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Martingales on noncompact manifolds: maximal inequalities and prescribed limits

by

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ABSTRACT. – A version of the Burkholder-Davis-Gundy inequalities is presented for Γ-martingales, with respect to an arbitrary connection Γ on a Riemannian manifold (M, g). Under convexity assumptions on the manifold, some limit results are derived for "H^p Γ-martingales", i.e. those whose total Riemannian quadratic variation is in $L^{p/2}$. These are applied to the extension to noncompact manifolds of Kendall's theorem on existence and uniqueness of Γ-martingales with a prescribed limit, which is related to the Dirichlet problem for harmonic maps.

Key words: Brownian motion, martingale, manifold, gamma-martingale, connection, harmonic map, convexity.

RÉSUMÉ. – Une version des inégalités de Burkholder-Davis-Gundy est présentée pour les Γ-martingales, où Γ est une connexion quelconque sur une variété riemannienne (M, g). Avec des hypothèses de convexité sur la variété, quelques resultats sont dérivés pour les limites des Γ-martingales dont la variation riemannienne quadratique apartient à $L^{p/2}$ (« H^p Γ-martingales »). Ces résultats sont appliqués à l'extension aux variétés noncompactes du théorème de Kendall au sujet de l'existence et l'unicité des Γ-martingales ayant une limite prescrite; ce théorème est relié au problème de Dirichlet pour les applications harmoniques.

A.M.S. (1980) Subject Classification : 58G32, 60G48, 60H20

CONTENTS

1. INTRODUCTION

A Γ-martingale is a kind of stochastic process with values in a manifold M, with connection Γ, which generalizes the notion of continuous local martingale on Euclidean space. For a general introduction to Γ-martingales, giving some of the basic references, *see* Emery and

Meyer [7]. This class of processes has attracted increasing interest in recent years because it offers the possibility of supplying probabilistic solutions to significant problems in nonlinear analysis, particularly the construction of a geometrically important class of mappings known as harmonic maps, in the same way that the ordinary theory of Brownian motion and martingales on R^n supplies a probabilistic construction of harmonic functions; for a fuller discussion *see* Kendall [12], [13], Picard [21], and especially Kendall [17].

In order to carry out such a program, it is necessary to solve a certain basic problem which, at the moment, still appears quite difficult. Given a filtered probability space $(\Omega, F, P, \{F_t, t \geq 0\})$ and a random variable Z in $L^1(\Omega, F_\infty, P; R^n)$, it is trivial to construct a martingale in R^n which converges to Z a.s. as $t \to \infty$: simply condition Z on the sequence of σ-fields $\{F_t, t \geq 0\}$. Unfortunately this procedure is specific to the Euclidean connection; for a general connection on R^n (or any manifold), the geometric analogue of taking conditional expectations may not be globally or unambiguously defined, and need not be associative when we condition on F_t and then on F_s (although *see* Emery and Mokobodzki [8] and Picard [22] for some ways around these difficulties). Thus a basic problem is the following:

1.1. Problem: Convergence to a Prescribed Limit

Given a filtration $\{F_t, t \geq 0\}$ (typically generated by a Wiener process, for the sake of path continuity of local martingales), an F_∞-measurable random variable Z on a manifold M, and a connection Γ on M, does there exist a Γ-martingale $\{X_t, F_t\}$ on M with $X_t \to Z$ a.s. as $t \to \infty$? If so, under what conditions is it unique?

One application of the solution to this problem is that the non-random X_0 tells us the value of a certain harmonic mapping, when Z is suitably defined; *see* Kendall [13], [18].

Uniqueness and other questions were considered by Emery [6]. Building on Emery's convexity ideas, Kendall [13] proved existence and uniqueness when M is a small ball in a Riemannian manifold, and in fact more generally when M is compact (with boundary) with "convex geometry" (*see* below for a similar notion); here Γ is the Levi-Civita connection. The case where there is a unique solution has been completely characterized by Kendall [14], [16] in the compact case. Independently Picard [20], [21], using Malliavin calculus and distance function estimates, proved existence on small convex domains in the Riemannian case, and when the Malliavin derivative process of the limit Z has a certain deterministic bound related to curvature, and uniqueness under certain other conditions; he also gives examples of non-uniqueness and non-existence, even when M is compact. Picard [22] gives results for more general "connectors", without the use of Malliavin calculus or the restriction to the Wiener filtration. Darling [5] gives a backwards SDE construction applicable to nonlinear connections. A different but related problem, the construction of a Γ-martingale whose limiting value has a given law, was considered by Emery and Mokobodzki [8].

This paper presents a set of techniques for dealing with Γ-martingales on noncompact manifolds. These include:

• A version of the Burkholder-Davis-Gundy inequalities for martingales on manifolds (Theorem 2.4), and application to Cartan-Hadamard manifolds (Proposition 2.5);

• A method for constructing a Γ-martingale knowing only its terminal value, if that terminal value is the limit of terminal values of known Γ-martingales (Proposition 4.4);

• Conditions under which a Γ-martingale must spend its whole lifetime contained in the same convex set as its terminal value (Proposition 6.1).

These methods are applied to the extension of Kendall's existence and uniqueness results to the case of unbounded terminal values on a noncompact manifold, under convexity assumptions (Theorem 5.2). It is conjectured in 6.3 that, under suitable compactness assumptions, it may be possible to construct a Γ-martingale with prescribed limit without any convexity or curvature conditions on the connection. Finally we examine some consequences for harmonic maps if this conjecture is true.

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2. HARDY SPACES OF GAMMA-MARTINGALES

We shall work with a triple (M, g, Γ), where (M, g) is either a complete, connected, Riemannian manifold or else a compact, connected, manifold-with-boundary, and is equipped with a connection Γ which is **not** necessarily the Levi-Civita connection for this metric, but is always torsion-free. For basic information on differential forms and connections, *see* for example Darling [4]. Let $\|.\|$ denote the norm on any tangent space, and $dist\ (.,.)$ denote the distance function on $M \times M$ induced by the Riemannian metric g. The topology of M is defined by this distance function.

2.1. Some Definitions

A filtered probability space $(\Omega, F, P, \{F_t, t \geq 0\})$ is given; all events in F_0 are assumed to have probability 0 or 1. Recall that a continuous semimartingale X on (M, Γ) is called a Γ-**martingale** if, for every $f \in C^\infty(M)$, the real semimartingale

$$ {}^f I(X)_t \equiv f(X_t) - f(X_0) - (1/2) \int_{(0,t]} \nabla df(X)(dX, dX) \tag{1} $$

belongs to M_{loc}^c, the space of continuous local martingales. In that case, the property just described extends to all $f \in C^2(M)$. For $M = R^m$ with the Euclidean connection, Γ-martingales are simply m-dimensional continuous local martingales.

Given a Riemannian metric g on M (not necessarily related to Γ), we may associate with any M-valued continuous semimartingale Y a **Riemannian quadratic variation process** $\int \langle dY | dY \rangle$, given in local coordinates by

$$ \int_{(0,t]} \langle dY | dY \rangle = \int_{(0,t]} \sum g_{ij}(Y) d[Y^i, Y^j] \tag{2} $$

(*see* Emery and Meyer [7]). Suppose $0 < p < \infty$. A Γ-martingale X is called an H^p Γ-martingale on (M, g), or is said to belong to the **Hardy space** H^p, if

$$ \int_{(0,\infty)} \langle dX | dX \rangle \in L^{p/2}. \tag{3} $$

Examples of H^p Γ-martingales, when Γ is the Levi-Civita connection, include:

- bounded Γ-martingales on Cartan-Hadamard manifolds (*see* Proposition 2.5);
- Brownian motion on (M, g), stopped at a stopping-time $\tau \in L^{p/2}$.

Obviously $H^p \supset H^{p'}$ if $p < p'$.

2.2. Some Classes of Functions on a Manifold

The expression $\nabla d\varphi \geq cg$, for $\varphi \in C^2(M)$ and $c \in R$, is an abbreviation for $\nabla d\varphi(x)(\xi, \xi) \geq c\|\xi\|^2$ for all x, and all $\xi \in T_x M$, and similarly for any other $(0, 2)$-tensor. Consider the following conditions on $\varphi \in C^2(M)$:

$$d\varphi \otimes d\varphi \leq \gamma g, \text{ some } \gamma > 0; \tag{4}$$

$$d\varphi \otimes d\varphi \leq \gamma |\varphi| g, \text{ some } \gamma > 0; \tag{5}$$

$$\nabla d\varphi \geq \alpha g, \text{ some } \alpha > 0; \tag{6}$$

$$-\beta g \leq \nabla d\varphi \leq \beta g, \text{ some } \beta > 0; \tag{7}$$

$$\varphi(x_0) = 0, \text{ some } x_0 \in M. \tag{8}$$

A function $\varphi \in C^2(M)$ is called **convex** with respect to Γ if $\nabla d\varphi \geq 0$, and **strictly convex** if (6) holds. We shall say that a function $\varphi \in C^2(M)$ has **bounded derivative** if (4) holds, and **bounded second covariant derivative** (with respect to g and Γ) if (7) holds. Note that (5) is equivalent to saying that $\varphi^{1/2}$ has bounded derivative, or that $\|grad\ \varphi^{1/2}\|$ is bounded. On R^m with the Euclidean metric and connection, $\varphi(x) = |x|^2$ satisfies (5), (6), and (7). Lemma 2.6 gives more examples.

We shall now give a characterization of H^p Γ-martingales which is applicable on any Riemannian manifold. For any real-valued process H, denote $\sup\{|H_t| : 0 \leq t \leq u\}$ by H_u^*.

2.3. Characterization Lemma

A Γ-martingale X on (M, g) belongs to the Hardy space H^p if and only if, for every $f \in C^2(M)$ of bounded derivative, the continuous local martingale ${}^f I(X)$ defined in (1) *satisfies:*

$${}^f I(X)_\infty^* \in L^p. \tag{9}$$

Proof. – First, suppose X is an H^p Γ-martingale. Since f has bounded derivative, there exists $c > 0$ such that $cg - df \otimes df$ is non-negative definite. Consequently the quadratic variation process of ${}^f I(X)$ satisfies:

$$\left[{}^f I(X), {}^f I(X)\right]_t = \int_{(0,t]} df \otimes df(dX, dX) \leq \int_{(0,t]} cg(dX, dX)$$
$$= c \int_{(0,t]} \langle dX | dX \rangle.$$

It follows from (3) that the continuous local martingale ${}^fI(X)$ satisfies

$$\left[{}^fI(X), {}^fI(X)\right]_\infty \in L^{p/2}. \tag{10}$$

Now the fact that ${}^fI(X)^*_\infty \in L^p$ follows from one of the Burkholder-Davis-Gundy inequalities (*see* for example Revuz and Yor [23]).

Conversely, suppose that for all $f \in C^2(M)$ of bounded derivative, ${}^fI(X)^*_\infty \in L^p$. From the other Burkholder-Davis-Gundy inequality, it follows that (10) holds for all such f. By the Nash embedding theorem, there exists an isometric embedding of (M, g) into the Euclidean space R^s for s sufficiently large. Define smooth functions $\{f_1, \ldots, f_s\}$ on M to be the coordinate functions on R^s composed with the embedding, so that $g = df_1 \otimes df_1 + \cdots + df_s \otimes df_s$; it follows that each of $\{f_1, \ldots, f_s\}$ has bounded derivative, and

$$\int_{(0,\infty)} \langle dX | dX \rangle = \sum_{1 \le j \le s} \left[{}^{f_j}I(X), {}^{f_j}I(X)\right]_\infty \in L^{p/2}.$$

Thus X belongs to the Hardy space H^p. □

We are now going to prove a version of the Burkholder-Davis-Gundy inequality. In the sequel, c, c', c'' will denote constants, usually depending on p, α, etc., which may vary from line to line.

2.4. Theorem: Burkholder-Davis-Gundy Inequality for Gamma-martingales

Suppose $0 < p < \infty$, *and* $\varphi \in C^2(M)$ *satisfies* $d\varphi \otimes d\varphi \le \gamma|\varphi| g$ *and* $\varphi(x_0) = 0$.

(i) *If furthermore* $-\beta g \le \nabla d\varphi \le \beta g$, *then there exists a constant* $c \equiv c(p, \beta, \gamma)$ *such that, for every* Γ*-martingale* X *on* (M, g) *with* $X_0 = x_0$,

$$E[\sup\{|\varphi(X_t)|^p : t \ge 0\}] \le cE\left[\left\{\int_{(0,\infty)} \langle dX | dX \rangle\right\}^p\right]. \tag{11}$$

(ii) *If on the other hand* $\nabla d\varphi \ge \alpha g$ *for some* $\alpha > 0$, *then there exists a constant* $c' \equiv c'(p, \alpha, \gamma)$ *such that, for every* Γ*-martingale* X *on* (M, g) *with* $X_0 = x_0$,

$$E\left[\left\{\int_{(0,\infty)} \langle dX | dX \rangle\right\}^p\right] \le c'E[\sup\{|\varphi(X_t)|^p : t \ge 0\}]. \tag{12}$$

Remark. – There exist Riemannian manifolds on which there exists no $\varphi \in C^2(M)$ such that $-\beta g \le \nabla d\varphi \le \beta g$ and $\varphi(x) \to \infty$ as $x \to \infty$; *see*

Counterexample 2.9. On a manifold with a closed geodesic, such as the sphere, no such φ as in part (ii) exists. Indeed for Brownian motion on a sphere, the left side of (12) is always infinite, while the right side is finite for every continuous φ. These estimates are not optimal for estimating passage time probabilities: *see* Darling [3].

Proof. – (i) Let $N \in M^c_{\text{loc}}$ be the process ${}^{\varphi}I(X)$ as in (1). Using (5), we have

$$d[N,N] = d\varphi \otimes d\varphi(dX, dX) \leq \gamma|\varphi|\langle dX|dX\rangle.$$

For any positive integer n, let $\tau(n) \equiv \inf\{t \geq 0 : |\varphi(X_t)| \geq n\}$. By the usual Burkholder-Davis-Gundy inequality,

$$E[\{N^*_{\tau(n)}\}^p] \leq c(p)E[[N,N]^{p/2}_{\tau(n)}]$$
$$\leq c(p,\gamma)E\left[\left\{\int_{(0,\tau(n)]} \langle dX|dX\rangle\right\}^{p/2}\right].$$

Take

$$u \equiv E[\{\varphi(X)^*_{\tau(n)}\}^p]^{1/2} \leq n^{p/2}$$

and $\nu \equiv E\left[\left\{\int_{(0,\tau(n)]} \langle dX|dX\rangle\right\}^p\right]^{1/2}$. Cauchy-Schwarz gives

$$E[\{N^*_{\tau(n)}\}^p] \leq c(p,\gamma)u\nu. \tag{13}$$

On the other hand if $A \equiv (1/2)\int \nabla d\varphi(dX, dX)$, (1) implies that

$$\varphi(X_t) = \varphi(X_0) + N_t + A_t, \tag{14}$$

with $\varphi(X_0) = 0$ by (8), and (7) implies that

$$A^*_{\tau(n)} \leq (\beta/2)\int_{(0,\tau(n)]} \langle dX|dX\rangle. \tag{15}$$

Since $|u+\nu|^p \leq c(p)(|u|^p + |\nu|^p)$, (13), (14), and (15) imply that

$$u^2 = E[\{\varphi(X)^*_{\tau(n)}\}^p] \leq c(p,\beta,\gamma)(E[\{N^*_{\tau(n)}\}^p]+\nu^2) \leq c'u\nu+c'\nu^2. \tag{16}$$

Either $\nu = \infty$, in which case (11) holds vacuously, or else $u^2 - c'u\nu - c'\nu^2 \leq 0$ with $u \leq n^{p/2}$, which implies u is bounded above by the positive root of the quadratic $u^2 - c'u\nu - c'\nu^2 = 0$, which is of the form $c\nu$, where

$c = c(p, \beta, \gamma)$; now let $n \to \infty$ and use monotone convergence to obtain (11) (this argument is taken from Revuz and Yor [23]).

(ii) Redefine $\tau(n)$ as $\inf\left\{t \geq 0 : \int_{(0,t]} \langle dX | dX \rangle \geq n\right\}$. By (14), $A_t = \varphi(X_t) - N_t$, and so (6) gives

$$\nu^2 \leq c(\alpha, p) E[\{A^*_{\tau(n)}\}^p] \leq c'(\alpha, p)(u^2 + E[\{N^*_{\tau(n)}\}^p]). \tag{17}$$

Now (13) shows that $\nu^2 \leq c''u^2 + c''u\nu$, where $c'' \equiv c''(p, \alpha, \gamma)$, and as in the previous part of the proof, $\nu \leq c'(p, \alpha, \gamma)u$; thus (12) follows on letting $n \to \infty$. $\square$

It is important to give examples of non-Euclidean manifolds on which the conditions of Theorem 2.4 can occur. Recall that a Cartan-Hadamard manifold (M, g) is one which is diffeomorphic to some R^m, and all of whose sectional curvatures are non-positive. Take a pole $o \in M$, and a polar coordinate system $(r, \theta^1, \ldots, \theta^{m-1})$ on $M - \{o\}$; thus $r(x) \equiv dist(x, o)$, and the metric tensor g can be expressed as $dr \otimes dr$ plus a metric on the sphere S^{m-1}.

2.5. Proposition: Maximal Inequalities on a Cartan-Hadamard Manifold

In the following, Γ *will be the Levi-Civita connection, and* $0 < p < \infty$.

(i) *There is a universal constant* $c(p)$ *such that for every Cartan-Hadamard manifold* (M, g) *with pole* $o \in M$, *and every* Γ*-martingale* X *on* M *with* $X_0 = o$,

$$E\left[\left\{\int_{(0,\infty)} \langle dX | dX \rangle\right\}^{p/2}\right] \leq c(p) E[\sup\{|r(X_t)|^p : t \geq 0\}]. \tag{18}$$

(ii) *There is a universal constant* $c(p, \kappa)$ *such that for every Cartan-Hadamard manifold* (M, g) *with pole* $o \in M$, *with sectional curvatures bounded below by* $-\kappa^2$, *and every* Γ*-martingale* X *on* M *with* $X_0 = o$,

$$E[\sup\{|\Psi(X_t)|^p : t \geq 0\}] \leq c(p, \kappa) E\left[\left\{\int_{(0,\infty)} \langle dX | dX \rangle\right\}^p\right] \tag{19}$$

where

$$\Psi = \left(r - \frac{3}{8}\right) 1_{\{r>1\}} + \left(\frac{6r^2 - r^4}{8}\right) 1_{\{r \leq 1\}}. \tag{20}$$

The proof follows immediately from Theorem 2.4 and the following Lemma.

2.6. Lemma: Properties of the Distance Function

(i) *On any Cartan-Hadamard manifold, the function* $\varphi \equiv r^2$ *is in* $C^2(M)$, *and satisfies* $d\varphi \otimes d\varphi \leq 4|\varphi|g$, $\nabla d\varphi \geq 2g$, *and* $\varphi(o) = 0$.

(ii) *On a Cartan-Hadamard manifold whose sectional curvatures are bounded below by* $-\kappa^2$, *the function* Ψ *given in (20) is in* $C^2(M)$, *and satisfies* $d\Psi \otimes d\Psi \leq (4|\Psi| \wedge 1)g$, $\Psi(o) = 0$, *and* $0 \leq \nabla d\Psi \leq \beta(\kappa)g$, *where the constant* $\beta(\kappa)$ *depends on* κ.

Proof. – (i) Smoothness of the squared distance function $\varphi \equiv r^2$ on a Cartan-Hadamard manifold is well known. Clearly $d(r^2) \otimes d(r^2) = 4r^2 dr \otimes dr \leq 4r^2 g$. Let r_0 and r_1 denote the distance functions on Euclidean R^m and on the Cartan-Hadamard manifold of constant sectional curvature $-\kappa^2$, respectively, with corresponding metrics g_0 and g_1. Recall from Greene and Wu [10] that

$$\begin{aligned} \nabla d(r_0^2) &= 2g_0, \\ \nabla d(r_1^2) &= 2\kappa r \, coth(\kappa r)(g_1 - dr_1 \otimes dr_1) + 2dr_1 \otimes dr_1. \end{aligned} \tag{21}$$

Now apply the Hessian Comparison Theorem of Greene and Wu [10] to see that $2g \leq \nabla d(r^2)$. The proof of (i) is now complete.

Under the assumptions of (ii), the Hessian Comparison Theorem gives

$$\nabla d(r^2) \leq 2\kappa r \, coth(\kappa r)(g - dr \otimes dr) + 2dr \otimes dr. \tag{22}$$

The function Ψ given in (20) is the composition with r^2 of the C^2 function

$$h(t) \equiv \left(t^{1/2} - \frac{3}{8}\right) 1_{\{t>1\}} + \left(\frac{6t - t^2}{8}\right) 1_{\{t \leq 1\}}. \tag{23}$$

Note that $\nabla d\Psi = \nabla d(h \bullet r^2) = h'(r^2)\nabla d(r^2) + h''(r^2)d(r^2) \otimes d(r^2)$; hence on the set $\{r > 1\}$, where $h'(r^2) = r^{-1}/2$ and $h''(r^2) = -r^{-3}/4$, (22) gives

$$\nabla d\Psi \leq \kappa \, coth(\kappa r)(g - dr \otimes dr) + r^{-1} dr \otimes dr - r^{-1} dr \otimes dr,$$

and therefore $\nabla d\Psi \leq \kappa \, coth(\kappa)g$ on $\{r > 1\}$. Obviously $\nabla d\Psi \leq \beta(\kappa)g$ follows by compactness of $\{r \leq 1\}$. Likewise the inequality $2g \leq \nabla d(r^2)$ gives

$$\nabla d\Psi \geq r^{-1} g - r^{-1} dr \otimes dr \geq 0.$$

Finally $d\Psi \otimes d\Psi = dr \otimes dr$ on $\{r > 1\}$, while on $\{r \leq 1\}$ we have

$$d\Psi \otimes d\Psi = \left(r(3 - r^2)/2\right)^2 dr \otimes dr \leq 4(r(6 - r)/8)^2 dr \otimes dr \leq 4|\Psi|g$$

which completes the proof that (ii) holds. □

Now we shall give some useful consequences of Theorem 2.4.

2.7. Corollary: Riemannian Quadratic Variation of Bounded Martingales

Suppose (M, g, Γ) is a manifold containing a compact set K such that, for each $y \in K$, there exists $\varphi_y \in C^2(M)$ satisfying $\varphi_y(y) = 0$, $\nabla d\varphi_y \geq \alpha g$ and $d\varphi_y \otimes d\varphi_y \leq \gamma|\varphi_y|g$ for some $\alpha > 0$ and $\gamma > 0$, uniformly in y. Then there exists for every $0 < p < \infty$ a universal constant $c(p, \alpha, \gamma, K)$ such that

$$E\left[\left\{\int_{(0,\infty)} \langle dX|dX\rangle\right\}^p\right] \leq c(p, \alpha, \gamma, K) \tag{24}$$

for every Γ-martingale X on M such that $P(X_t \in K, \forall t \geq 0) = 1$.

Proof. – Given such a Γ-martingale, there exists $y \in K$ such that $X_0 = y$ a.s. Apply Theorem 2.4 to $\varphi \equiv \varphi_y$. $\square$

The following simple fact will have several applications.

2.8. Corollary: Convex Functions and Submartingales

Suppose $\varphi \in C^2(M)$ is a convex function such that $0 \leq \nabla d\varphi \leq \beta g$ and $d\varphi \otimes d\varphi \leq \gamma g$ for some β, γ. If X is an H^p Γ-martingale on (M, g, Γ) for some $p \geq 2$, then $\varphi \bullet X$ is a uniformly integrable submartingale with

$$E[\sup\{|\varphi(X_t)|^{p/2} : t \geq 0\}] < \infty. \tag{25}$$

Proof. – Suppose $0 \leq \nabla d\varphi \leq \beta g$ and $d\varphi \otimes d\varphi \leq \gamma g$ hold. As in (14), we may write $\varphi(X_t) = \varphi(X_0) + N_t + A_t$, where the continuous local martingale $N \equiv {}^{\varphi}I(X)$ is actually a martingale, since $E[\{N^*_\infty\}^p] < \infty$ by (9); moreover $\varphi(X_0)$ is a.s. nonrandom, and convexity of φ ensures that A is an increasing process with $A_\infty \in L^{p/2}$ by (15). So $\varphi \bullet X$ is a submartingale satisfying (25), hence uniformly integrable. $\square$

Emery and Meyer [7] give the following counterexample (attributed to Prat) to show how finiteness of the Riemannian quadratic variation can fail to imply that a Γ-martingale has an almost sure limit in M. Here we extend it further to give a counterexample related to Theorem 2.4, showing how (11) can break down without appropriate assumptions on the function φ, *i.e.* how the Riemannian quadratic variation can fail to control the distance of the process for certain manifolds.

2.9. Counterexample: An Exploding Gamma-Martingale in Class $H(2)$

There exists a metric on the cylinder $M = R \times S^1$ such that (M, g) is complete, and Brownian motion B on (M, g) is an H^2 Γ-martingale (with respect to the Levi-Civita connection) such that $B_t \to \infty$ a.s. as t tends to an explosion time ζ.

Proof. – Take a metric on $M = R \times S^1$ whose expression in cylindrical coordinates (u, θ) is

$$g = du \otimes du + e^{2h(u)} d\theta \otimes d\theta. \tag{26}$$

Emery and Meyer [7], Section 5.40, point out that such a manifold is always complete, and that the radial part ξ of Brownian motion B on (M, g) is a diffusion satisfying the Itô stochastic differential equation $d\xi_t = dW_t + h'(\xi_t)dt$, where W is a one-dimensional Wiener process. From Exercise IX.2.15 of Revuz and Yor [23] we see that the explosion time ζ of ξ satisfies $E_u[\zeta] < \infty$ and $P_u(\zeta < \infty)$ for all $u \in R$ provided $\kappa(\infty) < \infty$ and $\kappa(-\infty) < \infty$, where

$$\kappa(z) \equiv 2 \int_{[0,z]} \int_{[0,z]} e^{-2(h(x)-h(y))} 1_{\{|y|<|x|\}} dx\, dy. \tag{27}$$

Thus $E_u[\zeta] < \infty$ for example if $h(u) = u^4$ (convert the integral in (27) to polar coordinates, change variables, and use properties of the Normal distribution). B is well known to be a Γ-martingale with respect to the Levi-Civita connection Γ, and in this case will be an H^2 Γ-martingale because

$$E_u\left[\int_{(0,\infty)} \langle dB | dB \rangle\right] = E_u\left[\int_{(0,\zeta)} 2dt\right] = E_u[2\zeta] < \infty. \tag{28}$$

On the other hand, $P_u(\zeta < \infty)$ implies that B has a finite explosion time almost surely, and so with probability one it converges to ∞. $\square$

3. CONVEX GEOMETRY

Recall that (M, g) is either a complete Riemannian manifold or else a compact manifold-with-boundary. Recall from Emery and Meyer [7] that, for any connection Γ on M, the product connection $\Gamma^{(2)}$ on $M \times M$ is the one whose geodesics are precisely of the form $t \to (\gamma(t), \delta(t))$ where γ

and δ are arbitrary geodesics on (M, Γ). Generalizing to the non-compact case a definition of Kendall [13], we shall say that the triple (M, g, Γ), has **Ψ-convex geometry** if there exists a C^2 non-negative function

$$\Psi : M \times M \to [0, \infty) \tag{29}$$

which is convex with respect to the product connection $\Gamma^{(2)}$, also written $\nabla^{(2)} d\Psi \geq 0$, and which has the following three properties. Here $g^{(2)} \equiv g \oplus g$ denotes the product metric on $M \times M$.

3.0.1. Bounded First Derivative

There exists $c_1 > 0$ such that $d\Psi \otimes d\Psi \leq c_1 g^{(2)}$.

3.0.2. Bounded Second Covariant Derivative

There exists $c_2 > 0$ such that $0 \leq \nabla^{(2)} d\Psi \leq c_2 g^{(2)}$.

3.0.3. Distance Goes to Zero as Ψ Goes to Zero

$\Psi(x, x) = 0$ for all x, and given $\epsilon > 0$, there exists $\delta > 0$ such that $\Psi(x, x') \leq \delta \Rightarrow dist(x, x') \leq \epsilon$, or in other words

$$\{(x, x') : dist(x, x') > \epsilon\} \subseteq \{(x, x') : \Psi(x, x') > \delta\}. \tag{30}$$

Note in particular that this implies that the diagonal $\Delta \equiv \{(x, x) : x \in M\}$ is precisely the set on which Ψ is zero.

Since the notion of Ψ-convex geometry will be assumed in most of our studies of H^p Γ-martingales, it is important to know something about how restrictive this condition is. Proposition (4.59) of Emery and Meyer [7] demonstrates Ψ-convex geometry in a sufficiently small neighborhood of any point in (M, g, Γ). Kendall [13], [15] has demonstrated that geodesic balls of radius less than $\pi/2$ in Riemannian manifolds with sectional curvature bounded above by 1 have Ψ-convex geometry for the Levi-Civita connection (he allows Ψ to be C^0 rather than C^2); there of course M is compact with boundary. For complete manifolds with the Levi-Civita connection, the existence of Ψ-convex geometry has severe topological consequences.

3.1. Proposition: Topological Triviality

If (M, g) is a non-compact, complete, connected, m-dimensional Riemannian manifold with Levi-Civita connection Γ, and if (M, g, Γ) has Ψ-convex geometry for some Ψ, then M is diffeomorphic to R^m.

Proof. – Fix $x_0 \in M$, and consider the convex function $\varphi(x) \equiv \Psi(x, x_0)$ on M. Since the set on which φ attains its minimum is $\{x_0\}$, which has empty interior, φ must be "locally nonconstant" in the sense of Greene and Shiohama [9], whose Proposition 2.1 shows that there exists a deformation retract from M to $\{x_0\}$. Since M is complete and noncompact, it must be diffeomorphic to R^m. □

We shall now give a straightforward class of examples of Riemannian manifolds with Ψ-convex geometry, including of course the Euclidean case.

3.2. Proposition: Cartan-Hadamard Manifolds with Convex Geometry

Suppose (M, g) is a Cartan-Hadamard manifold, i.e. complete, with non-positive sectional curvatures, and diffeomorphic to R^m, with the Levi-Civita connection Γ. If the sectional curvatures are bounded below by $-\kappa^2$ for some κ, then (M, g, Γ) has Ψ-convex geometry, using the function

$$\Psi(x, x') = \{(6\rho^2 - \rho^4)/8\}1_{\{\rho \leq 1\}} + \{\rho - 3/8\}1_{\{\rho > 1\}} \tag{31}$$

where $\rho \equiv dist(x, x')$.

Proof. – To show that Ψ is convex, it suffices to recall the classical result that for any Cartan-Hadamard manifold, the function $(x, x') \to dist(x, x')$ is convex with respect to $\Gamma^{(2)}$ (a proof of a more general result is given in Picard [21], Lemma 1.1.1). The boundedness of the first derivative follows from the fact that $x \to \Psi(x, y)$ and $y \to \Psi(x, y)$ have bounded derivative by Lemma 2.6. As for the second covariant derivative, we may write

$$\nabla^{(2)} d\Psi(x, y)(\zeta \oplus \xi, \zeta \oplus \xi) = \nabla^{(2)} d\Psi(x, y)(\zeta \oplus 0, \zeta \oplus 0)$$

$$+\nabla^{(2)} d\Psi(x, y)(0 \oplus \xi, 0 \oplus \xi) + 2\nabla^{(2)} d\Psi(x, y)(\zeta \oplus 0, 0 \oplus \xi).$$

The boundedness of the first two terms on the right follows from Lemma 2.6, and boundedness of the third term from Picard [21], Lemma 1.2.1; property 3.0.3 is immediate from (31). □

4. ASYMPTOTICS FOR MARTINGALES UNDER CONVEX GEOMETRY

Most of the results of this section are extensions to unbounded Γ-martingales of propositions for bounded Γ-martingales found in Emery [6] and Kendall [13].

4.1. Lemma: Convex Geometry Implies Martingale Limits Lie in M

Suppose (M, g, Γ) *has* Ψ*-convex geometry. If* Y *is an* H^p Γ*-martingale for some* $p > 0$*, then there exists a random variable* $Y_\infty \in M$ *such that* $Y_t \to Y_\infty$ a.s. as $t \to \infty$.

Remark. – Actually we shall only use the fact that $\int_{(0,\infty)} \langle dY|dY\rangle < \infty$ a.s.

Proof. – If M is compact with boundary, then the result follows from (3) and Darling [2] (but *see* the proof in Emery and Meyer [7], where the fact that Γ may be an arbitrary connection is apparent). When (M, g) is complete and non-compact, $\Psi(x, x_0) \to \infty$ as $dist(x, x_0) \to \infty$ for fixed $x_0 \in M$. This holds because, for any geodesic $\gamma : (-\infty, \infty) \to M$ with $\gamma(0) = x_0$, $f(t) \equiv \Psi(x_0, \gamma(t))$ is convex on $(-\infty, \infty)$, nonnegative, and zero only at $t = 0$; therefore $f(t) \to \infty$ as $t \to \pm\infty$. Since the function $\Psi(x) \equiv \Psi(x, x_0)$ has bounded derivative and bounded second covariant derivative, the result now follows from Proposition (5.37) of Emery and Meyer [7]. $\square$

4.2. Proposition: Cartesian Product of Two Martingales

Suppose X *and* Y *are* H^p Γ*-martingales on* (M, g, Γ)*. Then* $U \equiv (X, Y)$ *is an* H^p $\Gamma^{(2)}$*-martingale on* $(M \times M, g \oplus g)$*. Moreover if* (M, g, Γ) *has* Ψ*-convex geometry, and if* $p \geq 2$*, then* $\Psi(X, Y)$ *is a non-negative submartingale such that*

$$E[\sup\{\Psi(X_t, Y_t)^{p/2} : t \geq 0\}] < \infty. \tag{32}$$

Proof. – The Riemannian quadratic variation process of U is

$$\begin{aligned}\int_{(0,t]} \langle dU|dU\rangle &= \int_{(0,t]} \sum g_{ij}(X) d[X^i, X^j] + \int_{(0,t]} \sum g_{kl}(Y) d[Y^k, Y^\ell] \\ &= \int_{(0,t]} \langle dX|dX\rangle + \int_{(0,t]} \langle dY|dY\rangle\end{aligned}$$

so it is clear from the definition (3) that, if X and Y are H^p Γ-martingales, then so is $U \equiv (X, Y)$. The remainder of the proof follows immediately from Corollary 2.8 and properties 3.0.1 and 3.0.2 of Ψ. $\square$

The following result uses an idea due to Emery [6].

4.3. Corollary: Uniqueness of Martingales with Prescribed Limit

Suppose $p \geq 2$ *and* (M, g, Γ) *has* Ψ*-convex geometry. If* X *and* Y *are* H^p Γ*-martingales on* (M, g, Γ) *with a.s. limits* X_∞ *and* Y_∞ *(which must lie in* M *by Lemma* 4.1*) such that* $P(X_\infty = Y_\infty) = 1$*, then* $P(X_t = Y_t,\ \forall t \geq 0) = 1$*. In other words, an* H^2 Γ*-martingale on a manifold with* Ψ*-convex geometry is uniquely determined by its limiting value.*

Proof. – By (32), the non-negative random variables $\{\Psi(X_t, Y_t), t \geq 0\}$ are dominated by an integrable random variable, using the fact that $p \geq 2$; moreover these random variables converge to zero a.s. as $t \to \infty$, using property 3.0.3 of Ψ and the assumption that $\lim\limits_{t \to \infty} dist(X_t, Y_t) = 0$ a.s.; hence by dominated convergence,

$$\lim_{t \to \infty} E[\Psi(X_t, Y_t)] = 0.$$

The submartingale property implies that these expectations are non-decreasing in t, and so $E[\Psi(X_t, Y_t)] = 0$ for all t. Since the processes X and Y are continuous, it follows from property (30) of Ψ that

$$P(\sup\{dist(X_t, Y_t) : t \geq 0\} > 0) = 0. \quad \square$$

4.3.1. Counterexample for General Gamma-Martingales

If we drop the requirement that X and Y are in H^2, it is possible for different Γ-martingales to have the same terminal value. A trivial example is to take Γ to be the Euclidean connection on $M = R$, X to be identically 1, and Y to be Brownian motion started at 0 and stopped at $\tau(1)$, the first passage time to 1. These have the same limit because $\tau(1) < \infty$ a.s., but $\int \langle dY | dY \rangle = \tau(1)$ which is not integrable.

The next result plays a central role in this paper; it is in some ways an extension of Theorem (4.43) of Emery and Meyer [7], and follows the pattern of Theorem 5.5 of Kendall [13].

4.4. Proposition: a Sequence of Martingales Converges to a Limit Martingale

Suppose (M, g, Γ) *has* Ψ*-convex geometry, and* $\{Y^{(n)}\}$ *is a sequence of* H^p Γ*-martingales on* M*, where* $p \geq 2$*, having the following properties:*

(i) *The sequence of limiting values* $\{Y_\infty^{(n)}\}$ *(which exist a.s. in* M *by Lemma* 4.1*) converges in probability to some random variable* V *in* M *as* $n \to \infty$.

(ii) $E[\Psi(Y_\infty^{(n)}, Y_\infty^{(n')})] \to 0$ *as* $n, n' \to \infty$.

Then there exists a unique (up to indistinguishability) Γ-martingale Y such that $Y_t \to V$ a.s. as $t \to \infty$, and such that $\sup\{dist(Y_t^{(n)}, Y_t) : 0 \leq t \leq \infty\}$ *converges in probability to* 0 *as* $n \to \infty$.

Proof. – For fixed n and n', Proposition 4.2 shows that $\Psi(Y^{(n)}, Y^{(n')})$ is a non-negative submartingale such that

$$E[\sup\{\Psi(Y_t^{(n)}, Y_t^{(n')}) : t \geq 0\}] < \infty.$$

Given $\epsilon > 0$, choose $\delta > 0$ as in 3.0.3. Applying Doob's inequality (*see* Revuz and Yor [23]), we see that

$$P(\sup\{\Psi(Y_t^{(n)}, Y_t^{(n')}) : t \geq 0\} > \delta) \leq \delta^{-1} E[\Psi(Y_\infty^{(n)}, Y_\infty^{(n')})], \quad (33)$$

using the fact that $Y_t^{(n)}$ converges a.s. to $Y_\infty^{(n)}$ as $t \to \infty$. By (30) and assumption (ii), we have

$$P(\sup\{dist(Y_t^{(n)}, Y_t^{(n')}) : 0 \leq t \leq \infty\} > \epsilon) < \epsilon$$

for all sufficiently large n and n'. Thus the sequence of Γ-martingales $\{Y^{(n)}\}$ is a Cauchy sequence with respect to the topology of uniform convergence in probability on $[0, \infty]$. There exists a subsequence which is Cauchy with respect to uniform convergence a.s., and which therefore has a unique continuous adapted process Y on $[0, \infty]$ as its limit; all such subsequences have the same limit a.s. Necessarily we have, for all $\epsilon > 0$,

$$\lim_{n \to \infty} P(\sup\{dist(Y_t^{(n)}, Y_t) : 0 \leq t \leq \infty\} > \epsilon) = 0. \quad (34)$$

It follows from (34) that the a.s. limit of Y_t as $t \to \infty$ (which exists since Y is continuous) must be the V specified in assumption (i). The crucial fact that Y is a Γ-martingale follows from Theorem (4.43) of Emery and Meyer [7]. □

5. CONSTRUCTING MARTINGALES WITH PRESCRIBED LIMIT

Theorem 5.2 below gives a general procedure for extending results on existence and uniqueness of Γ-martingales with prescribed limit from compact to noncompact manifolds. There are now half a dozen such results for the compact case, under various geometric assumptions, and various assumptions about the filtration $\{F_t\}$, but it may be permissible to say that a definitive result has not yet been published. In anticipation of such a result, let us make the following definition.

5.1. Definition: Gamma-Martingale Dirichlet Property

A compact submanifold-with-boundary K of (M, g, Γ) will be said to have the Γ-**martingale Dirichlet property** *with respect to the filtration $\{F_t\}$ if, for every F_∞-measurable random variable V with values in K, there exists an $\{F_t\}$ Γ-martingale X on M whose almost sure limit is V, and such that $X_t \in K$ for all t, a.s.*

When $\{F_t\}$ is the Wiener filtration, Kendall [13] established the Γ-martingale Dirichlet property for the Levi-Civita Γ on a geodesic ball with convex geometry, and Picard [21] obtained a similar result under restrictions on the size of K or on the limiting value V; an entirely different approach, applicable to nonlinear connections, is given in Darling [5]. For results for more general filtrations, with restrictions on the limiting value V, *see* Picard [22].

Since the conditions of Theorem 5.2 seem somewhat abstruse, we present a concrete example in Corollary 5.3.

5.2. Theorem: Martingales with Prescribed Limit on Noncompact Manifolds

Suppose (M, g, Γ) has Ψ-convex geometry, and furthermore that there exist compact submanifolds-with-boundary $K_1 \subset K_2 \subset \ldots$ whose union is M, such that:

(i) *Each K_n has the Γ-martingale Dirichlet property with respect to $\{F_t\}$.*

(ii) *For each n and for each $y \in K_n$, there exists $\varphi_y \in C^2(K_n)$ satisfying $\varphi_y(y) = 0$, $\nabla d\varphi_y \geq \alpha_n g$ and $d\varphi_y \otimes d\varphi_y \leq \gamma_n |\varphi_y| g$ for some $\gamma_n > 0$, $\alpha_n > 0$, uniformly in y.*

Then for every F_∞-measurable random variable V on M such that, for some $o \in M$,

$$E[\Psi(V, o)] < \infty \tag{35}$$

there exists a Γ-martingale Y on M whose almost sure limit is V. For $p \geq 2$, there is at most one H^p Γ-martingale with this property.

Proof. – Let $V^{(n)}$ be the F_∞-measurable random variable which equals V when $V \in K_n$, and takes the value o otherwise. By Definition 5.1, there exists a Γ-martingale $Y^{(n)}$ with values in K_n, with $Y_\infty^{(n)} = V^{(n)}$ a.s. Take $y \equiv Y_0^{(n)}$; by compactness

$$\sup\{|\varphi_y(Y_t^{(n)})|^p : t \geq 0\} \leq c(p, n).$$

Using Corollary 2.7, valid since $d\varphi_y \otimes d\varphi_y \leq \gamma_n |\varphi_y| g$ and $\nabla d\varphi_y \geq \alpha_n g$,

$$E\left[\left\{\int_{(0,\infty)} \langle dY^{(n)} | dY^{(n)} \rangle\right\}^{p/2}\right] \leq c'(p,n),$$

and so $Y^{(n)}$ is an H^p Γ-martingale, for every $0 < p < \infty$. Observe that for $n \leq n'$,

$$E[\Psi(Y_\infty^{(n)}, Y_\infty^{(n')})] \leq E[\Psi(V, o) : V \notin K_n] \to 0$$

as n, $n' \to \infty$ by 3.0.3, (35), and dominated convergence. Evidently $Y_\infty^{(n)}$ converges in probability to V as $n \to \infty$. Now all the conditions are in place to apply Proposition 4.4, which gives the existence of a limiting Γ-martingale Y with $Y_\infty = V$ a.s. The uniqueness assertion is a restatement of Corollary 4.3. □

5.3. Corollary: Martingales with Prescribed Limit on CH Manifolds

Let (M, g, Γ) be a Cartan-Hadamard manifold with sectional curvatures bounded below by $-\kappa^2$, where Γ is the Levi-Civita connection, and let $r(x) \equiv dist(x, 0)$ for some pole $o \in M$. If $\{F_t\}$ is a Wiener filtration, then for every F_∞-measurable random variable V on M such that

$$E[r(V)] < \infty, \tag{36}$$

there exists a Γ-martingale Y on M whose almost sure limit is V. For $p \geq 2$, there is at most one H^p Γ-martingale with this property.

Proof. – To apply Theorem 5.2, we take K_n to be the closed geodesic ball in M of radius n, which is a compact submanifold-with-boundary for each n, and take Ψ to be the function described in Proposition 3.2. By Kendall [13], each K_n has the Γ-martingale Dirichlet property with respect to the Wiener filtration. On each K_n, the function $\varphi_y(x) \equiv dist(x, y)^2$ plays the role of φ_y, using Lemma 2.6. □

6. MARTINGALE CONTAINMENT AND A CONJECTURE

6.1. Proposition: Martingales Are Constrained by their Terminal Values

Suppose $\Phi \in C^2(M)$ has bounded derivative and bounded second covariant derivative, $G \equiv \{x \in M : \Phi(x) \leq 0\}$, and the restriction of Φ

to the complement of G is convex. If Y is an H^p Γ-martingale on (M, g, Γ), where $p \geq 2$, whose a.s. limit $Y_\infty \in G$ a.s., then $Y_t \in G$ for all t, with probability one.

Remark. – This result is closely related to Theorem 2.1.3 of Picard [21]. It can best be understood as a generalization to manifolds of the order-preserving property associated with taking the conditional expectation of real-valued random variables. In fact it is easy to show that Φ is convex on the whole of M, and hence $\{x \in M : \Phi(x) \leq 0\}$ is 'totally convex' in the sense that it contains every geodesic segment whose endpoints lie in G_a (*see* Bishop and O'Neill [1]). Thus a 'good' Γ-martingale is confined to the same totally convex set as its terminal value.

Proof. – Fix $n \geq 1$ and a time $r \geq 0$, and define a stopping-time σ by

$$\sigma = \inf\{t \geq r : \Phi(Y_t) \leq 1/n\}. \tag{37}$$

Observe that $P(\sigma < \infty) = 1$ and $\Phi(Y_\sigma) \leq 1/n$ because $\Phi(Y_\infty) \leq 0$ a.s. It follows from Corollary 2.8 that the process $\{\Phi(Y_{t\wedge\sigma}) : t \geq r\}$ is a uniformly integrable submartingale, (when $\sigma = r$ the process is constant) and therefore

$$\Phi(Y_r) \leq E[\Phi(Y_\sigma)|F_r] \leq 1/n, \text{ a.s.} \tag{38}$$

Since (38) holds for all n, we see that $P(Y_r \in G) = 1$; but r was arbitrary, so with probability one, $Y_t \in G$ for every rational t, and hence for all t, by path continuity. $\square$

Here is a simple application.

6.2. Corollary: Flatness Outside a Compact Set

Suppose Γ is a connection on Euclidean $(R^m, \langle .,. \rangle)$ whose Christoffel symbols vanish outside $\{x : |x - x_0| \leq a\}$. If $p \geq 2$, and X is an H^p Γ-martingale with terminal value $X_\infty \in \{x : |x - x_0| \leq a\}$ a.s., then $P(|X_t - x_0| \leq a, \forall t) = 1$.

Proof. – Let $x_0 = 0$, for brevity. All the conditions of the preceding proposition are satisfied by $\Phi(x) \equiv \{6|x/a|^2 - |x/a|^4 - 5\}1_{\{|x| \leq a\}} + 8\{|x/a| - 1\}1_{\{|x| > a\}}$, and $|x| \leq a \Leftrightarrow \Phi(x) \leq 0$. $\square$

Let us assume now that $0 \leq T \leq \infty$, and the filtration $\{F_t\}$ is generated by an ℓ-dimensional Wiener process $\{W(t), 0 \leq t < T\}$ $(0 \leq t \leq T$ if $T < \infty)$. A connection Γ on R^m will be called **compactly supported** if, in some coordinate system, all the Christoffel symbols $\{\Gamma^i_{jk}\}$ vanish outside a compact set. Take $\langle .|.\rangle$ to be the Euclidean metric on R^m in this coordinate system.

6.3. Conjecture for Compactly Supported Connections

For every compactly supported C^1 connection Γ on R^m, and every bounded F_T-measurable random variable V on R^m, there exists an H^2 Γ-martingale Y on $(R^m, \langle .|.\rangle)$ such that $Y_T = V$.

The one-dimensional case of the conjecture is true: take $Y_t \equiv h^{-1}(E[h(V)|F_t])$ where h is the invertible nonlinear transformation

$$h(y) \equiv \int_0^y e^{\lambda(x)} dx \tag{39}$$

and λ is any function such that $d\lambda/dx = \Gamma(x)$; using Itô's formula, it is possible to find an L^2 martingale N such that $dY = dN - (1/2)\Gamma(Y)d[N, N]$. This trick does not work in higher dimensions because of curvature. We have stated the problem in a Euclidean setting in order to emphasize its interpretation in terms of stochastic differential equations with prescribed terminal value, such as are described in Pardoux and Peng [19]. Unfortunately the non-Lipschitz character of the following equation (40) places it beyond the class of equations solved by these authors, although in certain cases a solution to (40) can be obtained as the limit of solutions of backwards SDE with Lipschitz coefficients (Darling [5]).

6.4. An Equivalent Conjecture for Pardoux-Peng S.D.E.

For Γ and V as above, there exists an adapted solution (Y, Z) to the stochastic differential equation

$$V = Y(t) + \int_{(t,T]} ZdW - (1/2) \int_{(t,T]} \Gamma(Y)\,(Z \cdot Z)ds, 0 \le t \le T, \tag{40}$$

such that $\int_{(0,T]} Tr(Z \cdot Z)ds \in L^1$.

Notation. – Here $Y(t) \in R^m$, $Z(t) \in R^{m\times\ell}$, $(Z \cdot Z)\,(t) \in R^m \otimes R^m$ is given by $(Z \cdot Z)^{jk}(t) \equiv \sum Z_q^j(t)Z_q^k(t)$, and $\sum \Gamma_{jk}^i(Y)\,(Z \cdot Z)^{jk}$ is the i-th component of $\Gamma(Y)\,(Z \cdot Z)$. The equation (40) says that

$$dY = ZdW - (1/2)\Gamma(Y)\,(Z \cdot Z)dt \tag{41}$$

(note the sign change from (40)) with unknown initial value, but known terminal value $Y(T) = V$.

6.5. Implications of Conjecture 6.3

Suppose Γ is a connection on R^m, and K is a compact subset of R^m, such that the pair (K, Γ) has the following property (*):

(*) There exist a compactly supported connection Γ' on R^m, and $\Phi \in C^2(R^m)$, with

- $K = \{x \in R^m : \Phi(x) \leq 0\}$;
- $\Gamma' = \Gamma$ on K;
- the restriction of Φ to the complement of K is Γ'-convex, with bounded derivative and bounded second covariant derivative (with respect to Γ').

Notice that this function Φ depends on Γ only through its values and derivatives on the boundary $\partial K \equiv \{x \in R^m : \Phi(x) = 0\}$; thus we see that: *Knowing the values of* Γ *on any neighborhood of* ∂K *is sufficient to determine whether* (*) *holds.*

If Conjecture 6.3 is true, then for every F_T-measurable random variable V with values in K, there exists a Γ'-martingale Y with $Y_T = V$, and Y is in H^p for all $1 < p < \infty$ (with respect to the Euclidean metric). Now Proposition 6.1 (martingale containment) implies that $Y_t \in K$ for all t, with probability one. However $\Gamma' = \Gamma$ on K, so Y is actually a Γ-martingale. Thus Conjecture 6.3 implies that:

If Γ *is a connection on* R^m, $K \subset R^m$ *is compact, and* (K, Γ) *has property* (*), *then for every* F_T*-measurable random variable* V *with values in* K, *there exists a* Γ*-martingale* Y *with* $Y_T = V$; *the same holds for any connection which agrees with* Γ *on some neighborhood of* ∂K.

Thus if Conjecture 6.3 is true, the Γ-martingale Dirichlet property holds under much weaker hypotheses than those discussed after Definition 5.1. For example the convexity and curvature properties of compact subsets of the interior of K can be altered without affecting the existence of Γ-martingales with prescribed limit.

6.6. Relevance to the Dirichlet Problem for Harmonic Maps

Suppose (N, g) is a Riemannian manifold-with-boundary, whose boundary is denoted ∂N, Γ is a connection on R^m, and the pair (K, Γ) has property (*). The Dirichlet problem for harmonic maps is as follows: given $\overline{\phi} : \partial N \to \partial K$ (regularity conditions are omitted), construct a harmonic map $\phi : (N, g) \to (K, \Gamma)$ which agrees with $\overline{\phi}$ on ∂N. Kendall [13] offers the following procedure: given $x \in N$, run Brownian motion $\{B_t^x, t \geq 0\}$ on (N, g), with $B_0^x = x$, until the time τ when it hits ∂N, and then construct a Γ-martingale Y^x on K, adapted to the Wiener filtration, with $Y_\infty^x = \overline{\phi}(B_\tau^x)$; then Y_0^x, being F_0-measurable, is non-random, and is taken to be the value of $\phi(x)$. The resulting map ϕ is "finely harmonic" in the sense that it sends Brownian motions to Γ-martingales; for a proof that it is smooth, and therefore harmonic in the usual sense, see Kendall [18].

The previous discussion now shows that, if Conjecture 6.3 is true: *If Γ is a connection on R^m, $K \subset R^m$ is compact, and (K, Γ) has property (*), then every suitably regular map $\overline{\phi} : \partial N \to \partial K$ may be extended to a finely harmonic map $\phi : (N, g) \to (K, \Gamma)$; the same holds for any connection which agrees with Γ on some neighborhood of ∂K.*

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