{m} P^{(x,y)}(\mathfrak{L}_{\mathfrak{s}^*} \geq \mathfrak{T}_M).$$

If $\mathfrak{J} = \{(0,1)\}$, then the first inequality says that we survive for a longer amount of time if we apply Steiner rearrangement, and the second says that the probability of surviving until arrival in $\{z \in \mathfrak{H} : \text{Re}\ z > M\}$ is increased, too.

The particularly interesting case is when $\mathfrak{s}$ takes values only in $\{0,1\}$ and $p = \frac{1}{2}$. In that case, $P^z(\mathfrak{L}_\mathfrak{s} \geq \mathfrak{T}_M)$ may be interpreted as a discrete harmonic measure at z in $U = \{w \in \mathfrak{H} : \mathfrak{s}_w = 1\}$ of the line $\{(M+1, y) : (M, y) \in U\}$. In that case, $U^* \stackrel{\text{def}}{=} \{w \in \mathfrak{H} : \mathfrak{s}_w^* = 1\}$ is a discrete one-sided Steiner rearrangement of the set U. Note that

$$U^* = \{(x, y) \in \mathfrak{H} : y \leq \text{Card}\ \{y' : (x, y') \in U\}\}.$$

If $\mathfrak{s}$ is allowed to take values in all of $[0, 1]$, then we have a certain generalization of harmonic measure, where we have in effect allowed the edges of the domain U to be fuzzy so that we need no longer have a sharp boundary at which the probability of termination is exactly 1.

Then, in this case of $\mathfrak{s}$ having values in $\{0, 1\}$ and of $p = \frac{1}{2}$, Theorem 7 is a discrete equivalent of classical Steiner symmetrization theorems for harmonic measures and for exit times in the special case where the domains are *a priori* symmetric about the real axis. These symmetrization theorems (in general and not just in the case of this *a priori* symmetry) can be proved by the methods of Haliste [6, Thm. 8.1] (which is the approach we use in our case, and which approach was used by Borell [3] to prove the analogous results). The continuous analogue of the second inequality in Theorem 7 for $\mathfrak{s}_z \in \{0, 1\}$ and $p = \frac{1}{2}$ can also be proved by Baernstein's $*$-function method [2].

The reason why Theorem 7 is analogous to symmetrization results for domains of $\mathbb{C}$ which are symmetric about the real axis is that if $p = \frac{1}{2}$ then we could easily redefine our random walk to be on all of $\mathbb{Z}^2$, and set $\mathfrak{s}_{(x,1-y)} = \mathfrak{s}_{(x,y)}$ and $\mathfrak{s}^*_{(x,1-y)} = \mathfrak{s}^*_{(x,y)}$ for $(x, y) \in \mathfrak{H}$. Under this definition, $\mathfrak{s}$ is symmetric about the axis $\{z \in \mathbb{C} : \text{Im}\ z = \frac{1}{2}\}$.

OPEN PROBLEM 3. – Find discrete equivalents of Theorem 7 for Steiner symmetrization on $\mathbb{Z}^2$ in interesting cases where the symmetry described in the above paragraph is missing.

Remark. – In the case $p = \frac{1}{2}$, a full analogue of the second inequality of Theorem 7 for Steiner symmetrization on $\mathbb{Z}^2$ without any *a priori* symmetry assumptions on $\mathfrak{s}$ was recently obtained [10]. The method of proof was similar to that of Theorem 7, except that the random walk was modified by in effect introducing a geometric delay between each time step. In the special case where $\mathfrak{s}$ takes on only the values 0 and 1, the methods of Quine [11] based on a discrete version of Baernstein's $*$-function also yield the same analogue of the second inequality of Theorem 7.

Outline of proof of Theorem 7. – Our proof basically uses the methods of [6], again. Without loss of generality set $x = 0$. We shall throughout assume that all our random walks are conditioned to start on $\{z \in \mathfrak{H} : \text{Re}\ z = 0\}$. For random walks $\mathfrak{r}$ and $\mathfrak{r}'$, we say that $\mathfrak{r} \sim \mathfrak{r}'$ provided

Re $\mathfrak{r}_i$ = Re $\mathfrak{r}'_i$ for every $i \in \mathbb{Z}_0^+$. This defines an equivalence relation on the set of random walks on $\mathfrak{H}$, and we may then split up all probabilities occurring in Theorem 7 into weighted sums over the equivalence classes of random walks under $\sim$. (To do all this rigorously, one might have to first consider random walks of length $\leq N$ and then take $N \to \infty$.)

Let $\mathfrak{M}$ be any one of the equivalence classes under $\sim$. We shall show that

$$\sum_{z \in \mathfrak{Z}} P^z(\mathfrak{L}_\mathfrak{s} > n \mid \mathfrak{r} \in \mathfrak{M}) \leq \sum_{y=1}^{m} P^{(0,y)}(\mathfrak{L}_{\mathfrak{s}^*} > n \mid \mathfrak{r} \in \mathfrak{M}) \tag{44}$$

and

$$\sum_{z \in \mathfrak{Z}} P^z(\mathfrak{L}_\mathfrak{s} \geq \mathfrak{T}_M \mid \mathfrak{r} \in \mathfrak{M}) \leq \sum_{y=1}^{m} P^{(0,y)}(\mathfrak{L}_{\mathfrak{s}^*} \geq \mathfrak{T}_M \mid \mathfrak{r} \in \mathfrak{M}). \tag{45}$$

The desired result will follow from this.

Now, let x_i = Re $\mathfrak{r}_i$ for some $\mathfrak{r} \in \mathfrak{M}$. By choice of $\mathfrak{M}$, the x_i do not depend on which $\mathfrak{r} \in \mathfrak{M}$ was used to define them. From now on we shall always assume that $\mathfrak{r}$ is in $\mathfrak{M}$. Let $n' = \inf \{i \geq 0 : x_i > M\}$. Then, (45) reduces to (44) under the assumption that $n = n' - 1$. To prove (44) in general, we let R_i = Im $\mathfrak{r}_i$. If $x_{i+1} = x_i$ then let $p^{(i)} = p$; otherwise, let $p^{(i)} = -\infty$. It is easy to see that R_i then has the transition probabilities which were given at the beginning of the present section. Let $s_k^{(i)} = \mathfrak{s}_{(x_i,k)}$. Clearly $(s^{(i)})_k^* = \mathfrak{s}^*_{(x_i,k)}$. Moreover, if L_s is defined as before, then, conditioning on the statement that $\mathfrak{r} \in \mathfrak{M}$, we must have $L_s = \mathfrak{L}_\mathfrak{s}$ and $L_{s^*} = \mathfrak{L}_{\mathfrak{s}^*}$. Then, (44) follows from Theorem 6. □

Remark. – If we were working with transition probabilities defined by α_1, α_2, α_3 and α_4 as in a remark above, then we would let $p^{(i)} = \alpha_1/(\alpha_1 + \alpha_2)$ where we had let $p^{(i)} = p$ in the above proof. There is no additional difficulty with handling the possibility of the α_k depending on i and/or on x_i

Note added in proof. – In connection with the Remark following Problem 3, even more general symmetrization inequalities than those in [10] can be found in an another preprint of the author [*Symmetrization inequalities for difference equations on graphs*, 1996] and in Chapter II of the author's doctoral dissertation [*Symmetrization, Green's functions, harmonic measures and difference equations*, University of British Columbia, Vancouver, B.C., Canada, 1996].

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