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Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory

by

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ABSTRACT. – A boundary theory for quantum Markov processes associated with nonconservative one parameter semigroups of completely positive linear contractions on a von Neumann algebra is initiated along the lines of Feller, Chung and Dynkin.

Key words: Completely positive maps, weak Markov flow, exit and entrance cocycles, exit time, boson Fock space.

RÉSUMÉ. – On développe une théorie des frontières pour des processus de Markov quantiques associés à des semi-groupes non conservatifs de contractions complètement positives sur une algèbre de von Neumann, parallèlement à la théorie classique de Feller, Chung et Dynkin.

1. INTRODUCTION

In classical probability theory it is well known that, to any one parameter semigroup of substochastic matrices or transition probability operators, one

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can associate a Markov process with an exit time which can be interpreted as a stop time at which the trajectory of the process goes out of the state space or hits a boundary. There are various possibilities for continuing the process after the exit time in such a manner that the Markov property and stationarity of transition probabilities are retained. Feller [Fe 1,2] initiated the study of this problem by a functional analytic approach based on resolvents or Laplace transforms of one parameter positive contraction semigroups whereas Chung [C1,2] and Dynkin [Dy] outlined a pathwise approach. The aim of the present paper is to investigate the same problem for quantum Markov flows when substochastic matrices or transition probability operators are replaced by completely positive linear contraction maps on a unital von Neumann algebra of operators in a Hilbert space.

In Section 2 it is shown how, by using the famous Stinespring's theorem [St, P1] and the GNS construction, one can associate a canonical weak Markov flow to any one parameter semigroup of completely positive contractions on a von Neumann or C^* algebra. To any such semigroup we introduce, in Section 3, the notion of entrance and exit cocycles and demonstrate how a Feller perturbation of the semigroup can be constructed using a pair (S, ω) where S is a cocycle and ω is a state. The resolvent of the perturbed semigroup is an exact quantum analogue of Feller's formula in [Fe 2]. This raises the basic open problem of constructing the Markov flow of the perturbed semigroup from the flow of the original semigroup. In order to study this problem we present in Section 4 a general procedure of gluing two quantum processes and their filtrations by using quantum stop times [H, PS] which are adapted spectral measures in the closed interval $[0, \infty]$. In Section 5 a quantum Markov flow for the perturbed semigroup is obtained by gluing countably many copies of a Markov flow for the unperturbed semigroup when the exit cocycle is the expectation of an exit time. We conclude the last section with several examples of nonconservative quantum Markov flows which admit exit times. Eventhough the presentation is done for the case of continuous time the reader can easily construct the discrete time analogue of many of our results.

Alternative approaches to dilations of quantum dynamical semigroups on a C^* algebra may be found in [EL], [Em], [Sa] and [Vi-S]. However, they are too weak especially because nothing concrete is mentioned about the conditional expectation of an observable at time t when the algebra of observables up to time s is fixed for some $0 < s < t$. Our approach in Theorem 2.12, 2.13 is much more direct and closer to the spirit of classical probability. This is illustrated by several examples in the course of the present exposition. A more elaborate and leisurely treatment of these ideas is included in the Ph.D. thesis of Bhat [B]. The case of dilation of a nonstationary quantum dynamical evolution is examined in the note [BP].

2. COMPLETELY POSITIVE SEMIGROUPS AND WEAK MARKOV FLOWS

Motivated by the notion of a quantum Markov process introduced by Accardi, Frigerio and Lewis [AFL] and influenced by the absence of conditional expectation in many situations in quantum probability we introduce here a weaker notion of a Markov flow and describe how such a weak Markov flow can be associated to any one parameter semigroup of completely positive linear maps of a von Neumann algebra into itself. This may be viewed as a continuous time version of Stinespring's theorem [St].

Let $\mathcal{H}$ be any complex Hilbert space with scalar product $< \cdot, \cdot >$ linear in the second variable. By a *weak filtration* F on $\mathcal{H}$ we mean a family $F = \{F(t), t \geq 0\}$ of orthogonal projection operators nondecreasing in the variable t. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ and write

$$\mathcal{B}^F_{t]} = \{F(t)XF(t), X \in \mathcal{B}(\mathcal{H})\}$$

for every t. Then $\{\mathcal{B}^F_{t]}, t \geq 0\}$ is a nondecreasing family of $*$ subalgebras of $\mathcal{B}(\mathcal{H})$. The map $\mathbb{E}^F_{t]} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}^F_{t]}$ defined by

$$\mathbb{E}^F_{t]}(X) = F(t)XF(t)$$

is called the *weak conditional expectation* with respect to F at time t.

PROPOSITION 2.1. – *The weak conditional expectation maps* $\{\mathbb{E}^F_{t]}, t \geq 0\}$ *satisfy the following :*

(i) $\mathbb{E}^F_{t]}$ *is a completely positive and contractive linear map;*

(ii) $\mathbb{E}^F_{t]}(I) = F(t)$;

(iii) $\mathbb{E}^F_{t]}(X) = X$ *for all* $X \in \mathcal{B}^F_{t]}$;

(iv) $\mathbb{E}^F_{t]}(XY) = X\mathbb{E}^F_{t]}(Y), \mathbb{E}^F_{t]}(YX) = \mathbb{E}^F_{t]}(Y)X$ *for all* $X \in \mathcal{B}^F_{t]}, Y \in \mathcal{B}(\mathcal{H})$;

(v) $\mathbb{E}^F_{s]}\mathbb{E}^F_{t]} = \mathbb{E}^F_{s\wedge t]}$ *where* $s \wedge t = \min(s,t)$.

Proof. – Immediate. ■

DEFINITION 2.2. – Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$ and let $\{T_t, t \geq 0\}$ be a one parameter semigroup of contractive and completely positive linear maps of $\mathcal{A}$ into itself with T_0 being identity. A triple $(\mathcal{H}, F, j_t)$ is called a *weak Markov flow* with *expectation semigroup* $\{T_t\}$ if $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_0$ as a subspace, F is a weak filtration on $\mathcal{H}$ with $F(0)$ having range $\mathcal{H}_0$ and $\{j_t, t \geq 0\}$ is a family of * homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ satisfying the following

(i) $\mathbb{E}^F_{0]} j_0(X) = XF(0)$ and $j_t(X)F(t) = F(t)j_t(X)F(t)$ for all $t \geq 0, X \in \mathcal{A}$;

(ii) $\mathbb{E}^F_{s]} j_t(X) = j_s(T_{t-s}(X))F(s)$ for all $0 \leq s \leq t < \infty, X \in \mathcal{A}$.

The flow is said to be *subordinate* if $j_t(I) \leq F(t)$ for all t. If $j_t(I) = F(t)$ for all t it is said to be *conservative.*

Condition (i) describes faithfulness of j_0 and adaptedness of the flow to the filtration F whereas condition (ii) describes the Markov property of the flow. In the case of a subordinate flow the factor $F(s)$ on the right hand side of (ii) may be dropped. It may be noted that if $(\mathcal{H}, F, j_t)$ is a weak Markov flow then $(\mathcal{H}, F, j_t(\cdot)F(t))$ is a subordinate weak Markov flow.

For any $X \in \mathcal{A}$ denote by L_X and R_X respectively the linear maps from $\mathcal{A}$ into itself defined by $L_X Y = XY$ and $R_X Y = YX$ for all $Y \in \mathcal{A}$. L_X and R_Y commute with each other for any X, Y. For any finite sequence $\underline{t} = (t_1, ..., t_n)$ in $\mathbb{R}_+$ and $\underline{X} = (X_1, ..., X_n)$ in $\mathcal{A}$ (of *length* n) write $j(\underline{t}, \underline{X}) = j(t_1, t_2, ..., t_n, X_1, ..., X_n) = j_{t_1}(X_1)j_{t_2}(X_2)...j_{t_n}(X_n)$. In particular, $j(t, X) = j_t(X)$. For $\underline{s} = (s_1, ..., s_m), \underline{X} = (X_1, ..., X_m), \underline{t} = (t_1, ..., t_n), \underline{Y} = (Y_1, ..., Y_n)$ we have $j(\underline{s}, \underline{X})j(\underline{t}, \underline{Y}) = j((\underline{s}, \underline{t}), (\underline{X}, \underline{Y}))$ where $(\underline{s}, \underline{t}) = (s_1, ..., s_m, t_1, ..., t_n), (\underline{X}, \underline{Y}) = (X_1, ..., X_m, Y_1, ..., Y_n)$. Since each j_t is a homomorphism we have $j(\underline{s}, t, \underline{X}, Y)j(t, Z) = j(\underline{s}, t, \underline{X}, YZ)$, and $j(s, X)j(s, \underline{t}, Y, \underline{Z}) = j(s, \underline{t}, XY, \underline{Z})$. With these conventions we shall establish a few elementary propositions concerning the operators $j(\underline{t}, \underline{X})$ and their expectation values.

PROPOSITION 2.3. – *Let $(\mathcal{H}, F, j_t)$ be a weak Markov flow with expectation semigroup $\{T_t\}$ on a von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$. Then the following holds :*

(i) *$j_t(X)F(t) = F(t)j_t(X) = F(t)j_t(X)F(t)$ for all $t \geq 0, X \in \mathcal{A}$;*

(ii) *If $0 \leq s \leq t_1 \leq \cdots \leq t_n, X_1, X_2, ..., X_n \in \mathcal{A}$ then*

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_1 - s} L_{X_1} T_{t_2 - t_1} \cdots L_{X_{n-1}} T_{t_n - t_{n-1}}(X_n))F(s);$$

(iii) *If $t_1 \geq t_2 \geq \cdots \geq t_n \geq s \geq 0$ then*

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_n - s} R_{X_n} T_{t_{n-1} - t_n} \cdots R_{X_2} T_{t_1 - t_2}(X_1))F(s).$$

Proof. – From property (i) in Definition 2.2 we have

$$\begin{aligned} F(t)j(t,X) &= \{j(t,X^*)F(t)\}^* \\ &= \{F(t)j(t,X^*)F(t)\}^* \\ &= F(t)j(t,X)F(t) \\ &= j(t,X)F(t). \end{aligned}$$

This proves (i). To prove (ii) we use property (i) of this proposition and the increasing nature of $F(t)$ repeatedly. Thus

$$\begin{aligned} &\mathbb{E}^F_{s]}j(\underline{t},\underline{X}) \\ &\quad = F(s)F(t_1)j(t_1,X_1)...j(t_n,X_n)F(t_{n-1})F(s) \\ &\quad = F(s)j(t_1,X_1)F(t_1)F(t_2)j(t_2,X_2)...j(t_n,X_n)F(t_{n-1})F(s) \\ &\quad = F(s)j(t_1,X_1)j(t_2,X_2)F(t_2)j(t_3,X_3)...j(t_n,X_n)F(t_{n-1})F(s) \\ &\quad = F(s)j(t_1,...,t_{n-1},X_1,...,X_{n-1})F(t_{n-1})j(t_n,X_n)F(t_{n-1})F(s) \\ &\quad = F(s)j(t_1,...,t_{n-1},X_1,...,X_{n-1})j(t_{n-1},T_{t_n-t_{n-1}}(X_n))F(s) \\ &\quad = \mathbb{E}^F_{s]}j(t_1,...,t_{n-1},X_1,...,X_{n-2},X_{n-1}T_{t_n-t_{n-1}}(X_n)). \end{aligned}$$

Now (ii) follows by induction on n. A similar argument yields (iii). ∎

PROPOSITION 2.4. – *Let $(\mathcal{H},F,j_t)$ be a subordinate weak Markov flow with expectation semigroup $\{T_t\}$ on a unital von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then the following holds:*

(i) *If* $0 \leq t_n \leq t_1 < t_2 < ... < t_{n-1}, X_1, ..., X_n \in \mathcal{A}$ *then* $j(\underline{t},\underline{X}) = j(t_1,t_n,Y,X_n)$ *where*

$$Y = L_{X_1}T_{t_2-t_1}L_{X_2}T_{t_3-t_2}\cdots L_{X_{n-2}}T_{t_{n-1}-t_{n-2}}(X_{n-1});$$

(ii) *If* $0 \leq t_1 < t_2 < \cdots < t_{i-1} \leq t_n \leq t_i < t_{i+1} < \cdots < t_{n-1}$ *then* $j(\underline{t},\underline{X}) = j(t_1,...,t_{i-1},t_n,X_1,...,X_{i-1},Y)$ *where*

$$Y = R_{X_n}T_{t_i-t_n}L_{X_i}T_{t_{i+1}-t_i}\cdots L_{X_{n-2}}T_{t_{n-1}-t_{n-2}}(X_{n-1}).$$

Proof. – First we prove (i). Since $F(t)$ is increasing and j_t is a homomorphism Definition 2.2 together with the hypothesis that $j(t,I) \leq F(t)$ implies

$$\begin{aligned} &j(\underline{t},\underline{X}) \\ &\quad = j(t_1,...,t_{n-2},X_1,...,X_{n-2})j(t_{n-2},I)j(t_{n-1},X_{n-1})j(t_n,I)j(t_n,X_n) \\ &\quad = j(t_1,...,t_{n-2},X_1,...,X_{n-2})F(t_{n-2})j(t_{n-1},X_{n-1})F(t_{n-2})j(t_n,X_n) \\ &\quad = j(t_1,...,t_{n-3},t_{n-2},t_n,X_1,...,X_{n-3},X_{n-2}T_{t_{n-1}-t_{n-2}}(X_{n-1}),X_n). \end{aligned}$$

Now (i) follows by induction on n. To prove (ii) we apply (i) to the sequence $t_n \leq t_i < t_{i+1} < \ldots < t_{n-1}$ and obtain

$$j(t_i, t_{i+1}, \ldots, t_n, X_i, X_{i+1}, \ldots, X_n) = j(t_i, t_n, Y', X_n)$$

where

$$Y' = L_{X_i} T_{t_{i+1}-t_i} \cdots L_{X_{n-2}} T_{t_{n-1}-t_{n-2}}(X_{n-1}).$$

Now observe that

$$\begin{aligned} &j(t_{i-1}, t_i, t_n, X_{i-1}, Y', X_n) \\ &\quad = j(t_{i-1}, X_{i-1}) F(t_n) j(t_i, Y') F(t_n) j(t_n, X_n) \\ &\quad = j(t_{i-1}, t_n, X_{i-1}, T_{t_i - t_n}(Y') X_n) \end{aligned}$$

which implies (ii). ■

The next theorem is of particular importance in reducing the computation of moments.

THEOREM 2.5. – *Let* $(\mathcal{H}, F, j_t)$ *satisfy the conditions of Proposition* 2.4. *Then for any sequence* $t_1, t_2, \ldots, t_n$ *in* $\mathbb{R}_+$ *and* $X_1, \ldots, X_n$ *in* $\mathcal{A}$ *there exists a sequence* $s_1, s_2, \ldots, s_m$ *in* $\mathbb{R}_+$ *and* $Y_1, \ldots, Y_m$ *in* $\mathcal{A}$ *such that* $m \leq n, s_1 = t_1, s_m = t_n$, *either* $s_1 < s_2 < \ldots < s_m$ *or* $s_1 > s_2 > \ldots > s_m$ *or* $s_1 > s_2 > \ldots > s_k < s_{k+1} < \ldots < s_m$ *for some* k *and* $j(\underline{t}, \underline{X}) = j(\underline{s}, \underline{Y})$.

Proof. – Without loss of generality we may assume that $t_1 \neq t_2 \neq \cdots \neq t_n$. If $\{t_i\}$ itself is either monotonic increasing or decreasing there is nothing to prove. If $t_1 < \cdots < t_i > t_{i+1}$ then either $t_{i+1} \leq t_1 < \ldots < t_i$ or $t_1 < \ldots < t_{k-1} \leq t_{i+1} \leq t_k < \ldots < t_i$ for some k. By Proposition 2.4 we may then express $j(t_1, \ldots, t_{i+1}, X_1, \ldots, X_{i+1})$ as $j(t_1, t_{i+1}, Y, X_{i+1})$ or $j(t_1, \ldots, t_{k-1}, t_{i+1}, X_1, \ldots, X_{k-1}, Y)$. In any case the length of the $\underline{t}$-sequence gets reduced in $j(\underline{t}, \underline{X})$. If $t_1 > \ldots > t_k < t_{k+1} < \ldots < t_{k+\ell} > t_{k+\ell+1}$ we may once again express $j(t_k, t_{k+1}, \ldots, t_{k+\ell+1}, X_k, X_{k+1}, \ldots, X_{k+\ell+1})$ in terms of a sequence of length not exceeding $\ell + 1$. Rest follows by induction on the length. ■

COROLLARY 2.6. – *Let* $(\mathcal{H}, F, j_t)$ *satisfy the conditions of Proposition* 2.4. *Then for any sequence* $t_1, t_2, \ldots, t_n$ *in* $\mathbb{R}_+$ *and* $X_1, X_2, \ldots, X_n$ *in* $\mathcal{A}$ *there exist* $t_1 = s_1 > s_2 > \ldots > s_m > 0, m \leq n$ *and* $Y_1, Y_2, \ldots, Y_m$ *in* $\mathcal{A}$ *such that* $j(\underline{t}, \underline{X}) F(0) = j(\underline{s}, \underline{Y}) F(0)$.

Proof. – In view of Theorem 2.5 we may assume without loss of generality that $t_1 > t_2 > \ldots > t_m < t_{m+1} < \ldots < t_n$. Now by Proposition 2.4 and the fact that $F(0) = j_0(I)$ we have

$$\begin{aligned} &j(t_m, t_{m+1}, \ldots, t_n, X_m, X_{m+1}, \ldots, X_n)F(0) \\ &\quad = j(t_m, t_{m+1}, \ldots, t_n, 0, X_m, X_{m+1}, \ldots, X_n, I) \\ &\quad = j(t_m, 0, Y, I) = j(t_m, Y)F(0) \end{aligned}$$

for some Y in $\mathcal{A}$. This completes the proof. ■

PROPOSITION 2.7. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose that it is also conservative. If* $s_1 > s_2 > \ldots > s_m \geq 0, t_1 > t_2 > \ldots > t_n \geq 0$ *and* $\{s_1, s_2, \ldots, s_m\} \subset \{t_1, t_2, \ldots, t_n\}$ *then for any* $X_1, X_2, \ldots, X_m$ *in* $\mathcal{A}$

$$j(\underline{s}, \underline{X})F(0) = j(\underline{t}, \underline{Y})F(0)$$

where

$$Y_i = \begin{cases} X_j & if\ t_i = s_j\ for\ some\ j, \\ I & otherwise. \end{cases}$$

Proof. – Let $t_{i_1} = s_1, \ldots, t_{i_m} = s_m$. Then

$$\begin{aligned} &j(s_r, X_r) = j(s_r, I)j(t_{i_r}, X_r) \\ &\quad = F(s_r)j(t_{i_r}, X_r) \\ &\quad = F(t_{i_{r-1}+1})F(t_{i_{r-1}+2}) \cdots F(t_{i_{r-1}})j(t_{i_r}, X_r) \\ &\quad = j(t_{i_{r-1}+1}, t_{i_{r-1}+2}, \ldots, t_{i_r-1}, t_{i_r}, I, I, \ldots, I, X_r) \end{aligned}$$

from which the required result follows. ■

PROPOSITION 2.8. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose* $t \geq s_1 > \ldots > s_k \geq 0, X_1, X_2, \ldots, X_k, Y, Z_1, Z_2, \ldots, Z_k \in \mathcal{A}$. *Then*

$$\begin{aligned} &F(0)j(s_k, s_{k-1}, \ldots, s_1, t, s_1, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)\{T_{s_k} L_{X_k} R_{Z_k} T_{s_{k-1}-s_k} \cdots L_{X_1} R_{Z_1} T_{t-s_1}(Y)\}F(0). \end{aligned}$$

Proof. – We have

$$\begin{aligned} &j(s_1, t, s_1, X_1, Y, Z_1) \\ &\quad = j(s_1, X_1)F(s_1)j(t, Y)F(s_1)j(s_1, Z_1) \\ &\quad = j(s_1, L_{X_1} R_{Z_1} T_{t-s_1}(Y)). \end{aligned}$$

Now the required result follows by repeating the same argument successively replacing the role of t by that of $s_1, s_2, \ldots, s_k$. ■

PROPOSITION 2.9. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose* $s_1 > s_2 > \ldots > s_k \geq t \geq 0, X_1, X_2, \ldots, X_k, Y, Z_1, Z_2, \ldots, Z_k \in \mathcal{A}$. *Then there exist elements* X_k', Z_k' *depending only on* $s_1, \ldots, s_k, X_1, \ldots, X_k$ *and* $Z_1, \ldots, Z_k$ *such that*

$$\begin{aligned} &F(0)j(s_k, \ldots, s_1, t, s_1, \ldots, s_k, X_k, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)T_t\{T_{s_k - t}(X_k')YT_{s_k - t}(Z_k')\}F(0). \end{aligned}$$

Proof. – Since $t \leq s_2 < s_1$ we have

$$\begin{aligned} &j(t, s_1, s_2, Y, Z_1, Z_2) \\ &\quad = j(t, Y)F(s_2)j(s_1, Z_1)F(s_2)j(s_2, Z_2) \\ &\quad = j(t, Y)j(s_2, T_{s_1 - s_2}(Z_1)Z_2). \end{aligned}$$

Repeating this argument we get

$$j(t, s_1, \ldots, s_k, Y, Z_1, \ldots, Z_k) = j(t, s_k, Y, Z_k')$$

where Z_k' depends only on $s_1, \ldots, s_k, Z_1, \ldots, Z_k$. Since $s_k < s_{k-1} < \ldots < s_1 \geq t$ and $t \leq s_k$ we have from (i) in Proposition 2.4

$$j(s_k, s_{k-1}, \ldots, s_1, t, X_k, X_{k-1}, \ldots, X_1, Y) = j(s_k, X_k')j(t, Y)$$

where X_k' depends only on $X_1, \ldots, X_k, s_1 \ldots, s_k$. Combining the two we obtain

$$\begin{aligned} &j(s_k, s_{k-1}, \ldots, s_1, t, s_1, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k) \\ &\quad = j(s_k, t, s_k, X_k', Y, Z_k'). \end{aligned}$$

Since $0 \leq s_k \geq t$ we have

$$\begin{aligned} &F(0)j(s_k, t, s_k, X_k', Y, Z_k')F(0) \\ &\quad = F(0)F(t)j(s_k, X_k')F(t)j(t, Y)F(t)j(s_k, Z_k')F(t)F(0) \\ &\quad = F(0)T_t(T_{s_k - t}(X_k')YT_{s_k - t}(Z_k'))F(0). \quad \blacksquare \end{aligned}$$

PROPOSITION 2.10. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose that* $s_1 > s_2 > \cdots > s_{i-1} \geq t \geq s_i > \cdots > s_k$, X_1, $X_2, \ldots, X_k$, Y, $Z_1, \ldots, Z_k \in \mathcal{A}$. *Then*

$$\begin{aligned} &F(0)j(s_k, s_{k-1}, \ldots, s_1, t, s_1, s_2, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)\{T_{s_k}L_{X_k}R_{Z_k}T_{s_{k-1} - s_k} \\ &\quad \cdots L_{X_i}R_{Z_i}T_{t - s_i}(T_{s_{i-1} - t}(X_{i-1}')YT_{s_{i-1} - t}(Z_{i-1}'))\}F(0) \end{aligned}$$

where X_{i-1}', Z_{i-1}' *depend only on* $s_1, \ldots, s_{i-1}, X_i, \ldots, X_{i-1}, Z_1, \ldots, Z_{i-1}$.

Proof. – By (i) in Proposition 2.4 we have

$$j(s_{i-1}, s_{i-2}, \ldots, s_1, t, X_{i-1}, X_{i-2}, \ldots, X_1, Y) = j(s_{i-1}, t, X'_{i-1}, Y) \quad (2.1)$$

where X'_{i-1} depends only on $s_1, \ldots, s_{i-1}, X_1, \ldots, X_{i-1}$. Since $t \leq s_2 < s_1$ we have

$$\begin{aligned} j(t, s_1, s_2, Y, Z_1, Z_2) &= j(t, Y)F(s_2)j(s_1, Z_1)F(s_2)j(s_2, Z_2) \\ &= j(t, s_2, Y, T_{s_1 - s_2}(Z_1)Z_2). \end{aligned}$$

Repeating this argument up to the pair s_{i-2}, s_{i-1} we get

$$j(t, s_1, s_2, \ldots, s_{i-1}, Y, Z_1, \ldots, Z_{i-1}) = j(t, s_{i-1}, Y, Z'_{i-1}). \quad (2.2)$$

Since $s_i \leq t \leq s_{i-1}$ we have

$$\begin{aligned} &j(s_i, s_{i-1}, t, s_{i-1}, s_i, X_i, X'_{i-1}, Y, Z'_{i-1}, Z_i) \\ &\quad = j(s_i, t, s_i, X_i, T_{s_{i-1}-t}(X'_{i-1})YT_{s_{i-1}-t}(X'_{i-1}), Z_i) \quad (2.3) \end{aligned}$$

Combining (2.1)-(2.3) and using Proposition 2.8 for the sequence $s_k, s_{k-1}, \ldots, s_i, t, s_i, s_{i+1}, \ldots, s_k$ we obtain the required result. ■

PROPOSITION 2.11. – *Suppose $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow with a strongly continuous expectation semigroup $\{T_t\}$ on a unital von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then for any $u, u' \in \mathcal{H}_0$, finite sequences $\underline{s} = (s_1, \ldots, s_k), \underline{s}' = (s'_1, \ldots, s'_{k'})$ in $\mathbb{R}_+$ and $X_1, \ldots, X_k, Y, X'_1, \ldots, X'_{k'} \in \mathcal{A}$ the function*

$$\phi(t) = \langle j(\underline{s}, \underline{X})u, j_t(Y)j(\underline{s}', \underline{X}')u' \rangle$$

is continuous in $t \in \mathbb{R}_+$.

Proof. – Since $F(0)u = u, F(0)u' = u'$ we can apply Corollary 2.6 and assume without loss of generality that $s_1 > s_2 > \cdots > s_k$ and $s'_1 > s'_2 > \cdots > s'_{k'}$. Since the flow is conservative we can apply Proposition

2.7 and assume without loss of generality that the sequences $\underline{s}$ and $\underline{s}'$ are same and strictly decreasing. Then $\phi(t)$ assumes the form

$$\phi(t) = \langle u, F(0)j(s_k, s_{k-1}, ..., s_1, t, s_1, ..., s_k, X_k^*, ..., X_1^*, Y, X_1', ..., X_k')F(0)u'\rangle.$$

Now the strong continuity and contractivity properties of $\{T_t\}$ together with Proposition 2.8, 2.9 and 2.10 respectively imply the continuity of $\phi(t)$ in the intervals $[s_1, \infty), [0, s_k]$ and $[s_i, s_{i-1}], i = k, k-1, ..., 2$. ■

THEOREM 2.12. – *Let $\mathcal{A}$ be a unital von Neumann algebra of operators in a Hilbert space $\mathcal{H}_0$ and let $\{T_t\}$ be a semigroup of completely positive linear maps of $\mathcal{A}$ into itself such that T_0 is identity and $T_t(I) = I$ for all t. Then there exists a conservative weak Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ satisfying the following :*

(i) *$\mathcal{H}_0 \subset \mathcal{H}$ and $\mathcal{H}_0$ is the range of $F(0)$;*

(ii) *The set $\{j(\underline{t}, \underline{X})u, u \in \mathcal{H}_0, \underline{t} = (t_1, t_2, ..., t_n), t_i \geq 0, \underline{X} = (X_1, X_2, ..., X_n), X_i \in \mathcal{A}, 1 \leq i \leq n, n = 1, 2, ...\}$ is total in $\mathcal{H}$;*

(iii) *The expectation semigroup of $(\mathcal{H}, F, j_t)$ is $\{T_t\}$;*

(iv) *If $(\mathcal{H}', F', j_t')$ is another subordinate weak Markov flow with expectation semigroup $\{T_t\}$ such that the range of $F'(0)$ is $\mathcal{H}_0$ and* (ii) *holds with $j, \mathcal{H}$ replaced by $j', \mathcal{H}'$ then there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ satisfying*

$$UF(t)U^{-1} = F'(t), Uj_t(X)U^{-1} = j_t'(X)$$

for all $t \geq 0, X \in \mathcal{A}$;

(v) *If $\{T_t\}$ is strongly continuous on the Banach space $\mathcal{A}$ then the maps $t \to F(t)$ and $t \to j_t(X)$ are strongly continuous for each $X \in \mathcal{A}$.*

Proof. – From [P2] it is known that there exists a family $\{h_t, t \geq 0\}$ of Hilbert spaces with $h_0 = \mathcal{H}_0$, $*$ unital homomorphisms $J_t : \mathcal{A} \to \mathcal{B}(h_t)$ and isometries $V(s,t) : h_s \to h_t$ for $0 \leq s \leq t < \infty$ such that (a) $J_0(X) = X$; (b) $V(s,s) = I$; (c) $V(s,t)^* J_t(X) V(s,t) = J_s(T_{t-s}(X))$; (d) $V(t,u)V(s,t) = V(s,u)$ for all $0 \leq s \leq t \leq u < \infty$. Let $\mathcal{M} = \bigcup_{s \geq 0} h_s$ be the disjoint union of all the h_s considered as abstract sets. Define the map $K : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ by

$$K(u,v) = \langle V(s, s \vee t)u, V(t, s \vee t)v\rangle_{h_{s \vee t}}$$

whenever $u \in h_s, v \in h_t, s \vee t$ denoting the maximum of s and t. We claim that K is a positive definite kernel on $\mathcal{M}$. Indeed, consider arbitrary scalars

$c_i \in \mathbb{C}, u_i \in h_{s_i}, 1 \leq i \leq n$ and put $s = s_1 \vee s_2 \vee \cdots \vee s_n$. Then

$$\begin{aligned}&\sum_{i,j} \bar{c}_i c_j K(u_i, u_j)\\ &= \sum_{i,j} \bar{c}_i c_j \langle V(s_i, s_i \vee s_j)u_i, V(s_j, s_i \vee s_j)u_j\rangle\\ &= \sum_{i,j} \bar{c}_i c_j \langle V(s_i \vee s_j, s)V(s_i, s_i \vee s_j)u_i, V(s_i \vee s_j, s)V(s_j, s_i \vee s_j)u_j\rangle\\ &= \sum_{i,j} \bar{c}_i c_j \langle V(s_i, s)u_i, V(s_j, s)u_j\rangle\\ &= \|\sum_i c_i V(s_i, s)u_i\|_{h_s}^2 \geq 0,\end{aligned}$$

which proves the claim. Hence by the GNS theorem there exists a Hilbert space $\mathcal{K}$ and a map $\lambda : \mathcal{M} \to \mathcal{K}$ such that $\{\lambda(u), u \in \mathcal{M}\}$ is total in $\mathcal{K}$ and

$$K(u, v) = \langle \lambda(u), \lambda(v)\rangle \text{ for all } u, v \in \mathcal{M}.$$

If $u, v \in h_t$ then

$$\langle \lambda(u), \lambda(v)\rangle = K(u, v) = \langle u, v\rangle_{h_t}.$$

Thus λ is an isometry from h_t onto a subspace $\mathcal{K}_t$ of $\mathcal{K}$. If $s < t$ and $u \in h_s$ then $V(s, t)u \in h_t$ and

$$\begin{aligned}&\|\lambda(u) - \lambda(V(s, t)u)\|^2\\ &= 2\|u\|^2 - 2 \operatorname{Re} K(u, V(s, t)u)\\ &= 2\|u\|^2 - 2\operatorname{Re} \langle V(s, t)u, V(s, t)u\rangle_{h_t} = 0.\end{aligned}$$

Thus $\mathcal{K}_s \subseteq \mathcal{K}_t$ whenever $s \leq t$. Denote by $E(t)$ the projection onto $\mathcal{K}_t$ and define j_t by

$$j_t(X) = \lambda J_t(X)\lambda^{-1}E(t) \qquad \text{for} \quad X \in \mathcal{A}, \quad t \geq 0$$

where λ^{-1} is the inverse of the map $\lambda : h_t \to \mathcal{K}_t$. Since the range of $j_t(X)$ is contained in $\mathcal{K}_t$ and J_t is a * homomorphism from $\mathcal{A}$ into $\mathcal{B}(h_t)$ it follows that $j_t(X) = E(t)\lambda J_t(X)\lambda^{-1}E(t), X \in \mathcal{A}$ is a * homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$ and $j_t(I) = E(t)$.

Now consider $u, v \in h_s, s < t$. Then

$$\begin{aligned}&\langle \lambda(u), j_t(X)\lambda(v)\rangle\\ &= \langle \lambda(V(s, t)u), j_t(X)\lambda(V(s, t)v)\rangle\\ &= \langle V(s, t)u, J_t(X)V(s, t)v\rangle\\ &= \langle u, V(s, t)^* J_t(X)V(s, t)v\rangle\\ &= \langle u, J_s(T_{t-s}(X))v\rangle\\ &= \langle \lambda(u), j_s(T_{t-s}(X))\lambda(v)\rangle.\end{aligned}$$

Thus

$$E(s)j_t(X)E(s) = j_s(T_{t-s}(X)) \quad \text{if} \quad X \in \mathcal{A}, \quad s \leq t.$$

Denote by $\mathcal{H} \subset \mathcal{K}$ the closed subspace spanned by the set M of all vectors of the form $j(\underline{t}, \underline{X})u, u \in \mathcal{H}_0, \underline{t} = (t_1, ..., t_n), \underline{X} = (X_1, ..., X_n), t_i \geq 0$, $X_i \in \mathcal{A}$, $n = 1, 2, ...$ Denote by $\mathcal{H}_t \subset \mathcal{H}$ the closed subspace spanned by the set $M_t \subset M$ of all vectors of the same form $j(\underline{t}, \underline{X})u$ with $t_i \leq t$ for every i. We now claim that $\mathcal{H}_t = \mathcal{H} \cap \mathcal{K}_t$. Indeed, let $\xi = j(\underline{s}, \underline{X})u$ where $\underline{s} = (s_1, ..., s_n), \underline{X} = (X_1, ..., X_n), t \geq s_i \geq 0, X_i \in \mathcal{A}, u \in \mathcal{H}_0$. Then ξ is in the range of $j(s_1, X_1)$ which is contained in $\mathcal{K}_{s_1} \subset \mathcal{K}_t$. Thus $M_t \subset \mathcal{H} \cap \mathcal{K}_t$ and therefore $\mathcal{H}_t \subset \mathcal{H} \cap \mathcal{K}_t$. Now consider an element of the form $\eta = E(t)j(\underline{s}, \underline{X})u$ where $u \in \mathcal{H}_0, \underline{s} = (s_1, ..., s_n)$ and $s_i \geq 0$ Then $\eta = j(t, s_1, ..., s_n, I, X_1, ..., X_n)u$. Since $(\mathcal{K}, E, j_t)$ is a conservative weak Markov flow it follows from Corollary 2.6 that we can express $\eta = j(t, s'_1, ..., s'_m, Y_0, Y_1, ..., Y_m)u$ where $t > s'_1 > \cdots > s'_m \geq 0$ and hence $\eta \in \mathcal{H}_t$. Thus $E(t)M \subset \mathcal{H}_t$ and therefore $\mathcal{H} \cap \mathcal{K}_t \subset \mathcal{H}_t$ proving the claim. Denote by $F(t)$ the projection on $\mathcal{H}_t$ in the Hilbert space $\mathcal{H}$ and $j_t(X)$ the restriction of $j_t(X)$ to $\mathcal{H}$. Then $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow satisfying properties (i) - (iii) of the theorem.

To prove (iv) we observe that the proofs of Theorem 2.5, Corollary 2.6 and Proposition 2.8 imply that

$$\langle j(\underline{s}, \underline{X})u, j(\underline{t}, \underline{Y})v \rangle = \langle j'(\underline{s}, \underline{X})u, j'(\underline{t}, \underline{Y})v \rangle$$

for all $u, v \in \mathcal{H}_0, \underline{s} = (s_1, ..., s_m), \underline{t} = (t_1, ..., t_n), \underline{X} = (X_1, ..., X_m), \underline{Y} = (Y_1, ..., Y_n)$. This shows that the correspondence $j(\underline{s}, \underline{X})u \to j'(\underline{s}, \underline{X})u$ is isometric and hence extends uniquely to a unitary isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$ satisfying (iv). Observe that cyclicity (property (ii)) forces j'_t to be conservative.

Property (v) of the theorem is immediate from Proposition 2.11 and the fact that j_t is a homomorphism for every $t \geq 0$. ■

Now we extend Theorem 2.12 to non-conservative contractive semigroups.

THEOREM 2.13. – *Let $\mathcal{A}$ be a unital von Neumann algebra of operators in a Hilbert space $\mathcal{H}_0$ and let $\{T_t\}$ be a semigroup of completely positive linear maps of $\mathcal{A}$ into itself such that T_0 is identity and $T_t(I) \leq I$ for all t. Then there exists a subordinate weak Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ satisfying* (i) - (v) *of Theorem* 2.12.

Proof. – Consider the extended von Neumann algebra $\widehat{\mathcal{A}} = \mathcal{A} \bigoplus \mathbb{C}$ acting on the Hilbert space $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 \bigoplus \mathbb{C}$. For convenience we denote the element

$X \bigoplus c$ of $\widehat{\mathcal{A}}$, for $X \in \mathcal{A}$ and $c \in \mathbb{C}$, by the column vector $\binom{X}{c}$. Define the maps $\widehat{T}_t : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ by

$$\widehat{T}_t\begin{pmatrix}X\\c\end{pmatrix} = \begin{pmatrix}T_t(X) + c(I - T_t(I))\\c\end{pmatrix}, \qquad X \in \mathcal{A}, \quad c \in \mathbb{C}. \tag{2.4}$$

Then $\{\widehat{T}_t\}$ is a conservative one parameter semigroup of completely positive linear maps. If $\{T_t\}$ is strongly continuous so is $\{\widehat{T}_t\}$. Thus Theorem 2.12 becomes applicable for $\{\widehat{T}_t\}$ and we have a conservative weak Markov flow $(\widehat{\mathcal{H}}, \widehat{F}, \hat{j}_t)$ on $\mathcal{A}$ with expectation semigroup $\{\widehat{T}_t\}$. Define the operators $F(t)$ and $j_t(X)$ on $\widehat{\mathcal{H}}$ by,

$$F(t) = \hat{j}_t\begin{pmatrix}I\\1\end{pmatrix} - \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix},$$

$$j_t(X) = \hat{j}_t\begin{pmatrix}X\\0\end{pmatrix} \qquad \text{for} \quad t \geq 0 \text{ and } X \in \mathcal{A}.$$

Before obtaining the required Markov flow we prove the following statements. For $0 \leq s \leq t, X \in \mathcal{A}$ and $c \in \mathbb{C}$

(a) $\left\{\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a family of projections nondecreasing in t;

(b) $j_t(X)\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix}j_t(X) = 0$;

(c) $\{F(t)\}$ is a family of projections nondecreasing in t;

(d) Range of $F(0)$ is $\mathcal{H}_0$ and range of $F(t)$ increases to the orthogonal complement of range of $\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix}$ as t increases to ∞;

(e) $F(t)\hat{j}_s\begin{pmatrix}X\\c\end{pmatrix} = j_s(X - cI)F(s) + cF(s)$.

Property (a) follows from the identity

$$\begin{aligned}\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix} &= \hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\widehat{F}(s)\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\widehat{F}(s)\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\\ &= \hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_s(\widehat{T}_{t-s}\begin{pmatrix}0\\1\end{pmatrix})\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\\ &= \hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_s\begin{pmatrix}I - T_{t-s}(1)\\1\end{pmatrix}\hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}\\ &= \hat{j}_s\begin{pmatrix}0\\1\end{pmatrix}.\end{aligned}$$

Now make use of (a) to obtain

$$j_t(X)\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = \hat{j}_t\begin{pmatrix}X\\0\end{pmatrix}\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\hat{j}_0\begin{pmatrix}0\\1\end{pmatrix} = 0 = \hat{j}_0\begin{pmatrix}0\\1\end{pmatrix}j_t(X)$$

and

$$\begin{aligned} F(t)F(s) &= \left(\hat{j}_t\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\left(\hat{j}_s\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \hat{j}_s\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = F(s) = F(s)F(t). \end{aligned}$$

Clearly $F(t)^* = F(t)$. This proves (b) and (c). The range of $\widehat{F}(0)$ is $\mathcal{H}_0 \bigoplus \mathbb{C}$ and hence the range of $F(0)$ is $\mathcal{H}_0$. The second part of (d) follows as $\hat{j}_t\binom{I}{1}$ increases to the identity operator in $\widetilde{\mathcal{H}}$ as t increases to ∞. Now from (a) and (b),

$$\begin{aligned} F(t)\hat{j}_s\begin{pmatrix} X \\ c \end{pmatrix} &= \left(\hat{j}_t\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\left(\hat{j}_s\begin{pmatrix} X \\ 0 \end{pmatrix} + \hat{j}_s\begin{pmatrix} 0 \\ c \end{pmatrix}\right) \\ &= \hat{j}_s\begin{pmatrix} X \\ 0 \end{pmatrix} + \hat{j}_s\begin{pmatrix} 0 \\ c \end{pmatrix} - 0 - \hat{j}_0\begin{pmatrix} 0 \\ c \end{pmatrix} \\ &= \hat{j}_s\begin{pmatrix} X - cI \\ 0 \end{pmatrix} + cF(s) \\ &= \hat{j}_s\begin{pmatrix} X - cI \\ 0 \end{pmatrix}F(s) + cF(s). \end{aligned}$$

Let $\mathcal{H}$ be the orthogonal complement of the range of $\hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\widehat{\mathcal{H}}$. Making use of (a)-(d) we can restrict $F(t)$ and $j_t(X)$ to $\mathcal{H}$ and verify that $(\mathcal{H}, F, j_t)$ is a subordinate weak Markov flow with expectation semigroup $\{T_t\}$ satisfying (i), (iii), and (v) of Theorem 2.12. Denote by $\mathcal{H}_t \subset \widehat{\mathcal{H}}$ the closed subspace spanned by the set M_t of all vectors of the form $j(\underline{t}, \underline{X})u$, with $t_i \leq t$ for every i and $u \in \mathcal{H}_0$. We now claim that the range of $F(t)$ is $\mathcal{H}_t$. Indeed, consider $\xi = j(\underline{t}, \underline{X})u$ with $t_i \leq t$ for every i and $u \in \mathcal{H}_0$. Then,

$$F(t)\xi = \left(\hat{j}_t\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\hat{j}_{t_1}\begin{pmatrix} X_1 \\ 0 \end{pmatrix}\hat{j}_{t_2}\begin{pmatrix} X_2 \\ 0 \end{pmatrix}\cdots\hat{j}_{t_n}\begin{pmatrix} X_n \\ 0 \end{pmatrix}u = \xi,$$

and hence the ragne of $F(t)$ contains $\mathcal{H}_t$. Now for $t \geq s_i \geq 0$, $X_i \in \mathcal{A}$, $c_i \in \mathbb{C}$ for $1 \leq i \leq n$ and $u \in \mathcal{H}_0, a \in \mathbb{C}$, consider $\eta = \hat{j}\left(\underline{s}, \begin{pmatrix} \underline{X} \\ \underline{c} \end{pmatrix}\right)\begin{pmatrix} u \\ a \end{pmatrix}$. From the statement (e) proved above,

$$\begin{aligned} F(t)\eta = {} & (j_{s_1}(X_1 - c_1 I)F(s_1) \\ & + c_1 F(s_1))\hat{j}_{s_2}\begin{pmatrix} X_2 \\ c_2 \end{pmatrix}\hat{j}_{s_3}\begin{pmatrix} X_3 \\ c_3 \end{pmatrix}\cdots\hat{j}_{s_n}\begin{pmatrix} X_n \\ c_n \end{pmatrix}\begin{pmatrix} u \\ a \end{pmatrix}. \end{aligned}$$

By induction on n we conclude that $F(t)\eta$ is a linear combination of elements in M_t. The closed linear span of all vectors η of the form above is the range of $\widehat{F}(t)$ and as the range of $F(t)$ is clearly contained in the range of $\widehat{F}(t)$ we conclude that $\mathcal{H}_t$ contains the whole of the range of $F(t)$. This proves properties (ii) and (iv) of Theorem 2.12 for the Markov flow $(\mathcal{H}, F, j_t)$. ■

Note that the construction in (2.4) is the quantum probabilistic analogue of associating with a substochastic semigroup $P_t = ((p_{ij}(t))), 1 \leq i, j < \infty$ of matrices the stochastic semigroup $\widehat{P}_t = ((\hat{p}_{ij}(t)), 0 \leq 1, j < \infty$ where

$$\hat{p}_{ij}(t) = \begin{cases} p_{ij}(t) & \text{if } i \geq 1,\ j \geq 1, \\ 0 & \text{if } i = 0,\ j \geq 1, \\ 1 & \text{if } i = 0,\ j = 0, \\ 1 - \sum_{j=1}^{\infty} p_{ij}(t) & \text{if } i \geq 1,\ j = 0. \end{cases}$$

In other words we have incorporated an absorbing boundary. This is reflected in the increasing nature of the family of projections $\{\hat{j}_t\begin{pmatrix}0\\1\end{pmatrix}\}$. It may also be noted that in general $\{\hat{j}_t\begin{pmatrix}I\\0\end{pmatrix}\}$ is not a commuting family of projections.

We conclude this section with three examples of the construction involved in Theorem 2.12 and 2.13.

Example 2.14. – Let $\mathcal{A}$ be the commutative von Neumann algebra of 2×2 diagonal matrices and let $T_t : \mathcal{A} \to \mathcal{A}$ be the semigroup defined by

$$T_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} e^{-ct}a + (1 - e^{-ct})b & 0 \\ 0 & b \end{bmatrix}$$

for $a, b \in \mathbb{C}, c > 0$ being a fixed constant. $\mathcal{A}$ acts on $\mathbb{C}^2$ in a natural way. Put $\mathcal{H} = \mathbb{C}^2 \bigoplus L^2(\mathbb{R}_+)$ with filtration F given by $F(t) = I \bigoplus \chi_t$ where I is the identity operator in $\mathbb{C}^2$ and χ_t denotes multiplication by the indicator function $\chi_{[0,t]}$ in $L^2(\mathbb{R}_+)$. Define $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$j_t\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = aQ(t) + b(F(t) - Q(t))$$

where $Q(t)$ is the rank one projection onto the subspace generated by the unit vector $e^{-\frac{c}{2}t}e_1 \bigoplus f_t$ with

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad f_t(x) = \sqrt{c}\, e^{-\frac{c}{2}(t-x)}\chi_{[0,t]}(x).$$

A routine computation shows that $F(s)j_t(X)F(s) = j_s(T_{t-s}(X))$ for all $X \in \mathcal{A}$ and $0 \leq s \leq t$. Thus $(\mathcal{H}, F, j_t)$ provides a weak Markov flow with expectation semigroup $\{T_t\}$ satisfying all the properties mentioned in Theorem 2.12. It is instructive to compare this with the Markov flow of classical probability theory associated with the one parameter semigroup of 2×2 stochastic matrices $\begin{bmatrix} e^{-ct} & 1 - e^{-ct} \\ 0 & 1 \end{bmatrix}$.

Example 2.15. – Let H be a positive selfadjoint operator in $\mathcal{H}_0$. Consider the nonconservative one parameter semigroup $\{T_t\}$ of completely positive maps on $\mathcal{B}(\mathcal{H}_0)$ defined by

$$T_t(X) = e^{-tH} X e^{-tH}, \qquad t \geq 0, \quad X \in \mathcal{B}(\mathcal{H}_0).$$

Following [HIP] introduce the unitary operators $\{U(s,t), 0 \leq s \leq t < \infty\}$ in the Hilbert space

$$\mathcal{H} = \mathcal{H}_0 \bigoplus L^2(\mathbb{R}_+, \mathcal{H}_0)$$

given by

$$U(s,t)\begin{pmatrix} u_0 \\ u \end{pmatrix} = \begin{pmatrix} A(s,t) & B(s,t) \\ C(s,t) & I + D(s,t) \end{pmatrix} \begin{pmatrix} u_0 \\ u \end{pmatrix}$$

where u_0 and $u = u(\cdot)$ are the components of an arbitrary element in $\mathcal{H}$ with respect to the direct sum decomposition in the definition of $\mathcal{H}$ and

$$\begin{aligned} A(s,t) &= e^{-(t-s)H} \\ B(s,t)u &= -(2H)^{1/2} \int_0^\infty \chi_{[s,t]}(x) e^{-(t-x)H} u(x)\,dx, \\ (C(s,t)u_0)(x) &= \chi_{[s,t]}(x)(2H)^{1/2} e^{-(x-s)H} u_0, \\ (D(s,t)u)(x) &= -2 \int_0^\infty \chi_{[s.t]}(y)\chi_{[y,t]}(x) H e^{-(x-y)H} u(y)\,dy. \end{aligned}$$

It is known from [HIP] that

$$U(s,t)U(r,s) = U(r,t) \quad \text{for all } 0 \leq r \leq s \leq t < \infty$$

and $U(s,t)$ is an operator of the form $V(s,t) \bigoplus I$ in the direct sum decomposition $\mathcal{H} = \mathcal{H}(s,t) \bigoplus \mathcal{H}(s,t)^{\perp}$ where $\mathcal{H}(s,t) = \mathcal{H}_0 \bigoplus L^2([s,t], \mathcal{H}_0)$. Define $F(t)$ to be the projection on the subspace $\mathcal{H}(0,t)$ and put

$$j_t(X) = U(0,t)^* \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U(0,t).$$

Then $j_t(I) = j_t(I)F(t) \leq F(t)$. From the fact that $\{U(s,t)\}$ is a time orthogonal dilation of the positive contraction semigroup $\{e^{-tH}\}$ it follows that $F(s)U(0,t)F(s) = U(0,s)\{e^{(t-s)H} \bigoplus I_{[0,s]} \bigoplus 0\}$ where the term in $\{ \quad \}$ on the right hand side is with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \bigoplus L^2([0,s],\mathcal{H}_0) \bigoplus L^2([s,\infty),\mathcal{H}_0)$. Thus

$$\begin{aligned}
&F(s)j_t(X)F(s)\\
&= U(0,s)^*F(s)U(s,t)^*F(s)\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}F(s)U(s,t)F(s)U(0,s)\\
&= U(0,s)^*\begin{pmatrix} T_{t-s}(X) & 0 \\ 0 & 0 \end{pmatrix}U(0,s)F(s)\\
&= j_s(T_{t-s}(X))F(s).
\end{aligned}$$

In other words $(\mathcal{H}, F, j_t)$ is a subordinate weak Markov flow with expectation semigroup $\{T_t\}$.

Example 2.16. – Let $\{J_t, t \geq 0\}$ be an Evans-Hudson flow [P1, Me] determined by structure maps $\{\theta^i_j, i, j \geq 0\}$ on a unital von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$ so that the quantum stochastic differential equations

$$dJ_t(X) = \sum_{i,j} J_t(\theta^i_j(X))d\Lambda^j_i(t), \qquad J_0(X) = X \otimes 1, \quad X \in \mathcal{A}$$

are fulfilled in the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes \ell^2), \Gamma$ indicating the boson Fock space over its argument. Let $F(t)$ denote the projection onto the subspace $\mathcal{H}_t = \mathcal{H}_0 \otimes \Gamma(L^2[0,t] \otimes \ell^2) \otimes \Phi_{[t} \subset \mathcal{H}$ where $\Phi_{[t}$ is the Fock vacuum in $\Gamma(L^2[t,\infty) \otimes \ell^2)$. Define $j_t(X) = J_t(X)F(t), t \geq 0, X \in \mathcal{A}$. Then $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow with expectation semigroup $T_t = e^{t\theta^0_0}, t \geq 0$. However, this need not satisfy the cyclicity condition (ii) of Theorem 2.12.

3. FELLER PERTURBATIONS OF POSITIVE SEMIGROUPS

In his analysis of Kolmogorov equations Feller [Fe 1,2,3] constructed a class of substochastic semigroups called minimal semigroups and outlined a method of constructing new semigroups including stochastic ones by perturbing their resolvents (or Laplace transforms) appropriately. The same goal was achieved more directly by a pathwise approach in the works of Chung [C1,2] and Dynkin [Dy]. In the context of quantum Markov processes minimal semigroups associated to Lindblad equations were introduced by Davies [Da] and their dilations to Evans-Hudson flows were studied by Mohari [Mo] and Fagnola [Fa]. Following the spirit of Feller and Chung we outline a general method of perturbation for positive semigroups on a von Neumann algebra.

Let $\mathcal{A}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$ and let $T_t : \mathcal{A} \rightarrow \mathcal{A}, t \geq 0$ be a strongly continuous positive semigroup of linear maps so that the following conditions are fulfilled : (i) $T_0(X) = X$ for all $X \in \mathcal{A}$; (ii) $T_s(T_t(X)) = T_{s+t}(X)$ for all $X \in \mathcal{A}, s, t \geq 0$; (iii) $\lim_{t \rightarrow s} T_t(X) = T_s(X)$ for all $X \in \mathcal{A}, s \geq 0$; (iv) $T_t(X) \geq 0$ for all $X \geq 0, X \in \mathcal{A}, t \geq 0$.

We consider two types of perturbations of $\{T_t\}$ which yield new semigroups. The first type arises from what we call an exit cocycle for the semigroup $\{T_t\}$. The second arises from a dualisation of the first and is based on an entrance cocycle for the same semigroup. The terminology is motivated from considerations of classical Markov processes.

DEFINITION 3.1. – Let $\mathcal{F}_b(\mathbb{R}_+)$ be the family of all bounded Borel subsets of $\mathbb{R}_+$. A map $S : \mathcal{F}_b(\mathbb{R}_+) \rightarrow \mathcal{A}_+$, the cone of nonnegative elements in $\mathcal{A}$, is called an *$\mathcal{A}_+$-valued Radon measure* on $\mathbb{R}_+$, if, for any sequence $\{E_i\}$ of disjoint elements in $\mathcal{F}_b(\mathbb{R}_+)$ such that $\bigcup_i E_i \in \mathcal{F}_b(\mathbb{R}_+), S(\bigcup_i E_i) = \sum_i S(E_i)$ where the right hand side converges in the strong sense.

DEFINITION 3.2. – An $\mathcal{A}_+$-valued Radon measure S on $\mathbb{R}_+$ is called an *exit cocycle* for the semigroup $\{T_t\}$ if

$$T_t(S(E)) = S(E + t) \quad \text{for all } E \in \mathcal{F}_b(\mathbb{R}_+), \qquad t \geq 0. \tag{3.1}$$

Remark 3.3. – The strong continuity of the semigroup $\{T_t\}$ and the fact that T_0 is identity imply that every exit cocycle is nonatomic, *i.e.*, $S(\{t\}) = 0$ for all $t \geq 0$.

Example 3.4. – Choose and fix an element B in $\mathcal{A}_+$. Define

$$S_B(E) = \int_E T_t(B)dt \quad \text{for } E \in \mathcal{F}_b(\mathbb{R}_+). \tag{3.2}$$

Then the semigroup property and positivity of $\{T_t\}$ imply that S_B is an exit cocycle.

Another class of exit cocycles is obtained by the following definition.

DEFINITION 3.5. – Let $A \in \mathcal{A}$. Then A is called *excessive* for the semigroup $\{T_t\}$ if $T_t(A) \leq A$ for all $t \geq 0$. If $T_t(A) = A$ for all $t \geq 0$, A is said to be *harmonic*.

Example 3.6. – Let $A \in \mathcal{A}$ be excessive for $\{T_t\}$. Define a Radon measure S by putting

$$S([a,b]) = T_a(A) - T_b(A) \quad \text{for } 0 \leq a \leq b < \infty. \tag{3.3}$$

Since A is excessive and T_t is positive we have $S([a,b]) = T_a(A - T_{b-a}(A)) \geq 0$. Since $T_t(S([a,b])) = S([a+t, b+t])$ it follows that S is an exit cocycle.

It should be noted that in this example if $\mathcal{L}$ is the generator of $\{T_t\}$ and A is in the domain of $\mathcal{L}$ then $S([a,b]) = \int_a^b T_s(-\mathcal{L}(A))ds$ reduces to Example 3.4. If $B \in \mathcal{A}_+$ is harmonic and μ denotes the Lebesgue measure in $\mathbb{R}_+$ then $S_B(E) = \mu(E)B$ which is a special case of Example 3.4.

Example 3.7. – If we replace the von Neumann algebra $\mathcal{A}$ by a C^*-algebra the definitions given in the preceding discussions are meaningful. For example let $\mathcal{A}$ denote the C^* algebra of bounded continuous functions on $\mathbb{R}_+$ and let $\{T_t\}$ be the semigroup of translation operators defined by

$$(T_t f)(x) = f(x+t), \qquad t \geq 0, \quad f \in \mathcal{A}.$$

Define the Radon measure S by

$$S([a,b])(x) = (b+x)^\delta - (a+x)^\delta, \qquad 0 \leq x < \infty$$

for some fixed $\delta, 0 < \delta < 1$. Then

$$\frac{d}{dx} S([a,b]) = \delta((b+x)^{\delta-1} - (a+x)^{\delta-1}) \leq 0$$

and hence $\sup_x S([a,b])(x) = b^\delta - a^\delta < \infty$. Clearly $S([a,b])(x) \geq 0$. The cocycle property is obvious. This cocycle if expressed as $\int_a^b \phi(x+s)ds$ then $\phi(x) = \delta x^{\delta-1}$ is unbounded and $\phi \notin \mathcal{A}$. On the other hand if $S([a,b]) = T_a\psi - T_b\psi$ then $\psi(x) = c - x^\delta$ for some constant c, is unbounded and does not belong to $\mathcal{A}$.

Example 3.8. – Let $\mathcal{A}$ be the C^* algebra of all bounded continuous functions on the real line $\mathbb{R}$ and $\{T_t\}$ be the semigroup defined by

$$(T_t f)(x) = \mathbb{E} f(x + B(t)), \qquad t \geq 0, \quad f \in \mathcal{A}$$

where $B(t)$ denotes the standard Brownian motion process on $\mathbb{R}$. Define S by

$$S([0,t])(x) = \mathbb{E}|x + B(t)| - |x|, \qquad x \in \mathbb{R}, \quad t \geq 0.$$

From Tanaka's formula (page 137 in [CW]) we know that

$$d|x + B(t)| = sgn(x + B(t))dB(t) + dL(t,x)$$

where $L(t,x)$ is the local time at $-x$. $L(t,x)$ is jointly continuous in the variables t and x and $L(t,x)$ is nondecreasing in t for fixed x. Thus $S([0,t])(x)$ is increasing in t and continuous in (t,x). Since $B(t)$ has a symmetric distribution it follows that $S([0,t])(x) = S([0,t])(-x)$. When $x \geq 0$ we have

$$\begin{aligned}
&S([0,t])(x)\\
&= \int_{-\infty}^{\infty} (|x + y\sqrt{t}| - |x|)(2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} dy\\
&= (2\pi)^{-\frac{1}{2}} \left\{ \int_{-xt^{-\frac{1}{2}}}^{\infty} y\sqrt{t} e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{-xt^{-\frac{1}{2}}} (2x + y\sqrt{t}) e^{-\frac{y^2}{2}} dy \right\}\\
&\leq (2\pi)^{-\frac{1}{2}} \sqrt{t} \left\{ \int_{-xt^{-\frac{1}{2}}}^{\infty} y e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{-xt^{-\frac{1}{2}}} y e^{-\frac{y^2}{2}} dy \right\}\\
&= \pi^{-\frac{1}{2}} \sqrt{2t} e^{-\frac{x^2}{2t}}
\end{aligned}$$

which shows that

$$\sup_x S([0,t])(x) < \infty.$$

The cocycle property is now immediate from the standard properties of Brownian motion.

We now go back to the semigroup $\{T_t\}$ on the von Neumann algebra $\mathcal{A}$ and associate a perturbation series with a pair (S, ω) where S is an exit cocycle for $\{T_t\}$ and ω is a state on $\mathcal{A}$. To this end we introduce the Radon measure μ defined by

$$\mu(E) = \omega(S(E)), E \in \mathcal{F}_b(\mathbb{R}_+) \tag{3.4}$$

and some notation. For any $t \geq 0, n = 0, 1, 2, \ldots$ define the linear maps $T_t^{(n)}$ on $\mathcal{A}$ by

$$T_t^{(n)}(X) = \begin{cases} T_t(X) & \text{if } n = 0,\\ \int_0^t S(dt_1)\omega(T_{t-t_1}(X)) & \text{if } n = 1,\\ \int_{\Delta_n(t)} S(dt_1)\mu(dt_2)\ldots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)) & \text{if } n \geq 2 \end{cases} \tag{3.5}$$

for all $X \in \mathcal{A}$, where

$$\Delta_n(t) = \{(t_1, t_2, ..., t_n) : t_i \geq 0 \text{ for each } i, t_1 + \cdots + t_n \leq t\}.$$

For $0 \leq s \leq t < \infty$ and $0 \leq m \leq n$ define

$$T_{s,t}^{m,n}(X) = \begin{cases} T_t(X) & \text{if } m = n = 0 \\ \displaystyle\int_{\Delta_{m,n}(s,t)} S(dt_1)\mu(dt_2)...\mu(dt_n) & \\ \quad \times \omega(T_{t-(t_1+\cdots+t_n)}(X)) & \text{otherwise} \end{cases} \tag{3.6}$$

for all $X \in \mathcal{A}$, where

$$\Delta_{m,n}(s,t) = \begin{cases} \{(t_1, t_2, ..., t_n) : & t_1 + \cdots + t_m \leq s < t_1 + \cdots + t_{m+1}, \\ & t_1 + \cdots + t_n \leq t, t_i \geq 0 \text{ for every } i\}, \text{ if } m < n, \\ \Delta_m(s) & \text{if } m = n. \end{cases}$$

Proposition 3.9. – *For each $X \in \mathcal{A}$ the infinite series*

$$\widehat{T}_t(X) = \sum_{n=0}^{\infty} T_t^{(n)}(X) \tag{3.7}$$

converges in operator norm. The convergence is uniform in t over bounded intervals.

Proof. – It follows from Remark 3.3 and the definition of μ in (3.4) that μ is nonatomic. Hence $\lim_{s \downarrow 0} \mu([0,s]) = \mu(\{0\}) = 0$. Choose and fix $t_0 > 0$ such that $\mu([0,t_0]) < 1$. We shall now estimate $\mu^{*^n}([0,t]) = \mu^{\otimes^n}(\{(t_1, ..., t_n) : t_1 + \cdots + t_n \leq t\})$. Let $t_1 + \cdots + t_n \leq t$ and $r = \#\{i : 1 \leq i \leq n, t_i \geq t_0\}$. Then $t \geq t_1 + \cdots + t_n \geq rt_0$ and, in particular, $r \leq \left[\frac{t}{t_0}\right] + 1 = j$, say. Hence

$$\begin{aligned} \mu^{*^n}([0,t]) &\leq \sum_{r=0}^{j} \binom{n}{r} \mu([0,t_0])^{n-r} \mu([t_0,t])^r \\ &\leq n^j \mu([0,t_0])^{n-j} \sum_{r=0}^{j} \binom{j}{r} \mu([0,t_0])^{j-r} \mu([t_0,t])^r \\ &= \mu([0,t])^j n^j \mu([0,t_0])^{n-j}. \end{aligned}$$

From (3.5) we have for $n \geq 1$

$$||T_t^{(n)}(X)|| \\ \leq ||\omega|| \, ||X|| \, ||S([0,t])|| \sup_{0 \leq s \leq t} ||T_s|| \mu([0,t])^j (n-1)^j \mu([0,t_0])^{n-1-j}$$

which implies the required result. ■

In order to show that $\{\widehat{T}_t\}$ is a semigroup we need the following lemma.

LEMMA 3.10. – *For any $s, t \in \mathbb{R}_+$ and $X \in \mathcal{A}$ the following holds:*
(i) $T_s^{(m)}(T_t^{(n)}(X)) = T_{s,s+t}^{m,m+n}(X)$ *for* $m, n \geq 0$*;*
(ii) $\sum_{m+n=k} T_s^{(m)}(T_t^{(n)}(X)) = T_{s+t}^{(k)}(X)$ *for* $k \geq 0$.

Proof. – First we prove (i). Clearly (i) holds when $m = n = 0$. When $m, n \geq 1$ we have

$$\begin{aligned} &T_s^{(m)}(T_t^{(n)}(X)) \\ &= \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)...\mu(ds_m)\omega(T_{s-(s_1+\cdots+s_m)}(T_t^{(n)}(X))) \\ &= \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)...\mu(ds_m) \\ &\times \int_{\Delta_n(t)} \omega(T_{s-(s_1+\cdots+s_m)}(S(dt_1)))\mu(dt_2)... \\ &\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)). \end{aligned}$$

Consider the change of variables

$$s_{m+1} = s - (s_1 + \cdots + s_m) + t_1, s_{m+2} = t_2, ..., s_{m+n} = t_n.$$

Then the cocycle property of S and the definition of μ imply

$$\omega(T_{s-(s_1+\cdots+s_m)}(S(dt_1))) = \mu(ds_{m+1})$$

and under the change of variables, the conditions $t_1 \geq 0$ and $t_1 + \cdots + t_n \leq t$ become $s \leq s_1 + \cdots + s_{m+1}$ and $s_1 + \cdots + s_{m+n} \leq s + t$ respectively. By the nonatomicity of S we may as well write $s < s_1 + \cdots + s_{m+1}$ so that

$$\begin{aligned} &T_s^{(m)}(T_t^{(n)}(X)) \\ &= \int_{\Delta_{m,m+n}(s,s+t)} S(ds_1)\mu(ds_2)\cdots\mu(ds_{m+n})\omega(T_{t+s-(s_1+\cdots+s_{m+n})}(X)) \end{aligned}$$

and (3.6) shows that the right hand side is the same as $T^{m,m+n}_{s,s+t}(X)$. When $m = 0, n \geq 1$ we have

$$\begin{aligned}
&T^{(m)}_s(T^{(n)}_t(X)) \\
&= T_s\Big(\int_{\Delta_n(t)} S(dt_1)\mu(dt_2)\cdots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X))\Big) \\
&= \int_{\Delta_n(t)} T_s(S(dt_1))\mu(dt_2)\cdots\mu(dt_n)\omega(T_{t-(t_1+\cdots+t_n)}(X)).
\end{aligned}$$

Changing the variables to $s_1 = s + t_1, s_2 = t_2, ..., s_n = t_n$ yields the required result as before. When $m \geq 1, n = 0$ the semigroup property of $\{T_t\}$ implies

$$T^{(m)}_s(T^{(n)}_t(X)) = \int_{\Delta_m(s)} S(ds_1)\mu(ds_2)\cdots\mu(ds_m)\omega(T_{s+t-(s_1+\cdots+s_m)}(X))$$

and completes the proof of (i).

Property (ii) is obvious for $k = 0$. When $k \geq 1$ property (i) together with the observation that $\Delta_k(s + t)$ is the disjoint union of $\{\Delta_{m,k}(s, s + t), 0 \leq m \leq k\}$ for all s and t implies (ii) and completes the proof of the lemma. ■

THEOREM 3.11. – *Let $T_t : \mathcal{A} \to \mathcal{A}$ be a positive strongly continuous semigroup of linear maps. Suppose ω is a state on $\mathcal{A}$ and S is an exit cocycle for $\{T_t\}$. Then the family $\{\widehat{T}_t\}$ defined by* (3.7) *is also a positive strongly continuous semigroup of linear maps on $\mathcal{A}$. If $\{T_t\}$ is completely positive so is $\{\widehat{T}_t\}$.*

Proof. – Clearly $\widehat{T}_0(X) = T_0(X) = X$ for all $X \in \mathcal{A}$. For $0 \leq s, t < \infty$ and $X \in \mathcal{A}$ we have from Lemma 3.10.

$$\begin{aligned}
\widehat{T}_s(\widehat{T}_t(X)) &= \sum_{m,n\geq 0} T^{(m)}_s(T^{(n)}_t(X)) \\
&= \sum_{k\geq 0}\sum_{m+n=k} T^{(m)}_s(T^{(n)}_t(X)) \\
&= \sum_{k\geq 0} T^{(k)}_{s+t}(X) = \widehat{T}_{s+t}(X).
\end{aligned}$$

Thus $\{\widehat{T}_t\}$ is a semigroup. By (3.5), $\{T^{(n)}_t\}$ is strongly continuous in t and linear on $\mathcal{A}$ for each n and Proposition 3.9 implies the same property for $\{\widehat{T}_t\}$. If $\{T_t\}$ is positive or completely positive so is each $\{T^{(n)}_t\}$ and hence $\{\widehat{T}_t\}$ also shares the same property. ■

The semigroup $\{\widehat{T}_t\}$ occurring in Theorem 3.11 is called the *Feller perturbation* of $\{T_t\}$ determined by the exit cocycle S and the state ω.

Remark 3.12. – Theorem 3.11 holds good when $\mathcal{A}$ is a C^* algebra and the proof remains the same.

Remark 3.13. – From the proof of Proposition 3.9 it follows that $\nu = \delta_0 + \mu + \mu^{*^2} + \ldots$ is a Radon measure on $\mathbb{R}_+$ where δ_0 is the Dirac measure at 0 and μ is defined by (3.4). This shows that the perturbed semigroup $\widehat{T}_t$ can be expressed as

$$\widehat{T}_t(X) = T_t(X) + \int_0^t (S * \nu)(ds)\omega(T_{t-s}(X))$$

when $S * \nu$ is the positive operator-valued Radon measure defined by

$$S * \nu([0,t]) = \int_0^t S(ds)\nu([0, t-s]).$$

If X is in the domain of the generator $\mathcal{L}$ of $\{T_t\}, u, v$ are elements of the Hilbert space $\mathcal{H}$ (with $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$) and $\langle u, S * \nu([0,t])v\rangle$ is differentiable at the origin then

$$\frac{d}{dt}\langle u, \widehat{T}_t(X)v\rangle|_{t=0} = \langle u, \mathcal{L}(X)v\rangle + \omega(X)\frac{d}{dt}\langle u, S * \nu([0,t])v\rangle|_{t=0}.$$

In particular, if $S(E) = S_B(E) = \int_E T_s(B)ds, B \in \mathcal{A}_+$ then

$$\frac{d}{dt}\langle u, \widehat{T}_t(X)v\rangle|_{t=0} = \langle u, \{\mathcal{L}(X) + \omega(X)B\}v\rangle.$$

In order to compare the perturbed semigroup $\{\widehat{T}_t\}$ with Feller's construction we shall compute its resolvent. At this stage it is useful to recollect the well known Hille-Yosida theorem which makes precise the one to one correspondence between a semigroup and its resolvent.

THEOREM 3.14 (Hille-Yosida [Y], [Dy]). – *Let $\mathcal{X}$ be a Banach space. Let $\{R_\lambda\ {}_{\lambda>\beta}$ be a family of operators in $\mathcal{X}$, with $\beta \geq 0$ a fixed scalar, satisfying the following:*

(i) $R_\lambda R_\mu = (\mu - \lambda)^{-1}(R_\lambda - R_\mu)$ *for* $\lambda, \mu > \beta, \lambda \neq \mu$;

(ii) $||R_\lambda|| \leq M(\lambda - \beta)^{-1}$ *for all* $\lambda > \beta$ *and some positive constant* M ;

(iii) $s - \lim_{\lambda \to \mu} R_\lambda(X) = R_\mu(X)$ *for all* $\mu > \beta, X \in \mathcal{X}$;

(iv) *Range of* R_λ *is dense in* $\mathcal{X}$ *for some* $\lambda > \beta$.

Then there exists a unique strongly continuous semigroup $\{T_t\}$ *of operators in* $\mathcal{X}$ *such that* $||T_t|| \leq Me^{\beta t}$ *and* $R_\lambda(X) = \int_0^\infty e^{-\lambda t} T_t(X)dt$ *for all* $\lambda > \beta, X \in \mathcal{X}$.

Conversely, if $T_t : \mathcal{X} \to \mathcal{X}, t \geq 0$ *is a strongly continuous semigroup of operators then there exist constants* $M, \beta \geq 0$ *such that* $||T_t|| \leq Me^{\beta t}$ *for all* $t \geq 0$ *and* $R_\lambda(X) = \int_0^\infty e^{-\lambda t} T_t(X)dt$ *defines a family of operators satisfying* (i) - (iv). *The semigroup* $\{T_t\}$ *is contractive if and only if* M *and* β *can be chosen to be* 1 *and* 0 *respectively.*

Proof. – We omit the proof (See page 30, Vol. I, [Dy]). ∎

THEOREM 3.15. – *Let the semigroup* $\{T_t\}$ *in Theorem* 3.11 *satisfy the inequalities* $||T_t|| \leq Me^{\beta t}$ *for* $M > 0, \beta \geq 0$ *and all* $t \geq 0$. *Let* $\{R_\lambda, \lambda > \beta\}$ *be its resolvent. Then there exists a* $\hat{\beta} \geq 0$ *such that the resolvent* $\{\widehat{R}_\lambda, \lambda > \hat{\beta}\}$ *is given by*

$$\widehat{R}_\lambda(X) = R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1 - \omega(A_\lambda)} A_\lambda \tag{3.8}$$

where

$$A_\lambda = \int_0^\infty e^{-\lambda t} S(dt), \quad \lambda > \hat{\beta}. \tag{3.9}$$

Proof. – Choose and fix a t_0 such that $\mu([0, t_0]) = a < 1$. Let $b = ||S([0, t_0])||$. By the cocycle property

$$||S([nt_0, (n+1)t_0])|| = ||T_{nt_0}(S([0, t_0]))|| \leq bMe^{\beta n t_0}, \qquad n \geq 0.$$

Hence

$$\begin{aligned}\left|\left|\int_0^\infty e^{-\lambda t} S(dt)\right|\right| &\leq \sum_{n \geq 0} \left|\left|\int_{nt_0}^{(n+1)t_0} e^{-\lambda t} S(dt)\right|\right| \\ &\leq b + \sum_{n=1}^{\infty} bMe^{-(\lambda-\beta)nt_0} \\ &\leq b + \frac{bMe^{-(\lambda-\beta)t_0}}{1 - e^{-(\lambda-\beta)t_0}} \\ &< \infty \text{ for all } \lambda > \beta.\end{aligned}$$

Since $\mu(E) = \omega(S(E))$ we have

$$\int_0^\infty e^{-\lambda t} \mu(dt) \leq a + \frac{bMe^{-(\lambda-\beta)t_0}}{1 - e^{-(\lambda-\beta)t_0}}.$$

Since $a < 1$ we conclude the existence of a constant $\hat{\beta} \geq 0$ such that

$$\omega(A_\lambda) = \int_0^\infty e^{-\lambda t} \mu(dt) < 1 \text{ for all } \lambda > \hat{\beta}.$$

Thus, for $X \in \mathcal{A}, \lambda > \hat{\beta}$, we have

$$\begin{aligned}
\widehat{R}_\lambda(X) &= \int_0^\infty e^{-\lambda t} \widehat{T}_t(X) \\
&= R_\lambda(X) + \sum_{n \geq 1} \int_0^\infty e^{-\lambda t} \\
&\quad \times \left\{ \int_{\Delta_n(t)} S(dt_1)\mu(dt_2) \cdots \mu(dt_n) \omega(T_{t-(t_1+\cdots+t_n)}(X)) \right\} dt \\
&= R_\lambda(X) + \sum_{n \geq 1} \omega(R_\lambda(X)) \omega(A_\lambda)^{n-1} A_\lambda \\
&= R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1 - \omega(A_\lambda)} A_\lambda.
\end{aligned}$$

Remark 3.16. – As a direct consequence of the cocycle property of S it follows that

$$R_\lambda(A_\mu) = \frac{A_\lambda - A_\mu}{\mu - \lambda} \quad \text{for } \lambda, \mu > \beta, \quad \lambda \neq \mu.$$

Using this relation we can verify that $\widehat{R}_\lambda$ satisfies the resolvent identity. So we could as well have defined $\widehat{R}_\lambda$ directly by (3.8) and (3.9), used the Hille-Yosida theorem and recovered the semigroup $\widehat{T}_t$. Indeed, Feller [Fe 2] had taken this approach. We shall compare the formula for $\widehat{R}_\lambda$ with that of Feller.

Let $A \in \mathcal{A}_+$ be excessive for $\{T_t\}$ and let S be defined as in Example 3.6 so that $S(t) = S([0, t]) = A - T_t(A)$. Then

$$A_\lambda = \int_0^\infty e^{-\lambda t} S(dt) = \lambda \int_0^\infty e^{-\lambda t} (A - T_t(A)) dt,$$

i.e., $A_\lambda = A - \lambda R_\lambda(A)$. Hence (3.8) becomes

$$\widehat{R}_\lambda(X) = R_\lambda(X) + \frac{\omega(R_\lambda(X))}{1 - \omega(A) + \lambda \omega(R_\lambda(A))} (A - \lambda R_\lambda(A)). \tag{3.10}$$

Now consider the special case $A = qI$ for some $q > 0$ and $\omega(X) = \text{tr}\, \rho X$ for some density matrix ρ describing the state. Then

$$\begin{aligned}
\widehat{R}_\lambda(X) &= R_\lambda(X) + \frac{\text{tr}\, \rho R_\lambda(X)}{1 - q + q\lambda \,\text{tr}\, \rho R_\lambda(I)} q(I - \lambda R_\lambda(I)) \\
&= R_\lambda(X) + \frac{\text{tr}\, \rho R_\lambda(X)}{m + \lambda \,\text{tr}\, \rho R_\lambda(I)} (I - \lambda R_\lambda(I)) \qquad (3.11)
\end{aligned}$$

where $m = \frac{1-q}{q}$. If $\{T_t\}$ is contractive and $m \geq 0$ then $\{\widehat{T}_t\}$ is also contractive. When $m = 0$ and $\{T_t\}$ is not conservative it follows that $\lambda \widehat{R}_\lambda(I) = I$ and hence $\{\widehat{T}_t\}$ is conservative. When $\mathcal{A} = \ell_\infty$ and $\{T_t\}$ is the minimal semigroup of substochastic matrices associated with a Kolmogorov equation, formula (3.11) coincides with the expression (8.1) in [Fe 2]. This suggests that the density matrix ρ in (3.11) mediates the transition from a "boundary point" back into the "state space" of the Markov flow and $\frac{m}{m+1}$ is the probability that it is stuck in the boundary. Of course, it is desirable to have a clearer picture of the manner in which ρ mediates the transition.

We now proceed to a brief discussion on entrance cocycles. Let $\mathcal{A}_*$ denote the predual of the von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Then $\mathcal{A}_*$ is a subalgebra of the algebra of all trace class operators on $\mathcal{H}$.

DEFINITION 3.17. – An $\mathcal{A}_*$-valued nonnegative Radon measure ψ on $\mathbb{R}_+$ is called an *entrance cocycle* for the semigroup $\{T_t\}$ if

$$\begin{aligned} &\operatorname{tr} \psi(E) T_s(X) \\ &\quad = \operatorname{tr} \psi(E+s) X \text{ for all } s \geq 0, E \in \mathcal{F}_b(\mathbb{R}_+), X \in \mathcal{A}. \end{aligned} \tag{3.12}$$

Imitating Example 3.4 and 3.6 we can obtain examples of entrance cocycles provided there exists a semigroup $\{\pi_t\}$ in $\mathcal{A}_*$ satisfying

$$\operatorname{tr} \pi_t(\rho) X = \operatorname{tr} \rho T_t(X) \text{ for all } X \in \mathcal{A}, \rho \in \mathcal{A}_*, t \geq 0. \tag{3.13}$$

In such a case we have the following examples.

Example 3.18. – Let $\rho \in \mathcal{A}_*$ be positive. Define the Radon measure ψ_ρ by

$$\psi_\rho(E) = \int_E \pi_s(\rho) ds \quad \text{for} \quad E \in \mathcal{F}_b(\mathbb{R}_+). \tag{3.14}$$

Then ψ_ρ is an entrance cocycle for the semigroup $\{T_t\}$.

Example 3.19. – Suppose $\rho_0 \in \mathcal{A}_*$ is excessive for $\{\pi_t\}$. Define ψ by

$$\psi([a,b]) = \pi_a(\rho_0) - \pi_b(\rho_0). \tag{3.15}$$

Then ψ is an entrance cocycle.

Let ψ be an entrance cocycle for the semigroup $\{T_t\}$ and let Z be a fixed positive element in $\mathcal{A}$. In analogy with (3.7) define

$$\begin{aligned} \check{T}_t(X) = &T_t(X) + \sum_{n\geq 1} \int_{\Delta_n(t)} \operatorname{tr}(\psi(dt_1)X) \operatorname{tr}(\psi(dt_2)Z) \\ &\cdot \operatorname{tr}(\psi(dt_n)Z) T_{t-(t_1+\cdots+t_n)}(Z) \end{aligned} \tag{3.16}$$

for $t \geq 0, X \in \mathcal{A}$.

THEOREM 3.20. – *The series on the right hand side of* (3.16) *converges in norm and* $\{\check{T}_t\}$ *is a strongly continuous positive semigroup. If* $\{T_t\}$ *is completely positive so is* $\{\check{T}_t\}$.

Proof. – This is exactly along the same lines of the proof of Theorem 3.11. ■

THEOREM 3.21. – *Let* R_λ *and* $\check{R}_\lambda$ *be the resolvents of* $\{T_t\}$ *and* $\{\check{T}_t\}$ *respectively for* $\lambda > \gamma$ *for some* $\gamma \geq 0$. *Then*

$$\check{R}_\lambda(X) = R_\lambda(X) + \frac{\alpha_\lambda(X)}{1 - \alpha_\lambda(Z)} R_\lambda(Z) \text{ for } X \in \mathcal{A}$$

where α_λ *is the positive linear functional on* $\mathcal{A}$ *given by*

$$\alpha_\lambda(X) = \int_0^\infty e^{-\lambda t} \operatorname{tr} (\psi(dt)X)$$

Proof. – This is obtained by a direct computation. ■

We conclude this section with some remarks on perturbations of direct sums and tensor products of semigroups. Suppose $\mathcal{A}_i$ is a von Neumann algebra of operators in a Hilbert space $\mathcal{H}_i$ and $\{T_t^{(i)}\}$ is a positive strongly continuous semigroup of linear maps on $\mathcal{A}_i$ for each $i = 1, 2$. Let ω_i be a state in $\mathcal{A}_i$ and let S_i be an exit cocycle for each i. For the semigroup $T_t = T_t^{(1)} \bigoplus T_t^{(2)}, S = S_1 \bigoplus S_2$ is an exit cocycle and for any $0 \leq p \leq 1, \omega = p\omega_1 \bigoplus (1-p)\omega_2$ is a state on $\mathcal{A} = \mathcal{A}_1 \bigoplus \mathcal{A}_2$. Expressing any element of $\mathcal{A}$ as a column vector $\binom{X}{Y}, X \in \mathcal{A}_1, Y \in \mathcal{A}_2$ we see that the perturbed semigroup $\{\widehat{T}_t\}$ associated with the pair (S, ω) has its resolvent $\widehat{R}_\lambda$ given by

$$\widehat{R}_\lambda \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} R_\lambda^{(1)}(X) \\ R_\lambda^{(2)}(Y) \end{pmatrix} + \frac{p\omega_1(R_\lambda^{(1)}(X)) + (1-p)\omega_2(R_\lambda^{(2)}(Y))}{1 - \{p\omega_1(A_\lambda^{(1)}) + (1-p)\omega_2(A_\lambda^{(2)})\}} \begin{pmatrix} A_\lambda^{(1)} \\ A_\lambda^{(2)} \end{pmatrix} \tag{3.17}$$

where $R_\lambda^{(i)}$ is the resolvent of $\{T_t^{(i)}\}$ and $A_\lambda^{(i)} = \int_0^\infty e^{-\lambda t} S_i(dt)$. When $\mathcal{A}_2 = \mathbb{C}, p = 0, \omega_2(c) = c$ and $S_1([0,t]) = I - T_t^{(1)}(I), \{T_t^{(1)}\}$ being contractive (3.17) reduces to

$$\widehat{R}_\lambda \begin{pmatrix} X \\ c \end{pmatrix} = \begin{pmatrix} R_\lambda^{(1)}(X) \\ \lambda^{-1}c \end{pmatrix} + \lambda^{-1}c \begin{pmatrix} I - \lambda R_\lambda^{(1)}(I) \\ 0 \end{pmatrix}.$$

This is the resolvent of a semigroup which is the quantum probabilistic analogue of a Markov chain with an absorbing boundary point as described after Theorem 2.13.

Just like direct sums we can also perturb tensor products of semigroups. Indeed, let $T_t = T_t^{(1)} \otimes T_t^{(2)}$ in $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Then there exists an exit cocycle S for $\{T_t\}$ such that $S([0,t]) = S_1([0,t]) \otimes S_2([0,t])$ for all t. It should be noted that $S([a,b]) \neq S_1([a,b]) \otimes S_2([a,b])$. It is also interesting to note that $A_1 \otimes A_2$ is excessive for $\{T_t\}$ if A_i is excessive for $\{T_t^{(i)}\}, i = 1, 2$. Indeed,

$$\begin{aligned}(T_t^{(1)} \otimes T_t^{(2)})(A_1 \otimes A_2) &= T_t^{(1)}(A_1) \otimes T_t^{(2)}(A_2)\\ &\leq A_1 \otimes T_t^{(2)}(A_2)\\ &\leq A_1 \otimes A_2.\end{aligned}$$

If H is harmonic for $\{T_t^{(2)}\}$ then $S([0,t]) = S_1([0,t]) \otimes H$ defines an exit cocycle for $\{T_t\}$. If $T_t^{(2)} \equiv$ identity we can express the resolvent of the perturbed semigroup $\{\widehat{T}_t\}$ associated with the exit cocycle S and any state ω on $\mathcal{A}$ as

$$\widehat{R}_\lambda(X \otimes Y) = R_\lambda^{(1)}(X) \otimes Y + \frac{\omega(R_\lambda^{(1)}(X) \otimes Y)}{1 - \omega(A_\lambda^{(1)} \otimes H)} A_\lambda^{(1)} \otimes H.$$

A similar analysis can be done with entrance cocycles.

4. GLUING ADAPTED PROCESSES USING STOP TIMES

In Section 2 we saw how it is possible to construct weak Markov flows out of one parameter semigroups of completely positive contractive linear maps on a von Neumann algebra. Given such a semigroup $\{T_t\}$ and its Feller perturbation $\{\widehat{T}_t\}$ based on an exit cocycle S and a state ω it is natural to examine the relationship between the flows associated with $\{T_t\}$ and $\{\widehat{T}_t\}$. If we follow the classical approach of Chung [C1] it is not difficult to see the possibility of obtaining the flow associated with $\{\widehat{T}_t\}$ by appropriately "gluing" independent copies of the flow associated with $\{T_t\}$ at suitable stop times.

Just as a classical stochastic process is a family of random variables $\{\xi(t)\}$ a quantum stochastic process may be viewed as a family of operators $\{X(t)\}$ in some Hilbert space. Given two classical stochastic processes

$\{\xi(t)\}$ and $\{\eta(t)\}$ with $t \geq 0$ and a stop time τ for $\{\xi(t)\}$ we can glue them at time τ and obtain a new process $\{\zeta(t)\}$ by defining

$$\zeta(t) = \begin{cases} \xi(t) & \text{if } t < \tau, \\ \eta(t-\tau) & \text{if } t \geq \tau. \end{cases}$$

Already from the papers of Hudson [H] and Parthasarathy and Sinha [PS] the fruitfulness of looking upon stop time as an adapted spectral measure on $\mathbb{R}_+$ is evident. Our aim in the present section is to outline a method of gluing operator-valued processes in different Hilbert spaces by using appropriate spectral measures and obtain the glued process in their tensor product. To this end we begin with the definition of an integral of an operator-valued function with respect to a spectral measure. Since this notion will be used extensively in the sequel we present a list of its basic properties for ready reference.

Let $(\Omega, \mathcal{F}, \mu)$ be a totally finite standard measure space and let P^μ denote the canonical spectral measure on Ω so that $P^\mu(E)$ is the operator of multiplication by the indicator function χ_E of $E \in \mathcal{F}$ in the Hilbert space $L^2(\mu)$. Suppose k is a Hilbert space and $X : \Omega \to \mathcal{B}(k)$ is a map satisfying the following:

(i) the map $\omega \to \langle u, X(\omega)v\rangle$ on Ω is measurable for every $u, v \in k$;

(ii) $\sup_\mu ||X(\cdot)||_k < \infty$.

Note that the Hilbert space $L^2(\mu) \otimes k$ is isomorphic to the Hilbert space $L^2(\mu, k)$ where

$$L^2(\mu, k) = \left\{ f | f : \Omega \to k, \int_\Omega ||f(\omega)||_k^2 \mu(d\omega) < \infty \right\}$$

with

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle_k \mu(d\omega),$$

$f, g, \ldots$ denoting equivalence classes modulo μ-null sets. Making use of this identification between $L^2(\mu) \otimes k$ and $L^2(\mu, k)$ we define the operator $\int_\Omega P^\mu(d\omega)X(\omega)$ on $L^2(\mu) \otimes k$ by

$$\left\{ \int_\Omega P^\mu(d\omega)X(\omega)f \right\}(\omega') = X(\omega')f(\omega'),\ \omega' \in \Omega, \tag{4.2}$$

Then $\int_\Omega P^\mu(d\omega)X(\omega)$ is a bounded operator on $L^2(\mu) \otimes k$ with

$$\left|\left| \int_\Omega P^\mu(d\omega)X(\omega) \right|\right| \leq \sup_\mu ||X(\cdot)||_k. \tag{4.3}$$

It is natural to denote the operator given by (4.2) as $\int_\Omega P^\mu(d\omega) \otimes X(\omega)$ but we drop the symbol $\otimes$ for notational convenience.

Now suppose that P is any spectral measure on the standard Borel space $(\Omega, \mathcal{F})$ with values in the lattice of orthogonal projections in a Hilbert space $\mathcal{H}$. By a part of the Hahn-Hellinger theorem [P1] there exist totally finite measures $\{\mu_\alpha, \alpha \in S\}$ on $(\Omega, \mathcal{F})$ and a unitary operator $U : \mathcal{H} \to \bigoplus_\alpha L^2(\mu_\alpha)$ such that $UPU^{-1} = \bigoplus_\alpha P^{\mu_\alpha}$. Let now $X : \Omega \to \mathcal{B}(k)$ be a weakly measurable map satisfying $\sup_P ||X(\cdot)||_k < \infty$. Then we define the integral of $X(\cdot)$ with respect to P by

$$\int_\Omega P(d\omega)X(\omega) = U^{-1}\left\{\bigoplus_\alpha \int_\Omega P^{\mu_\alpha}(d\omega)X(\omega)\right\}U. \tag{4.4}$$

Then the left hand side yields an operator on $\mathcal{H} \otimes k$ with

$$\left|\left|\int_\Omega P(d\omega)X(\omega)\right|\right| \leq \sup_\mu ||X(\cdot)||_k. \tag{4.5}$$

PROPOSITION 4.1. – *Let $(\Omega, \mathcal{F})$ be a standard Borel space and let $\mathcal{H}, k$ be Hilbert spaces. Suppose P is a spectral measure on $\mathcal{F}$ with values in the lattice of orthogonal projections in $\mathcal{H}$. Let $\mathcal{N}$ be the $*$ unital algebra of all weakly measurable maps of the form $X : \Omega \to \mathcal{B}(k)$ satisfying the condition $\sup_P ||X(\cdot)||_k < \infty$. Then the following holds:*

(i) *the map $X \to \int_\Omega P(d\omega)X(\omega)$ is a $*$ unital homomorphism from $\mathcal{N}$ into $\mathcal{B}(\mathcal{H} \otimes k)$ such that* (4.3) *holds;*

(ii) *for any $u, u' \in \mathcal{H}, v, v' \in k$*

$$\left\langle u \otimes v, \int_\Omega P(d\omega)X(\omega)u' \otimes v' \right\rangle = \int_\Omega \langle u, P(d\omega)u'\rangle\langle v, X(\omega)v'\rangle.$$

Proof. – This is immediate when $\mathcal{H} = L^2(\mu)$ and $P = P^\mu$. Rest follows from (4.4) and (4.5) . ■

DEFINITION 4.2. – By a *bounded process* $X = \{X(t), t \geq 0\}$ in a Hilbert space $\mathcal{H}$ we mean a family of bounded operators in $\mathcal{H}$ satisfying the following: (i) the map $t \to X(t)$ is weakly measurable; (ii) $\sup_{0 \leq s \leq t} ||X(s)|| < \infty$ for every t. Such a process is called *contractive, isometric* or *coisometric* according as all the operators $X(t), t \geq 0$ possess the same property. If F is a weak filtration in $\mathcal{H}$ then X is said to be *adapted* to F if

$$X(t)F(t) = F(t)X(t)F(t) \quad \text{for every } t.$$

A *stop time* in $\mathcal{H}$ is a spectral measure on the closed interval $[0, \infty] = \mathbb{R}_+ \cup \{\infty\}$ with values in the lattice of projections in $\mathcal{H}$. A stop time P is called a *stop time for the bounded process* X in $\mathcal{H}$ if $X(t)P([0,t]) = P([0,t])X(t)$ for every t. P is called an *F-adapted stop time* for the bounded process X if, in addition,

$$P([0,t])F(t) = F(t)P([0,t]) \quad \text{for every } t.$$

We shall now introduce a quantum analogue for the construction in (4.1). Let X_i be a bounded process in the Hilbert space $\mathcal{H}_i, i = 1, 2$ and let P_1 be a stop time for X_1. Then the *glued process* $X_1 \circ_{P_1} X_2$ is defined by

$$X_1 \circ_{P_1} X_2(t) = X_1(t)(1 - P_1(t)) + \int_{[0,t]} P_1(dt_1)X_2(t - t_1) \tag{4.6}$$

where the first term is actually the ampliated operator $X_1(t)(1 - P_1(t)) \otimes I_2, P_1(t) = P_1([0,t])$ and I_2 is the identity operator in $\mathcal{H}_2$. By Proposition 4.1 it follows that $X_1 \circ_{P_1} X_2$ is a bounded process in $\mathcal{H}_1 \otimes \mathcal{H}_2$. When the stop time P_1 is clear in a context we shall write $X_1 \circ X_2$ for $X_1 \circ_{P_1} X_2$. The event that the process X_1 is stopped at a time not exceeding t is described by the projection $P_1(t)$. Since $X_1(t)$ and $P_1(t)$ commute with each other we may express the first term on the right hand side of (4.6) also as $P_1((t, \infty])X_1(t)P_1((t, \infty])$.

Normally $P_1(\{0\}) = 0$ so that $X_1 \circ X_2(0) = X_1(0)P_1((0, \infty]) = X_1(0)$, *i.e.*, the glued process starts at $X_1(0)$. Otherwise $X_1 \circ X_2(0) = X_1(0)P_1((0, \infty]) + P_1(\{0\})X_2(0)$. This may be interpreted as an instantaneous change from $X_1(0)$ to $X_2(0)$ (with some probability in a given state).

PROPOSITION 4.3. – *Let P_i be a stop time in $\mathcal{H}_i, i = 1, 2$. Then $P_1 \circ_{P_1} P_2$ is a stop time in $\mathcal{H}_1 \otimes \mathcal{H}_2$. If, in addition, P_i is a stop time for the bounded process $X_i, i = 1, 2$ then $P_1 \circ_{P_1} P_2$ is a stop time for the glued process $X_1 \circ_{P_1} X_2$.*

Proof. – We have from (4.6)

$$\begin{aligned}P_1 \circ_{P_1} P_2(t) &= P_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)P_2(t-t_1)\\ &= \int_{0\le t_1+t_2\le t} P_1(dt_1)P_2(dt_2)\\ &= P_1 \otimes P_2(\{(t_1,t_2) : 0 \le t_1+t_2 \le t; t_1,t_2 \ge 0\}). \quad (4.7)\end{aligned}$$

This proves the first part. The second part is immediate from Proposition 4.1 and the definition of a stop time for a bounded process. ■

We denote the stop time $P_1 \circ_{P_1} P_2$ by $P_1 \circ P_2$ and call it the *cumulative stop time* of P_1 followed by P_2. In other words we wait till the stop time P_1 first and subsequently wait till P_2 so that the total waiting time is $P_1 \circ P_2$. Such a view is useful in gluing more than two processes.

PROPOSITION 4.4. – *Let X_i be a bounded process in $\mathcal{H}_i, i = 1,2,3$ and let P_i be a stop time for $X_i, i = 1,2$. Then*

$$\begin{aligned}&\{(X_1 \circ_{P_1} X_2) \circ_{P_1\circ P_2} X_3\}(t)\\ &\quad = \{X_1 \circ_{P_1} (X_2 \circ_{P_2} X_3)\}(t)\\ &\quad = X_1(t)P_1((t,\infty]) + \int_{0\le t_1\le t} P_1(dt_1)P_2((t-t_1,\infty])X_2(t-t_1)\\ &\quad\quad + \int_{0\le t_1+t_2\le t} P_1(dt_1)P_2(dt_2)X_3(t-t_1-t_2) \quad (4.8)\end{aligned}$$

for all $t \ge 0$ in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

Proof. – By repeated application of (4.6) we have

$$\begin{aligned}&\{X_1 \circ_{P_1} (X_2 \circ_{P_2} X_3)\}(t)\\ &\quad = X_1(t)P_1((t,\infty]) + \int_{[0,t]} P_1(dt_1)(X_2 \circ_{P_2} X_3)(t-t_1)\\ &\quad = X_1(t)P_1((t,\infty])\\ &\quad\quad + \int_{[0,t]} P_1(dt_1)\{X_2(t-t_1)P_2((t-t_1,\infty])\\ &\quad\quad + \int_{[0,t-t_1]} P_2(dt_2)X_3(t-t_1-t_2)\}\end{aligned}$$

which agrees with the right hand side of (4.8) owing to the fact that P_2 is a stop time for X_2. Similarly by Proposition 4.1 we have

$$\begin{aligned}
&\{(\boldsymbol{X}_1 \circ_{P_1} \boldsymbol{X}_2) \circ_{P_1 \circ P_2} \boldsymbol{X}_3\}(t) \\
&\quad = (\boldsymbol{X}_1 \circ_{P_1} \boldsymbol{X}_2)(t)(I - P_1 \circ P_2(t)) + \int_{[0,t]} (P_1 \circ P_2)(dt_2) X_3(t - t_2) \\
&\quad = \{X_1(t) P_1((t, \infty]) \\
&\qquad + \int_{[0,t]} P_1(dt_1) X_2(t - t_1)\} \left\{ I - \int_{[0,t]} P_1(dt_1) P_2(t - t_1) \right\} \\
&\qquad + \int_{0 \leq t_1 + t_2 \leq t} P_1(dt_1) P_2(dt_2) X_3(t - t_1 - t_2) \\
&\quad = X_1(t) P_1((t, \infty]) + \int_{[0,t]} P_1(dt_1)(I - P_2(t - t_1)) X_2(t - t_1) \\
&\qquad + \int_{0 \leq t_1 + t_2 \leq t} P_1(dt_1) P_2(dt_2) X_3(t - t_1 - t_2),
\end{aligned}$$

which once again agrees with the right hand side of (4.8). ■

In view of Proposition 4.4 we can now take the liberty of denoting the left hand side of (4.8) as $\boldsymbol{X}_1 \circ \boldsymbol{X}_2 \circ \boldsymbol{X}_3$ whenever the concerned stop times P_1 and P_2 are unambiguously fixed.

Consider a sequence of triples $\mathcal{H}_n, \boldsymbol{X}_n, P_n, n = 1, 2, \ldots$ where $\mathcal{H}_n$ is a Hilbert space, $\boldsymbol{X}_n$ is a bounded process and P_n is a stop time for $\boldsymbol{X}_n$ for each n. Let $\mathcal{H}_{n]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. Define the operators $\widehat{X}_{n+1}(t), X^0_{n+1}(t)$ and $X_{n+1]}(t)$ in $\mathcal{H}_{n+1]}$ by

$$\begin{aligned}
&\widehat{X}_{n+1}(t) \\
&\quad = \int_{t_1 + \cdots + t_n \leq t < t_1 + \cdots + t_{n+1}} P_1(dt_1) \cdots P_{n+1}(dt_{n+1}) X_{n+1}(t - \overline{t_1 + \cdots t_n}) \\
&\quad = \int_{t_1 + \cdots + t_n \leq t} P_1(dt_1) \ldots P_n(dt_n) \\
&\qquad \times X_{n+1}(t - \overline{t_1 + \cdots t_n}) P_{n+1}(t - \overline{t_1 + \cdots t_n}, \infty])
\end{aligned} \tag{4.9}$$

for $n \geq 1$,

$$\widehat{X}_1(t) = \int_{t < t_1} P_1(dt_1) X_1(t) = X_1(t) P_1((t, \infty]), \tag{4.10}$$

$$X^0_{n+1}(t) = \int_{t_1 + \cdots + t_n \leq t} P_1(dt_1) \cdots P_n(dt_n) X_{n+1}(t - \overline{t_1 + \cdots t_n}) \tag{4.11}$$

$$X_{n+1]}(t) = \sum_{m=0}^{n-1} \widehat{X}_{m+1}(t) + X^0_{n+1}(t) \tag{4.12}$$

for $n \geq 1$, where the m-th term which looks like an operator in $\mathcal{H}_{m+1]}$ is, indeed, ampliated to $\mathcal{H}_{n+1]}$. It is to be emphasized that $X_{n+1]} = X_1 \circ X_2 \circ ... X_{n+1}$, the glued process obtained from the sequence $X_1, X_2, ..., X_{n+1}$ through the stop times $P_1, P_2, ..., P_n$. Define the spectral measures $P_{n]}$ in $\mathcal{H}_{n]}$ by

$$P_{n]}(E) = \int_{\substack{t_1+\cdots+t_n \in E \\ 0 \leq t_i \leq \infty}} P_1(dt_1)...P_n(dt_n), \tag{4.13}$$

for any Borel set $E \subset [0, \infty]$, and denote their ampliations by the same symbols. Then, for any fixed t, $P_{n]}(t) = P_{n]}([0, t])$ is a decreasing sequence in n and

$$\widehat{X}_{n+1}(t) = (P_{n]}(t) - P_{n+1]}(t))\widehat{X}_{n+1}(t)(P_{n]}(t) - P_{n+1]}(t)), \tag{4.14}$$

$$X^0_{n+1}(t) = P_{n]}(t)X^0_{n+1}(t)P_{n]}(t) \tag{4.15}$$

with the understanding that $P_{0]}(t) \equiv I$. We have the estimates

$$||\widehat{X}_{n+1}(t)|| \leq \sup_{0 \leq s \leq t} ||X_{n+1}(s)||, \tag{4.16}$$

$$||X_{n+1]}(t)|| \leq \sup_{1 \leq j \leq n+1} \sup_{0 \leq s \leq t} ||X_j(s)||. \tag{4.17}$$

Let now ϕ_n be a unit vector in $\mathcal{H}_n, n = 1, 2, ...$ Consider the countable tensor product $\mathcal{H} = \bigotimes_{n=1}^{\infty} \mathcal{H}_n$ defined with respect to the stabilizing sequence $\{\phi_n\}$. Assume that

$$\sup_n \sup_{0 \leq s \leq t} ||X_n(s)|| < \infty \quad \text{for all } t \geq 0. \tag{4.18}$$

On ampliating $\widehat{X}_{n+1}(t)$ to $\mathcal{H}$ we see from (4.14) and (4.16) that the infinite series

$$X_{\infty]}(t) = \sum_{m=0}^{\infty} \widehat{X}_{m+1}(t) \tag{4.19}$$

converges strongly and

$$||X_{\infty]}(t)|| \leq \sup_n \sup_{0 \leq s \leq t} ||X_n(s)||. \tag{4.20}$$

Roughly speaking, $X_{\infty]}$ is the glued process $X_1 o X_2 \circ ...$ Note that the infinitely glued process $X_{\infty]}$ depends on the stabilizing sequence $\{\phi_n\}$. The next two propositions describe the basic properties of the operation of gluing a finite or countable number of bounded processes.

PROPOSITION 4.5. – *Let* X_n, Y_n *be bounded processes in the Hilbert space* $\mathcal{H}_n$ *for each* $n = 1, 2, \ldots$ *satisfying* (4.18) *and let* P_n *be a stop time for both* X_n *and* Y_n *for each* n. *Then the following holds for all* $2 \leq n \leq \infty$:

(i) $X_{n]} + Y_{n]} = (X + Y)_{n]}$;

(ii) $X_{n]}Y_{n]} = (XY)_{n]}$;

(iii) $(X_{n]})^* = (X^*)_{n]}$;

(iv) $X_{n]}$ *is positive or contractive according as each* X_i *is positive or contractive.*

Proof. – Immediate from Proposition 4.1 and the definition of glued processes. ■

PROPOSITION 4.6. – *In Proposition* 4.5 *suppose that* X_n *is the process* I_n *where* $I_n(t) \equiv I$ *in* $\mathcal{H}_n$ *for each* n. *Then*

$$I_{n]}(t) \equiv I \qquad for \quad 2 \leq n < \infty.$$

Define the probability measures $\{\nu_n\}$ *on* $[0, \infty]$ *associated with the stabilizing sequence* $\{\phi_n\}$ *by*

$$\nu_n(E) = \langle \phi_n, P_n(E)\phi_n \rangle \quad for\ every\ Borel\ set\ E \subset [0, \infty].$$

Then

$$I_{\infty]}(t) \equiv I$$

if and only if

$$\lim_{n\to\infty} (\nu_1 * \nu_2 * \cdots * \nu_n)([0, t]) = 0 \tag{4.21}$$

for all $0 \leq t < \infty$.

Proof. – The first part is immediate from the relations $\hat{I}_{n+1}(t) = P_{n]}(t) - P_{n+1]}(t)$, $P_{0]}(t) = I$ and the fact that $X^0_{n+1}(t)$ in (4.11) becomes $P_{n]}(t)$. To prove the sufficiency in the second part consider an element $u = u_1 \otimes u_2 \otimes \cdots \otimes u_k \otimes \phi_{k+1} \otimes \phi_{k+2} \otimes \cdots$ in $\mathcal{H}$ and observe that (4.11) yields

$$||X^0_{n+1}(t)u||^2 = (\lambda_1 * \cdots * \lambda_k * \nu_{k+1} * \nu_{k+2} * \cdots * \nu_n)([0, t]) \tag{4.22}$$

when $X_i(t) \equiv I$ for all $i, n > k, \lambda_i$ being the measure defined by $\lambda_i(E) = \langle u_i, P_i(E)u_i \rangle, i = 1, 2, \ldots, k$ and E any Borel subset of $[0, \infty]$.

Now (4.21) implies that the left hand side of (4.22) converges to 0 as $n \to \infty$. Since vectors of the form u are total in $\mathcal{H}$ it follows that $X^0_{n+1}(t) \to 0$ strongly as $n \to \infty$ for every t. Now (4.12) and (4.19) together with the first part imply that $\hat{I}_{\infty]}(t) = I$ for all $t \geq 0$.

To prove the necessity of (4.21) observe that $\nu_1 * \nu_2 * \cdots * \nu_n([0,t])$ decreases monotonically in n for every fixed $t \geq 0$.

Suppose that $\lim\limits_{n\to\infty} \nu_1 * \cdots * \nu_n([0,t_0]) = \delta > 0$ for some $t_0 > 0$. Then (4.12) implies that for the unit vector $u = \phi_1 \otimes \phi_2 \otimes \cdots$ in $\mathcal{H}$

$$\begin{aligned}||I_{\infty]}(t_0)u||^2 &= \lim_{n\to\infty} ||\sum_{m=0}^{n-1} \hat{I}_{m+1}(t_0)u||^2\\ &= 1 - \delta < 1.\end{aligned}$$

In other words $I_{\infty]}(t_0)$ is a proper projection. ■

Remark 4.7. – From Proposition 4.5 and 4.6 it is clear that for $2 \leq n < \infty$ the bounded process $X_{n]}$ is isometric, coisometric or unitary according as each $X_i, i = 1, 2, \ldots$ has the same property. If the measures $\{\nu_n\}$ defined in Proposition 4.6 satisfy the condition (4.21) then $X_{\infty]}$ is isometric, coisometric or unitary according as each $X_i, i = 1, 2, \ldots$ has the same property.

PROPOSITION 4.8. – *Let $\mathcal{H}_n, X_n, P_n, n = 1, 2, \ldots$ be as in Proposition 4.5. Suppose that the maps $t \to X_n(t)$ are strongly right continuous for each n. Then $X_{n]}(t)$ is strongly right continuous in t for every $2 \leq n \leq \infty$. If $X_n(t)$ is strongly continuous in t and P_n has no atoms in $\mathbb{R}_+$ for every n then $X_{n]}(t)$ is strongly continuous in t for every $2 \leq n \leq \infty$.*

Proof. – Consider an element $u = u_1 \otimes u_2 \otimes \ldots$ in $\mathcal{H}$ where each u_n is a unit vector and $u_n = \phi_n$ for all n exceeding some n_0. From (4.14) and (4.19) we have

$$||X_{\infty]}(t)u||^2 = \sum_{n=0}^{\infty} ||\hat{X}_{n+1}(t)u||^2. \tag{4.23}$$

Consider a fixed bounded interval $[0,T]$ and observe that (4.14), (4.16) and (4.18) imply the existence of a positive constant C depending on T such that

$$\begin{aligned}&||\hat{X}_{n+1}(t)u||^2\\ &\leq C||(P_{n]}(t) - P_{n+1]}(t))u||^2\\ &= C\{(\mu_1 * \cdots * \mu_n)([0,t]) - (\mu_1 * \cdots * \mu_{n+1})([0,t])\}\end{aligned} \tag{4.24}$$

for all $0 \leq t \leq T$ where $\mu_i(\cdot) = \langle u_i, P_i(\cdot)u_i\rangle$. Note that when $n = 0$ the right hand side of the inequality above is to be interpreted as $C(1 - \mu_1([0, t]))$. It follows from (4.23) and (4.24) that the right hand side of (4.23) converges uniformly in $t \in [0, T]$. Thus, in order to prove the first part of the proposition, it suffices to show that the map $t \to \widehat{X}_{n+1}(t)u$ is strongly right continuous. We have

$$\begin{aligned}
&\{\widehat{X}_{n+1}(t+h) - \widehat{X}_{n+1}(t)\}u \\
&\quad = \left\{\int_{[0,t+h]} P_{n]}(ds)X_{n+1}(t+h-s)P_{n+1}((t+h-s,\infty])\right\}u \\
&\quad\quad - \left\{\int_{[0,t+h]} P_{n]}(ds)X_{n+1}(t-s)P_{n+1}((t-s,\infty])\right\}u \\
&\quad = \int_{[0,t]} P_{n]}(ds)\{X_{n+1}(t+h-s)P_{n+1}((t+h-s,\infty]) \\
&\quad\quad - X_{n+1}(t-s)P_{n+1}((t-s,\infty])\}u \\
&\quad\quad + \int_{(t,t+h]} P_{n]}(ds)X_{n+1}(t+h-s)P_{n+1}((t+h-s,\infty])u.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\{\widehat{X}_{n+1}(t+h) - \widehat{X}_{n+1}(t)\}u\|^2 \\
&\quad \leq \int_{[0,t]} < u_{n]}, P_{n]}(ds)u_{n]} > \|\{X_{n+1}(t+h-s)P_{n+1}((t+h-s,\infty]) \\
&\qquad\qquad\qquad - X_{n+1}(t-s)P_{n+1}((t-s,\infty])\}u_{n+1}\|^2 \\
&\quad\quad + C\langle u_{n]}, P_{n]}((t,t+h])u_{n]}\rangle,
\end{aligned}$$

where C is the positive constant mentioned earlier, $u_{n]} = u_1 \otimes \cdots \otimes u_n$, $h > 0, t+h \leq T$. Since $X_{n+1}(t)$ and $P_{n+1}((t, \infty])$ are both right continuous for every n the right continuity of $\widehat{X}_n(t)$ in t follows from the inequality above. It is to be noted that we have used the fact that vectors of the form u described at the beginning are total in $\mathcal{H}$. The second part of the proposition is proved in the same manner. ■

Now we proceed to define the glued filtration which may be considered as the natural filtration for glued processes. To this end we consider a sequence $(\mathcal{H}_n, F_n), n = 1, 2, \ldots$ where $\mathcal{H}_n$ is a Hilbert space and F_n is a weak filtration in $\mathcal{H}_n$ such that the map $t \to F_n(t)$ is strongly right continuous. Let X_n be a bounded process adapted to F_n and let P_n be a stop time for X_n so that P_n is adapted to F_n in the sense of Definition 4.2. We choose the stabilizing sequence of unit vectors $\{\phi_n\}$ such that ϕ_n is in the range of $F_n(0)$ for each $n = 1, 2, ...$ Denote by $\Phi([m, n]) = |\phi_m\rangle\langle\phi_m| \otimes |\phi_{m+1}\rangle\langle\phi_{m+1}| \otimes \cdots \otimes |\phi_n\rangle\langle\phi_n|$ the one dimensional projection ampliated to $\mathcal{H}_{n]}$ or $\mathcal{H}$ whenever necessary and using the same symbol for operators and their ampliations introduce the operators $\widetilde{F}_{n+1]}(t)$ by

$$\begin{aligned}
&\widetilde{F}_{n+1]}(t) \\
&\quad = \sum_{m=0}^{n-1} \widehat{F}_{m+1}(t)\Phi([m+2, n+1]) \\
&\qquad + \int_{t_1+\cdots+t_n \leq t} P_1(dt_1)...P_n(dt_n)F_{n+1}(t - \overline{t_1 + \cdots + t_n})
\end{aligned} \tag{4.25}$$

for $1 \leq n < \infty$ and the operators $\widetilde{F}_{\infty]}(t)$ in $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes ...$ by

$$\widetilde{F}_{\infty]}(t) = \sum_{m=0}^{\infty} \widehat{F}_{m+1}(t)\Phi([m+2, \infty)). \tag{4.26}$$

Here we have used the notations (4.9) and (4.10). It is important to compare (4.25) and (4.26) with (4.12) and (4.19).

PROPOSITION 4.9. – *$\widetilde{F}_{n+1]}$ defined by* (4.25) *is a right continuous weak filtration in $\mathcal{H}_{n+1]}$ for every $2 \leq n < \infty$. If X_n is an F_n-adapted bounded process in $\mathcal{H}_n$ and P_n is an F_n-adapted stoptime for every n then $X_{n+1]}$ is $\widetilde{F}_{n+1]}$-adapted for every $1 \leq n < \infty$.*

Proof. – We prove the result first when $n = 1$. From Proposition 4.1 it follows easily that the operator

$$\widetilde{F}_{2]}(t) = F_1(t)(1 - P_1(t))|\phi_2\rangle\langle\phi_2| + \int_{[0,t]} P_1(dt_1)F_2(t - t_1)$$

with ϕ_2 in the range of $F_2(0)$ is a projection. For $0 \leq s \leq t < \infty$ we have

$$\begin{aligned}
&\widetilde{F}_{2]}(t)\widetilde{F}_{2]}(s) \\
&\quad = F_1(t)(1 - P_1(t))(1 - P_1(s))F_1(s)|\phi_2\rangle\langle\phi_2| \\
&\quad + F_1(t)\int_{(t,\infty]} P_1(dt_1)|\phi_2\rangle\langle\phi_2| \int_{[0,s]} P_1(ds_1)F_2(s - s_1) \\
&\quad + \int_{[0,t]} P_1(dt_1)F_2(t - t_1) \int_{(s,\infty]} P_1(ds_1)F_1(s)|\phi_2\rangle\langle\phi_2| \\
&\quad + \int_{[0,t]} P_1(dt_1)F_2(t - t_1) \int_{[0,s]} P_1(ds_1)F_2(s - s_1) \\
&\quad = (1 - P_1(t))F_1(s)|\phi_2\rangle\langle\phi_2| + 0 + \int_{(s,t]} P_1(ds_1)F_1(s)|\phi_2\rangle\langle\phi_2| \\
&\quad + \int_{[0,s]} P_1(ds_1)F_2(s - s_1) \\
&\quad = \widetilde{F}_{2]}(s)
\end{aligned}$$

which shows that $\widetilde{F}_{2]}(t)$ is increasing in t. In other words $\widetilde{F}_{2]}$ is a filtration. To prove the adaptedness of $X_{2]}$ with respect to $\widetilde{F}_{2]}$ observe that

$$\begin{aligned}
&X_{2]}(t)\widetilde{F}_{2]}(t) \\
&\quad = \{X_1(t)(1 - P_1(t)) + \int_{[0,t]} P_1(dt_1)X_2(t - t_1)\}\widetilde{F}_{2]}(t) \\
&\quad = X_1(t)(1 - P_1(t))F_1(t)|\phi_2\rangle\langle\phi_2| + \int_{[0,t]} P_1(dt_1)X_2(t - t_1)F_2(t - t_1) \\
&\quad = |\phi_2\rangle\langle\phi_2|F_1(t)X_1(t)(1 - P_1(t))F_1(t)|\phi_2\rangle\langle\phi_2| \\
&\quad + \int_{[0,t]} P_1(dt_1)F_2(t - t_1)X_2(t - t_1)F_2(t - t_1) \\
&\quad = \widetilde{F}_{2]}(t)X_{2]}(t)\widetilde{F}_{2]}(t),
\end{aligned}$$

which proves the claim for $n = 1$. Now assume that the Proposition is true for $n \leq k$. Then on gluing $\widetilde{F}_{k+1]}$ with F_{k+2} using the cumulative stop time $P_{k+1]} = P_1 \circ P_2 \circ \ldots \circ P_{k+1}$ given by

$$P_{k+1]}(t) = \int_{t_1+\cdots+t_{k+1}\leq t} P_1(dt_1)\cdots P_{k+1}(dt_{k+1})$$

we have a new filtration G given by

$$\begin{aligned}
G(t) = &\widetilde{F}_{k+1]}(t)(I - P_{k+1]}(t))|\phi_{k+2}\rangle\langle\phi_{k+2}| \\
&+ \int_{[0,t]} P_{k+1]}(dt_{k+2})F_{k+2}(t - t_{k+2})
\end{aligned}$$

which is easily verified to be the same as $\widetilde{F}_{k+2]}(t)$. Hence $\widetilde{F}_{k+2]}$ is also a filtration. Since $X_{k+2]} = X_{k+1]} \circ X_{k+2}$ a repetition of the earlier argument shows that $X_{k+2]}$ is $\widetilde{F}_{k+2]}$-adapted.

The strong right continuity of $\widetilde{F}_{n]}(t)$ in t is proved exactly as in Proposition 4.8.

PROPOSITION 4.10. – *In Proposition* 4.9 *suppose that the sequence of probability measures* $\{\nu_n\}$ *in the closed interval* $[0, \infty]$ *defined by* $\nu_n(\cdot) = \langle \phi_n, P_n(\cdot)\phi_n \rangle, n = 1, 2, \ldots$ *satisfies* (4.21). *Then* $\widetilde{F}_{\infty]}$ *defined by* (4.26) *is a strongly right continuous weak filtration in* $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ *If* X_n *is an* F_n*-adapted process and* P_n *is an* F_n*-adapted stop time for* X_n *for every* n *then* $X_{\infty]}$ *is* $\widetilde{F}_{\infty]}$*-adapted.*

Proof. – Proceeding along the lines of the proof of Proposition 4.6 we conclude that $\widetilde{F}_{\infty]}(t) = s.\lim_{n\to\infty} \widetilde{F}_{n+1]}(t), X_{\infty]}(t) = s.\lim_{n\to\infty} X_{n+1]}(t)$. The strong right continuity of $\widetilde{F}_{\infty]}(t)$ in t is proved exactly as in Proposition 4.8. Rest is immediate from Proposition 4.9. ■

5. GLUING MARKOV FLOWS

In classical probability theory a Markov process governed by a nonconservative or substochastic semigroup of transition probability operators on a state space Ω is interpreted as a Markov process whose trajectories may get out of the space Ω (or hit the boundary) at an exit time depending on the individual trajectory. Such an exit time provides a natural stop time at which the trajectory may be stopped at the boundary with probability p or continued with probability $q = 1 - p$ along a new independent trajectory of the original flow starting from a point $x \in \Omega$ chosen according to a suitable entrance probability law. Such a procedure can be repeated ad infinitum. The aim of the present section is to quantize this idea or, equivalently, express it in the language of operators in a Hilbert space by adopting the gluing mechanism described in Section 4 with respect to suitable exit times for a nonconservative Markov flow mediated by a one parameter semigroup $\{T_t\}$ of completely positive and contractive linear maps on a unital von Neumann algebra and thereby obtain a new Markov flow whose expectation semigroup $\{\widehat{T}_t\}$ is a Feller perturbation of $\{T_t\}$.

DEFINITION 5.1. – Let $(\mathcal{H}, F, j_t)$ be a weak Markov flow on a unital von Neumann algebra $\mathcal{A}$ of operators in a Hilbert space $\mathcal{H}_0$ with expectation semigroup $\{T_t\}$. A spectral measure P on the closed interval $[0, \infty]$ with values in the lattice of orthogonal projections in $\mathcal{H}$ is called an *exit time* for the flow $(\mathcal{H}, F, j_t)$ if the following conditions hold:

$$\text{(i)} \qquad j_t(X)P([0,t]) = 0 \quad \text{for all } t \geq 0, X \in \mathcal{A}; \tag{5.1}$$

$$\text{(ii)} \qquad P([0,t])F(t) = F(t)P([0,t]) \quad \text{for all } t \geq 0;$$

(iii) If S_P denotes the positive operator-valued Radon measure defined on $\mathbb{R}_+$ by

$$S_P(\{0\}) = 0, S_P((a,b]) = \mathbb{E}_{0]}^F P((a,b])|_{\mathcal{H}_0} \tag{5.2}$$

then $S_P(E) \in \mathcal{A}$ and

$$j_s(S_P(E)) = \mathbb{E}_{s]}^F P(E+s) \quad \text{for all } s \geq 0, E \in \mathcal{F}_b(\mathbb{R}_+). \tag{5.3}$$

Condition (ii) expresses the adaptedness of the stop time P and for any initial state λ on $\mathcal{A}, \lambda(S_P([0,t]))$ is the probability that "hitting the boundary" occurs at or before time t. Condition (i) can be interpreted as the fact that if the system or flow goes out of $\mathcal{A}$ before time t the event $j_t(X)$ for any projection X in $\mathcal{A}$ cannot occur at time t. Condition (iii) emphasizes the covariant nature of the exit time under the flow.

Proposition 5.2. – *If $\mathcal{H}, F, j_t$ and P are as in Definition* 5.1 *then the Radon measure S_P satisfying* (5.2) *and* (5.3) *is an exit cocycle for the expectation semigroup $\{T_t\}$ of the flow $(\mathcal{H}, F, j_t)$.*

Proof. – Taking conditional expectation $\mathbb{E}_{0]}^F$ in (5.3) we have from the Markov property of the flow

$$\begin{aligned} T_s(S_P(E))F(0) &= \mathbb{E}_{0]}^F j_s(S_P(E)) = \mathbb{E}_{0]}^F \mathbb{E}_{s]}^F P(E+s) \\ &= \mathbb{E}_{0]}^F P(E+s) = S_P(E+s)F(0). \quad \blacksquare \end{aligned}$$

Let $\mathcal{H}_n, F_n, j_t^{(n)}, P_n, n = 1, 2, \ldots$ be copies of $\mathcal{H}, F, j_t, P$ in Definition 5.1. Note that equation (5.1) together with its adjoint and condition (ii) of Definition 5.1 imply that the exit time P is also an F-adapted stop time for the bounded process $\{j_t(X), t \geq 0\}$ for every $X \in \mathcal{A}$. Choose and fix a unit vector ϕ in the range of $F(0)$ in $\mathcal{H}$. Let $\widehat{\mathcal{H}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ where the countably infinite tensor product is taken with respect to the stabilizing sequence $\{\phi_n\}$ with $\phi_n = \phi$ in the n-th copy for each n. Using $\{P_n\}$ we make an infinite gluing of the processes $\{j_t^{(n)}(X)\}$ for each $X \in \mathcal{A}$ as in Section 4 to obtain the processes

$$\hat{j}_t(X) = j_t^{(1)}(X)P_1((t,\infty])$$

$$+\sum_{n\geq 1}\int_{t_1+\cdots+t_n\leq t<t_1+\cdots+t_{n+1}} P_1(dt_1)\cdots$$
$$P_{n+1}(dt_{n+1})j^{(n+1)}_{t-(t_1+\cdots+t_n)}(X). \tag{5.4}$$

By (5.1) we have $j_t(X)P((t,\infty]) = \dot{j}_t(X)$ and (5.4) can be expressed as

$$\hat{j}_t(X) = \sum_{n\geq 0}\hat{j}_t^{(n)}(X) = \sum_{n\geq 0}\sum_{m=0}^{n}\hat{j}_{s,t}^{m,n}(X) \quad \text{for } 0\leq s\leq t, X\in\mathcal{A} \tag{5.5}$$

where

$$\hat{j}_t^{(n)}(X) = \begin{cases} j_t^{(1)}(X) & \text{if } n=0,\\ \displaystyle\int_{\Delta_n(t)} P_1(dt_1)\ldots P_n(dt_n)j^{(n+1)}_{t-(t_1+\cdots+t_n)}(X) & \text{if } n\geq 1\end{cases}$$

and for $0\leq m\leq n$

$$\hat{j}_{s,t}^{(m,n)}(X)$$
$$=\begin{cases} j_t^{(1)}(X) & \text{if } m=n=0,\\ \displaystyle\int_{\Delta_{m,n}(s,t)} P_1(ds_1)\ldots P_n(ds_n)j^{(n+1)}_{t-(s_1+\cdots+s_n)}(X) & \text{otherwise}\end{cases}$$

where Δ_n and $\Delta_{m,n}(s,t)$ are as in (3.5) - (3.7). It is useful to compare the two expressions above with (3.5) and (3.6) and interpret $\hat{j}_t^{(n)}$ as a description of the glued process at time t under the knowledge that exactly n exits have occurred upto time t. Similarly $\hat{j}_{s,t}^{m,n}$ describes the glued process under the knowledge that exactly n exits upto time t and m exits upto time s have been made.

THEOREM 5.3. – *Let $\mathcal{H}, F, j_t, T_t, P, S_P$ be as in Definition 5.1 and let ϕ be a unit vector in the range of $F(0)$. Define the maps $\hat{j}_t : \mathcal{A}\to\mathcal{B}(\widehat{\mathcal{H}})$ by (5.4). Let $\widehat{F} = \widetilde{F}_{\infty]}$ be the glued filtration in $\widehat{\mathcal{H}}$ defined by (4.26). Then $(\widehat{\mathcal{H}}, \widehat{F}, \hat{j}_t)$ is a weak Markov flow with expectation semigroup $\{\widehat{T}_t\}$ which is the Feller perturbation of $\{T_t\}$ determined by the exit cocycle S_P and the vector state ω with density matrix $|\phi\rangle\langle\phi|$.*

Proof. – It follows from Proposition 4.5 that for each t, $\hat{j}_t$ is a $*$ homomorphism from $\mathcal{A}$ into $\mathcal{B}(\widehat{\mathcal{H}})$. From (5.4) and (5.1) we have $\hat{j}_0(X) = j_0^{(1)}(X)(1-P_1(0)) = j_0^{(1)}(X)$ which is $j_0(X)$ in the first copy of $\mathcal{H}$ ampliated to $\widehat{\mathcal{H}}$. Thus

$$\begin{aligned}\widehat{F}(0)\hat{j}_0(X)\widehat{F}(0) &= F(0)j_0(X)F(0)\otimes\Phi([2,\infty))\\ &= X\widehat{F}(0).\end{aligned}$$

Since the measure μ defined by $\mu(\cdot) = \langle \phi, P(\cdot)\phi \rangle$ is not degenerate at 0 it is clear that $\lim\limits_{n\to\infty} \mu^{*^n}([0,t]) = 0$ for every $t \geq 0$. Hence by Proposition 4.10 the process $\{\hat{j}_t(X)\}$ is adapted to the filtration $\widehat{F}$ for every $X \in \mathcal{A}$.

Fixing $0 \leq s \leq t$ and using (5.4), (5.5) and (4.26) we obtain

$$
\begin{aligned}
&\widehat{F}(s)\hat{j}_t(X)\widehat{F}(s) \\
&= \{\sum_{k\geq 0} \widehat{F}_{k+1}(s)\Phi([k+2,\infty))\} \\
&\quad \times \{ \sum_{0\leq m\leq n<\infty} \hat{j}^{m,n}_{s,t}(X)\}\{\sum_{k\geq 0} \widehat{F}_{k+1}(s)\Phi([k+2,\infty))\} \\
&= \sum_{0\leq m\leq n<\infty} Z_{m,n} \qquad (5.6)
\end{aligned}
$$

where

$$
\begin{aligned}
Z_{n,n} = &\int_{t_1+\cdots+t_n\leq s} P_1(dt_1)\cdots P_n(dt_n)\mathbb{E}^{(n+1)}_{s-\overline{t_1+\cdots+t_n}]} \\
&\times (j^{(n+1)}_{t-\overline{t_1+\cdots+t_n}}(X))F_{n+1}(s-\overline{t_1+\cdots+t_n})\Phi([n+2,\infty)) \quad (5.7)
\end{aligned}
$$

and for $m < n$

$$
\begin{aligned}
Z_{m,n} = &\int_{\substack{t_1+\cdots+t_m\leq s<t_1+\cdots+t_{m+1} \\ t_1+\cdots+t_n\leq t}} P_1(dt_1)\cdots P_m(dt_m) \\
&\times F_{m+1}(s-\overline{t_1+\cdots+t_m})P_{m+1}(dt_{m+1})F_{m+1}(s-\overline{t_1+\cdots+t_m}) \\
&\times \mu(dt_{m+2})\cdots\mu(dt_n)\langle\phi, T_{t-\overline{t_1+\cdots+t_n}}(X)\phi\rangle\Phi([m+2,\infty)). \quad (5.8)
\end{aligned}
$$

From the Markov property of j_t it follows that (5.7) can be expressed as

$$
\begin{aligned}
Z_{n,n} = &\int_{t_1+\cdots+t_n\leq s} P_1(dt_1)\cdots P_n(dt_n)j^{(n+1)}_{s-\overline{t_1+\cdots+t_n}} \\
&\times (T_{t-s}(X))F_{n+1}(s-\overline{t_1+\cdots+t_n})\Phi([n+2,\infty)) \quad (5.9)
\end{aligned}
$$

In (5.8) make the change of variables:

$$
s_1 = t_1 + \cdots + t_{m+1} - s, s_2 = t_{m+2}, \ldots, s_{n-m} = t_n
$$

and use (5.3) in the form

$$\mathbb{E}^{(m+1)}_{s-\overline{t_1+\cdots+t_m}]}\{P_{m+1}(dt_{m+1})\} = j^{(m+1)}_{s-\overline{t_1+\cdots+t_m}}(S_P(ds_1)).$$

Then we obtain

$$\begin{aligned} Z_{m,n} = \int_{t_1+\cdots+t_m\leq s} & P_1(dt_1)\cdots P_m(dt_m) \\ & \times j^{(m+1)}_{s-\overline{t_1+\cdots+t_m}}\left(\int_{s_1+\cdots+s_{n-m}\leq t-s} S_P(ds_1)\mu(ds_2)\cdots\right. \\ & \left.\times\, \mu(ds_{n-m})\langle\phi, T_{t-s-\overline{s_1+\ldots+s_{n-m}}}(X)\phi\rangle\right) \\ & \times F_{m+1}(s-\overline{t_1+\cdots+t_m})\Phi([m+2,\infty)). \end{aligned} \tag{5.10}$$

Plugging the expressions (5.9) and (5.10) in (5.6), first summing over the variable $n-m$ from 0 to ∞ and then over the variable m from 0 to ∞ we obtain

$$\widehat{F}(s)\hat{j}_t(X)\widehat{F}(s) = \hat{j}_s(\widehat{T}_{t-s}(X))\widehat{F}(s),$$

where $\{\widehat{T}_t\}$ is the Feller perturbation of $\{T_t\}$ based on the exit cocycle S_P and the vector state ϕ. ■

Remark 5.4. – Theorem 5.3 can be easily adapted to the case of Feller perturbations based on S_P and a state determined by a density matrix of the form $\rho = \sum_n p_n|\phi_n\rangle\langle\phi_n|$ where $\{\phi_n\}$ is an orthonormal sequence in the range of $F(0)$ and $\{p_n\}$ is a probability sequence, *i.e.*, $p_n \geq 0$ for each n and $\sum_n p_n = 1$. We do this as follows. Put $\mathcal{H}'_0 = \mathcal{H}_0 \otimes \mathcal{H}_0, \mathcal{A}' = \{I_o \otimes X, X \in \mathcal{A}\}, T'_t(I_0 \otimes X) = I_0 \otimes T_t(X)$ for all $t \varepsilon 0, X \in \mathcal{A}$ where I_0 is the identity operator in $\mathcal{H}_0$. Let $\mathcal{H}' = \mathcal{H}_0 \otimes \mathcal{H}, F'(t) = I_0 \otimes F(t), j'_t(I_0 \otimes X) = I_0 \otimes j_t(X), P' = I_0 \otimes P, S'_{P'} = I_0 \otimes S_P$. Then $(\mathcal{H}', F', j'_t)$ is a weak Markov flow on $\mathcal{A}'$ with exit time P' and expectation semigroup $\{T'_t\}$. $S'_{P'}$ is an exit cocycle for $\{T'_t\}$. Now consider the vector state on $\mathcal{A}'$ determined by the unit vector $\phi' = \sum_n \sqrt{p}_n\phi_n \otimes \phi_n$ in $\mathcal{H}'_0$. We may view ϕ' as an element in the range of $F'(0)$ and construct the infinitely glued flow $(\widehat{\mathcal{H}}', \widehat{F}', \hat{j}'_t)$ according to Theorem 5.3. This glued flow is a weak Markov flow on $\mathcal{A}'$ with expectation semigroup $\{\widehat{T}'_t\}$ where $\widehat{T}'_t(I_0 \otimes X) = I_0 \otimes \widehat{T}_t(X), \{\widehat{T}_t\}$ being the Feller perturbation of $\{T_t\}$ based on (S_P, ρ).

It is interesting to note that for any initial state ω on $\mathcal{A}$ and projection $Q \in \mathcal{A}$ the probability that according to the glued flow $(\widehat{\mathcal{H}}', \widehat{F}', \hat{j}'_t)$ exactly

m exits occur upto time s, n exits occur upto time t and the event Q occurs at time t is equal to

$$\int_{\substack{s_1+\ldots+s_m \leq s < s_1+\ldots+s_{m+1} \\ s_1+\ldots+s_n \leq t}} tr\rho(T_{t-(s_1+\ldots+s_n)}(Q)) \\ \times (\omega \circ S_P)(ds_1)\mu(ds_2)\ldots\mu(ds_n)$$

where $0 \leq s \leq t < \infty$ and $\mu = tr\rho S_P$.

Example 5.5. – Using isometric cocycles arising naturally from the theory of quantum stochastic differential equations $(q.s.d.e.)$ in the Fock spaces one can construct many examples of nonconservative flows with an exit time. Indeed, let $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k)$ as in [Mo]. Consider an isometric cocycle $U = \{U(s,t), 0 \leq s \leq t < \infty\}$ obeying the q.s.d.e

$$U(s,s) = 1, \qquad dU(s,t) = U(s,t)\{\sum_{i,j} L^i_j d\Lambda^j_i(t)\}$$

where $\{L^i_j\}$ is a family of operators in $\mathcal{H}_0$. (See [Mo]). By the cocycle property $U(0,s)U(s,t) = U(0,t)$ for all $0 \leq s \leq t < \infty$. Define

$$j^0_t(X) = U(0,t)XU(0,t)^*, \qquad X \in \mathcal{B}(\mathcal{H}_0)$$

where we denote an operator and its ampliation by the same symbol. Then $j^0_t(I) = U(0,t)U(0,t)^*$ is a projection. For any $0 \leq s \leq t < \infty, \psi \in \mathcal{H}$ we have

$$\begin{aligned}\langle \psi, j^0_t(I)\psi\rangle &= \|U(0,t)^*\psi\|^2 \\ &= \|U(s,t)^*U(0,s)^*\psi\|^2 \\ &\leq \|U(0,s)^*\psi\|^2 \\ &= \langle \psi, j^0_s(I)\psi\rangle.\end{aligned}$$

This shows that $\{j^0_t(I)\}$ is a family of projections decreasing in t. Using the strong continuity of $j^0_t(X)$ in t we conclude the existence of a spectral measure P on $[0,\infty]$ such that $P([0,t]) = 1 - j^0_t(I)$ for all t where 1 and I denote the identity operators in $\mathcal{H}$ and $\mathcal{H}_0$ respectively. Let $\{T_t\}$ be the semigroup of completely positive linear maps on $\mathcal{B}(\mathcal{H}_0)$ satisfying $T_t(X) = \mathbb{E}_{0]}j^0_t(X)$ for all $t \geq 0, X \in \mathcal{A}$ where $\mathbb{E}_{0]}$ is the Fock vacuum conditional expectation. Let S be the positive operator-valued Radon measure determined by $S([0,t]) = I - T_t(I)$ for all $t \geq 0$. Denoting by $\mathbb{E}_{s]}$ the usual conditional expectation with respect to the Fock vacuum vector $\Phi_{[s}$ in $\Gamma(L^2([s,\infty)\otimes k)$ we have

$$\begin{aligned}\mathbb{E}_s P((s,t+s]) &= \mathbb{E}_{s]}(P([0,t+s]) - P([0,s])) \\ &= \mathbb{E}_{s]}(j_s^0(I) - j_{s+t}^0(I)) \\ &= j_s^0(I - T_t(I)) \\ &= j_s^0(S([0,t])). \end{aligned} \tag{5.11}$$

Let $F(t)$ denote the projection on to the subspace $\mathcal{H}_0 \otimes \Gamma(L^2[0,t] \otimes k) \otimes \Phi_{[t} \subset \mathcal{H}$. Define

$$j_t(X) = j_t^0(X)F(t).$$

Then $(\mathcal{H}, F, j_t)$ is a subordinate weak Markov flow on $\mathcal{B}(\mathcal{H}_0)$ with expectation semigroup $\{T_t\}$. Furthermore

$$\begin{aligned} j_t(X)P([0,t]) &= j_t^0(X)F(t)(1 - j_t^0(I)) \\ &= F(t)j_t^0(X)(1 - j_t^0(I)) \\ &= 0 \end{aligned}$$

and (5.11) implies

$$\begin{aligned} j_s(S([0,t])) &= j_s^0(S([0,t]))F(s) \\ &= \{\mathbb{E}_{s]}P((s,s+t])\}F(s) \\ &= \mathbb{E}_{s]}^F P((s,s+t]). \end{aligned}$$

In other words P is an exit time for $(\mathcal{H}, F, j_t)$.

More generally, consider a family of non-conservative Evans-Hudson flows $j_{s,t}^0$, $s \le t$, on a unital von Neumann algebra $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$, taking values in $\mathcal{A}_{[s} = \mathcal{A}_0 \otimes \mathcal{B}(\Gamma(L_2([s,\infty),k]))$ with structure maps $\{\theta_j^i\}$ so that

$$d_t j_{s,t}^0(X) = j_{s,t}^0(\theta_j^i(X))d\Lambda_i^j(t), \quad j_{s,s}^0(X) = X$$

for $s \le t$. Extend the domain of definition of $j_{0,s}^0$ from $\mathcal{A}_0$ to $A_{[s}$ by putting

$$j_s^0(X \otimes Z) = j_{0,s}^0(X)\widehat{Z}$$

for $X \in \mathcal{A}_0$ and $Z \in \mathcal{B}(\Gamma(L_2([s,\infty),k))$, where $\widehat{Z}$ is the ampliation of Z to an element of $\mathcal{A}_{[0} = \mathcal{A}_0 \otimes \mathcal{B}(\Gamma(L_2([0,s),k))) \otimes \mathcal{B}(\Gamma(L_2([s,\infty),k)))$.

Then

$$j_t^0(X) = j_{0,t}^0(X), \quad X \in \mathcal{A}_0$$
$$j_t^0(X) = j_s^0(j_{s,t}^0(X)), \qquad X \in \mathcal{A}_0, \quad 0 \leq s \leq t.$$

This shows, in particular, that

$$j_t^0(I) = j_{s,t}^0(j_s^0(I)), \qquad 0 \leq s \leq t.$$

Since $j_{s,t}^0$ is a contractive * homomorphism it follows that $\{j_t^0(I)\}$ is a family of projections which is decreasing and strongly continuous in t. Thus there exists a spectral measure P on $[0, \infty]$ such that $P([0,t]) = 1 - j_t^0(I)$, 1 being the identity operator in $\mathcal{H}$. As before define $j_t(X) = j_t^0(X)F(t)$. Then $(\mathcal{H}, F, j_t)$ yields a weak Markov flow with exit time P.

Example 5.6. – The simplest example of a nonconservative flow is constructed from a given conservative flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ with expectation semigroup $\{T_t\}$ as follows. Consider a classical Poisson process with intensity λ_0 whose probability measure μ in the path space yields the Hilbert space $\mathcal{H}_1 = L^2(\mu)$.

Let $P_1([0,t])$ be the projection in $\mathcal{H}_1$ which is multiplication by the indicator function of the event that the Poisson path undergoes a jump in the interval $[0,t]$. Let $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_1$ and let $\widetilde{P}$ be the spectral measure in $[0, \infty]$ determined by

$$\widetilde{P}([0,t]) = 1 \otimes P_1([0,t]) \quad \text{for all } t \geq 0.$$

Define

$$\widetilde{j}_t(X) = j_t(X) \otimes \widetilde{P}((t, \infty]), \quad t \geq 0, \quad X \in \mathcal{A}.$$

Note that $\widetilde{P}(\{\infty\}) = 0$. If $F_1(t)$ is the projection on to the subspace of functions of the Poisson path upto time t and $\widetilde{F}(t) = F(t) \otimes F_1(t)$ it follows from the fact that the Poisson process has independent increments, that

$$\widetilde{F}(s)\widetilde{j}_t(X)\widetilde{F}(s) = j_s(T_{t-s}(X))e^{-\lambda_0(t-s)}$$

for all $0 \leq s \leq t < \infty$. In other words we have a weak Markov flow $(\widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{j}_t)$ with expectation semigroup $\{e^{-\lambda_0 t}T_t\}$. It is easily verified that $\widetilde{P}$ is an exit time for this flow.

Example 5.7. – If $\mathcal{A}$ is a unital von Neumann algebra of operators in $\mathcal{H}_0$ and $\{T_t\}$ is a uniformly continuous contraction semigroup of completely positive linear maps on $\mathcal{A}$ then by the method outlined in [P1] it can be shown that the infinitesimal generator $\mathcal{L}$ of $\{T_t\}$ has the form

$$\mathcal{L}(X) = \mathcal{L}_0(X) - \frac{1}{2}(XB + BX), \quad X \in \mathcal{A}$$

where $\mathcal{L}_0$ is the generator of a conservative and uniformly continuous contraction semigroup of completely positive linear maps on $\mathcal{A}$ and B is a positive element of $\mathcal{A}$.

As a special case when $\mathcal{A} = \mathcal{B}(\mathcal{H}_0)$ it follows that $\mathcal{L}$ has the form

$$\begin{aligned}\mathcal{L}(X) =& i[H_0, X] - \frac{1}{2}\sum_j (L_j^* L_j X + X L_j^* L_j - 2L_j^* X L_j) \\ & - \frac{1}{2}(BX + XB) \end{aligned} \tag{5.12}$$

where H_0, L_j and B are bounded operators in $\mathcal{H}_0, H_0$ is selfadjoint, B is positive and $\sum_j L_j^* L_j$ is strongly converent. We shall now construct a concrete Markov flow whose expectation semigroup has generator $\mathcal{L}$. To this end consider $\mathcal{B}(\mathcal{H}_0 \bigoplus \mathcal{H}_0)$ and represent any element in it in the form of a matrix $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ where $X_{ij} \in \mathcal{B}(\mathcal{H}_0)$ for each i, j. Define the operators

$$H = \begin{pmatrix} H_0 & -\frac{1}{2i}\sqrt{B} \\ \frac{1}{2i}\sqrt{B} & 0 \end{pmatrix}$$

$$L^{(1)} = \begin{pmatrix} 0 & 0 \\ \sqrt{B} & I \end{pmatrix}, \qquad L^{(j)} = \begin{pmatrix} L_{j-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad j = 2, 3, \ldots$$

Consider the standard Evans-Hudson flow $\tilde{j}_t$ induced by a unitary cocycle in the Hilbert space

$$\tilde{\mathcal{H}} = (\mathcal{H}_0 \bigoplus \mathcal{H}_0) \otimes \Gamma(L^2(\mathbb{R}_+) \otimes \ell^2)$$

satisfying $\mathbb{E}_{s]}\tilde{j}_t(\tilde{X}) = \tilde{j}_s(\tilde{T}_{t-s}(\tilde{X}))$ for all $\tilde{X} \in \mathcal{B}(\mathcal{H}_0 \bigoplus \mathcal{H}_0)$, $0 \leq s \leq t < \infty$ where $\tilde{T}_t$ has generator $\tilde{\mathcal{L}}$ given by

$$\tilde{\mathcal{L}}(\tilde{X}) = i[H, \tilde{X}] - \frac{1}{2}\sum_j (L^{(j)^*} L^{(j)} \tilde{X} + \tilde{X} L^{(j)^*} L^{(j)} - 2L^{(j)^*} \tilde{X} L^{(j)})$$

and $\mathbb{E}_{s]}$ is Fock vacuum conditional expectation. When $\tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ an easy computation shows that

$$\tilde{\mathcal{L}}(\tilde{X}) = \begin{pmatrix} \mathcal{L}(X) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\mathcal{L}(X)$ is given by (5.12). Let $F(t) = 1_{t]} \otimes |\Phi_{[t}\rangle\ \langle\Phi_{[t}|$ where $1_{t]}$ is the identity operator in $(\mathcal{H}_0 \bigoplus \mathcal{H}_0) \otimes \Gamma(L^2[0,t] \otimes \ell^2)$ and $\Phi_{[t}$ is the vacuum vector in $\Gamma(L^2[t,\infty) \otimes \ell^2)$. Put

$$j_t(X) = \tilde{j}_t\left(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right)F(t), \quad X \in \mathcal{A}.$$

Then we get a weak Markov flow $(\widetilde{\mathcal{H}}, F, j_t)$ with expectation semigroup $\{T_t\}$. It is not clear whether the projection

$$P_t = 1 - \tilde{j}_t\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)$$

is increasing in t. If it were so, it would determine an exit time for the flow $(\widetilde{\mathcal{H}}, F, j_t)$.

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