ANNALES DE L'I. H. P., SECTION B

MICHAEL B. MARCUS

JAY ROSEN

Laws of the iterated logarithm for intersections of random walks on $\mathbb{Z}^4$

Annales de l'I. H. P., section B, tome 33, n° 1 (1997), p. 37-63

http://www.numdam.org/item?id=AIHPB_1997__33_1_37_0

© Gauthier-Villars, 1997, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Laws of the iterated logarithm for intersections of random walks on Z^4

by

Michael B. MARCUS*

Department of Mathematics, The City College of CUNY, New York, NY 10031.

and

Jay ROSEN†

Department of Mathematics, College of Staten Island, CUNY, Staten Island, NY 10301.

ABSTRACT. – Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ be two independent copies of a symmetric random walk in Z^4 with finite third moment. In this paper we study the asymptotics of I_n , the number of intersections up to step n of the paths of X and X' as $n \rightarrow \infty$. Our main result is

$$(1) \quad \limsup \frac{I_n}{\log(n) \log_3(n)} = \frac{1}{2\pi^2 |Q|^{1/2}} \quad \text{a.s.}$$

where Q denotes the covariance matrix of X_1 . A similar result holds for J_n , the number of points in the intersection of the ranges of X and X' up to step n .

* This research was supported, in part, by grants from the National Science Foundation, the Guggenheim Foundation, PSC-CUNY and an Scholar Incentive Award from The City College of CUNY. M. B. Marcus is grateful as well to Université Louis Pasteur and C.N.R.S., Strasbourg and the Statistical Laboratory and Clare Hall, Cambridge University for the support and hospitality he received while much of this work was carried out.

† This research was supported, in part, by grants from the National Science Foundation, PSC-CUNY and the Lady Davis Fellowship Trust. J. Rosen is grateful as well to the Institute of Mathematics of the Hebrew University, Jerusalem for the support and hospitality he received while much of this work was carried out.

RÉSUMÉ. – Soient $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ deux copies indépendantes d'une marche aléatoire symétrique dans Z^4 avec un moment d'ordre trois. Dans cet article, nous étudions le comportement asymptotique de I_n , le nombre de couples de temps d'intersection jusqu'au temps n des trajectoires de X et X' . Notre principal résultat donne

$$(1) \quad \limsup \frac{I_n}{\log(n) \log_3(n)} = \frac{1}{2\pi^2|Q|^{1/2}} \quad \text{p.s.}$$

où Q désigne la matrice de covariance de X_1 . Un résultat analogue est vrai pour J_n , le nombre de points d'intersection des trajectoires jusqu'au temps n .

1. INTRODUCTION

Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ be two independent copies of a symmetric random walk in Z^4 with finite variance. In this paper we study the asymptotics of the number of intersections up to step n of the paths of X and X' as $n \rightarrow \infty$, both the number of “intersection times”

$$(1.1) \quad I_n = \sum_{i=1}^n \sum_{j=1}^n 1_{\{X_i=X'_j\}}$$

and the number of “intersection points”

$$(1.2) \quad J_n = |X(1, n) \cap X'(1, n)|$$

where $X(1, n)$ denotes the range of X up to time n and $|A|$ denotes the cardinality of the set A . For random walks with finite variance, dimension four is the “critical case” for intersections, since $I_n, J_n \uparrow \infty$ almost surely but two independent Brownian motions in R^4 do not intersect.

We assume that X_n is adapted, which means that X_n does not live on any proper subgroup of Z^4 . In the terminology of Spitzer [7] X_n is aperiodic.

We have the following two limit theorems.

THEOREME 1. – Assume that $E(|X_1|^3) < \infty$. Then

$$(1.3) \quad \limsup_{n \rightarrow \infty} \frac{I_n}{\log(n) \log_3(n)} = \frac{1}{2\pi^2|Q|^{1/2}} \quad \text{a.s.}$$

where Q denotes the covariance matrix of X_1 .

As usual, $\log_j$ denotes the j -fold iterated logarithm.

In the particular case of the simple random walk on Z^4 , where $Q = \frac{1}{4}I$, Theorem 1 states that

$$(1.4) \quad \limsup_{n \rightarrow \infty} \frac{I_n}{\log(n) \log_3(n)} = \frac{8}{\pi^2} \quad \text{a.s.}$$

A similar result holds for J_n :

THEOREME 2. – Assume that $E(|X_1|^3) < \infty$. Then

$$(1.5) \quad \limsup_{n \rightarrow \infty} \frac{J_n}{\log(n) \log_3(n)} = \frac{q^2}{2\pi^2|Q|^{1/2}} \quad \text{a.s.}$$

where q denotes the probability that X will never return to its initial point.

Le Gall [2] proved that $(\log n)^{-1}J_n$ converges in distribution to the square of a normal random variable. In this paper we use some of the ideas of [2] along with techniques developed in [5], [6].

2. PROOF OF THEOREM 1

We use $p_n(x)$ to denote the transition function for X_n . Recall

$$(2.1) \quad \begin{aligned} I_n &= \sum_{i=1}^n \sum_{j=1}^n 1_{\{X_i = X'_j\}} \\ &= \sum_{x \in \mathbf{Z}^4} \left\{ \left(\sum_{i=1}^n 1_{\{X_i = x\}} \right) \left(\sum_{j=1}^n 1_{\{X'_j = x\}} \right) \right\}. \end{aligned}$$

We set

$$(2.2) \quad \begin{aligned} h(n) &= E(I_n) = \sum_{x \in \mathbf{Z}^4} \left\{ \left(\sum_{i=1}^n p_i(x) \right) \left(\sum_{j=1}^n p_j(x) \right) \right\} \\ &= \sum_{i=1}^n \sum_{j=1}^n p_{i+j}(0) \end{aligned}$$

where in the last step we used the fact that our random walk X is symmetric.

As shown in [7] the random walk X_n is adapted if and only if the origin is the unique element of T^4 satisfying $\phi(p) = 1$ where $\phi(p)$ is the characteristic function of X_1 and $T^4 = (-\pi, \pi]^4$ is the usual four

dimensional torus. We use τ to denote the number of elements in the set $\{p \in T^4 \mid |\phi(p)| = 1\}$. According to the local central limit theorem, *see e.g.* Prop. 2.4 of [3], we have that

$$p_j(0) = 0 \quad \text{if } j \not\equiv 0 \pmod{\tau}$$

while

$$(2.3) \quad p_{n\tau}(0) \sim \frac{1}{(2\pi)^{2\tau}|Q|^{1/2}} \frac{1}{n^2}$$

where Q denotes the covariance matrix of X_1 .

When $\tau = 1$ we see from (2.2) and (2.3) that

$$\begin{aligned} (2.4) \quad h(n) &= \sum_{i=1}^n \sum_{j=1}^n p_{i+j}(0) \\ &= \sum_{k=1}^n k p_k(0) + \sum_{k=n+1}^{2n} (2n-k) p_k(0) \\ &\sim \sum_{k=1}^n k p_k(0) \\ &\sim \frac{1}{(2\pi)^2 |Q|^{1/2}} \log n. \end{aligned}$$

The same sort of calculation shows that this holds in general:

$$\begin{aligned} (2.5) \quad h(n) &= \sum_{i=1}^n \sum_{j=1}^n p_{i+j}(0) \\ &\sim \sum_{m=0}^{\tau-1} \sum_{i=1}^{\lfloor n/\tau \rfloor} \sum_{j=1}^{\lfloor n/\tau \rfloor} p_{(i\tau+m)+(j\tau-m)}(0) \\ &\sim \tau \sum_{k=1}^{\lfloor n/\tau \rfloor} k p_{k\tau}(0) \\ &\sim \frac{1}{(2\pi)^2 |Q|^{1/2}} \log n. \end{aligned}$$

Thus the assertion of Theorem 1 can be written as

$$(2.6) \quad \limsup_{n \rightarrow \infty} \frac{I_n}{2h(n) \log_2 h(n)} = 1 \quad \text{a.s.}$$

We begin our proof with some moment calculations.

$$\begin{aligned}
 (2.7) \quad E(I_t^n) &= \sum_{x_1, \dots, x_n} \left\{ E \left(\prod_{i=1}^n \sum_{r_i=1}^t 1_{\{X_{r_i}=x_i\}} \right) \right\}^2 \\
 &= \sum_{x_1, \dots, x_n} \left\{ \sum_{\pi} \sum_{r_1 \leq r_2 \leq \dots \leq r_n \leq t} E \left(\prod_{i=1}^n 1_{\{X_{r_i}=x_{\pi(i)}\}} \right) \right\}^2 \\
 &= \sum_{x_1, \dots, x_n} \left(\sum_{\pi} \sum_{r_1 \leq r_2 \leq \dots \leq r_n \leq t} \prod_{i=1}^n p_{r_i - r_{i-1}}(x_{\pi(i)} - x_{\pi(i-1)}) \right)^2 \\
 &= n! \sum_{x_1, \dots, x_n} \left(\sum_{r_1 \leq r_2 \leq \dots \leq r_n \leq t} \prod_{i=1}^n p_{r_i - r_{i-1}}(x_i - x_{i-1}) \right) \\
 &\quad \left(\sum_{\pi} \sum_{s_1 \leq s_2 \leq \dots \leq s_n \leq t} \prod_{j=1}^n p_{s_j - s_{j-1}}(x_{\pi(j)} - x_{\pi(j-1)}) \right)
 \end{aligned}$$

where $\sum_{\pi}$ runs over the set of permutations π of $\{1, 2, \dots, n\}$. Set

$$u_t(x) = \sum_{r=1}^t p_r(x).$$

Then we see from (2.7) that

$$\begin{aligned}
 (2.8) \quad E(I_t^n) &\leq n! \sum_{x_1, \dots, x_n} \left(\prod_{i=1}^n u_t(x_i - x_{i-1}) \right) \\
 &\quad \left(\sum_{\pi} \prod_{j=1}^n u_t(x_{\pi(j)} - x_{\pi(j-1)}) \right),
 \end{aligned}$$

while

$$\begin{aligned}
 (2.9) \quad E(I_t^n) &\geq n! \sum_{x_1, \dots, x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1}) \right) \\
 &\quad \left(\sum_{\pi} \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right).
 \end{aligned}$$

We note here that by Lemma 5 of the Appendix we have

$$(2.10) \quad u_t(x) \leq \sum_{j=1}^{\infty} p_j(x) \leq \frac{C}{1 + |x|^2}.$$

On the other hand, using $E(|X_1|^3) < \infty$ we have

$$(2.11) \quad p_j(x) = P(X_j = x) \leq P(|X_j| \geq |x|) \leq \frac{Cj^3}{|x|^3}$$

so that

$$(2.12) \quad u_t(x) \leq C \frac{t^4}{|x|^3}$$

giving us the bound

$$(2.13) \quad u_t(x) \leq \frac{C}{1 + |x|^{5/2}} \quad \text{for all } |x| > t^8.$$

LEMMA 1. – For all integers $n, t \geq 0$ and for any $\epsilon > 0$

$$(2.14) \quad E(I_t^n) \leq (1 + \epsilon)(2n)!!h^n(t) + R(n, t)$$

where

$$(2.15) \quad 0 \leq R(n, t) \leq C(n!)^4 h^{n-1/2}(t)$$

Here $(2n)!! = \prod_{j=1}^n (2j-1)$ denotes the odd factorial.

Proof of Lemma 1. – We will make use of several ideas of Le Gall [2]. We begin by rewriting (2.8) as

$$(2.16) \quad E(I_t^n) \leq n! \sum_{y_1, \dots, y_n} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\sum_{\pi} \prod_{j=1}^n u_t(v_{\pi, j}) \right),$$

where $y_i = x_i - x_{i-1}$,

$$(2.17) \quad v_{\pi, j} = x_{\pi(j)} - x_{\pi(j-1)} = \sum_{k \in]\pi(j-1), \pi(j)]} y_j,$$

and (with a slight abuse of notation), $k \in]\pi(j-1), \pi(j)]$ means

$$k \in [\min(\pi(j-1), \pi(j)), \max(\pi(j-1), \pi(j))].$$

In view of (2.16), in order to prove our lemma it suffices to show that

$$(2.18) \quad \begin{aligned} n! \sum_{y_1, \dots, y_n} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\sum_{\pi} \prod_{j=1}^n u_t(v_{\pi, j}) \right) \\ = (1 + \epsilon)(2n)!!h^n(t) + R(n, t) \end{aligned}$$

with $R(n, t)$ as in (2.15). For each permutation σ of $\{1, 2, \dots, n\}$ we define

$$\Delta_\sigma = \{(y_1, \dots, y_n) \mid |y_{\sigma(1)}| \leq |y_{\sigma(2)}| \leq \dots \leq |y_{\sigma(n)}|\}$$

and rewrite the left hand side of (2.18) as

$$(2.19) \quad n! \sum_{\sigma, \pi} \sum_{\Delta_\sigma} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi, j}) \right).$$

Note that by (2.10)

$$(2.20) \quad \begin{aligned} & \sum_{y \leq |x| \leq 4y} (u_t(x))^2 \\ & \leq \sum_{y \leq |x| \leq 4y} C \frac{1}{1 + |x|^4} \\ & \leq C(\log 4y - \log y) = C \log(4) \end{aligned}$$

and that by (2.2)

$$(2.21) \quad \sum_x u_t^2(x) = h(t).$$

Let $A_{\sigma, k} = \{(y_1, \dots, y_n) \mid |y_{\sigma_{k-1}}| \leq |y_{\sigma_k}| \leq 4|y_{\sigma_{k-1}}|\}$. Using the Cauchy-Schwarz inequality we have

$$(2.22) \quad \begin{aligned} & \sum_{(y_1, \dots, y_n) \in A_{\sigma, k}} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi, j}) \right) \\ & \leq \left(\sum_{(y_1, \dots, y_n) \in A_{\sigma, k}} \prod_{i=1}^n (u_t(y_i))^2 \right)^{1/2} h^{n/2}(t) \\ & \leq C h^{n-1/2}(t). \end{aligned}$$

Set

$$\hat{\Delta}_\sigma = \{(y_1, \dots, y_n) \mid 4|y_{\sigma(k-1)}| < |y_{\sigma(k)}|, \forall k\}.$$

We see that the sum in (2.19) differs from the sum over $\hat{\Delta}_\sigma$ by an error term which can be incorporated into $R(n, t)$. Up to the error terms described above, we can write the sum in (2.19) as

$$(2.23) \quad n! \sum_{\sigma, \pi} \sum_{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi, j}) \right).$$

For given σ, π define the map $\phi = \phi_{\sigma, \pi} : \{1, 2, \dots, n\} \mapsto \{1, 2, \dots, n\}$ by

$$\phi(j) = \sigma(k_{\sigma, \pi, j}),$$

where

$$k_{\sigma, \pi, j} = \max\{k \mid \sigma(k) \in]\pi(j-1), \pi(j)]\}.$$

Note that on $\hat{\Delta}_\sigma$, $\phi(j)$ is the unique integer in $]\pi(j-1), \pi(j)]$ such that $|y_{\phi(j)}| = \sup_{k \in]\pi(j-1), \pi(j)]} |y_k|$. Furthermore, on $\hat{\Delta}_\sigma$, we see that $\frac{1}{2}|v_{\pi, j}| < |y_{\phi(j)}| < 2|v_{\pi, j}|$. Using the Cauchy-Schwarz inequality, and the bounds (2.10), (2.13) we have

$$\begin{aligned} (2.24) \quad & \sum_{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi, j}) \right) \\ & \leq \left(\sum_{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma} \prod_{j=1}^n (u_t(v_{\pi, j}))^2 \right)^{1/2} h^{n/2}(t) \\ & \leq \left(\sum_{\substack{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma \\ |v_{\pi, j}| \leq t^8, \forall j}} \prod_{j=1}^n (u_t(v_{\pi, j}))^2 \right)^{1/2} h^{n/2}(t) + Ch^{n-1/2}(t) \\ & \leq C \left(\sum_{\substack{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma \\ |y_j| \leq 2t^8, \forall j}} \prod_{j=1}^n \frac{1}{1 + |y_{\phi(j)}|^4} \right)^{1/2} h^{n/2}(t) + Ch^{n-1/2}(t). \end{aligned}$$

We now show that

$$(2.25) \quad \sum_{\substack{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma \\ |y_j| \leq 2t^8, \forall j}} \prod_{j=1}^n \frac{1}{1 + |y_{\phi(j)}|^4} \leq Ch^{n-1}(t)$$

unless $\phi = \phi_{\sigma, \pi} : \{1, 2, \dots, n\} \mapsto \{1, 2, \dots, n\}$ is bijective.

To begin, we note that by (2.17) both $\{y_j, j = 1, \dots, n\}$ and $\{v_{\pi, j}, j = 1, \dots, n\}$ generate $\{x_j, j = 1, \dots, n\}$ in the sense of linear combinations, so that both sets consist of n linearly independent vectors. Furthermore, from (2.17) we see that each $v_{\pi, j}$ is a sum of vectors from $\{y_j, j = 1, \dots, n\}$. However, from the definitions, we see that when we write out any vector in $\{v_{\pi, j} \mid k_{\sigma, \pi, j} \leq m\}$ as such a sum, the sum will only involve vectors from $\{y_{\sigma(j)} \mid j \leq m\}$. Hence $\{v_{\pi, j} \mid k_{\sigma, \pi, j} \leq m\}$ will contain at most m linearly independent vectors. Therefore, for each $m = 0, 1, \dots, n-1$, the set $\{v_{\pi, j} \mid k_{\sigma, \pi, j} > m\}$ will contain at least

$n - m$ elements. Equivalently, for each $m = 0, 1, \dots, n - 1$, the set $\{j \mid \sigma^{-1}\phi(j) > m\}$ will contain at least $n - m$ elements. This shows that for each $m = 0, 1, \dots, n - 1$, the product

$$\prod_{j=1}^n \frac{1}{1 + |y_{\phi(j)}|^4}$$

will contain at least $n - m$ factors of the form

$$\frac{1}{1 + |y_{\sigma(j)}|^4}$$

with $j > m$. We now return to (2.25) and sum in turn over the variables $y_{\sigma(n)}, y_{\sigma(n-1)}, \dots, y_{\sigma(1)}$ using the fact that

$$(2.26) \quad \sum_{\{y_{\sigma(j)} \in \mathbf{Z}^4 \mid 4|y_{\sigma(j-1)}| \leq |y_{\sigma(j)}| \leq t^8\}} \frac{1}{1 + |y_{\sigma(j)}|^4} \leq Ch(t)$$

while for any $k > 1$

$$(2.27) \quad \sum_{\{y_{\sigma(j)} \in \mathbf{Z}^4 \mid 4|y_{\sigma(j-1)}| \leq |y_{\sigma(j)}| \leq t^8\}} \frac{1}{1 + |y_{\sigma(j)}|^{4k}} \leq C \frac{1}{1 + |y_{\sigma(j-1)}|^{4(k-1)}}.$$

The above considerations show that as we sum successively over the variables $y_{\sigma(n)}, y_{\sigma(n-1)}, \dots, y_{\sigma(1)}$, at the stage when we sum over $y_{\sigma(j)}$, we will be summing a factor of the form $\frac{1}{1 + |y_{\sigma(j)}|^{4k}}$ for some $k \geq 1$, while if $\phi = \phi_{\sigma, \pi} : \{1, 2, \dots, n\} \mapsto \{1, 2, \dots, n\}$ is not bijective we must have $k > 1$ at some stage. These considerations, together with (2.26) and (2.27) establish (2.25).

Let Ω_n be the set of (σ, π) for which $\phi_{\sigma, \pi}$ is a bijection. Up to the error terms described above, we can write the sum in (2.23) as

$$(2.28) \quad n! \sum_{(\sigma, \pi) \in \Omega_n} \sum_{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi, j}) \right).$$

Since on $\hat{\Delta}_\sigma$, we have that $|y_{\phi(j)}| > 2|v_{\pi, j} - y_{\phi(j)}|$, we can then replace each occurrence of $v_{\pi, j}$ in (2.28) by $y_{\phi(j)}$, bounding the error terms using

$$(2.29) \quad \sum_{\{|x| > 2|a|\}} (u_t(x + a) - u_t(x))^2 \\ \leq C \sum_{\{|x| > 2|a|\}} \left(\frac{|a|^2}{1 + |x|^6} + \frac{1}{1 + |x|^5} \right) \leq C$$

which comes from (2.13) and Lemma 6 of the Appendix.

Thus, up to error terms described which can be incorporated into $R(n, t)$, we can write the sum in (2.28) as

$$(2.30) \quad n! \sum_{(\sigma, \pi) \in \Omega_n} \sum_{(y_1, \dots, y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t^2(y_i) \right).$$

Proceeding as above, up to the error terms described above, we can replace (2.30) by

$$(2.31) \quad n! \sum_{(\sigma, \pi) \in \Omega_n} \sum_{(y_1, \dots, y_n) \in \Delta_\sigma} \left(\prod_{i=1}^n u_t^2(y_i) \right).$$

Since

$$n! \sum_{(y_1, \dots, y_n) \in \Delta_\sigma} \left(\prod_{i=1}^n u_t^2(y_i) \right) \sim h^n(t),$$

and as by the remark following Lemma 2.5 of [2] we have $|\Omega_n| = (2n)!!$, the lemma is proved. $\square$

We will use $E^{v,w}$ to denote expectation with respect to the random walks X, X' where $X_0 = v$ and $X'_0 = w$. We define

$$(2.32) \quad a(v, w, t) = \frac{h(v, w, t)}{h(t)}$$

where

$$(2.33) \quad \begin{aligned} h(v, w, t) &= E^{v,w}(I_t) \\ &= \sum_{x \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^t p_i(x - v) \right) \left(\sum_{j=1}^t p_j(x - w) \right) \right\} \\ &= \sum_{i,j=1}^t p_{i+j}(v - w). \end{aligned}$$

We will need the following lower bound.

LEMMA 2. – For all integers $n, t \geq 0$ and for any $\epsilon > 0$

$$(2.34) \quad E^{v,w}(I_t^n) \geq (1 - \epsilon)(2n)!! a(v, w, t/n) h^n(t/n) - R'(n, t)$$

where

$$(2.35) \quad 0 \leq R'(n, t) \leq C(n!)^4 h^{n-1/2}(t).$$

Proof of Lemma 2. – We first note that as in (2.9)

$$(2.36) \quad E^{v,w}(I_t^n) \geq n! \sum_{x_1, \dots, x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1}) \right) \left(\sum_{\pi} \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right)$$

where now we use the convention $x_0 = v, x_{\pi(0)} = w$. We then use (2.18), observing that if $\phi_{\sigma, \pi}$ is bijective we must have $\phi_{\sigma, \pi}(j) = 1$ for some j and this must be $j = 1$ since $1 \in]\pi(j-1), \pi(j)]$ is possible only for $j = 1$. Thus, $v_{\pi,1}$ is replaced in (2.23) by y_1 . $\square$

LEMMA 3. – For all $t \geq 0$ and $x = O(\log \log h(t))$ we have

$$(2.37) \quad P\left(\frac{I_t}{2h(t)} \geq x\right) \leq C\sqrt{x}e^{-x}.$$

Proof of Lemma 3. – We first note that if $n = O(\log \log h(t))$ then

$$(2.38) \quad \frac{(n!)^4}{h^{1/2}(t)} \rightarrow 0$$

as $t \rightarrow \infty$, so that by Lemma 1 we have

$$(2.39) \quad E(I_t^n) \leq C(2n)!!h^n(t).$$

Then Chebyshev's inequality gives us

$$(2.40) \quad P\left(\frac{I_t}{2h(t)} \geq x\right) \leq C \frac{(2n)!!}{(2x)^n} = C \frac{\sqrt{n}n^n e^{-n}}{x^n} (1 + O(1/n))$$

for any $n = O(\log \log h(t))$. Taking $n = [x]$ then yields (2.37). $\square$

LEMMA 4. – For all $\epsilon > 0$ there exists an x_0 and a $t' = t'(\epsilon, x_0)$ such that for all $t \geq t'$ and $x_0 \leq x = O(\log \log h(t))$ we have

$$(2.41) \quad P\left(\frac{I_t}{2h(t)} \geq (1 - \epsilon)x\right) \geq C_\epsilon e^{-x}$$

and

$$(2.42) \quad P^{v,w}\left(\frac{I_t}{2h(t)} \geq (1 - \epsilon)x\right) \geq C_\epsilon(a(v, w, 2t/(3x))e^{-x} - e^{-(1+\epsilon')x})$$

for some $\epsilon' > 0$.

Proof of Lemma 4. – This follows from Lemmas 2, 3 and (2.38) by the methods used in the proof of Lemma 2.7 in [5]. $\square$

Proof of Theorem. 1. – For $\theta > 1$ we define the sequence $\{t_n\}$ by

$$(2.43) \quad h(t_n) = \theta^n.$$

By Lemma 3 we have that for all integers $n \geq 2$ and all $\epsilon > 0$

$$(2.44) \quad P\left(\frac{I_{t_n}}{2h(t_n) \log \log h(t_n)} \geq (1 + \epsilon)\right) \leq C e^{-(1+\epsilon) \log n}.$$

Therefore, by the Borel-Cantelli lemma

$$(2.45) \quad \limsup_{n \rightarrow \infty} \frac{I_{t_n}}{2h(t_n) \log \log h(t_n)} \leq 1 + \epsilon \quad \text{a.s.}$$

By taking θ arbitrarily close to 1 it is simple to interpolate in (2.45) to obtain

$$(2.46) \quad \limsup_{n \rightarrow \infty} \frac{I_n}{2h(n) \log \log h(n)} \leq 1 + \epsilon \quad \text{a.s.}$$

We now show that for any $\epsilon > 0$

$$(2.47) \quad \limsup_{n \rightarrow \infty} \frac{I_{t_n}}{2h(t_n) \log \log h(t_n)} \geq 1 - \epsilon \quad \text{a.s.}$$

for all θ sufficiently large. It is sufficient to show that

$$(2.48) \quad \limsup_{n \rightarrow \infty} \frac{I_{t_n} - I_{t_{n-1}}}{2h(t_n) \log \log h(t_n)} \geq 1 - \epsilon \quad \text{a.s.}$$

Let $s_n = t_n - t_{n-1}$ and note that, as in (2.60) of [5], we have $h(s_n) \sim h(t_n)$.

We also note that

$$(2.49) \quad |I_{t_n} - I_{t_{n-1}} - I_{s_n} \circ \Theta_{t_{n-1}}| \leq I_{t_n, t_{n-1}} + I_{t_{n-1}, t_n}$$

where

$$(2.50) \quad I_{n,m} = \sum_{x \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^n 1_{\{X_i=x\}} \right) \left(\sum_{j=1}^m 1_{\{X'_j=x\}} \right) \right\}.$$

As in Lemma 1, we can show that for $t \geq t'$, and for all integers $n \geq 0$ and any $\epsilon > 0$

$$(2.51) \quad \begin{aligned} E(I_{t,t'}^n) &\leq (1 + \epsilon)(2n)!! h^{n/2}(t) h^{n/2}(t') \\ &\quad + O((n!)^4 h^{n/2}(t) h^{n/2-1/2}(t')) \end{aligned}$$

which, as before, leads to

$$\begin{aligned}
 (2.52) \quad & \limsup_{n \rightarrow \infty} \frac{I_{t_n, t_{n-1}}}{2h(t_n) \log \log h(t_n)} \\
 &= \limsup_{n \rightarrow \infty} \frac{I_{t_n, t_{n-1}}}{2\sqrt{\theta h(t_n)h(t_{n-1})} \log \log h(t_n)} \\
 &\leq \frac{1+\epsilon}{\sqrt{\theta}} \quad \text{a.s.}
 \end{aligned}$$

Using this for θ large, (2.49), Levy's Borel-Cantelli lemma (see Corollary 5.29 in [1]) and the Markov property, we see that (2.48) will follow from

$$(2.53) \quad \sum_{n=1}^{\infty} P^{X_{t_{n-1}}, X'_{t_{n-1}}} \left(\frac{I_{s_n}}{2h(s_n) \log \log h(s_n)} \geq 1 - \epsilon \right) = \infty \quad \text{a.s.}$$

If we apply Lemma 4 with $t = s_n$ and $x = \log \log s_n$ we see that (2.53) will follow from

$$(2.54) \quad \sum_{n=1}^{\infty} a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n) \frac{1}{n^{1-\epsilon'}} = \infty \quad \text{a.s.}$$

We begin by showing

$$(2.55) \quad \sum_{n=1}^{\infty} E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n)) \frac{1}{n^{1-\epsilon'}} = \infty.$$

To see this we note that

$$\begin{aligned}
 (2.56) \quad & E(a(X_t, X'_t, k)) \\
 &= \frac{\sum_{x \in \mathbb{Z}^d} \left\{ \left(\sum_{i=1}^k p_{i+t}(x) \right) \left(\sum_{j=1}^k p_{j+t}(x) \right) \right\}}{h(k)}
 \end{aligned}$$

so that

$$\begin{aligned}
 (2.57) \quad & E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n)) \\
 &= \frac{\sum_{x \in \mathbb{Z}^d} \left\{ \left(\sum_{i=1}^{s_n / \log n} p_{i+t_{n-1}}(x) \right) \left(\sum_{j=1}^{s_n / \log n} p_{j+t_{n-1}}(x) \right) \right\}}{h(s_n / \log n)} \\
 &= \frac{h(t_{n-1} + s_n / \log n) - h(t_{n-1})}{h(s_n / \log n)} \\
 &\quad - \frac{2 \sum_{x \in \mathbb{Z}^d} \left\{ \left(\sum_{i=1}^{t_{n-1}} p_i(x) \right) \left(\sum_{j=1}^{s_n / \log n} p_{j+t_{n-1}}(x) \right) \right\}}{h(s_n / \log n)}.
 \end{aligned}$$

Also note that

$$(2.58) \quad \frac{h(t_{n-1} + s_n / \log n) - h(t_{n-1})}{h(s_n / \log n)} \geq \frac{h(s_n / \log n) - h(t_{n-1})}{h(s_n / \log n)} \sim 1 - \frac{1}{\theta}.$$

This follows fairly easily since $h(t) \sim c \log(t)$. (For the details, in a more general setting, see the proof of Theorem 1.1 of [5], especially that part of the proof surrounding (2.82)). Furthermore, we have by the Cauchy-Schwarz inequality

$$(2.59) \quad \frac{\sum_{x \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{t_{n-1}} p_i(x) \right) \left(\sum_{j=1}^{s_n / \log n} p_{j+t_{n-1}}(x) \right) \right\}}{h(s_n / \log n)} \leq \frac{\sqrt{h(t_{n-1})h(t_n)}}{h(s_n / \log n)} \sim \frac{1}{\sqrt{\theta}}.$$

Taking θ large establishes 2.55.

Furthermore, since $a(v, w, t) \leq 1$ (compare (2.4) and (2.33)), we see that for any $\epsilon' < 1/2$

$$(2.60) \quad \sum_{n=1}^{\infty} E \left(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n) \frac{1}{n^{1-\epsilon'}} \right)^2 < \infty.$$

(2.54) will now follow from the Paley-Zygmund lemma once we show that

$$(2.61) \quad \frac{E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n) a(X_{t_{m-1}}, X'_{t_{m-1}}, s_m / \log m))}{E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n / \log n)) E(a(X_{t_{m-1}}, X'_{t_{m-1}}, s_m / \log m))} \leq 1 + 2\epsilon$$

for all $\epsilon > 0$, when $n > m \geq N(\epsilon)$ for some $N(\epsilon)$ sufficiently large. To prove (2.61) we begin by noting that as in (2.56)

$$(2.62) \quad \begin{aligned} E(h(X_t, X'_t, s)) &= \sum_{x \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^s p_{i+t}(x) \right) \left(\sum_{j=1}^s p_{j+t}(x) \right) \right\} \\ &= \sum_{i,j=1}^s p_{i+j+2t}(0) \end{aligned}$$

and for $t' < t$

$$\begin{aligned}
 (2.63) \quad & E(h(X_{t'}, X'_{t'}, s')h(X_t, X'_t, s)) \\
 &= \sum_{x, y, x', y'} h(x, x', s')p_{t'}(x)p_{t'}(x')h(y, y', s)p_{t-t'}(y-x)p_{t-t'}(y'-x') \\
 &= \sum_{x, x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \\
 &\quad \cdot \sum_{u \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^s p_{i+t-t'}(u-x) \right) \left(\sum_{j=1}^s p_{j+t-t'}(u-x') \right) \right\} \\
 &= \sum_{x, x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \sum_{i,j=1}^s p_{i+j+2(t-t')}(x-x') \\
 &\leq \sum_{x, x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \sum_{i,j=1}^s p_{i+j+2(t-t')}(0) \\
 &= \sum_{x \in \mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{s'} p_{i+t'}(x) \right) \left(\sum_{j=1}^{s'} p_{j+t'}(x) \right) \right\} \sum_{i,j=1}^s p_{i+j+2(t-t')}(0) \\
 &= \sum_{i,j=1}^{s'} p_{i+j+2t'}(0) \sum_{i,j=1}^s p_{i+j+2(t-t')}(0).
 \end{aligned}$$

From (2.62), (2.63) we see that

$$\begin{aligned}
 (2.64) \quad & \frac{E(h(X_{t'}, X'_{t'}, s')h(X_t, X'_t, s))}{E(h(X_{t'}, X'_{t'}, s'))E(h(X_t, X'_t, s))} \\
 & \leq \frac{\sum_{i=1}^s \sum_{j=1}^s p_{i+j+2(t-t')}(0)}{\sum_{i=1}^s \sum_{j=1}^s p_{i+j+2t}(0)}.
 \end{aligned}$$

Now let us assume that $t - t' > (1 - \epsilon)t$. (This will certainly hold in our case where $t = t_{n-1}, t' = t_{m-1}$ with $m < n$). Then $i + j + 2(t - t') > (1 - \epsilon)(i + j + 2t)$. Assume first that $\tau = 1$. Since by (2.3) we have that $p_{\cdot}(0)$ is regularly varying at infinity of order -2 , we see that if t is sufficiently large, then

$$(2.65) \quad p_{i+j+2(t-t')}(0) \leq (1 + 2\epsilon)p_{i+j+2t}(0)$$

so that (2.64) is $\leq 1 + 2\epsilon$. This completes the proof of (2.61) when $\tau = 1$. The general case is easily handled if instead of t_n we work with $t'_n \sim t_n$ satisfying $t'_n = 0 \pmod{\tau}$. This completes the proof of Theorem 1. $\square$

3. PROOF OF THEOREM 2

We begin with some moment calculations. Recall

$$(3.1) \quad \begin{aligned} J_n &= |X(1, n) \cap X'(1, n)| \\ &= \sum_{x \in \mathbf{Z}^4} 1_{\{x \in X(1, n)\}} 1_{\{x \in X'(1, n)\}}. \end{aligned}$$

As usual set

$$T_x = \inf\{k \mid X_k = x\},$$

and note that

$$(3.2) \quad \begin{aligned} E(J_t^n) &= E\left\{\left(\sum_x 1_{\{x \in X(1, t)\}} 1_{\{x \in X'(1, t)\}}\right)^n\right\} \\ &= \sum_{x_1, \dots, x_n} E\left(\prod_{i=1}^n 1_{\{x_i \in X(1, t)\}} 1_{\{x_i \in X'(1, t)\}}\right) \\ &= \sum_{x_1, \dots, x_n} \left\{E\left(\prod_{i=1}^n 1_{\{x_i \in X(1, t)\}}\right)\right\}^2 \\ &\leq \sum_{x_1, \dots, x_n} \left\{\sum_{\pi} P(T_{x_{\pi(1)}} \leq T_{x_{\pi(2)}} \leq \dots \leq T_{x_{\pi(n)}} \leq t)\right\}^2 \\ &= n! \sum_{x_1, \dots, x_n} (P(T_{x_1} \leq T_{x_2} \leq \dots \leq T_{x_n} \leq t)) \\ &\quad \cdot \left(\sum_{\pi} P(T_{x_{\pi(1)}} \leq T_{x_{\pi(2)}} \leq \dots \leq T_{x_{\pi(n)}} \leq t)\right) \end{aligned}$$

where $\sum_{\pi}$ runs over the set of permutations π of $\{1, 2, \dots, n\}$. Set

$$v_t(x) = P(T_x \leq t).$$

Then we see from 3.2 that

$$(3.3) \quad \begin{aligned} E(J_t^n) &\leq n! \sum_{x_1, \dots, x_n} \left(\prod_{i=1}^n v_t(x_i - x_{i-1})\right) \\ &\quad \left(\sum_{\pi} \prod_{j=1}^n v_t(x_{\pi(j)} - x_{\pi(j-1)})\right), \end{aligned}$$

while

$$(3.4) \quad E(J_t^n) \geq n! \sum_{\substack{x_1, x_2, \dots, x_n \\ \text{distinct}}} \left(\prod_{i=1}^n v_{t/n}(x_i - x_{i-1}) \right) \\ \left(\sum_{\pi} \prod_{j=1}^n v_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right).$$

Here we used the fact that the inequality in 3.2 is due to the possible double counting if $x_i = x_j$ for some i, j .

Let

$$f_r(x) = P(T_x = r)$$

so that

$$v_t(x) = \sum_{r=1}^t f_r(x).$$

We have

$$(3.5) \quad p_j(x) = \sum_{i=1}^j f_i(x) p_{j-i}(0)$$

where as usual we set $p_0(x) = 1_{\{x=0\}}$. From this we see that

$$(3.6) \quad u_t(x) = \sum_{j=1}^t p_j(x) \\ = \sum_{j=1}^t \sum_{i=1}^j f_i(x) p_{j-i}(0) \\ = \sum_{i=1}^t \sum_{j=i}^t f_i(x) p_{j-i}(0) \\ = \sum_{i=1}^t f_i(x) (1 + u_{t-i}(0)).$$

Consequently we have

$$(3.7) \quad u_t(x) \leq v_t(x)(1 + u_t(0))$$

and

$$(3.8) \quad u_{2t}(x) \geq v_t(x)(1 + u_t(0)).$$

Now it is well known that

$$(3.9) \quad \frac{1}{1 + u_t(0)} \downarrow q$$

so that for any $\epsilon > 0$ we can find $t_0 < \infty$ such that

$$(3.10) \quad qu_t(x) \leq v_t(x) \leq (q + \epsilon)u_{2t}(x)$$

for all $t \geq t_0$ and x . Hence (3.3) and (3.4) give us

$$(3.11) \quad E(J_t^n) \leq (q + \epsilon)^{2n} n! \sum_{x_1, \dots, x_n} \left(\prod_{i=1}^n u_{2t}(x_i - x_{i-1}) \right) \\ \left(\sum_{\pi} \prod_{j=1}^n u_{2t}(x_{\pi(j)} - x_{\pi(j-1)}) \right),$$

and

$$(3.12) \quad E(J_t^n) \geq q^{2n} n! \sum_{\substack{x_1, \dots, x_n \\ \text{distinct}}} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1}) \right) \\ \left(\sum_{\pi} \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right).$$

The proof of Theorem 2 now follows exactly along the lines of the proof of Theorem 1. $\square$

4. APPENDIX

LEMMA 5. — *Let X_n be a mean-zero adapted random walk in Z^4 . Assume that $E(|X_1|^2 \log_+ |X_1|) < \infty$. Then for some $C < \infty$*

$$(4.1) \quad u(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} p_n(x) \leq \frac{C}{1 + |x|^2}$$

for all x .

In the proof of Lemma 5 we actually show that

$$(4.2) \quad |u(x) - G(x)| = o(1/|x|^2)$$

where $G(x)$ is the Green's function of the non-isotropic Brownian motion in R^4 with covariance matrix equal to that of X_1 .

In a recent paper [4], Lawler shows that 4.1 does not hold for all mean zero finite variance random walks. He also proves Lemma 5. We present here a different proof of Lemma 5 because our method of proof will be used, in Lemma 6, to obtain a bound for $|G(x+a) - G(x)|$.

Proof of Lemma 5. – Let

$$\phi(p) = E(e^{ipX_1})$$

denote the characteristic function of X_1 . We have

$$(4.3) \quad u(x) = \frac{1}{(2\pi)^2} \int_{[-\pi, \pi]^4} \frac{e^{ipx}}{1 - \phi(p)} dp.$$

Let $Q = \{Q_{i,j}\}$ denote the covariance matrix of $X_1 = (X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)})$, i.e. $Q_{i,j} = E(X_1^{(i)} X_1^{(j)})$. We write

$$Q(p) = \frac{1}{2} \sum_{i,j=1}^4 Q_{i,j} p_i p_j$$

for $p \in [-\pi, \pi]^4$. Let $q_t(x)$ denote the transition density for Brownian motion in R^4 and set

$$(4.4) \quad v_\delta(x) = \int_\delta^\infty q_t(x) dt = \frac{1}{(2\pi)^2} \int_{R^4} e^{ipx} \frac{e^{-\delta|p|^2/2}}{|p|^2/2} dp.$$

We have

$$(4.5) \quad \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2} \int_{R^4} e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp.$$

Note that

$$(4.6) \quad v_\delta(x) \uparrow v_0(x) = \int_0^\infty q_t(x) dt = \frac{1}{(2\pi)^2 |x|^2}$$

as $\delta \rightarrow 0$ and thus to prove (4.1) it suffices to show that

$$(4.7) \quad \lim_{\delta \rightarrow 0} \left| u(x) - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} \right| \leq \frac{c}{|x|^2}.$$

If $x = (x_1, x_2, x_3, x_4)$, we can assume, without loss of generality, that $|x| \neq 0$ and that $|x_1| = \max_j |x_j|$. We have

$$(4.8) \quad \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2} \int_A e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp \\ + \frac{1}{(2\pi)^2} \int_B e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp + \frac{1}{(2\pi)^2} \int_C e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp$$

where $A = [-\pi, \pi]^4$, $B = [-\pi, \pi]^c \times [-\pi, \pi]^3$, and $C = R \times ([-\pi, \pi]^3)^c$. Note that

$$(4.9) \quad C = \bigcup_{j=2}^4 \{|p_j| > \pi\}.$$

We have

$$(4.10) \quad u(x) - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2} \int_A e^{ipx} \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ - \frac{1}{(2\pi)^2} \int_B e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp - \frac{1}{(2\pi)^2} \int_C e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp.$$

We first show that

$$(4.11) \quad \lim_{\delta \rightarrow 0} \left| \frac{1}{(2\pi)^2} \int_C e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp \right| \leq \frac{c}{|x|^3}.$$

To see this we integrate by parts three times in the p_1 direction to see that

$$(4.12) \quad \frac{1}{(2\pi)^2} \int_C e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp = \frac{i^3}{x_1^3} \frac{1}{(2\pi)^2} \int_C e^{ipx} D_1^3 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp$$

and

$$(4.13) \quad D_1^3 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) = D_1^3(e^{-\delta Q(p)}) \frac{1}{Q(p)} + 3D_1^2(e^{-\delta Q(p)}) D_1^1 \left(\frac{1}{Q(p)} \right) \\ + 3D_1(e^{-\delta Q(p)}) D_1^2 \left(\frac{1}{Q(p)} \right) + e^{-\delta Q(p)} D_1^3 \left(\frac{1}{Q(p)} \right)$$

Note that $\inf_{p \in B \cup C} Q(p) \geq d > 0$. Also, $D_1^j(\frac{1}{Q(p)})$ is homogeneous in p of degree $-(2+j)$, so that the last term in (4.13) is integrable on C even when we take $\delta = 0$. Since

$$(4.14) \quad D_1(e^{-\delta Q(p)}) = -\delta Q_1(p) e^{-\delta Q(p)}$$

and $Q_1(p)D_1^2\left(\frac{1}{Q(p)}\right)$ is homogeneous in p of degree -3 , scaling out δ shows that the integral of the absolute value of the third term in (4.13) is bounded by

$$(4.15) \quad \delta^{1/2} \int \frac{e^{-Q(p)}}{|p|^3} dp \leq c\delta^{1/2}.$$

The first two terms in (4.13) are handled similarly, proving (4.11).

We next integrate the first two terms in (4.10), by parts, twice in the p_1 direction, to get

$$(4.16) \quad \begin{aligned} & \frac{1}{(2\pi)^2} \int_A e^{ipx} \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp - \frac{1}{(2\pi)^2} \int_B e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)} dp \\ &= \frac{i^2}{x_1^2 (2\pi)^2} \int_A e^{ipx} D_1^2 \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & \quad - \frac{i^2}{x_1^2 (2\pi)^2} \int_B e^{ipx} D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \end{aligned}$$

where we have used the fact that the boundary terms coming from the integrals over A and B cancel. (These boundary terms are easily seen to be finite). Arguing as in the proof of 4.11) we see that

$$(4.17) \quad \lim_{\delta \rightarrow 0} \left| \frac{1}{(2\pi)^2} \int_B e^{ipx} D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \right| \leq c.$$

(In fact, a further integration by parts shows that

$$(4.18) \quad \lim_{\delta \rightarrow 0} \left| \frac{1}{(2\pi)^2} \int_B e^{ipx} D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \right| \leq c/x_1$$

as in the proof of (4.11).)

We now write

$$(4.19) \quad \begin{aligned} & D_1^2 \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) \\ &= D_1^2 \left(\frac{1}{1-\phi(p)} - \frac{1}{Q(p)} \right) + (1 - e^{-\delta Q(p)}) D_1^2 \left(\frac{1}{Q(p)} \right) \\ & \quad - 2D_1(e^{-\delta Q(p)}) D_1 \left(\frac{1}{Q(p)} \right) - D_1^2(e^{-\delta Q(p)}) \frac{1}{Q(p)}. \end{aligned}$$

As before, we see that the last three terms in (4.19) give rise to bounded integrals over A . (In fact, they vanish as $\delta \rightarrow 0$). More care will be needed to handle the first term

$$(4.20) \quad D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{1}{Q(p)} \right) = \left(\frac{\phi_{1,1}(p)}{(1 - \phi(p))^2} + \frac{Q_{1,1}}{(Q(p))^2} \right) + 2 \left(\frac{(\phi_1(p))^2}{(1 - \phi(p))^3} - \frac{(Q_1(p))^2}{(Q(p))^3} \right).$$

We write out the first term on the right hand side of (4.20) as

$$(4.21) \quad \begin{aligned} & \left(\frac{\phi_{1,1}(p)}{(1 - \phi(p))^2} + \frac{Q_{1,1}}{(Q(p))^2} \right) \\ &= \frac{\phi_{1,1}(p)(Q(p))^2 + Q_{1,1}(1 - \phi(p))^2}{(Q(p))^2(1 - \phi(p))^2} \\ &= \frac{(\phi_{1,1}(p) + Q_{1,1})(Q(p))^2}{(Q(p))^2(1 - \phi(p))^2} \\ &+ \frac{Q_{1,1}((1 - \phi(p))^2 - (Q(p))^2)}{(Q(p))^2(1 - \phi(p))^2}. \end{aligned}$$

Observe that for $|p| \leq 1$

$$(4.22) \quad \begin{aligned} & |1 - \phi(p) - Q(p)| \\ &= |E(1 - e^{ip \cdot X} + ip \cdot X + (1/2)(ip \cdot X)^2)| \\ &\leq c|p|^3 E(1_{\{|X| \leq 1/|p|\}} |X|^3) + c|p|^2 E(1_{\{|X| > 1/|p|\}} |X|^2) \\ &\leq c|p|^2 / \log_+(1/|p|) = o(|p|^2). \end{aligned}$$

Hence, we can bound (4.21) by

$$(4.23) \quad \frac{c|\phi_{1,1}(p) + Q_{1,1}|}{|p|^4} + \frac{c|1 - \phi(p) - Q(p)|}{|p|^6}.$$

Using the second line of (4.22) we see that

$$(4.24) \quad \begin{aligned} & \int_{|p| \leq 1} \frac{|1 - \phi(p) - Q(p)|}{|p|^6} dp \\ &\leq cE \left(\left(\int_{\{|p| \leq 1/|X|\}} \frac{1}{|p|^3} dp \right) |X|^3 \right) \\ &\quad + cE \left(\left(\int_{\{|p| > 1/|X|\}} \frac{1}{|p|^4} dp \right) |X|^2 \right) \\ &\leq cE(|X|^2 \log_+ |X|) < \infty. \end{aligned}$$

Similarly, we see that

$$(4.25) \quad |\phi_{1,1}(p) + Q_{1,1}| = |E(-X_1^2(e^{ip \cdot X} - 1))| \\ \leq c|p|E(1_{\{|X| \leq 1/|p|\}}|X|^3) + cE(1_{\{|X| > 1/|p|\}}|X|^2)$$

and using this as in (4.24) we see that

$$(4.26) \quad \int_{|p| \leq 1} \frac{|\phi_{1,1}(p) + Q_{1,1}|}{|p|^4} dp < \infty$$

The same methods apply to the second term on the right hand side of (4.20), completing the proof of the lemma.

Remark 1. – As $\delta \rightarrow 0$ we see that

$$(4.27) \quad u(x) - \frac{1}{(2\pi)^2 |Q|^{1/2} (x \cdot Q^{-1}x)} \\ = \frac{i^2}{x_1^2 (2\pi)^2} \int_A e^{ipx} D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{1}{Q(p)} \right) dp + O(1/|x|^3)$$

which together with the Riemann-Lebesgue lemma establishes 4.2.

LEMMA 6. – *Let X be a mean-zero adapted random walk in Z^4 . Assume that $E(|X_1|^3) < \infty$. Then for some $C < \infty$*

$$(4.28) \quad |u(x+a) - u(x)| \leq \frac{C|a|}{1 + |x|^3},$$

for all a, x satisfying $|a| < |x|/8$.

Furthermore, for some $C < \infty$

$$(4.29) \quad |u_t(x+a) - u_t(x)| \leq \frac{C|a|}{1 + |x|^3},$$

for all a, x, t satisfying $|a| < |x|/8$ and $|x|^{1/8} < t$.

Proof of Lemma 6. – As in the proof of the previous lemma we may assume that $|x_1| = \max_j |x_j|$ and we have

$$(4.30) \quad u(x+a) - u(x) - \left(\frac{v_\delta(Q^{-1/2}(x+a))}{|Q|^{1/2}} - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} \right) \\ = \frac{1}{(2\pi)^2} \int_A (e^{ip(x+a)} - e^{ipx}) \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ - \frac{1}{(2\pi)^2} \int_B (e^{ip(x+a)} - e^{ipx}) \frac{e^{-\delta Q(p)}}{Q(p)} dp \\ - \frac{1}{(2\pi)^2} \int_C (e^{ip(x+a)} - e^{ipx}) \frac{e^{-\delta Q(p)}}{Q(p)} dp.$$

It suffices to show that in the limit as $\delta \rightarrow 0$ the right hand side is $O\left(\frac{c|a|}{|x|^3}\right)$. By (4.11) we see immediately that this holds for the last integral in (4.30). For the first two integrals on the right hand side of (4.30) we obtain as in (4.16)

$$(4.31) \quad \begin{aligned} & \frac{i^2}{(x_1 + a_1)^2} \frac{1}{(2\pi)^2} \int_A e^{ip(x+a)} D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & - \frac{i^2}{x_1^2} \frac{1}{(2\pi)^2} \int_A e^{ipx} D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & - \frac{i^2}{(x_1 + a_1)^2} \frac{1}{(2\pi)^2} \int_B e^{ip(x+a)} D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & + \frac{i^2}{x_1^2} \frac{1}{(2\pi)^2} \int_B e^{ipx} D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \end{aligned}$$

Using the fact that

$$\left| \frac{1}{(x_1 + a_1)^2} - \frac{1}{x_1^2} \right| \leq c|a|/|x|^3$$

and the arguments used to bound (4.16) it is easily seen that (4.31) is equal to

$$(4.32) \quad \begin{aligned} & \frac{i^2}{x_1^2} \frac{1}{(2\pi)^2} \int_A e^{ipx} (e^{ipa} - 1) D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & - \frac{i^2}{x_1^2} \frac{1}{(2\pi)^2} \int_B e^{ipx} (e^{ipa} - 1) D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ & + O_\delta(|a|/|x|^3) \end{aligned}$$

where $O_\delta(|a|/|x|^3)$ denotes a term whose $\delta \rightarrow 0$ limit is $O(|a|/|x|^3)$. To bound the integrals in (4.32) we now integrate by parts once more in the p_1 direction to obtain

$$(4.33) \quad \begin{aligned} & = \frac{i^3}{x_1^3} \frac{1}{(2\pi)^2} \int_A e^{ipx} D_1 \left\{ (e^{ipa} - 1) D_1^2 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) \right\} dp \\ & - \frac{i^3}{x_1^3} \frac{1}{(2\pi)^2} \int_B e^{ipx} D_1 \left\{ (e^{ipa} - 1) D_1^2 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) \right\} dp + O_\delta(|a|/|x|^3). \end{aligned}$$

Once again, the (finite) boundary terms cancel. (Actually, each boundary term is $O(1/|x|^3)$.) As before, we easily see that (4.33) equals

$$(4.34) = \frac{i^3}{x_1^3} \frac{1}{(2\pi)^2} \int_A e^{ipx} (e^{ipa} - 1) D_1^3 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ - \frac{i^3}{x_1^3} \frac{1}{(2\pi)^2} \int_B e^{ipx} (e^{ipa} - 1) D_1^3 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \\ + O(|a|/|x|^3)$$

As in the proof of (4.11), we see that

$$(4.35) \quad \lim_{\delta \rightarrow 0} \left| \frac{1}{(2\pi)^2} \int_B e^{ipx} (e^{ipa} - 1) D_1^3 \left(\frac{e^{-\delta Q(p)}}{Q(p)} \right) dp \right| \leq c.$$

To handle the first integral in (4.34) we note that

$$(4.36) \quad (e^{ipa} - 1) D_1^3 \left(\frac{1}{1 - \phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) \\ = (e^{ipa} - 1) D_1^3 \left(\frac{1}{1 - \phi(p)} - \frac{1}{Q(p)} \right) \\ + (e^{ipa} - 1) (1 - e^{-\delta Q(p)}) D_1^3 \left(\frac{1}{Q(p)} \right) \\ - 3(e^{ipa} - 1) D_1(e^{-\delta Q(p)}) D_1^2 \left(\frac{1}{Q(p)} \right) \\ - 3(e^{ipa} - 1) D_1^2(e^{-\delta Q(p)}) D_1 \left(\frac{1}{Q(p)} \right) \\ - (e^{ipa} - 1) D_1^3(e^{-\delta Q(p)}) \frac{1}{Q(p)}$$

Once again it is easy to control the last four terms in (4.36), while for the first term we use

$$(4.37) \quad D_1^3 \left(\frac{1}{1 - \phi(p)} - \frac{1}{Q(p)} \right) = \frac{\phi_{1,1,1}(p)}{(1 - \phi(p))^2} \\ + 4 \left(\frac{\phi_1(p) \phi_{1,1}(p)}{(1 - \phi(p))^3} - \frac{Q_1(p) Q_{1,1}}{(Q(p))^3} \right) \\ + 6 \left(\frac{(\phi_1(p))^3}{(1 - \phi(p))^4} + \frac{(Q_1(p))^3}{(Q(p))^4} \right).$$

The assumptions of our lemma give

$$(4.38) \quad 1 - \phi(p) = Q(p) + O(|p|^3),$$

$$(4.39) \quad \phi_1(p) = -Q_1(p) + O(|p|^2),$$

$$(4.40) \quad \phi_{1,1}(p) = -Q_{1,1} + O(|p|),$$

and

$$(4.41) \quad \phi_{1,1,1}(p) \leq C < \infty.$$

These show that

$$(4.42) \quad |(e^{ipa} - 1)D_1^3 \left(\frac{1}{1 - \phi(p)} - \frac{1}{Q(p)} \right)| \leq \frac{c|a|}{|p|^3}$$

completing the proof of (4.28).

To prove (4.29) we first note that

$$(4.43) \quad u_{n-1}(x) = \frac{1}{(2\pi)^2} \int_{[-\pi, \pi]^4} \frac{e^{ipx}(1 - \phi^n(p))}{1 - \phi(p)} dp.$$

Set

$$(4.44) \quad v_\delta^n(x) = \int_\delta^n q_t(x) dt = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4} e^{ipx} \frac{e^{-\delta|p|^2/2} - e^{-n|p|^2/2}}{|p|^2/2} dp.$$

We note that by the mean-value theorem

$$(4.45) \quad |q_t(x+a) - q_t(x)| \leq C|a| \sup_{0 \leq \theta \leq 1} \frac{|x + \theta a|}{t} q_t(x + \theta a) \\ \leq C|a| \frac{|x|}{t} q_{2t}(x)$$

where we have used the fact that under our assumptions

$$\frac{1}{2}|x| \leq |x + \theta a| \leq \frac{3}{2}|x|.$$

Since $t^{-1}q_t(x)$ is, up to a constant multiple, the transition density for Brownian motion in R^6 , which has Green's function $C|x|^{-4}$, we have

$$(4.46) \quad |v_\delta^n(x+a) - v_\delta^n(x)| \leq C|a| \int_0^\infty \frac{|x|}{t} q_t(x) dt \leq C \frac{|a|}{|x|^3}.$$

Therefore, it suffices to bound as before an expression of the form (4.30) where u is replaced by u_{n-1} and v_δ is replaced by v_δ^n . All bounds involving

v_δ^n are handled exactly as before. We only point out that whereas in the proof of the previous lemma we were often satisfied with a bound such as (4.15), since we are taking $\delta \rightarrow 0$, we now make use of the extra factor $e^{ip(x+a)} - e^{ipx}$ with the bound

$$|e^{ip(x+a)} - e^{ipx}| \leq |a||p|$$

to guarantee that after scaling no (divergent) factors involving n will remain.

The terms involving u_{n-1} will be handled similarly, after we make several observations. First of all, using Spitzer's trick, in Section 26 of [7], it suffices to assume that $\tau = 1$, (in Spitzer's terminology this means that X is strongly aperiodic) so that $|\phi(p)| = 1$ if and only if $p = 0$. Hence for any $\epsilon > 0$ we have that $\sup_{|p| \geq \epsilon} |\phi(p)| \leq \gamma$ for some $\gamma < 1$, so that, using our assumption that $n-1 > |x|^{1/8}$, we find that the factor $\phi^n(p)$ together with all its derivatives gives us rapid falloff in $|x|$. Taking ϵ sufficiently small, and using (4.38)-(4.41), we see that in the region $|p| \leq \epsilon$, the integrals involving $\phi^n(p)$ and its derivatives can be handled as in the preceeding paragraphs.

REFERENCES

- [1] L. BREIMAN, *Probability*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.
- [2] J.-F. LE GALL, *Propriétés d'intersection des marches aléatoires, II*, Comm. Math. Phys., Vol. **104**, 1986, pp. 509-528.
- [3] J.-F. LE GALL and J. ROSEN, *The range of stable random walks*, Ann. Probab., Vol. **19**, 1991, pp. 650-705.
- [4] G. LAWLER, *A note on the Green's function for random walk in four dimensions*, Duke Math. Department, preprint.
- [5] M. MARCUS and J. ROSEN, *Laws of the iterated logarithm for the local times of recurrent random walks on Z^2 and of Lévy processes and recurrent random walks in the domain of attraction of Cauchy random variables*, Ann. Inst. H. Poincaré Prob. Stat., Vol. **30**, 1994, pp. 467-499.
- [6] M. MARCUS and J. ROSEN, *Laws of the iterated logarithm for the local times of symmetric Lévy processes and recurrent random walks*, Ann. Probab., Vol. **22**, 1994, pp. 626-658.
- [7] F. SPITZER, *Principles of random walk*, Springer Verlag, New York, 1976.

(Manuscript received January 2, 1995;

Revised September 1, 1995.)