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Laws of the iterated logarithm for intersections of random walks on Z^4

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ABSTRACT. – Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ be two independent copies of a symmetric random walk in Z^4 with finite third moment. In this paper we study the asymptotics of I_n, the number of intersections up to step n of the paths of X and X' as $n \to \infty$. Our main result is

$$(1) \qquad \limsup \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{2\pi^2|Q|^{1/2}} \qquad \text{a.s.}$$

where Q denotes the covariance matrix of X_1. A similar result holds for J_n, the number of points in the intersection of the ranges of X and X' up to step n.

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RÉSUMÉ. – Soient $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ deux copies indépendantes d'une marche aléatoire symétrique dans Z^4 avec un moment d'ordre trois. Dans cet article, nous étudions le comportement asymptotique de I_n, le nombre de couples de temps d'intersection jusqu'au temps n des trajectoires de X et X'. Notre principal résultat donne

$$\limsup \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{2\pi^2|Q|^{1/2}} \qquad \text{p.s.} \tag{1}$$

où Q désigne la matrice de covariance de X_1. Un résultat analogue est vrai pour J_n, le nombre de points d'intersection des trajectoires jusqu'au temps n.

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1. INTRODUCTION

Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ be two independent copies of a symmetric random walk in Z^4 with finite variance. In this paper we study the asymptotics of the number of intersections up to step n of the paths of X and X' as $n \to \infty$, both the number of "intersection times"

$$I_n = \sum_{i=1}^{n}\sum_{j=1}^{n} 1_{\{X_i = X'_j\}} \tag{1.1}$$

and the number of "intersection points"

$$J_n = |X(1,n) \cap X'(1,n)| \tag{1.2}$$

where $X(1,n)$ denotes the range of X up to time n and $|A|$ denotes the cardinality of the set A. For random walks with finite variance, dimension four is the "critical case" for intersections, since $I_n, J_n \uparrow \infty$ almost surely but two independent Brownian motions in R^4 do not intersect.

We assume that X_n is adapted, which means that X_n does not live on any proper subgroup of Z^4. In the terminology of Spitzer [7] X_n is aperiodic.

We have the following two limit theorems.

THEOREME 1. – *Assume that $E(|X_1|^3) < \infty$. Then*

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{2\pi^2|Q|^{1/2}} \qquad a.s. \tag{1.3}$$

where Q denotes the covariance matrix of X_1.

As usual, $\log_j$ denotes the j-fold iterated logarithm.

In the particular case of the simple random walk on Z^4, where $Q = \frac{1}{4}I$, Theorem 1 states that

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{8}{\pi^2} \qquad \text{a.s.} \tag{1.4}$$

A similar result holds for J_n:

THEOREME 2. – *Assume that $E(|X_1|^3) < \infty$. Then*

$$\limsup_{n\to\infty} \frac{J_n}{\log(n)\log_3(n)} = \frac{q^2}{2\pi^2|Q|^{1/2}} \qquad a.s. \tag{1.5}$$

where q denotes the probability that X will never return to its initial point.

Le Gall [2] proved that $(\log n)^{-1}J_n$ converges in distribution to the square of a normal random variable. In this paper we use some of the ideas of [2] along with techniques developed in [5], [6].

2. PROOF OF THEOREM 1

We use $p_n(x)$ to denote the transition function for X_n. Recall

$$\begin{aligned} I_n &= \sum_{i=1}^{n}\sum_{j=1}^{n} 1_{\{X_i = X'_j\}} \\ &= \sum_{x\in\mathbf{Z}^4}\left\{\left(\sum_{i=1}^{n} 1_{\{X_i=x\}}\right)\left(\sum_{j=1}^{n} 1_{\{X'_j=x\}}\right)\right\}. \end{aligned} \tag{2.1}$$

We set

$$\begin{aligned} h(n) = E(I_n) &= \sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{n} p_i(x)\right)\left(\sum_{j=1}^{n} p_j(x)\right)\right\} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} p_{i+j}(0) \end{aligned} \tag{2.2}$$

where in the last step we used the fact that our random walk X is symmetric.

As shown in [7] the random walk X_n is adapted if and only if the origin is the unique element of T^4 satisfying $\phi(p) = 1$ where $\phi(p)$ is the characteristic function of X_1 and $T^4 = (-\pi,\pi]^4$ is the usual four dimensional torus. We use τ to denote the number of elements in the set $\{p \in T^4 | \ |\phi(p)| = 1\}$. According to the local central limit theorem, *see* e.g. Prop. 2.4 of [3], we have that

$$p_j(0) = 0 \quad \text{if } j \neq 0 \ (\text{mod } \tau)$$

while

$$p_{n\tau}(0) \sim \frac{1}{(2\pi)^2 \tau |Q|^{1/2}} \frac{1}{n^2} \tag{2.3}$$

where Q denotes the covariance matrix of X_1.

When $\tau = 1$ we see from (2.2) and (2.3) that

$$\begin{aligned} h(n) &= \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i+j}(0) \\ &= \sum_{k=1}^{n} k p_k(0) + \sum_{k=n+1}^{2n} (2n-k) p_k(0) \\ &\sim \sum_{k=1}^{n} k p_k(0) \\ &\sim \frac{1}{(2\pi)^2 |Q|^{1/2}} \log n. \end{aligned} \tag{2.4}$$

The same sort of calculation shows that this holds in general:

$$\begin{aligned} h(n) &= \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i+j}(0) \\ &\sim \sum_{m=0}^{\tau-1} \sum_{i=1}^{[n/\tau]} \sum_{j=1}^{[n/\tau]} p_{(i\tau+m)+(j\tau-m)}(0) \\ &\sim \tau \sum_{k=1}^{[n/\tau]} k p_{k\tau}(0) \\ &\sim \frac{1}{(2\pi)^2 |Q|^{1/2}} \log n. \end{aligned} \tag{2.5}$$

Thus the assertion of Theorem 1 can be written as

$$\limsup_{n \to \infty} \frac{I_n}{2h(n) \log_2 h(n)} = 1 \qquad \text{a.s.} \tag{2.6}$$

We begin our proof with some moment calculations.

$$
\begin{aligned}
(2.7)\quad E(I_t^n) &= \sum_{x_1,\dots,x_n} \left\{ E\Big(\prod_{i=1}^{n}\sum_{r_i=1}^{t} 1_{\{X_{r_i}=x_i\}}\Big)\right\}^2 \\
&= \sum_{x_1,\dots,x_n} \left\{ \sum_{\pi} \sum_{r_1\le r_2\le\dots\le r_n\le t} E\Big(\prod_{i=1}^{n} 1_{\{X_{r_i}=x_{\pi(i)}\}}\Big)\right\}^2 \\
&= \sum_{x_1,\dots,x_n} \Big(\sum_{\pi} \sum_{r_1\le r_2\le\dots\le r_n\le t} \prod_{i=1}^{n} p_{r_i-r_{i-1}}(x_{\pi(i)}-x_{\pi(i-1)})\Big)^2 \\
&= n! \sum_{x_1,\dots,x_n} \Big(\sum_{r_1\le r_2\le\dots\le r_n\le t} \prod_{i=1}^{n} p_{r_i-r_{i-1}}(x_i-x_{i-1})\Big) \\
&\qquad \Big(\sum_{\pi} \sum_{s_1\le s_2\le\dots\le s_n\le t} \prod_{j=1}^{n} p_{s_j-s_{j-1}}(x_{\pi(j)}-x_{\pi(j-1)})\Big)
\end{aligned}
$$

where $\sum_\pi$ runs over the set of permutations π of $\{1,2,\dots,n\}$. Set

$$u_t(x) = \sum_{r=1}^{t} p_r(x).$$

Then we see from (2.7) that

$$
(2.8)\qquad E(I_t^n) \le n! \sum_{x_1,\dots,x_n} \Big(\prod_{i=1}^{n} u_t(x_i-x_{i-1})\Big) \Big(\sum_{\pi}\prod_{j=1}^{n} u_t(x_{\pi(j)}-x_{\pi(j-1)})\Big),
$$

while

$$
(2.9)\qquad E(I_t^n) \ge n! \sum_{x_1,\dots,x_n} \Big(\prod_{i=1}^{n} u_{t/n}(x_i-x_{i-1})\Big) \Big(\sum_{\pi}\prod_{j=1}^{n} u_{t/n}(x_{\pi(j)}-x_{\pi(j-1)})\Big).
$$

We note here that by Lemma 5 of the Appendix we have

$$
(2.10)\qquad u_t(x) \le \sum_{j=1}^{\infty} p_j(x) \le \frac{C}{1+|x|^2}.
$$

On the other hand, using $E(|X_1|^3) < \infty$ we have

$$p_j(x) = P(X_j = x) \leq P(|X_j| \geq |x|) \leq \frac{Cj^3}{|x|^3} \tag{2.11}$$

so that

$$u_t(x) \leq C\frac{t^4}{|x|^3} \tag{2.12}$$

giving us the bound

$$u_t(x) \leq \frac{C}{1+|x|^{5/2}} \quad \text{for all } |x| > t^8. \tag{2.13}$$

LEMMA 1. – *For all integers $n, t \geq 0$ and for any $\epsilon > 0$*

$$E(I_t^n) \leq (1+\epsilon)(2n)!!h^n(t) + R(n,t) \tag{2.14}$$

where

$$0 \leq R(n,t) \leq C(n!)^4 h^{n-1/2}(t) \tag{2.15}$$

Here $(2n)!! = \prod_{j=1}^n (2j-1)$ denotes the odd factorial.

Proof of Lemma 1. – We will make use of several ideas of Le Gall [2]. We begin by rewriting (2.8) as

$$E(I_t^n) \leq n! \sum_{y_1,\ldots,y_n} \left(\prod_{i=1}^n u_t(y_i)\right)\left(\sum_\pi \prod_{j=1}^n u_t(v_{\pi,j})\right), \tag{2.16}$$

where $y_i = x_i - x_{i-1}$,

$$v_{\pi,j} = x_{\pi(j)} - x_{\pi(j-1)} = \sum_{k\in]\pi(j-1),\pi(j)]} y_j, \tag{2.17}$$

and (with a slight abuse of notation), $k \in]\pi(j-1), \pi(j)]$ means

$$k \in] \min(\pi(j-1), \pi(j)), \max(\pi(j-1), \pi(j))].$$

In view of (2.16), in order to prove our lemma it suffices to show that

$$\begin{aligned} n! \sum_{y_1,\ldots,y_n} \left(\prod_{i=1}^n u_t(y_i)\right)\left(\sum_\pi \prod_{j=1}^n u_t(v_{\pi,j})\right) \\ = (1+\epsilon)(2n)!!h^n(t) + R(n,t) \end{aligned} \tag{2.18}$$

with $R(n,t)$ as in (2.15). For each permutation σ of $\{1,2,\dots,n\}$ we define

$$\Delta_\sigma = \{(y_1,\cdots,y_n)|\ |y_{\sigma(1)}| \le |y_{\sigma(2)}| \le \cdots \le |y_{\sigma(n)}|\}$$

and rewrite the left hand side of (2.18) as

$$n!\sum_{\sigma,\pi}\sum_{\Delta_\sigma}\left(\prod_{i=1}^n u_t(y_i)\right)\left(\prod_{j=1}^n u_t(v_{\pi,j})\right). \tag{2.19}$$

Note that by (2.10)

$$\begin{aligned}\sum_{y\le|x|\le 4y}(u_t(x))^2 &\le \sum_{y\le|x|\le 4y} C\frac{1}{1+|x|^4}\\ &\le C(\log 4y-\log y) = C\log(4)\end{aligned} \tag{2.20}$$

and that by (2.2)

$$\sum_x u_t^2(x) = h(t). \tag{2.21}$$

Let $A_{\sigma,k} = \{(y_1,\dots,y_n)\ |\ |y_{\sigma_{k-1}}| \le |y_{\sigma_k}| \le 4|y_{\sigma_{k-1}}|\}$. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{(y_1,\dots,y_n)\in A_{\sigma,k}}&\left(\prod_{i=1}^n u_t(y_i)\right)\left(\prod_{j=1}^n u_t(v_{\pi,j})\right)\\ &\le \left(\sum_{(y_1,\dots,y_n)\in A_{\sigma,k}}\prod_{i=1}^n(u_t(y_i))^2\right)^{1/2} h^{n/2}(t)\\ &\le Ch^{n-1/2}(t).\end{aligned} \tag{2.22}$$

Set

$$\hat\Delta_\sigma = \{(y_1,\cdots,y_n)|\ 4|y_{\sigma(k-1)}| < |y_{\sigma(k)}|,\ \forall k\}.$$

We see that the sum in (2.19) differs from the sum over $\hat\Delta_\sigma$ by an error term which can be incorporated into $R(n,t)$. Up to the error terms described above, we can write the sum in (2.19) as

$$n!\sum_{\sigma,\pi}\sum_{(y_1,\dots,y_n)\in\hat\Delta_\sigma}\left(\prod_{i=1}^n u_t(y_i)\right)\left(\prod_{j=1}^n u_t(v_{\pi,j})\right). \tag{2.23}$$

For given σ, π define the map $\phi = \phi_{\sigma,\pi} : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ by

$$\phi(j) = \sigma(k_{\sigma,\pi,j}),$$

where

$$k_{\sigma,\pi,j} = \max\{k | \, \sigma(k) \in]\pi(j-1), \pi(j)]\}.$$

Note that on $\hat{\Delta}_\sigma$, $\phi(j)$ is the unique integer in $]\pi(j-1), \pi(j)]$ such that $|y_{\phi(j)}| = \sup_{k \in]\pi(j-1), \pi(j)]} |y_k|$. Futhermore, on $\hat{\Delta}_\sigma$, we see that $\frac{1}{2}|v_{\pi,j}| < |y_{\phi(j)}| < 2|v_{\pi,j}|$. Using the Cauchy-Schwarz inequality, and the bounds (2.10), (2.13) we have

$$\begin{aligned}
(2.24) \quad & \sum_{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\prod_{j=1}^n u_t(v_{\pi,j}) \right) \\
& \leq \left(\sum_{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma} \prod_{j=1}^n (u_t(v_{\pi,j}))^2 \right)^{1/2} h^{n/2}(t) \\
& \leq \left(\sum_{\substack{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma \\ |v_{\pi,j}| \leq t^8, \forall j}} \prod_{j=1}^n (u_t(v_{\pi,j}))^2 \right)^{1/2} h^{n/2}(t) + C h^{n-1/2}(t) \\
& \leq C \left(\sum_{\substack{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma \\ |y_j| \leq 2t^8, \forall j}} \prod_{j=1}^n \frac{1}{1 + |y_{\phi(j)}|^4} \right)^{1/2} h^{n/2}(t) + C h^{n-1/2}(t).
\end{aligned}$$

We now show that

$$(2.25) \qquad \sum_{\substack{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma \\ |y_j| \leq 2t^8, \forall j}} \prod_{j=1}^n \frac{1}{1 + |y_{\phi(j)}|^4} \leq C h^{n-1}(t)$$

unless $\phi = \phi_{\sigma,\pi} : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ is bijective.

To begin, we note that by (2.17) both $\{y_j, \ j = 1, \ldots, n\}$ and $\{v_{\pi,j}, \ j = 1, \ldots, n\}$ generate $\{x_j, \ j = 1, \ldots, n\}$ in the sense of linear combinations, so that both sets consist of n linearly independent vectors. Furthermore, from (2.17) we see that each $v_{\pi,j}$ is a sum of vectors from $\{y_j, \ j = 1, \ldots, n\}$. However, from the definitions, we see that when we write out any vector in $\{v_{\pi,j} \mid k_{\sigma,\pi,j} \leq m\}$ as such a sum, the sum will only involve vectors from $\{y_{\sigma(j)} \mid j \leq m\}$. Hence $\{v_{\pi,j} \mid k_{\sigma,\pi,j} \leq m\}$ will contain at most m linearly independent vectors. Therefore, for each $m = 0, 1, \ldots, n-1$, the set $\{v_{\pi,j} \mid k_{\sigma,\pi,j} > m\}$ will contain at least

$n - m$ elements. Equivalently, for each $m = 0, 1, \ldots, n-1$, the set $\{j \mid \sigma^{-1}\phi(j) > m\}$ will contain at least $n - m$ elements. This shows that for each $m = 0, 1, \ldots, n-1$, the product

$$\prod_{j=1}^{n} \frac{1}{1 + |y_{\phi(j)}|^4}$$

will contain at least $n - m$ factors of the form

$$\frac{1}{1 + |y_{\sigma(j)}|^4}$$

with $j > m$. We now return to (2.25) and sum in turn over the variables $y_{\sigma(n)}, y_{\sigma(n-1)}, \ldots, y_{\sigma(1)}$ using the fact that

$$\sum_{\{y_{\sigma(j)} \in \mathbf{Z}^4 \mid \, 4|y_{\sigma(j-1)}| \le |y_{\sigma(j)}| \le t^8\}} \frac{1}{1 + |y_{\sigma(j)}|^4} \le Ch(t) \tag{2.26}$$

while for any $k > 1$

$$\sum_{\{y_{\sigma(j)} \in \mathbf{Z}^4 \mid \, 4|y_{\sigma(j-1)}| \le |y_{\sigma(j)}| \le t^8\}} \frac{1}{1 + |y_{\sigma(j)}|^{4k}} \le C \frac{1}{1 + |y_{\sigma(j-1)}|^{4(k-1)}}. \tag{2.27}$$

The above considerations show that as we sum successively over the variables $y_{\sigma(n)}, y_{\sigma(n-1)}, \ldots, y_{\sigma(1)}$, at the stage when we sum over $y_{\sigma(j)}$, we will be summing a factor of the form $\frac{1}{1+|y_{\sigma(j)}|^{4k}}$ for some $k \ge 1$, while if $\phi = \phi_{\sigma,\pi} : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ is not bijective we must have $k > 1$ at some stage. These considerations, together with (2.26) and (2.27) establish (2.25).

Let Ω_n be the set of (σ, π) for which $\phi_{\sigma,\pi}$ is a bijection. Up to the error terms described above, we can write the sum in (2.23) as

$$n! \sum_{(\sigma,\pi) \in \Omega_n} \sum_{(y_1,\ldots,y_n) \in \hat{\Delta}_\sigma} \left(\prod_{i=1}^{n} u_t(y_i) \right) \left(\prod_{j=1}^{n} u_t(v_{\pi,j}) \right). \tag{2.28}$$

Since on $\hat{\Delta}_\sigma$, we have that $|y_{\phi(j)}| > 2|v_{\pi,j} - y_{\phi(j)}|$, we can then replace each occurence of $v_{\pi,j}$ in (2.28) by $y_{\phi(j)}$, bounding the error terms using

$$\begin{aligned} &\sum_{\{|x| > 2|a|\}} (u_t(x+a) - u_t(x))^2 \\ &\quad \le C \sum_{\{|x| > 2|a|\}} \left(\frac{|a|^2}{1 + |x|^6} + \frac{1}{1 + |x|^5} \right) \le C \end{aligned} \tag{2.29}$$

which comes from (2.13) and Lemma 6 of the Appendix.

Thus, up to error terms described which can be incorporated into $R(n,t)$, we can write the sum in (2.28) as

$$n! \sum_{(\sigma,\pi)\in\Omega_n} \sum_{(y_1,\dots,y_n)\in\hat{\Delta}_\sigma} \left(\prod_{i=1}^{n} u_t^2(y_i)\right). \tag{2.30}$$

Proceeding as above, up to the error terms described above, we can replace (2.30) by

$$n! \sum_{(\sigma,\pi)\in\Omega_n} \sum_{(y_1,\dots,y_n)\in\Delta_\sigma} \left(\prod_{i=1}^{n} u_t^2(y_i)\right). \tag{2.31}$$

Since

$$n! \sum_{(y_1,\dots,y_n)\in\Delta_\sigma} \left(\prod_{i=1}^{n} u_t^2(y_i)\right) \sim h^n(t),$$

and as by the remark following Lemma 2.5 of [2] we have $|\Omega_n| = (2n)!!$, the lemma is proved. □

We will use $E^{v,w}$ to denote expectation with respect to the random walks X, X' where $X_0 = v$ and $X'_0 = w$. We define

$$a(v,w,t) = \frac{h(v,w,t)}{h(t)} \tag{2.32}$$

where

$$\begin{aligned} h(v,w,t) &= E^{v,w}(I_t) \\ &= \sum_{x\in\mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{t} p_i(x-v)\right)\left(\sum_{j=1}^{t} p_j(x-w)\right)\right\} \\ &= \sum_{i,j=1}^{t} p_{i+j}(v-w). \end{aligned} \tag{2.33}$$

We will need the following lower bound.

LEMMA 2. – *For all integers $n, t \geq 0$ and for any $\epsilon > 0$*

$$E^{v,w}(I_t^n) \geq (1-\epsilon)(2n)!!a(v,w,t/n)h^n(t/n) - R'(n,t) \tag{2.34}$$

where

$$0 \leq R'(n,t) \leq C(n!)^4 h^{n-1/2}(t). \tag{2.35}$$

Proof of Lemma 2. – We first note that as in (2.9)

$$E^{v,w}(I_t^n) \geq n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1})\right) \left(\sum_\pi \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)})\right) \tag{2.36}$$

where now we use the convention $x_0 = v, x_{\pi(0)} = w$. We then use (2.18), observing that if $\phi_{\sigma,\pi}$ is bijective we must have $\phi_{\sigma,\pi}(j) = 1$ for some j and this must be $j = 1$ since $1 \in]\pi(j-1), \pi(j)]$ is possible only for $j = 1$. Thus, $v_{\pi,1}$ is replaced in (2.23) by y_1. □

LEMMA 3. – *For all* $t \geq 0$ *and* $x = O(\log\log h(t))$ *we have*

$$P\left(\frac{I_t}{2h(t)} \geq x\right) \leq C\sqrt{x}e^{-x}. \tag{2.37}$$

Proof of Lemma 3. – We first note that if $n = O(\log\log h(t))$ then

$$\frac{(n!)^4}{h^{1/2}(t)} \to 0 \tag{2.38}$$

as $t \to \infty$, so that by Lemma 1 we have

$$E(I_t^n) \leq C(2n)!!h^n(t). \tag{2.39}$$

Then Chebyshev's inequality gives us

$$P\left(\frac{I_t}{2h(t)} \geq x\right) \leq C\frac{(2n)!!}{(2x)^n} = C\frac{\sqrt{n}n^n e^{-n}}{x^n}(1 + O(1/n)) \tag{2.40}$$

for any $n = O(\log\log h(t))$. Taking $n = [x]$ then yields (2.37). □

LEMMA 4. – *For all* $\epsilon > 0$ *there exists an* x_0 *and a* $t' = t'(\epsilon, x_0)$ *such that for all* $t \geq t'$ *and* $x_0 \leq x = O(\log\log h(t))$ *we have*

$$P\left(\frac{I_t}{2h(t)} \geq (1-\epsilon)x\right) \geq C_\epsilon e^{-x} \tag{2.41}$$

and

$$P^{v,w}\left(\frac{I_t}{2h(t)} \geq (1-\epsilon)x\right) \geq C_\epsilon(a(v,w,2t/(3x))e^{-x} - e^{-(1+\epsilon')x}) \tag{2.42}$$

for some $\epsilon' > 0$.

Proof of Lemma 4. – This follows from Lemmas 2, 3 and (2.38) by the methods used in the proof of Lemma 2.7 in [5]. □

Proof of Theorem. 1. – For $\theta > 1$ we define the sequence $\{t_n\}$ by

$$h(t_n) = \theta^n. \tag{2.43}$$

By Lemma 3 we have that for all integers $n \geq 2$ and all $\epsilon > 0$

$$P\left(\frac{I_{t_n}}{2h(t_n)\log\log h(t_n)} \geq (1+\epsilon)\right) \leq Ce^{-(1+\epsilon)\log n}. \tag{2.44}$$

Therefore, by the Borel-Cantelli lemma

$$\limsup_{n\to\infty} \frac{I_{t_n}}{2h(t_n)\log\log h(t_n)} \leq 1+\epsilon \qquad \text{a.s.} \tag{2.45}$$

By taking θ arbitrarily close to 1 it is simple to interpolate in (2.45) to obtain

$$\limsup_{n\to\infty} \frac{I_n}{2h(n)\log\log h(n)} \leq 1+\epsilon \qquad \text{a.s.} \tag{2.46}$$

We now show that for any $\epsilon > 0$

$$\limsup_{n\to\infty} \frac{I_{t_n}}{2h(t_n)\log\log h(t_n)} \geq 1-\epsilon \qquad \text{a.s.} \tag{2.47}$$

for all θ sufficiently large. It is sufficient to show that

$$\limsup_{n\to\infty} \frac{I_{t_n} - I_{t_{n-1}}}{2h(t_n)\log\log h(t_n)} \geq 1-\epsilon \qquad \text{a.s.} \tag{2.48}$$

Let $s_n = t_n - t_{n-1}$ and note that, as in (2.60) of [5], we have $h(s_n) \sim h(t_n)$. We also note that

$$|I_{t_n} - I_{t_{n-1}} - I_{s_n} \circ \Theta_{t_{n-1}}| \leq I_{t_n,t_{n-1}} + I_{t_{n-1},t_n} \tag{2.49}$$

where

$$I_{n,m} = \sum_{x\in\mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{n} 1_{\{X_i = x\}} \right) \left(\sum_{j=1}^{m} 1_{\{X'_j = x\}} \right) \right\}. \tag{2.50}$$

As in Lemma 1, we can show that for $t \geq t'$, and for all integers $n \geq 0$ and any $\epsilon > 0$

$$\begin{aligned} E(I^n_{t,t'}) &\leq (1+\epsilon)(2n)!!h^{n/2}(t)h^{n/2}(t') \\ &\quad + O((n!)^4 h^{n/2}(t)h^{n/2-1/2}(t')) \end{aligned} \tag{2.51}$$

which, as before, leads to

$$\limsup_{n\to\infty} \frac{I_{t_n,t_{n-1}}}{2h(t_n)\log\log h(t_n)} = \limsup_{n\to\infty} \frac{I_{t_n,t_{n-1}}}{2\sqrt{\theta h(t_n)h(t_{n-1})}\log\log h(t_n)} \le \frac{1+\epsilon}{\sqrt{\theta}} \quad \text{a.s.} \tag{2.52}$$

Using this for θ large, (2.49), Levy's Borel-Cantelli lemma (*see* Corollary 5.29 in [1]) and the Markov property, we see that (2.48) will follow from

$$\sum_{n=1}^{\infty} P^{X_{t_{n-1}},X'_{t_{n-1}}}\left(\frac{I_{s_n}}{2h(s_n)\log\log h(s_n)} \ge 1-\epsilon\right) = \infty \qquad \text{a.s.} \tag{2.53}$$

If we apply Lemma 4 with $t = s_n$ and $x = \log\log s_n$ we see that (2.53) will follow from

$$\sum_{n=1}^{\infty} a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n)\frac{1}{n^{1-\epsilon'}} = \infty \qquad \text{a.s.} \tag{2.54}$$

We begin by showing

$$\sum_{n=1}^{\infty} E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n))\frac{1}{n^{1-\epsilon'}} = \infty. \tag{2.55}$$

To see this we note that

$$E(a(X_t, X'_t, k)) = \frac{\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{k} p_{i+t}(x)\right)\left(\sum_{j=1}^{k} p_{j+t}(x)\right)\right\}}{h(k)} \tag{2.56}$$

so that

$$\begin{aligned} E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n)) &= \frac{\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{s_n/\log n} p_{i+t_{n-1}}(x)\right)\left(\sum_{j=1}^{s_n/\log n} p_{j+t_{n-1}}(x)\right)\right\}}{h(s_n/\log n)} \\ &= \frac{h(t_{n-1}+s_n/\log n) - h(t_{n-1})}{h(s_n/\log n)} \\ &\quad - \frac{2\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{t_{n-1}} p_i(x)\right)\left(\sum_{j=1}^{s_n/\log n} p_{j+t_{n-1}}(x)\right)\right\}}{h(s_n/\log n)}. \end{aligned} \tag{2.57}$$

Also note that

$$(2.58)\qquad \frac{h(t_{n-1}+s_n/\log n)-h(t_{n-1})}{h(s_n/\log n)} \geq \frac{h(s_n/\log n)-h(t_{n-1})}{h(s_n/\log n)} \sim 1-\frac{1}{\theta}.$$

This follows fairly easily since $h(t) \sim c\log(t)$. (For the details, in a more general setting, see the proof of Theorem 1.1 of [5], especially that part of the proof surrounding (2.82)). Furthermore, we have by the Cauchy-Schwarz inequality

$$(2.59)\qquad \frac{\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{t_{n-1}} p_i(x)\right)\left(\sum_{j=1}^{s_n/\log n} p_{j+t_{n-1}}(x)\right)\right\}}{h(s_n/\log n)} \leq \frac{\sqrt{h(t_{n-1})h(t_n)}}{h(s_n/\log n)} \sim \frac{1}{\sqrt{\theta}}.$$

Taking θ large establishes 2.55.

Furthermore, since $a(v,w,t) \leq 1$ (compare (2.4) and (2.33)), we see that for any $\epsilon' < 1/2$

$$(2.60)\qquad \sum_{n=1}^{\infty} E\left(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n)\frac{1}{n^{1-\epsilon'}}\right)^2 < \infty.$$

(2.54) will now follow from the Paley-Zygmund lemma once we show that

$$(2.61)\qquad \frac{E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n)a(X_{t_{m-1}}, X'_{t_{m-1}}, s_m/\log m))}{E(a(X_{t_{n-1}}, X'_{t_{n-1}}, s_n/\log n))E(a(X_{t_{m-1}}, X'_{t_{m-1}}, s_m/\log m))} \leq 1+2\epsilon$$

for all $\epsilon > 0$, when $n > m \geq N(\epsilon)$ for some $N(\epsilon)$ sufficiently large. To prove (2.61) we begin by noting that as in (2.56)

$$(2.62)\qquad E(h(X_t, X'_t, s)) = \sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{s} p_{i+t}(x)\right)\left(\sum_{j=1}^{s} p_{j+t}(x)\right)\right\} = \sum_{i,j=1}^{s} p_{i+j+2t}(0)$$

and for $t' < t$

(2.63)

$$\begin{aligned}
&E(h(X_{t'}, X'_{t'}, s')h(X_t, X'_t, s)) \\
&\quad = \sum_{x,y,x',y'} h(x, x', s')p_{t'}(x)p_{t'}(x')h(y, y', s)p_{t-t'}(y - x)p_{t-t'}(y' - x') \\
&\quad = \sum_{x,x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \\
&\qquad \cdot \sum_{u\in\mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{s} p_{i+t-t'}(u - x) \right) \left(\sum_{j=1}^{s} p_{j+t-t'}(u - x') \right) \right\} \\
&\quad = \sum_{x,x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \sum_{i,j=1}^{s} p_{i+j+2(t-t')}(x - x') \\
&\quad \leq \sum_{x,x'} h(x, x', s')p_{t'}(x)p_{t'}(x') \sum_{i,j=1}^{s} p_{i+j+2(t-t')}(0) \\
&\quad = \sum_{x\in\mathbf{Z}^d} \left\{ \left(\sum_{i=1}^{s'} p_{i+t'}(x) \right) \left(\sum_{j=1}^{s'} p_{j+t'}(x) \right) \right\} \sum_{i,j=1}^{s} p_{i+j+2(t-t')}(0) \\
&\quad = \sum_{i,j=1}^{s'} p_{i+j+2t'}(0) \sum_{i,j=1}^{s} p_{i+j+2(t-t')}(0).
\end{aligned}$$

From (2.62), (2.63) we see that

$$\frac{E(h(X_{t'}, X'_{t'}, s')h(X_t, X'_t, s))}{E(h(X_{t'}, X'_{t'}, s'))E(h(X_t, X'_t, s))} \leq \frac{\sum_{i=1}^{s} \sum_{j=1}^{s} p_{i+j+2(t-t')}(0)}{\sum_{i=1}^{s} \sum_{j=1}^{s} p_{i+j+2t}(0)}. \tag{2.64}$$

Now let us assume that $t - t' > (1 - \epsilon)t$. (This will certainly hold in our case where $t = t_{n-1}, t' = t_{m-1}$ with $m < n$). Then $i + j + 2(t - t') > (1 - \epsilon)(i + j + 2t)$. Assume first that $\tau = 1$. Since by (2.3) we have that $p_{\cdot}(0)$ is regularly varying at infinity of order -2, we see that if t is sufficiently large, then

$$p_{i+j+2(t-t')}(0) \leq (1 + 2\epsilon)p_{i+j+2t}(0) \tag{2.65}$$

so that (2.64) is $\leq 1 + 2\epsilon$. This completes the proof of (2.61) when $\tau = 1$. The general case is easily handled if instead of t_n we work with $t'_n \sim t_n$ satisfying $t'_n = 0 \bmod \tau$. This completes the proof of Theorem 1. □

3. PROOF OF THEOREM 2

We begin with some moment calculations. Recall

$$(3.1)\qquad \begin{aligned} J_n &= |X(1,n) \cap X'(1,n)| \\ &= \sum_{x \in \mathbf{Z}^4} 1_{\{x \in X(1,n)\}} 1_{\{x \in X'(1,n)\}}. \end{aligned}$$

As usual set

$$T_x = \inf\{k |\, X_k = x\},$$

and note that

$$(3.2)\qquad \begin{aligned} E(J_t^n) &= E\{(\sum_x 1_{\{x \in X(1,t)\}} 1_{\{x \in X'(1,t)\}})^n\} \\ &= \sum_{x_1,\ldots,x_n} E\left(\prod_{i=1}^n 1_{\{x \in X(1,t)\}} 1_{\{x \in X'(1,t)\}}\right) \\ &= \sum_{x_1,\ldots,x_n} \left\{E\left(\prod_{i=1}^n 1_{\{x_i \in X(1,t)\}}\right)\right\}^2 \\ &\le \sum_{x_1,\ldots,x_n} \{\sum_\pi P(T_{x_{\pi(1)}} \le T_{x_{\pi(2)}} \le \cdots \le T_{x_{\pi(n)}} \le t)\}^2 \\ &= n! \sum_{x_1,\ldots,x_n} (P(T_{x_1} \le T_{x_2} \le \cdots \le T_{x_n} \le t)) \\ &\quad \cdot (\sum_\pi P(T_{x_{\pi(1)}} \le T_{x_{\pi(2)}} \le \cdots \le T_{x_{\pi(n)}} \le t)) \end{aligned}$$

where $\sum_\pi$ runs over the set of permutations π of $\{1, 2, \ldots, n\}$. Set

$$v_t(x) = P(T_x \le t).$$

Then we see from 3.2 that

$$(3.3)\qquad \begin{aligned} E(J_t^n) \le n! \sum_{x_1,\ldots,x_n} &\left(\prod_{i=1}^n v_t(x_i - x_{i-1})\right) \\ &\left(\sum_\pi \prod_{j=1}^n v_t(x_{\pi(j)} - x_{\pi(j-1)})\right), \end{aligned}$$

while

$$E(J_t^n) \geq n! \sum_{\substack{x_1, x_2, \ldots, x_n \\ distinct}} \left(\prod_{i=1}^{n} v_{t/n}(x_i - x_{i-1}) \right) \tag{3.4}$$
$$\left(\sum_{\pi} \prod_{j=1}^{n} v_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right).$$

Here we used the fact that the inequality in 3.2 is due to the possible double counting if $x_i = x_j$ for some i, j.

Let

$$f_r(x) = P(T_x = r)$$

so that

$$v_t(x) = \sum_{r=1}^{t} f_r(x).$$

We have

$$p_j(x) = \sum_{i=1}^{j} f_i(x) p_{j-i}(0) \tag{3.5}$$

where as usual we set $p_0(x) = 1_{\{x=0\}}$. From this we see that

$$\begin{aligned} u_t(x) &= \sum_{j=1}^{t} p_j(x) \\ &= \sum_{j=1}^{t} \sum_{i=1}^{j} f_i(x) p_{j-i}(0) \\ &= \sum_{i=1}^{t} \sum_{j=i}^{t} f_i(x) p_{j-i}(0) \\ &= \sum_{i=1}^{t} f_i(x)(1 + u_{t-i}(0)). \end{aligned} \tag{3.6}$$

Consequently we have

$$u_t(x) \leq v_t(x)(1 + u_t(0)) \tag{3.7}$$

and

$$u_{2t}(x) \geq v_t(x)(1 + u_t(0)). \tag{3.8}$$

Now it is well known that

$$\frac{1}{1+u_t(0)} \downarrow q \tag{3.9}$$

so that for any $\epsilon > 0$ we can find $t_0 < \infty$ such that

$$qu_t(x) \leq v_t(x) \leq (q+\epsilon)u_{2t}(x) \tag{3.10}$$

for all $t \geq t_0$ and x. Hence (3.3) and (3.4) give us

$$E(J_t^n) \leq (q+\epsilon)^{2n} n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^n u_{2t}(x_i - x_{i-1})\right) \left(\sum_\pi \prod_{j=1}^n u_{2t}(x_{\pi(j)} - x_{\pi(j-1)})\right), \tag{3.11}$$

and

$$E(J_t^n) \geq q^{2n} n! \sum_{\substack{x_1,\dots,x_n \\ distinct}} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1})\right) \left(\sum_\pi \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)})\right). \tag{3.12}$$

The proof of Theorem 2 now follows exactly along the lines of the proof of Theorem 1. □

4. APPENDIX

LEMMA 5. – *Let X_n be a mean-zero adapted random walk in Z^4. Assume that $E(|X_1|^2 \log_+ |X_1|) < \infty$. Then for some $C < \infty$*

$$u(x) \stackrel{\text{def}}{=} \sum_{n=1}^\infty p_n(x) \leq \frac{C}{1+|x|^2} \tag{4.1}$$

for all x.

In the proof of Lemma 5 we actually show that

$$|u(x) - G(x)| = o(1/|x|^2) \tag{4.2}$$

where $G(x)$ is the Green's function of the non-isotropic Brownian motion in R^4 with covariance matrix equal to that of X_1.

In a recent paper [4], Lawler shows that 4.1 does not hold for all mean zero finite variance random walks. He also proves Lemma 5. We present here a different proof of Lemma 5 because our method of proof will be used, in Lemma 6, to obtain a bound for $|G(x+a) - G(x)|$.

Proof of Lemma 5. – Let

$$\phi(p) = E(e^{ipX_1})$$

denote the characteristic function of X_1. We have

$$u(x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^4} \frac{e^{ipx}}{1-\phi(p)}\, dp. \tag{4.3}$$

Let $Q = \{Q_{i,j}\}$ denote the covariance matrix of $X_1 = (X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)})$, *i.e.* $Q_{i,j} = E(X_1^{(i)} X_1^{(j)})$. We write

$$Q(p) = \frac{1}{2} \sum_{i,j=1}^{4} Q_{i,j} p_i p_j$$

for $p \in [-\pi, \pi]^4$. Let $q_t(x)$ denote the transition density for Brownian motion in R^4 and set

$$v_\delta(x) = \int_\delta^\infty q_t(x)\, dt = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4} e^{ipx} \frac{e^{-\delta|p|^2/2}}{|p|^2/2}\, dp. \tag{4.4}$$

We have

$$\frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4} e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)}\, dp. \tag{4.5}$$

Note that

$$v_\delta(x) \uparrow v_0(x) = \int_0^\infty q_t(x)\, dt = \frac{1}{(2\pi)^2 |x|^2} \tag{4.6}$$

as $\delta \to 0$ and thus to prove (4.1) it suffices to show that

$$\lim_{\delta \to 0} |u(x) - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}}| \le \frac{c}{|x|^2}. \tag{4.7}$$

If $x = (x_1, x_2, x_3, x_4)$, we can assume, without loss of generality, that $|x| \neq 0$ and that $|x_1| = \max_j |x_j|$. We have

$$\text{(4.8)} \quad \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2}\int_A e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp + \frac{1}{(2\pi)^2}\int_B e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp + \frac{1}{(2\pi)^2}\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp$$

where $A = [-\pi,\pi]^4$, $B = [-\pi,\pi]^c \times [-\pi,\pi]^3$, and $C = R \times ([-\pi,\pi]^3)^c$. Note that

$$\text{(4.9)} \quad C = \bigcup_{j=2}^{4}\{|p_j| > \pi\}.$$

We have

$$\text{(4.10)} \quad u(x) - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^2}\int_A e^{ipx}\left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)}\right)dp - \frac{1}{(2\pi)^2}\int_B e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp - \frac{1}{(2\pi)^2}\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp.$$

We first show that

$$\text{(4.11)} \quad \lim_{\delta\to 0}\left|\frac{1}{(2\pi)^2}\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp\right| \le \frac{c}{|x|^3}.$$

To see this we integrate by parts three times in the p_1 direction to see that

$$\text{(4.12)} \quad \frac{1}{(2\pi)^2}\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp = \frac{i^3}{x_1^3}\frac{1}{(2\pi)^2}\int_C e^{ipx}D_1^3\Big(\frac{e^{-\delta Q(p)}}{Q(p)}\Big)\,dp$$

and

$$\text{(4.13)} \quad D_1^3\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right) = D_1^3(e^{-\delta Q(p)})\frac{1}{Q(p)} + 3D_1^2(e^{-\delta Q(p)})D_1^1\left(\frac{1}{Q(p)}\right) + 3D_1(e^{-\delta Q(p)})D_1^2\Big(\frac{1}{Q(p)}\Big) + e^{-\delta Q(p)}D_1^3\left(\frac{1}{Q(p)}\right)$$

Note that $\inf_{p\in B\bigcup C} Q(p) \ge d > 0$. Also, $D_1^j(\frac{1}{Q(p)})$ is homogeneous in p of degree $-(2+j)$, so that the last term in (4.13) is integrable on C even when we take $\delta = 0$. Since

$$\text{(4.14)} \quad D_1(e^{-\delta Q(p)}) = -\delta Q_1(p)e^{-\delta Q(p)}$$

and $Q_1(p)D_1^2(\frac{1}{Q(p)})$ is homogeneous in p of degree -3, scaling out δ shows that the integral of the absolute value of the third term in (4.13) is bounded by

$$\delta^{1/2}\int\frac{e^{-Q(p)}}{|p|^3}\,dp\leq c\delta^{1/2}. \tag{4.15}$$

The first two terms in (4.13) are handled similarly, proving (4.11).

We next integrate the first two terms in (4.10), by parts, twice in the p_1 direction, to get

(4.16)

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_A e^{ipx}\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp-\frac{1}{(2\pi)^2}\int_B e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp\\&=\frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_A e^{ipx}D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\\&-\frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_B e^{ipx}D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\end{aligned}$$

where we have used the fact that the boundary terms coming from the integrals over A and B cancel. (These boundary terms are easily seen to be finite). Arguing as in the proof of 4.11) we see that

$$\lim_{\delta\to 0}\left|\frac{1}{(2\pi)^2}\int_B e^{ipx}D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\right|\leq c. \tag{4.17}$$

(In fact, a further integration by parts shows that

$$\lim_{\delta\to 0}\left|\frac{1}{(2\pi)^2}\int_B e^{ipx}D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\right|\leq c/x_1 \tag{4.18}$$

as in the proof of (4.11).)

We now write

$$\begin{aligned}D_1^2&\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)\\&=D_1^2\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right)+(1-e^{-\delta Q(p)})D_1^2\left(\frac{1}{Q(p)}\right)\\&-2D_1(e^{-\delta Q(p)})D_1\left(\frac{1}{Q(p)}\right)-D_1^2(e^{-\delta Q(p)})\frac{1}{Q(p)}.\end{aligned} \tag{4.19}$$

As before, we see that the last three terms in (4.19) give rise to bounded integrals over A. (In fact, they vanish as $\delta \to 0$). More care will be needed to handle the first term

$$D_1^2\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right)=\left(\frac{\phi_{1,1}(p)}{(1-\phi(p))^2}+\frac{Q_{1,1}}{(Q(p))^2}\right)+2\left(\frac{(\phi_1(p))^2}{(1-\phi(p))^3}-\frac{(Q_1(p))^2}{(Q(p))^3}\right). \tag{4.20}$$

We write out the first term on the right hand side of (4.20) as

$$\begin{aligned}&\left(\frac{\phi_{1,1}(p)}{(1-\phi(p))^2}+\frac{Q_{1,1}}{(Q(p))^2}\right)\\&=\frac{\phi_{1,1}(p)(Q(p))^2+Q_{1,1}(1-\phi(p))^2}{(Q(p))^2(1-\phi(p))^2}\\&=\frac{(\phi_{1,1}(p)+Q_{1,1})(Q(p))^2}{(Q(p))^2(1-\phi(p))^2}\\&\quad+\frac{Q_{1,1}((1-\phi(p))^2-(Q(p))^2)}{(Q(p))^2(1-\phi(p))^2}.\end{aligned} \tag{4.21}$$

Observe that for $|p| \le 1$

$$\begin{aligned}&|1-\phi(p)-Q(p)|\\&=|E(1-e^{ip\cdot X}+ip\cdot X+(1/2)(ip\cdot X)^2)|\\&\le c|p|^3E(1_{\{|X|\le 1/|p|\}}|X|^3)+c|p|^2E(1_{\{|X|>1/|p|\}}|X|^2)\\&\le c|p|^2/\log_+(1/|p|)=o(|p|^2).\end{aligned} \tag{4.22}$$

Hence, we can bound (4.21) by

$$\frac{c|\phi_{1,1}(p)+Q_{1,1}|}{|p|^4}+\frac{c|1-\phi(p)-Q(p)|}{|p|^6}. \tag{4.23}$$

Using the second line of (4.22) we see that

$$\begin{aligned}&\int_{|p|\le 1}\frac{|1-\phi(p)-Q(p)|}{|p|^6}\,dp\\&\le cE\left(\left(\int_{\{|p|\le 1/|X|\}}\frac{1}{|p|^3}\,dp\right)|X|^3\right)\\&\quad+cE\left(\left(\int_{\{|p|>1/|X|\}}\frac{1}{|p|^4}\,dp\right)|X|^2\right)\\&\le cE(|X|^2\log_+|X|)<\infty.\end{aligned} \tag{4.24}$$

Similarly, we see that

$$(4.25)\quad |\phi_{1,1}(p) + Q_{1,1}| = |E(-X_1^2(e^{ip\cdot X} - 1))| \\ \leq c|p|E(1_{\{|X|\leq 1/|p|\}}|X|^3) + cE(1_{\{|X|>1/|p|\}}|X|^2)$$

and using this as in (4.24) we see that

$$(4.26)\qquad \int_{|p|\leq 1} \frac{|\phi_{1,1}(p) + Q_{1,1}|}{|p|^4}\, dp < \infty$$

The same methods apply to the second term on the right hand side of (4.20), completing the proof of the lemma.

Remark 1. – As $\delta \to 0$ we see that

$$(4.27)\quad u(x) - \frac{1}{(2\pi)^2|Q|^{1/2}(x\cdot Q^{-1}x)} \\ = \frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_A e^{ipx} D_1^2\left(\frac{1}{1-\phi(p)} - \frac{1}{Q(p)}\right) dp + O(1/|x|^3)$$

which together with the Riemann-Lebesgue lemma establishes 4.2.

LEMMA 6. – *Let X be a mean-zero adapted random walk in Z^4. Assume that $E(|X_1|^3) < \infty$. Then for some $C < \infty$*

$$(4.28)\qquad |u(x+a) - u(x)| \leq \frac{C|a|}{1+|x|^3},$$

for all a, x satisfying $|a| < |x|/8$.

Furthermore, for some $C < \infty$

$$(4.29)\qquad |u_t(x+a) - u_t(x)| \leq \frac{C|a|}{1+|x|^3},$$

for all a, x, t satisfying $|a| < |x|/8$ and $|x|^{1/8} < t$.

Proof of Lemma 6. – As in the proof of the previous lemma we may assume that $|x_1| = \max_j |x_j|$ and we have

$$(4.30)\quad u(x+a) - u(x) - \left(\frac{v_\delta(Q^{-1/2}(x+a))}{|Q|^{1/2}} - \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}}\right) \\ = \frac{1}{(2\pi)^2}\int_A (e^{ip(x+a)} - e^{ipx})\left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)}\right) dp \\ - \frac{1}{(2\pi)^2}\int_B (e^{ip(x+a)} - e^{ipx})\frac{e^{-\delta Q(p)}}{Q(p)}\, dp \\ - \frac{1}{(2\pi)^2}\int_C (e^{ip(x+a)} - e^{ipx})\frac{e^{-\delta Q(p)}}{Q(p)}\, dp.$$

It suffices to show that in the limit as $\delta \to 0$ the right hand side is $O\left(\frac{c|a|}{|x|^3}\right)$. By (4.11) we see immediately that this holds for the last integral in (4.30). For the first two integrals on the right hand side of (4.30) we obtain as in (4.16)

$$(4.31)\quad \frac{i^2}{(x_1+a_1)^2}\frac{1}{(2\pi)^2}\int_A e^{ip(x+a)}D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$-\frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_A e^{ipx}D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$-\frac{i^2}{(x_1+a_1)^2}\frac{1}{(2\pi)^2}\int_B e^{ip(x+a)}D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$+\frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_B e^{ipx}D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$

Using the fact that

$$\left|\frac{1}{(x_1+a_1)^2}-\frac{1}{x_1^2}\right| \le c|a|/|x|^3$$

and the arguments used to bound (4.16) it is easily seen that (4.31) is equal to

$$(4.32)\quad \frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_A e^{ipx}(e^{ipa}-1)D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$-\frac{i^2}{x_1^2}\frac{1}{(2\pi)^2}\int_B e^{ipx}(e^{ipa}-1)D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$+O_\delta(|a|/|x|^3)$$

where $O_\delta(|a|/|x|^3)$ denotes a term whose $\delta \to 0$ limit is $O(|a|/|x|^3)$. To bound the integrals in (4.32) we now integrate by parts once more in the p_1 direction to obtain

(4.33)

$$=\frac{i^3}{x_1^3}\frac{1}{(2\pi)^2}\int_A e^{ipx}D_1\left\{(e^{ipa}-1)D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)\right\}dp$$
$$-\frac{i^3}{x_1^3}\frac{1}{(2\pi)^2}\int_B e^{ipx}D_1\left\{(e^{ipa}-1)D_1^2\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)\right\}dp+O_\delta(|a|/|x|^3).$$

Once again, the (finite) boundary terms cancel. (Actually, each boundary term is $O(1/|x|^3)$.) As before, we easily see that (4.33) equals

$$
\begin{aligned}
(4.34)\quad &= \frac{i^3}{x_1^3}\frac{1}{(2\pi)^2}\int_A e^{ipx}(e^{ipa}-1)D_1^3\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp \\
&\quad - \frac{i^3}{x_1^3}\frac{1}{(2\pi)^2}\int_B e^{ipx}(e^{ipa}-1)D_1^3\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp \\
&\quad + O(|a|/|x|^3)
\end{aligned}
$$

As in the proof of (4.11), we see that

$$
(4.35)\qquad \lim_{\delta\to 0}\left|\frac{1}{(2\pi)^2}\int_B e^{ipx}(e^{ipa}-1)D_1^3\left(\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\right| \le c.
$$

To handle the first integral in (4.34) we note that

$$
\begin{aligned}
(4.36)\qquad &(e^{ipa}-1)D_1^3\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right) \\
&= (e^{ipa}-1)D_1^3\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right) \\
&\quad + (e^{ipa}-1)(1-e^{-\delta Q(p)})D_1^3\left(\frac{1}{Q(p)}\right) \\
&\quad - 3(e^{ipa}-1)D_1(e^{-\delta Q(p)})D_1^2\left(\frac{1}{Q(p)}\right) \\
&\quad - 3(e^{ipa}-1)D_1^2(e^{-\delta Q(p)})D_1\left(\frac{1}{Q(p)}\right) \\
&\quad - (e^{ipa}-1)D_1^3(e^{-\delta Q(p)})\frac{1}{Q(p)}
\end{aligned}
$$

Once again it is easy to control the last four terms in (4.36), while for the first term we use

$$
\begin{aligned}
(4.37)\qquad D_1^3\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right) &= \frac{\phi_{1,1,1}(p)}{(1-\phi(p))^2} \\
&\quad + 4\left(\frac{\phi_1(p)\phi_{1,1}(p)}{(1-\phi(p))^3}-\frac{Q_1(p)Q_{1,1}}{(Q(p))^3}\right) \\
&\quad + 6\left(\frac{(\phi_1(p))^3}{(1-\phi(p))^4}+\frac{(Q_1(p))^3}{(Q(p))^4}\right).
\end{aligned}
$$

The assumptions of our lemma give

$$1 - \phi(p) = Q(p) + O(|p|^3), \tag{4.38}$$

$$\phi_1(p) = -Q_1(p) + O(|p|^2), \tag{4.39}$$

$$\phi_{1,1}(p) = -Q_{1,1} + O(|p|), \tag{4.40}$$

and

$$\phi_{1,1,1}(p) \le C < \infty. \tag{4.41}$$

These show that

$$|(e^{ipa} - 1)D_1^3\left(\frac{1}{1-\phi(p)} - \frac{1}{Q(p)}\right)| \le \frac{c|a|}{|p|^3} \tag{4.42}$$

completing the proof of (4.28).

To prove (4.29) we first note that

$$u_{n-1}(x) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^4} \frac{e^{ipx}(1-\phi^n(p))}{1-\phi(p)}\, dp. \tag{4.43}$$

Set

$$v_\delta^n(x) = \int_\delta^n q_t(x)\, dt = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^4} e^{ipx}\frac{e^{-\delta|p|^2/2} - e^{-n|p|^2/2}}{|p|^2/2}\, dp. \tag{4.44}$$

We note that by the mean-value theorem

$$\begin{aligned} |q_t(x+a) - q_t(x)| &\le C|a| \sup_{0\le\theta\le 1} \frac{|x+\theta a|}{t} q_t(x+\theta a) \\ &\le C|a|\frac{|x|}{t} q_{2t}(x) \end{aligned} \tag{4.45}$$

where we have used the fact that under our assumptions

$$\frac{1}{2}|x| \le |x + \theta a| \le \frac{3}{2}|x|.$$

Since $t^{-1}q_t(x)$ is, up to a constant multiple, the transition density for Brownian motion in R^6, which has Green's function $C|x|^{-4}$, we have

$$|v_\delta^n(x+a) - v_\delta^n(x)| \le C|a|\int_0^\infty \frac{|x|}{t} q_t(x)\, dt \le C\frac{|a|}{|x|^3}. \tag{4.46}$$

Therefore, it suffices to bound as before an expression of the form (4.30) where u is replaced by u_{n-1} and v_δ is replaced by v_δ^n. All bounds involving

v_δ^n are handled exactly as before. We only point out that whereas in the proof of the previous lemma we were often satisfied with a bound such as (4.15), since we are taking $\delta \to 0$, we now make use of the extra factor $e^{ip(x+a)} - e^{ipx}$ with the bound

$$|e^{ip(x+a)} - e^{ipx}| \leq |a||p|$$

to guarantee that after scaling no (divergent) factors involving n will remain.

The terms invoving u_{n-1} will be handled similarly, after we make several observations. First of all, using Spitzer's trick, in Section 26 of [7], it suffices to assume that $\tau = 1$, (in Spitzer's terminology this means that X is strongly aperiodic) so that $|\phi(p)| = 1$ if and only if $p = 0$. Hence for any $\epsilon > 0$ we have that $\sup_{|p| \geq \epsilon} |\phi(p)| \leq \gamma$ for some $\gamma < 1$, so that, using our assumtion that $n - 1 > |x|^{1/8}$, we find that the factor $\phi^n(p)$ together with all its derivatives gives us rapid falloff in $|x|$. Taking ϵ sufficiently small, and using (4.38)-(4.41), we see that in the region $|p| \leq \epsilon$, the integrals involving $\phi^n(p)$ and its derivatives can be handled as in the preceeding paragraphs.

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