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Ann. Inst. Henri Poincaré, (In Memoriam C. KIPNIS)

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Variance of number of lattice points in random narrow elliptic strip

by

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Dedicated to the memory of Claude Kipnis

ABSTRACT. – Let $N(x;\, s)$ be the number of integral points inside an elliptic strip of area s, bounded by the ellipses $E(x)$ and $E(x+s)$, where

$$E(x) = \{ (q_1,\, q_2) \,:\, (q_1^2 + \mu\, q_2^2)\, \pi \mu^{-1/2} = x \}.$$

We prove that if $\mu > 0$ is a diophantine number and $s = s(T) = \text{const}\, T^\gamma$, with $0 < \gamma < 1/2$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = 4$$

(with the factor 4 coming from symmetry considerations). Contrariwise, if μ is rational then

$$\lim_{T\to\infty} \frac{1}{T \log T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = \left(\frac{1}{2} - \gamma\right) c(\mu)$$

with some $c(\mu) > 0$, and if μ is a Liouville number then

$$\limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = \infty.$$

Key words: Lattice points, random strip.

Classification A.M.S. : 10 J 25, 10 E 35.

RÉSUMÉ. – Soit $N(x;\, s)$ le nombre de points entiers dans une bande elliptique d'aire s, comprise entre les ellipses $E(x)$ et $E(x+s)$, où

$$E(x) = \{ q_1,\, q_2) \,:\, (q_1^2 + \mu\, q_1^2)\, \pi \mu^{-1/2} = x \}$$

Nous montrons que, si $\mu > 0$ est un nombre diophantien et $s = s(T) = \mathrm{const}\, T^\gamma$, with $0 < \gamma < 1/2$, alors

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = 4$$

(le facteur 4 vient de considérations de symmétrie). Au contraire si μ est rationnel alors

$$\lim_{T\to\infty} \frac{1}{T \log T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = \left(\frac{1}{2} - \gamma\right) c(\mu)$$

où $c(\mu) > 0$, et si μ est un nombre de Liouville, alors,

$$\limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = \infty.$$

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1. INTRODUCTION

The statistical properties of the eigenvalues

$$0 = E_0 < E_1 \le E_2 \le \ldots, \qquad \lim_{n\to\infty} E_n = \infty,$$

of the Laplace-Beltrami operator $(-\Delta)$ on a compact d-dimensional Riemannian manifold M^d, $d \ge 2$, is a problem of both physical and mathematical interest. According to the Weyl law,

$$\left.\begin{aligned} N(E) &\equiv \sharp\{ E_n \le E \} \\ &= C E^{d/2} + n(E), \qquad n(E) = o(E^{d/2}), \\ C &= \frac{(\mathrm{Vol}\, M^d)\, \omega_d}{(2\pi)^d}, \qquad \omega_d = \mathrm{Vol}\{ x \in \mathbb{R}^d \,:\, |x| \le 1 \}. \end{aligned}\right\} \tag{1.1}$$

Defining rescaled eigenvalues by

$$x_n = (C^{-1} E_n)^{2/d}, \qquad 0 = x_0 < x_1 \le x_2 \le \ldots,$$

(1.1) is equivalent to

$$N(x) \equiv \sharp\{ x_n \le x \} = x + n(x), \qquad n(x) = o(x),$$

so that the density of the rescaled eigenvalues x_n is equal to 1,

$$\lim_{x \to \infty} x^{-1} N(x) = 1.$$

We are interested in statistical properties of the sequence $x_0 < x_1 \leq x_2 \dots$

To characterize the statistical properties of $\{x_n\}$ consider the asymptotic behavior as $T \to \infty$ of the "temporal" moments

$$M_k(T;\, s) = \frac{1}{T} \int_T^{2T} N^k(x;\, s)\, dx, \qquad k = 0,\, 1,\, 2,\, \dots,$$

of $N(x;\, s) = N(x+s) - N(x)$, the number of x_n's lying in the window $[x,\, x+s]$. A striking conjecture proposed by Berry and Tabor [BT] is that for a "generic" Riemannian manifold with completely integrable geodesic flow the limiting moments of $N(x;\, s)$ coincide with those of the Poisson process, in which points are thrown randomly and independently on the line wih density one,

$$\lim_{T \to \infty} M_k(T;\, s) = \left. \frac{d^k}{d\xi^k}\, e^{s\,(e^{\xi}-1)} \right|_{\xi=0} \equiv \gamma_k(s). \tag{1.2}$$

Numerical evidence and impressive analytic arguments in the favor of this conjecture were given in [BT] and also in subsequent papers of Berry [Ber], Casati, Chirikov and Guarneri [CCG], Sinai [Sin], Major [Maj], Cheng and Lebowitz [CL], Cheng, Lebowitz and Major [CLM] and others; *see* also monographs of Gutzwiller [Gu], Ozorio de Almeida [OdA], and Tabor [Tab]. By contrast, for a generic Riemannian manifold of negative curvature the limiting moments of $N(x;\, s)$ are expected to coincide with those of the Wigner-Dyson gaussian orthogonal ensemble of random matrices. The proof of this as well as of (1.2) remains a challenging open problem.

Keeping s fixed while $T \to \infty$ in $M_k(T;\, s)$ measures only local correlations among the x_n's. To characterize long range correlations between different x_n consider windows of size $s = s(T)$ which grow with T. The question is then, for what $s(T)$ does (1.2) still hold? Casati, Chirikov and Guarneri [CCG] studied this problem for a simple model system of a torus $\mathbb{T}^2$ with the metric

$$dq^2 = dq_1^2 + \mu^{-1}\, dq_2^2, \tag{1.3}$$

and they discovered a saturation of the Poisson asymptotics. Berry [Ber] gave an analytic explanation of the phenomenon of saturation using a heuristic trace formula for $N(E)$ and a diagonal approximation for the second moment. As a matter of fact, Berry computed the saturation of rigidity, a more complicated statistical mean of the second order, but this does not matter for our discussion.

In the present paper we continue the rigorous study of the phenomenon of saturation which was begun in our paper [BL] (see also [BDL]). To describe the phenomenon of saturation we consider the variance

$$D(T;\, s) = \frac{1}{T} \int_T^{2T} |N(x;\, s) - s|^2 dx. \tag{1.4}$$

Then the claim is that for a generic integrable system,

$$D(T;\, s) \sim \begin{cases} s & \text{if } s \ll T^{1/2}, \\ cT^{1/2} & \text{if } s \gg T^{1/2}, \end{cases} \tag{1.5}$$

or more precisely,

$$\lim_{T\to\infty;\, s/T^{1/2}\to 0} s^{-1} D(T;\, s) = 1, \tag{1.6}$$

$$\lim_{T\to\infty;\, s/T^{1/2}\to \infty} (cT^{1/2})^{-1} D(T;\, s) = 1. \tag{1.7}$$

In other words, $D(T;\, s)$ grows linearly, as a function of s, for $s \ll T^{1/2}$, like for the Poisson process, and then it saturates at $cT^{1/2}$ for $s \gg T^{1/2}$. The crossover behavior of $D(T;\, s)$ is described by a scaling function $V(\lambda)$, so that

$$\lim_{T\to\infty;\, s/T^{1/2}\to 0} T^{-1/2} D(T;\, s) = V(\lambda). \tag{1.8}$$

The validity of (1.7), (1.8) was established in [BL] for a number of systems. In the present work we study (1.6).

We will consider the CCG (Casati-Chirikov-Guarneri) model system of a torus $\mathbb{T}^2$ with the metric (1.3), i.e., we have a periodic box with sides of length 1 and $\mu^{-1/2}$. The eigenvalues of the Laplace operator are then

$$\{\, (2\pi)^2 (n_1^2 + \mu n_2^2),\ n_1,\, n_2 \in \mathbb{Z} \,\},$$

so that the scaled eigenvalues $x_0 < x_1 \le x_2 \le \ldots$ are elements of the set

$$X = \{\, (n_1^2 + \mu n_2^2)\, \mu^{-1/2},\ n_1,\, n_2 \in \mathbb{Z} \,\},$$

ordered in increasing order. Hence

$$\begin{aligned} N(x;\, s) &= N(x + S) - N(x) \\ &= \sharp\{\, (n_1,\, n_2) \in \mathbb{Z}^2 \,:\, x < (n_1^2 + \mu n_2^2)\, \pi\mu^{-1/2} \le x + s \,\}. \end{aligned}$$

$N(x)$ and $N(x;\, s)$ have the obvious geometric interpretation of the number of integral points inside of the ellipse $E(s) = \{\, (q_1^2 + \mu q_2^2)\, \pi\mu^{-1/2} = x \,\}$ and inside of the elliptic strip between $E(x)$ and $E(x + s)$, respectively.

A number $\mu > 0$ is called diophantine if $\exists\, C,\ M > 0$ such that for every rational p/q,

$$\left| \mu - \frac{p}{q} \right| \geq \frac{C}{q^M}.$$

μ is a Liouville number if there exist a sequence of rationals $\frac{p_i}{q_i}$ and a sequence of positive numbers c_i, $i = 1, 2, \ldots$, such that $\lim_{i \to \infty} c_i = \infty$ and

$$\left| \mu - \frac{p_i}{q_i} \right| \leq e^{-c_i q_i}, \qquad i = 1, 2, \ldots$$

In what follows we will assume that $s = s(T)$ is a function which satisfies

$$\exists\, \omega > 0 : \lim_{T \to \infty} T^{-\omega} s = \infty; \qquad \lim_{T \to \infty} T^{-1/2} s = 0. \tag{1.9}$$

For the sake of definiteness, one may take $s = \mathrm{const}\, T^{\gamma}$ with an arbitrary $0 < \gamma < 1/2$. Let

$$d = \frac{s}{2\, T^{1/2}};$$

then d characterizes the width the elliptic strip between $E(x)$ and $E(x+s)$, $T \leq x \leq 2\,T$, and the second condition in (1.9) is equivalent to $d \to 0$ as $T \to \infty$. We prove the following theorems.

THEOREM 1.1. – *If μ is diophantine and s satisfies* (1.9) *then*

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = 4. \tag{1.10}$$

THEOREM 1.2. – *If $\mu = 1$ and s satisfies* (1.9) *then*

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s |\log d|}\, dx = \frac{4}{\pi}. \tag{1.11}$$

THEOREM 1.3. – *If μ is rational and s satisfies* (1.9) *then $\exists\, c(\mu) > 0$ such that*

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s |\log d|}\, dx = c(\mu). \tag{1.12}$$

Remarks. – 1. An exact value of $c(\mu)$ follows from the formulae (4.8) and (A.2) below.

2. For $s = \mathrm{const}\, T^{\gamma}$ (1.12) reduces to

$$\lim_{T \to \infty} \frac{1}{T \log T} \int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = c(\mu) \left(\frac{1}{2} - \gamma \right).$$

In our last theorem we will need a more restrictive condition on $s = s(T)$ than (1.9):

$$\exists\,\omega > 0 : \lim_{T\to\infty} T^{-\omega}\, s = \infty, \qquad \lim_{T\to\infty} T^{-(1/2)+\omega}\, s = 0. \tag{1.9'}$$

THEOREM 1.4. – *If* μ *is a Liouville number and* s *satisfies* $(1.9')$ *then*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \frac{|N(x;\, s) - s|^2}{s}\, dx = \infty. \tag{1.13}$$

Before passing to the proof of the theorems we make some comments. The "temporal" averaging $\frac{1}{T}\int_T^{2T} \ldots dx$ in Theorems 1.1-1.4 can be replaced by more general averaging

$$\frac{1}{T}\int_{Ta}^{Tb} \ldots p(x/T)\, dx, \tag{1.14}$$

with arbitrary $b > a > 0$ and arbitrary bounded probability density $p(t)$ on $[a, b]$. The proof remains the same for the uniform density $p(t) = (b-a)^{-1}$, and then by linearity it can be extended to stepwise densities. Since any bounded probability density is approximated by a sequence of stepwise densities the general case follows; for details *see* [BCDL] and [Ble] (*see* also Theorem 2.1 below). The advantage of having $p(t)$ in (1.14) is that this allows to make change of variable in averaging. For instance, when $p(t) = (1/2)(b^{1/2} - a^{1/2})^{-1}\, t^{-1/2}$, (1.14) reduces to averaging with respect to dR where $R = x^{1/2}$ (*cf.* [BL]).

Note that if $\mu > 0$ is irrational then every rescaled eigenvalue

$$x_{n_1 n_2} = (n_1^2 + \mu\, n_2^2)\, \mu^{-1/2} \tag{1.15}$$

with $n_1, n_2 > 0$ is fourfold degenerate since in this case $x_{m_1 m_2} = x_{n_1 n_2}$ if and only if $m_1 = \pm n_1$, $m_2 = \pm n_2$. This fourfold degeneracy of eigenvalues is the origin of the 4 in the RHS of (1.10). More precisely, if we get rid of the degeneracy by defining

$$N_+(x;\, s) = \sharp\{(n_1, n_2) \in \mathbb{Z}^2 : n_1, n_2 > 0,$$

$$x < (n_1^2 + \mu n_2^2)^{-1/2} \le x + s\},$$

then (1.10) is equivalent to

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \frac{|N(x;\, s) - (1/4)\, s|^2}{(1/4)\, s}\, dx = 1.$$

If μ is rational then the degeneracy can be higher than 4, since for some $k \in \mathbb{Z}$ there exist many solutions of the equation

$$qn_1^2 + pn_2^2 = k. \tag{1.16}$$

On the average the number of solutions of (1.16) grows like $c \log k$ (*see* appendix), which gives rise to the logarithmic multiplier in the denominator in the LHS of (1.11) and (1.12). Finally, if μ is a Liouville number then it is well approximated by rationals p_i/q_i. We will show that this implies that we can choose T_i such that $|N(x;\, s) - s|^2$ is on the average of order $c(p_i/q_i)\, s|\log d|$ on the interval $T_i < x < 2T_i$ and $c(p_i/q_i)|\log d| \to \infty$, hence (1.13) follows.

As we pointed out above, the second condition in (1.9) is equivalent to $d \to 0$ as $T \to \infty$, where $d = (2T^{1/2})^{-1} s$ characterizes the width of the elliptic strip between $E(x)$ and $E(x+s)$, $T \leq x \leq 2T$. This condition has a simple geometric origin. Namely, if d is bounded from below by a positive constant, then $N(x;\, s)$ "feels" the rigidity of the lattice and for typical (diophantine) μ this leads to deviation from the Poisson-like behavior of $D(T;\, s)$ as $T \to \infty$ and this leads also to saturation when $d \to \infty$ as $T \to \infty$ [*cf.* (1.7), (1.8)]. Theorem 1.1 can be interpreted then as an indication of the fact that if $d \to 0$ as $T \to \infty$ then for typical μ, $N(x;\, s)$ does not feel the rigidity of the lattice (modulo obvious symmetry with respect to the axes), and $D(T;\, s)$ behaves as if the lattice points were randomly and independently thrown on the positive quadrant with density one and then every point were symmetrically reflected with respect to the axes. An interesting open problem remains to prove (or to disprove) Theorem 1.1 for s fixed.

Theorem 1.1 can be extended to the case when the elliptic strip is bounded by ellipses shifted in a fixed vector $\alpha = (\alpha_1,\, \alpha_2)$ in the plane. Namely, let $N_\alpha(x;\, s)$ be the number of lattice points between ellipses $\alpha + E(x)$ and $\alpha + E(x+s)$. A minor modification of the proof of Theorem 1.1 below allows to prove the following extension of this theorem:

THEOREM 1.1′. – *If μ is diophantine and s satisfies* (1.9) *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N_\alpha(x;\, s) - s|^2}{s}\, dx = m_\alpha,$$

where m_α is the symmetry factor,

$$m_\alpha = \begin{cases} 1 & if\ \ both\ 2\,\alpha_1\ and\ 2\,\alpha_2\ are\ not\ integers, \\ 2 & if\ \ only\ one\ of\ the\ numbers\ 2\,\alpha_1,\ 2\,\alpha_2\ is\ an\ integer, \\ 4 & if\ \ both\ 2\,\alpha_1\ and\ 2\,\alpha_2\ are\ integers. \end{cases}$$

The case of rational μ and nonzero α can be also treated by our methods; in this case the asymptotic behavior of the variance of $N_\alpha(x;\, s)$ depends on arithmetic properties of α: for α diophantine,

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N_\alpha(x;\, s) - s|^2}{s}\, dx = 1,$$

while for α rational a $\log T$-multiplier appears in the normalization factor, like in Theorem 1.3 (*cf.* [BL]).

We would like to mention here an interesting paper of Luo and Sarnak [LS] who proved some estimates on the variance for the eigenvalues of the Laplace-Beltrami operator on arithmetic hyperbolic surfaces. The situation they study is certainly very different from ours (hyperbolic geodesic flow versus integrable) but because of an arithmetic degeneracy, the behavior of the variance turns out to be somewhat similar to the integrable case.

Theorem 1.1 is proved is Section 2. The proof utilizes a uniform estimate on the average number of integral points in a narrow elliptic strip, which is proved in Section 3. In Section 4 we prove Theorems 1.2-1.4 and in the appendix we evaluate the average squared number of solutions of the equation (1.16).

2. PROOF OF THEOREM 1.1

By (1.4) the equation (1.10) reads

$$\lim_{T\to\infty} s^{-1}\, D(T;\, s) = 4. \tag{2.1}$$

It is useful to generalize this to

$$\lim_{T,\, V\to\infty} s^{-1}\, D(T,\, V;\, s) = 4, \tag{2.2}$$

where

$$D(T,\, V;\, s) = \frac{1}{V} \int_T^{T+V} |N(x;\, s) - s|^2\, dx. \tag{2.3}$$

We will prove (2.2) under the assumption that

$$(1/4)\, T^{1-\xi} \le V \le 4\, T^{1-\xi}, \tag{2.4}$$

where $\xi > 0$ is a fixed small number which will be chosen later. Then (2.1) follows. Indeed, let L be the integral part of T^ξ, $L = [T^\xi]$. Divide the interval $T \le x \le 2T$ into L subintervals $[T_j,\, T_{j+1}]$, where

$$T_j = T + j\, L^{-1}\, T, \qquad j = 0,\, 1,\, \ldots,\, L-1.$$

Then

$$V = \Delta T = T_{j+1} - T_j = L^{-1}\, T = [T^{\xi}]^{-1}\, T$$

satisfies (2.4), hence (2.2) holds which implies that $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$ such that if $T \geq T(\varepsilon)$ then

$$|s^{-1}\, D(T_j,\, \Delta T;\, s) - 4| \leq \varepsilon, \qquad j = 0,\, 1,\, \ldots,\, L - 1.$$

Now,

$$\begin{aligned} |s^{-1}\, D(T;\, s) - 4| &= \left| L^{-1}\, s^{-1} \sum_{j=0}^{L-1} D(T_j,\, \Delta T;\, s) - 4 \right| \\ &\leq L^{-1} \sum_{j=0}^{L-1} |s^{-1}\, D(T_j,\, \Delta T;\, s) - 4| \leq \varepsilon, \end{aligned}$$

so that (2.1) indeed follows from (2.2) with the restriction (2.4).

The advantage of the restriction (2.4) is that it makes V much smaller than T (for large T) and this enables us to pass from averaging over x to averaging over $R = x^{1/2}$, which is useful for the application of the Hardy-Voronoi summation formula and for all subsequent considerations. So we can rewrite $D(T,\, V;\, s)$ as

$$D(T, V; s) = \frac{1}{V_0} \int_{T_0}^{T_0 + V_0} |N_0(R + d(R)) - N_0(R) - s|^2 \frac{2\, R V_0}{V}\, dR \quad (2.5)$$

where $N_0(R) = N(R^2)$ and

$$T_0 = T^{1/2}, \qquad T_0 + V_0 = (T + V)^{1/2}, \qquad (R + d(R))^2 = R^2 + s.$$

Observe that

$$V_0 = (T + V)^{1/2} - T^{1/2} = \frac{V}{2\, T^{1/2}}\, (1 + O(V/T)), \qquad (2.6)$$

so

$$\begin{aligned} \frac{2\, R V_0}{V} &= \frac{2\, T_0\, V_0}{V}\, (1 + O(V_0/T_0)) = \frac{2\, T^{1/2}\, V\, (2\, T^{1/2})^{-1}}{V}\, (1 + O(V/T)) \\ &= 1 + O(V/T), \end{aligned}$$

which implies that we can drop the multiplier $\frac{2\,RV_0}{V}$ in the RHS of (2.5), in the limit when $T \to \infty$ and $V/T \to 0$ [here and in what follows $f = O(g)$ means $|f| \leq \text{const.}\, g$]. More precisely, define

$$D_0(T_0,\, V_0;\, s) = \frac{1}{V_0}\int_{T_0}^{T_0+V_0} |N_0(R + d(R)) - N_0(R) - s|^2 dR \quad (2.7)$$

Then

$$D(T,\, V;\, s) = D_0(T_0,\, V_0;\, s)\,(1 + O(V/T)). \quad (2.8)$$

By (2.4) and (2.6), when T is large,

$$T_0^{1-2\xi} \leq V_0 \leq 2\,T_0^{1-2\xi}. \quad (2.9)$$

Put now

$$d = d(T_0) = (T_0^2 + s)^{1/2} - T_0 = \frac{2}{2\,T_0}\,(1 + O(s/T_0^2)). \quad (2.10)$$

We want to replace $d(R)$ in (2.7) by d. Define to that end

$$D_1(T_0,\, V_0;\, d) = \frac{1}{V_0}\int_{T_0}^{T_0+V_0} |N_0(R + d) - N_0(R) - 2\,R\,d - d^2|^2\, dR.$$

We will prove

THEOREM 2.1. – *Assume that μ is diophantine, $d = d(T_0)$ satisfies*

$$\exists\,\omega > 0 \,:\, T_0^{\omega-1} < d, \qquad \lim_{T_0\to\infty} d = 0 \quad (2.11)$$

and $V_0 = V_0(T_0)$ satisfies (2.9) *with $0 < 2\xi < \omega$. Then*

$$\lim_{T_0\to\infty}\,(2\,T_0\,d)^{-1}\,D_1(T_0,\, V_0;\, d) = 4. \quad (2.12)$$

Theorems 1.1 and 2.1 are very close and the difference between these two theorems is that in Theorem 1.1 we fix the area s of the elliptic strip while in Theorem 2.1 we fix its width d. To reduce Theorem 1.1 to Theorem 2.1 we prove also

LEMMA 2.2.

$$(T_0\,d)^{-1}\,[D_1(T_0,\, V_0;\, d(T_0)) - D_0(T_0,\, V_0;\, s)] = o(1), \quad T \to \infty. \quad (2.13)$$

Theorem 2.1, Lemma 2.2 and (2.8) imply obviously Theorem 1.1.

Proof of Theorem 2.1. – To simplify notations we redenote T_0, V_0 and $N_0(R)$ by T, V and $N(R)$, respectively, so that now

$$N(R) = \sharp\{(n_1, n_2) \in \mathbb{Z}^2 \,:\, (n_1^2 + \mu n_2^2)\,\pi\mu^{-1/2} \le R^2\},$$

and (2.12) reads

$$\lim_{T\to\infty} \frac{1}{V}\int_T^{T+V} \frac{|N(R+d) - N(R) - 2Rd - d^2|^2}{2Td}\, dR = 4; \tag{2.14}$$

(2.9) and (2.11) read

$$T^{1-2\xi} \le V \le 2T^{1-2\xi}, \qquad T^{\omega-1} < d. \tag{2.15}$$

Note that

$$N(R) = \sum_{n\in\mathbb{Z}^2} \chi(n;\, R),$$

where

$$\chi(x;\, R) = \begin{cases} 1 & \text{if } |x|_\mu \le R, \\ 0 & \text{if } |x|_\mu > R, \end{cases}$$

and

$$|x|_\mu^2 = (x_1^2 + \mu x_2^2)\,\pi\mu^{-1/2}.$$

Let

$$\chi_\delta(x;\, R) = \int_{|y|_\mu \le R} \frac{1}{\delta^2}\,\varphi\left(\frac{x-y}{\delta}\right) dy,$$

where $\varphi(x) \ge 0$ is a C^∞ function with $\int \varphi(x)\,dx = 1$ and $\varphi(x) = 0$ when $|x|_\mu > 1$. For what follows we put

$$\delta = T^{-\zeta}$$

where ζ satisfies

$$1 \ge \zeta > 1 + \xi - \frac{\omega}{2}. \tag{2.16}$$

Note that (2.15) and (2.16) imply

$$\frac{\delta}{d} \le T^{-\zeta+1-\omega} \le T^{-\xi-(\omega/2)},$$

hence δ is much smaller than d when T is large. Define

$$N_\delta(R) = \sum_{n \in \mathbb{Z}^2} \chi_\delta(n;\, R).$$

LEMMA 2.3. – $\exists\, \varepsilon_0 > 0$ *such that*

$$\frac{1}{V} \int_T^{T+V} \frac{|N_\delta(R) - N(R)|^2}{T\,d}\, dR = O(T^{-\varepsilon_0}), \qquad T \to \infty.$$

Proof. – By (2.15),

$$V T\, d \geq T^{1-2\xi+\omega}$$

hence it suffices to show that

$$\frac{1}{T^{1-2\xi+\omega}} \int_T^{T+V} |N_\delta(R) - N(R)|^2 dR = O(T^{-\varepsilon_0}).$$

Now,

$$N_\delta(R) - N(R) = \sum_n [\chi_\delta(n;\, R) - \chi(n;\, R)],$$

hence

$$I \equiv \int_T^{T+V} |N_\delta(R) - N(R)|^2\, dR = \sum_{n,\, n'} I(n,\, n'),$$

where

$$I(n,\, n') = \int_T^{T+V} [\chi_\delta(n;\, R) - \chi(n;\, R)]\,[\chi_\delta(n';\, R) - \chi(n';\, R)]\, dR.$$

By symmetry we can assume $|n|_\mu \geq |n'|_\mu$. Observe that $\chi_\delta(x; R) - \chi(x; R)$ has support in the δ-neighborhood of the ellipse $E(R)$. This implies that $I(n,\, n') = 0$ if $|n|_\mu - |n'|_\mu > 2\,\delta = 2\,T^{-\zeta}$ or $|n|_\mu < T$ or $|n|_\mu > T + 2\,V$. In addition,

$$I(n,\, n') \leq \int_{|n|_\mu - \delta}^{|n|_\mu + \delta} dR = 2\,\delta = 2\,T^{-\zeta},$$

hence

$$I \leq c\,T^{-\zeta} \sum_{n\,:\, T \leq |n|_\mu \leq T+V} \; \sum_{n'\,:\, 0 \leq |n|_\mu - |n'|_\mu \leq 2T^{-\zeta}} 1 \equiv c\,T^{-\zeta}\, J.$$

We will prove in the next section that

$$J = O\,(T^{3-\zeta+\varepsilon}), \qquad \forall\, \varepsilon > 0 \tag{2.17}$$

(*see* Lemma 3.1). This gives

$$\int_T^{T+V} |N_\delta\,(R) - N\,(R)|^2 dR = O\,(T^{3-2\zeta+\varepsilon}), \qquad \forall\, \varepsilon > 0.$$

Hence

$$\frac{1}{T^{1-2\xi+\omega}} \int_T^{2T} |N_\delta\,(R) - N\,(R)|^2 dR = O\,(T^{2-2\zeta+2\xi-\omega+\varepsilon}), \qquad \forall\, \varepsilon > 0.$$

Since by (2.16) $2 - 2\,\zeta + 2\,\xi - \omega < 0$, Lemma 2.3 follows.

By the triangle inequality Lemma 2.3 implies

$$\lim_{T\to\infty} \left\{ \left(\frac{1}{V} \int_T^{T+V} \frac{|N_\delta\,(R+d) - N_\delta\,(R) - 2\,R\,d - d^2|^2}{2\,T\,d}\, dR \right)^{1/2} - \left(\frac{1}{V} \int_T^{T+V} \frac{|N\,(R+d) - N\,(R) - 2\,R\,d - d^2|^2}{2\,T\,d}\, dR \right)^{1/2} \right\} = 0,$$

hence it suffices to prove

$$\lim_{T\to\infty} \frac{1}{V} \int_T^{T+V} \frac{|N_\delta\,(R+d) - N_\delta\,(R) - 2\,R\,d - d^2|^2}{2\,T\,d}\, dR = 4. \tag{2.18}$$

Let

$$\psi\,(\xi) = \int_{\mathbb{R}^2} e^{2\pi i x \xi}\, \varphi\,(x)\, dx$$

and

$$|n| = \mu^{1/4}\, \pi^{-1/2}\, (n_1^2 + \mu^{-1}\, n_2^2)^{1/2}, \tag{2.19}$$

which is, up to a multiplier, a dual norm to $|\cdot|_\mu$. The Hardy-Voronoi summation formula gives

$$\frac{N_\delta\,(R) - R^2}{R^{1/2}} = \pi^{-2} \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |n|^{-3/2} \psi\,(n\,\delta) \times \cos\left(2\,\pi |n| R - \frac{3\,\pi}{4}\right) + O\,(R^{-1}), \tag{2.20}$$

with the remainder $O(R^{-1})$ uniform in δ. From (2.20)

$$\begin{aligned}\frac{N_\delta(R+d)-N_\delta(R)-2Rd-d^2}{R^{1/2}} \\ = -2\pi^{-2}F(R;\,T)+O(R^{-1}),\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}F(R;\,T) = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |n|^{-3/2}\psi(n\,\delta)\sin(\pi|n|d) \\ \times \sin\left(2\pi|n|\left(R+\frac{d}{2}\right)-\frac{3\pi}{4}\right),\end{aligned} \tag{2.22}$$

so that (2.18) is equivalent to

$$\lim_{T\to\infty}\frac{2\pi^{-4}}{Vd}\int_T^{T+V}|F(R;\,T)|^2dR = 4. \tag{2.23}$$

By symmetry we can rewrite $F(R;\,T)$ as

$$\begin{aligned}F(R;\,T) = \sum_{n\in\Lambda_+} r(n)|n|^{-3/2}\psi(n\,\delta)\sin(\pi|n|d) \\ \times \sin\left(2\pi|n|\left(R+\frac{d}{2}\right)-\frac{3\pi}{4}\right),\end{aligned} \tag{2.24}$$

where

$$\Lambda_+ = \{n=(n_1,\,n_2)\in\mathbb{Z}^2\,:\,n_1,\,n_2\geq 0;\;n\neq 0\}$$

and

$$r(n) = \begin{cases} 4 & \text{if } n_1,\,n_2\neq 0, \\ 2 & \text{if } n_1\,n_2 = 0, \end{cases}$$

By (2.24),

$$\frac{2\pi^{-4}}{Vd}\int_T^{T+V}|F(R;\,T)|^2dR = \sum_{n,\,n'} a(n)\,a(n')\,K(n,\,n'), \tag{2.25}$$

where

$$a(n) = r(n)|n|^{-3/2}\psi(n\,\delta)\sin(\pi|n|d) \tag{2.26}$$

and

$$K(n, n') = \frac{2\pi^{-4}}{Vd} \int_T^{T+V} \sin\left(2\pi|n|\left(R+\frac{d}{2}\right) - \frac{3\pi}{4}\right) \times \sin\left(2\pi|n'|\left(R+\frac{d}{2}\right) - \frac{3\pi}{4}\right) dR. \tag{2.27}$$

The idea to prove (2.23) is to show that the diagonal sum

$$I_d = \sum_n a(n)^2 K(n, n)$$

converges to 4 as $T \to \infty$, while the off-diagonal sum

$$I_o = \sum_{n \neq n'} a(n)\, a(n')\, K(n, n')$$

converges to 0. The difficult part is to show the convergence of the off-diagonal sum. First we consider the diagonal sum.

Observe that

$$K(n, n) = \pi^{-4} d^{-1} (1 + O(V^{-1}|n|^{-1})),$$

so

$$I_d = \pi^{-4} d^{-1} \sum_{n \in \Lambda_+} r(n)^2 |n|^{-3} \psi(n\delta)^2 \sin^2(\pi|n|d)\,(1 + O(V^{-1}|n|^{-1})),$$

which is an approximating sum of the integral

$$M = 16\pi^{-4} \int_{\mathbb{R}^2_+} |x|^{-3} \sin^2(\pi|x|)\, dx,$$

so that $\lim_{T\to\infty} I_d = M$. A direct computation gives $M = 4$ (*see* the appendix to [BL]), hence

$$\lim_{T\to\infty} I_d = 4.$$

We turn now to the evaluation of I_o.

From (2.27),

$$|K(n, n')| \leq cd^{-1} \min\{1, V^{-1}|\,|n| - |n'|\,|^{-1}\}. \tag{2.28}$$

Define

$$S = \{(n, n') \in \Lambda_+ \oplus \Lambda_+ : 0 < |\,|n| - |n'|\,| \leq 1\},$$
$$S^c = \{(n, n') \in \Lambda_+ \oplus \Lambda_+ : |\,|n| - |n'|\,| > 1\}.$$

Then

$$\begin{aligned} I^c &= \sum_{(n,\, n') \in S^c} a(n)\, a(n')\, K(n,\, n') \\ &\leq c\,(V\,d)^{-1} \sum_{(n,\, n') \in S^c} |n|^{-3/2}\, |n'|^{-3/2} \\ &\quad \times\ \psi(n\,\delta)\, \psi(n'\,\delta)|\, |n| - |n'|\, |^{-1}. \end{aligned} \tag{2.29}$$

The sum

$$\sum_{(n,\, n') \in S^c} |n|^{-3/2}\, |n'|^{-3/2} |\, |n| - |n'|\, |^{-1}$$

diverges as $\log^2$, *i.e.*,

$$\sum_{(n,\, n') \in S^c\, :\, |n|,\, |n'| \leq L} |n|^{-3/2}\, |n'|^{-3/2} |\, |n| - |n'|\, |^{-1} \sim c \log^2 L.$$

The function $\psi(n\,\delta)\, \psi(n'\,\delta)$ produces a smooth cut-off at the scale $\delta^{-1} = T^{\zeta}$ in the RHS of (2.29) hence we obtain

$$|I^c| \leq c\,(V\,d)^{-1}\, T^{\varepsilon} \leq c_0\, T^{2\xi - \omega + \varepsilon}, \qquad \forall \varepsilon > 0.$$

This implies $\lim\limits_{T \to \infty} I^c = 0$.

To estimate

$$I = \sum_{(n,\, n') \in S} a(n)\, a(n')\, K(n,\, n')$$

we slice S into layers

$$S_j = \{ j\, T^{-1} \leq |\, |n| - |n'|\, | \leq (j+1)\, T^{-1} \}$$

and estimate

$$I_j(N) = \sum_{(n,\, n') \in S_j\, :\, (N/2) \leq |n| \leq N} a(n)\, a(n')\, K(n,\, n').$$

Let us fix some ζ' such that $\zeta < \zeta' < 1$ and assume first that $N < T^{\zeta}$. By (2.28), for any $(n,\, n') \in S_j$,

$$|K(n,\, n')| \leq c\,(j+1)^{-1}\, (V\,d)^{-1}\, T,$$

and by (2.26),

$$|a(n)|,\ |a(n')|\ \leq c|n|^{-3/2} g(|n|d)$$

where

$$g(t) = \min\{1,\ t\},$$

hence

$$I_j(N) \leq c(j+1)^{-1} N^{-3} (V d)^{-1} T g^2 (N d) \sum_{(n,\, n') \in S_j\, :\, (N/2) \leq |n| \leq N} 1.$$

In the next section we will prove the estimate

$$\sum_{(n,\, n') \in S_j\, :\, (N/2) \leq |n| \leq N} 1 \leq c N^{\kappa}, \qquad \kappa < 2, \tag{2.30}$$

(*see* Lemma 3.2) which implies

$$I_j(N) \leq c\,(j+1)^{-1} N^{-3+\kappa} (V d)^{-1} T g^2 (N d). \tag{2.31}$$

When $N \leq d^{-1}$ this gives

$$I_j(N) \leq c\,(j+1)^{-1} N^{-1+\kappa} V^{-1} T d.$$

Summing this last estimate over $j = 0, 1, \ldots, T$ and $N = [2^{-i} d^{-1}]$, $i = 0, 1, \ldots, [\,|\log_2 d|]$, we obtain

$$\begin{aligned} \sum_{(n,\, n') \in S\, :\, |n| \leq d^{-1}} & |a(n)\, a(n')\, K(n,\, n')| \\ & \leq c d^{2-\kappa} V^{-1} T \log T \\ & \leq c_0 T^{-(2-\kappa)\,\omega + 2\xi} \log T \to 0 \quad \text{as } T \to \infty, \end{aligned} \tag{2.32}$$

if we take ξ such that

$$0 < \xi < \frac{(2-\kappa)\,\omega}{2}. \tag{2.33}$$

When $N > d^{-1}$, (2.31) gives

$$I_j(N) \leq c(j+1)^{-1} N^{-3+\kappa} (V d)^{-1} T.$$

Summing this estimate over $j = 0, 1, \ldots, T$ and $N = [2^i d^{-1}]$, $i = 0, 1, \ldots, [\log_2 (T^{\zeta'} d)]$, we obtain

$$\begin{aligned}\sum_{(n,\, n') \in S \,:\, d^{-1} \le |n| \le T^{\zeta'}} &| a(n)\, a(n')\, K(n,\, n')| \\ &\le c d^{2-\kappa}\, V^{-1}\, T \log T \\ &\le c_0\, T^{-(2-\kappa)\,\omega + 2\xi} \log T \to 0 \quad \text{as } T \to \infty,\end{aligned}$$

if (2.33) holds. When $|n| \ge T^{\zeta'}$, $a(n)$ is very small due to the cut-off function $\psi(n\,\delta)$. To see this note that $\forall M > 0$, $\exists T_0 > 0$ such that if $T > T_0$ then

$$|a(n)| \le \psi(n T^{-\zeta}) \le (|n| T^{-\zeta})^{-M}, \qquad \forall n : |n| \ge T^{\zeta'},$$

which implies

$$|a(n)| \le T^{-3} |n|^{-3} \qquad \text{(say)} \qquad \forall n : |n| \ge T^{\zeta'}; \quad T > T_0.$$

Therefore

$$\begin{aligned}\sum_{(n,\, n') \in S \,:\, |n| \ge T^{\zeta'}} &| a(n)\, a(n')\, K(n,\, n')| \\ &\le c\, T^{-6+\gamma} \sum_{(n,\, n') \in S \,:\, |n| \ge T^{\zeta'}} |n|^{-6} \to 0 \quad \text{as } T \to \infty.\end{aligned}$$

Thus we have proved that

$$I = \sum_{(n,\, n') \in S} a(n)\, a(n')\, K(n,\, n') \to 0 \quad \text{as } T \to \infty,$$

which finishes the proof of Theorem 2.1.

Proof of Lemma 2.2. – For any two positive functions $f(R)$ and $g(R)$ on $[T,\, T+V]$, define

$$\rho(f,\, g) = \left(\frac{1}{s V} \int_T^{T+V} |n(R + f(R)) - n(R + g(R))|^2 dR \right)^{1/2},$$

where

$$\begin{aligned} n(R) &= N(R) - R^2 \\ &= \sharp\{ (n_1,\, n_2) \in \mathbb{Z}^2 : (n_1^2 + \mu n_2^2)\, \pi \mu^{-1/2} \le R^2 \} - R^2. \quad (2.34)\end{aligned}$$

Obviously,

$$\rho(f, h) \leq \rho(f, g) + \rho(g, h).$$

To prove (2.13) it suffices to show that

$$\rho^2(f, g) = o(1), \qquad T \to \infty, \tag{2.35}$$

with

$$f(R) = d(R), \quad g(R) \equiv d = d(T); \qquad d(R) = (R^2+s)^{1/2} - R. \tag{2.36}$$

Consider the sequence of stepwise functions

$$f_j(R) = f(T + k\,2^{-j}\,V) \quad \text{when } T + k\,2^{-j}\,V \leq R < T + (k+1)\,2^{-j}\,V,$$
$$k = 0, 1, \ldots, 2^j - 1.$$

Then $f_0(R) = g(R)$ and $f_j(R) \to f(R)$ as $j \to \infty$. In addition,

$$|f'(R)| = |d'(R)| = |R(R^2+s)^{-1/2} - 1| \sim R^{-2}\,s \sim T^{-1}\,d,$$

where $f \sim g$ means $cf \leq g \leq c' f$, which implies

$$d_j \equiv \sup_{T \leq R \leq T+V} |f(R) - f_j(R)| \sim 2^j\,VT^{-1}\,d. \tag{2.37}$$

LEMMA 2.4. – *If $d_j > T^{-1}$ then*

$$\rho^2(f, f_j) = O(2^{-2j}\,VT^{\varepsilon}\,d), \qquad \forall \varepsilon > 0.$$

Proof. – Let us show that

$$\left.\begin{aligned} &\frac{1}{sV}\int_T^{T+V} |N(R+f(R)) - N(R+f_j(R))|^2 dR \\ &\qquad \leq c_\varepsilon\,2^{-2j}\,VT^{\varepsilon}\,d, \qquad \forall \varepsilon > 0. \end{aligned}\right\} \tag{2.38}$$

The proof is the same as the one of Lemma 2.3. Namely,

$$I = \int_T^{T+V} |N(R+f(R)) - N(R+f_j(R))|^2 dR = \sum_{n, n'} I(n, n'),$$

where

$$I(n, n') = \int_T^{T+V} \chi(n; R)\,\chi(n'; R)\,dR$$

where $\chi(x; R)$ is the characteristic function of the elliptic strip between $E(R+f(R))$ and $E(R+f_j(R))$. Observe that the width of this strip is $O(d_j)$. Now using the same arguments as we used in the proof of Lemma 2.3, we obtain the estimate

$$I \leq c_\varepsilon T^{3+\varepsilon} d_j^2, \qquad \forall \varepsilon > 0,$$

which implies (2.38) [use (2.37)]. In addition to (2.38),

$$\frac{1}{sV} \int_T^{T+V} |(R+f(R))^2 - (R+f_j(R))^2|^2 dR \leq c\, 2^{-2j} V d, \tag{2.39}$$

[use again (2.37)], hence Lemma 2.4 follows.

Take

$$j_0 = \max\{ j : d_j > T^{-1} \}.$$

Then by (2.37),

$$2^{-j_0} V d \sim 1, \tag{2.40}$$

hence

$$\rho^2(f, f_{j_0}) \leq c(Vd)^{-1} T^\varepsilon = o(1), \qquad T \to \infty. \tag{2.41}$$

By (2.40), $j_0 \sim \log T$. Let us estimate now $\rho^2(f_j, f_{j+1})$ when $j \leq j_0$. Define

$$\rho_\delta(f, g) = \left(\frac{1}{sV} \int_T^{T+V} |n_\delta(R+f(R)) - n_\delta(R+f(R))|^2 dR \right)^{1/2},$$

where $n_\delta(R) = N_\delta(R) - R^2$. Put $\delta = T^{-\zeta}$ with some ζ satisfying (2.16). Lemma 2.3 gives then

$$\rho_\delta^2(f_j, f_{j+1}) - \rho^2(f_j, f_{j+1}) = O(T^{-\varepsilon_0}). \tag{2.42}$$

LEMMA 2.5. – $\exists\, \varepsilon_0 > 0$ *such that*

$$\rho_\delta^2(f_j, f_{j+1}) \leq c T^{-\varepsilon_0}, \qquad J = 0, 1, \ldots, j_0.$$

Proof. – With the help of the Hardy-Voronoi summation formula we obtain (as in the proof of Theorem 2.1 above)

$$\rho_\delta^2(f_j, f_{j+1}) = \sum_{k=0}^{2^{j+1}-1} \sum_{n, n'} a_k(n)\, a_k(n')\, K_k(n, n') + O(T^{-\varepsilon_0}),$$

where

$$a_k(n) = r(n)|n|^{-3/2}\psi(n\delta)\sin(\pi|n|\Delta_k), \qquad \Delta_k = f_{j+1}(R_k) - f_j(R_k),$$
$$R_k = T + k\,2^{-j-1}V,$$

and

$$K_k(n, n') = \frac{2\pi^{-3}}{Vd}\int_{R_k}^{R_{k+1}} \sin\left(2\pi|n|\left(R + \frac{\Delta_k}{2}\right) - \frac{3\pi}{4}\right)$$
$$\times \sin\left(2\pi|n'|\left(R + \frac{\Delta_k}{2}\right) - \frac{3\pi}{4}\right) dR$$

[*cf.* (2.26), (2.27)]. Now with the help of the same arguments that we used in the proof of Theorem 1.2 we obtain that

$$\sum_{k=0}^{2^{j+1}-1} \sum_{n, n'} a_k(n)\, a_k(n')\, K_k(n, n') = O\left(T^{-(2-\kappa)\omega+2\xi}\log T\right)$$

[*cf.* (2.32)]. This proves Lemma 2.5

Lemma 2.5 combined with (2.41), (2.42) proves (2.35) and thus Lemma 2.2 is proved.

3. UNIFORM ESTIMATE OF THE NUMBER OF CLOSE PAIRS

In this section we put $|n| = (n_1^2 + \mu n_2^2)^{1/2}$. We call a pair $(n, n') \in \mathbb{Z}^2 \oplus \mathbb{Z}^2$ close if $|\,|n| - |n'|\,|$ is small. We are interested in evaluation of the number of close pairs in the annulus $T \leq |n|, |n'| \leq 2T$.

LEMMA 3.1. – *$\forall \varepsilon > 0$, $\exists c_\varepsilon > 0$ such that $\forall T > 1$ and $\forall 1 > \delta > T^{-1}$,*

$$J \equiv \sum_{n : T \leq |n| \leq 2T} \;\sum_{n' : 0 < ||n| - |n'|| \leq \delta} 1 \leq c_\varepsilon T^{3+\varepsilon}\delta.$$

Proof. – By symmetry we may assume $|n| > |n'|$ and also that n_1, n_1', $n_2, n_2' \geq 0$ and estimate the sum J with these restrictions. We have

$$|n| - |n'| = \frac{|n|^2 - |n'|^2}{|n| + |n'|} \geq \frac{(n_1^2 - n_1'^2) + \mu(n_2^2 - n_2'^2)}{T},$$

hence

$$J \leq \sum_{n\,:\,T\leq|n|\leq 2T} \;\; \sum_{n'\,:\,0<(n_1^2-n_1'^2)-\mu\,(n_2^2-n_2'^2)\leq T\,\delta} 1.$$

Let

$$n_1^2 - n_1'^2 = m_1, \qquad n_2^2 - n_2'^2 = m_2.$$

Then

$$0 < m_1 - \mu m_2 \leq T\,\delta. \tag{3.1}$$

We can count all pairs n, n' first counting m_2 from $-4\,T^2$ to $4\,T^2$, then counting m_1 which satisfy (3.1) and then counting the divisors of m_1, m_2. Assume first that $m_1\,m_2 \neq 0$. This part of J is estimated as

$$J_{\{m_1\,m_2\neq 0\}} \leq c \sum_{m_2=1}^{4\,T^2} \sum_{m_1} d\,(m_1)\,d\,(m_2) \leq c \sum_{m_2=1}^{4\,T^2} \sum_{m_1} [d\,(m_1)^2 + d\,(m_2)^2]$$

$$\leq c_0\,T\,\delta \sum_{m=1}^{c_1\,T^2} d\,(m)^2 \leq c_\varepsilon\,T\,\delta\,T^{2+\varepsilon} = c_\varepsilon\,T^{3+\varepsilon}\,\delta.$$

Assume second that $m_2 = 0$, $m_1 \neq 0$. This part is estimated as

$$J_{\{m_2=0\}} \leq c \sum_{n_2=0}^{2T} \sum_{m_1} d\,(m_1) \leq c_\varepsilon\,T^{2+\varepsilon}\,\delta.$$

Similarly we estimate the part with $m_1 = 0$, $m_2 \neq 0$. Lemma 3.1 is proved.

LEMMA 3.2. – *If μ is diophantine and $\zeta > 1$ then $\exists\,\kappa < 2$ such that*

$$J = O\,(T^k), \qquad T \to \infty.$$

Proof. – If μ is diophantine and $\zeta > 1$, then the number of m_2, $|m_2| \leq 4\,T^2$, which satisfy (3.1) for at least one m_1, is $O\,(T^{\kappa'})$ with some $\kappa' < 2$. This implies $J = O\,(T^{\kappa'+\varepsilon})$, $\forall\,\varepsilon > 0$. Lemma 3.2 is proved.

LEMMA 3.3. – *If μ is a rational and $\zeta > 1$ then for large T,*

$$J = \sum_{n\,:\,T\leq|n|\leq 2T} \;\; \sum_{n'\,:\,0<|\,|n|\,-\,|n'|\,|\leq T^{-\zeta}} 1 = 0.$$

Proof. – Let $\mu = \dfrac{p}{q}$ Then

$$|\,|n|\,-\,|n'|\,| = \frac{|\,|n|^2\,-\,|n'|^2\,|}{|n|\,+\,|n'|} \geq \frac{q^{-1}}{4T} > T^{-\zeta},$$

hence $J = 0$ for $T^{\zeta-1} > 4\,q$. Lemma 3.3 is proved.

4. PROOF OF THEOREMS 1.2-1.4

Proof of Theorem 1.2. – There are two places where the proof of Theorem 1.2 differs from the proof of Theorem 1.1: the first place concerns the computation of the diagonal sum and the second one concerns the estimation of the off-diagonal sum. Let us first discuss the diagonal sum.

When $\mu = 1$, (2.19) reduces to

$$|n| = \pi^{-1/2} (n_1^2 + n_2^2)^{1/2}.$$

we rewrite (2.22) as

$$F(R;\, T) = \sum_{k=1}^{\infty} r_2(k)\, \underline{k}^{-3/4}\, \psi(\underline{k}^{1/2}\, \delta)\, \sin(\pi\, \underline{k}^{1/2}\, d) \times \sin\left(2\,\pi\, \underline{k}^{1/2}\left(R + \frac{d}{2}\right) - \frac{3\,\pi}{4}\right), \tag{4.1}$$

where $r_2(k)$ is the number of representations of k as a sum of two squares, $k = n_1^2 + n_2^2$, and $\underline{k} = \pi^{-1} k = |n|^2$. As is well known, $r_2(k) \leq c_\varepsilon\, k^\varepsilon$, $\forall\, \varepsilon > 0$. To prove Theorem 1.2 we have to show that

$$\lim_{T \to \infty} \frac{2\,\pi^{-4}}{V\, d |\log d|} \int_T^{T+V} |F(R;\, T)|^2\, dR = \frac{4}{\pi} \tag{4.2}$$

[*cf.* (2.23)]. By (4.1),

$$\frac{2\,\pi^{-4}}{V\, d |\log d|} \int_T^{T+V} |F(R;\, T)|^2\, dR = \sum_{k,\, k'} a(k)\, a(k')\, K(k,\, k'),$$

where

$$a(k) = r_2(k)\, \underline{k}^{-3/4}\, \psi(\underline{k}^{1/2}\, \delta)\, \sin(\pi\, \underline{k}^{1/2}\, d) \tag{4.3}$$

and

$$K(k,\, k') = \frac{2\,\pi^{-4}}{V\, d |\log d|} \int_T^{T+V} \sin\left(2\,\pi\, \underline{k}^{1/2}\left(R + \frac{d}{2}\right) - \frac{3\,\pi}{4}\right) \times \sin\left(2\,\pi\, \underline{k}'^{1/2}\left(R + \frac{d}{2}\right) - \frac{3\,\pi}{4}\right) dR.$$

The diagonal sum reduces to

$$I_d = \frac{\pi^{-4}}{d|\log d|} \sum_{k=1}^{\infty} r_2(k)^2 \underline{k}^{-3/2} \psi(\underline{k}^{1/2} \delta) \\ \times \sin^2(\pi \underline{k}^{1/2} d)(1 + O(V^{-1} k^{-1/2})). \tag{4.4}$$

As shown in [BD] (*see* also the appendix to the present paper),

$$\lim_{T\to\infty} (N \log N)^{-1} \sum_{k=1}^{N} r_2(k)^2 = 4. \tag{4.5}$$

Substituting $x_k = \pi^{-1} kd^2$ we can reduce (4.4) to an approximating sum

$$I_d = \pi^{-3} |\log d|^{-1} \sum_{k=1}^{\infty} r_2(k)^2 x_k^{-3/2} \sin^2(\pi x_k^{1/2}) \Delta x (1 + O(V^{-1} k^{-1/2})),$$

$$\Delta x = x_{k+1} - x_k,$$

of the integral

$$M = 8\pi^{-3} \int_0^{\infty} x^{-3/2} \sin^2(\pi x^{1/2})\, dx = \frac{4}{\pi}.$$

Indeed, when x_k is order of 1, k is of order of d^{-2}, hence by (4.5) $r_2(k)^2$ is on the average of order of $4 \log d^{-2}$, so that

$$\lim_{T\to\infty} I_d = M = \frac{4}{\pi}.$$

The off-diagonal sum I_o is estimated in the same way as in the proof of Theorem 1.1, except instead of Lemma 3.2 we use Lemma 3.3. Theorem 1.2 is proved.

Proof of Theorem 1.3. – For a general $\mu = \frac{p}{q}$, (2.19) reduces to

$$|n| = \left(\frac{p}{q}\right)^{1/4} \pi^{-1/2} \left(n_1^2 + \frac{q}{p} n_2^2\right)^{1/2} = \pi^{-1/2} (pq)^{-1/4} (pn_1^2 + qn_2^2)^{1/2},$$

and (4.1) reads

$$F(R; T) = \sum_{k=1}^{\infty} r_2(k; p, q) \underline{k}^{-3/4} \psi(\underline{k}^{1/2} \delta) \sin(\pi \underline{k}^{1/2} d) \\ \times \sin\left(2\pi \underline{k}^{1/2} \left(R + \frac{d}{2}\right) - \frac{3\pi}{4}\right),$$

where $r_2(k;\ p,\ q)$ is the number of representation of k as

$$k = pn_1^2 + qn_2^2,$$

and

$$\underline{k} = \pi^{-1}(pq)^{-1/2}\,k = |n|^2.$$

We will prove in the appendix that

$$\lim_{N\to\infty}(N\log N)^{-1}\sum_{k=1}^{N} r_2(k;\ p,\ q)^2 = \sigma(p,\ q) > 0, \tag{4.6}$$

which is a generalization of (4.5). The diagonal sum (4.4) now reads

$$I_d = \frac{\pi^{-3}}{d|\log d|}\sum_{k=1}^{\infty} r_2(k;\ p,\ q)^2\,\underline{k}^{-3/2}\,\psi(\underline{k}^{1/2}\,\delta) \times \sin^2(\pi\,\underline{k}^{1/2}\,d)\,(1 + O(V^{-1}\,\underline{k}^{-1/2})). \tag{4.7}$$

Substituting $x_k = \underline{k}d^2$ we can reduce (4.7) to an approximating summ of the integral

$$M = \sigma(p,q)\,(pq)^{1/2}\,\pi^{-3}\,2\int_0^{\infty} x^{-3/2}\sin^2(\pi\,x^{1/2})\,dx = \frac{\sigma(p,\ q)\,(pq)^{1/2}}{\pi},$$

hence

$$\lim_{T\to\infty} I_d = c(\mu), \qquad c(\mu) = \frac{\sigma(p,\ q)\,(pq)^{1/2}}{\pi}, \tag{4.8}$$

where $\sigma(p,\ q)$ is defined in (A.2).

The off-diagonal sum I_o is again estimated in the same way as in the proof of Theorem 1.1 except instead of Lemma 3.2 we use Lemma 3.3. Theorem 1.3 is proved.

Proof of Theorem 1.4. – Assume

$$\left|\mu - \frac{p_i}{q_i}\right| \le e^{-c_i q_i}, \quad i = 1, 2, \ldots; \qquad \lim_{i\to\infty} c_i = \infty. \tag{4.9}$$

Put $T_i = e^{b_i q_i}$ with some b_i such that $\lim_{i\to\infty} b_i = \infty$. By Theorem 1.3 when T is large,

$$\frac{1}{T}\int_T^{2T}\frac{|N(x;\ s,\ \mu_i) - s|^2}{s}\,dx \ge \frac{c(\mu_i)\,\omega}{2}\log T, \qquad \mu_i = \frac{p_i}{q_i}, \tag{4.10}$$

where $N(x;\ s,\ \mu)$ stands for $N(x;\ s)$ referred to an ellipse with the value of parameter μ. A careful inspection of the proof of Theorem 1.3 shows that (4.10) holds when $T = T_i$. In addition (4.8), (A.2) imply

$$c(\mu_i) \geq \frac{\text{const.}}{q_i},$$

so that

$$\frac{1}{T_i}\int_{T_i}^{2T_i} \frac{|N(x;\ s,\ \mu_i) - s|^2}{s}\,dx \geq \text{const.}\, b_i. \tag{4.11}$$

Now, a simple estimate gives

$$\frac{1}{T}\int_{T}^{2T} |N(x;\ s,\ \mu) - N(x;\ s,\ \mu_i)|^2 dx \leq \text{const.}\, T^4 |\mu - \mu_i|,$$

which implies

$$\begin{aligned}\frac{1}{T_i}\int_{T_i}^{2T_i} &|N(x;\ s,\ \mu) - N(x;\ s,\ \mu_i)|^2 dx \\ &\leq \text{const. } \exp(4\, b_i\, q_i - c_i\, q_i).\end{aligned} \tag{4.12}$$

If we take $b_i = (1/5)\, c_i$, then (4.12) combined with (4.11) give

$$\frac{1}{T_i}\int_{T_i}^{2T_i} \frac{|N(x;\ s,\ \mu) - s|^2}{s}\,dx \geq \text{const.}\, b_i \to \infty, \qquad i \to \infty.$$

Theorem 1.4 is proved.

APPENDIX

Let $r_2(k;\ p,\ q)$ with $\gcd(p,\ q) = 1$ be the number of representations of k as $k = qn_1^2 + pn_2^2$.

LEMMA A.1.

$$\lim_{N\to\infty} (N \log N)^{-1} \sum_{k=1}^{N} r_2(k;\ p,\ q)^2 = \sigma(p,\ q) \tag{A.1}$$

with

$$\sigma(p,\, q) = \begin{cases} \dfrac{4\, d(p)\, d(q)}{pq} & \text{if } p \equiv q \equiv 1 \mod 2; \\ \dfrac{(6\, l + 2)\, d(p')\, d(q)}{pq} & \text{if } p = 2^l\, p', \quad p' \equiv q \equiv 1 \mod 2, \end{cases} \tag{A.2}$$

where $d(p)$ is the number of divisors of p.

Proof. – We follow the appendix to [BCDL] (*see* also [BD1], [BD2]) where (A.1) was proved for $p = q = 1$. Consider the exponential sum

$$S(b) = \sum_{k=1}^{\infty} r_2(k;\, p,\, q)^2\, e^{-k/b}. \tag{A.3}$$

We will show that

$$\lim_{b \to \infty} \frac{1}{b \log b} S(b) = \sigma(p,\, q). \tag{A.4}$$

By the tauberian theorem of Hardy and Littlewood [HL], (A.1) follows from (A.4). We can rewrite $S(b)$ as

$$S(b) = \sum_{m,\, m' \in \mathbb{Z}^2 \setminus \{0\}} e^{-|m|^2/b}, \tag{A.5}$$

summed over integral vectors m, m' with

$$|m|^2 = |m'|^2, \qquad |m|^2 = q m_1^2 + p m_2^2. \tag{A.6}$$

We want to convert (A.5) into an unrestricted sum. To that end let us analyze (A.6). Assume first that

$$p \equiv q \equiv 1 \mod 2. \tag{A.7}$$

Obviously (A.6) is equivalent to

$$q\,(m_1 + m_1')\,(m_1 - m_1') = p\,(m_2 + m_2')\,(-m_2 + m_2'). \tag{A.8}$$

This is satisfied when

$$\left.\begin{array}{ll} m_1 + m_1' = p_1\, jk, & m_1 - m_1' = p_2\, hl, \\ \text{or } k \equiv m_2 + m_2' = q_1\, jl, & -m_2 + m_2' = q_2\, hk, \end{array}\right\} \tag{A.9}$$

where

$$p_1\, p_2 = p, \qquad q_1\, q_2 = q. \tag{A.10}$$

We would like to have a one-to-one correspondence between pairs m, $m' \in \mathbb{Z}^2$ which satisfy (A.8) and the numbers h, j, k, l, p_1, q_1. From (A.9),

$$4|m|^2 = 4\,|m'|^2 = (p_1\,q_1\,j^2 + p_2\,q_2\,h^2)\,(p_1\,q_2\,k^2 + p_2\,q_1\,l^2) \qquad \text{(A.11)}$$

and

$$jk \equiv hl \mod 2, \qquad jl \equiv hk \mod 2 \qquad \text{(A.12)}$$

[use (A.7)]. This implies that

$$j^2 + h^2 \neq 0, \qquad k^2 + l^2 \neq 0 \qquad \text{(A.13)}$$

(otherwise $|m| = |m'| = 0$) and

$$\left.\begin{array}{c} \text{either} \quad j \equiv h \equiv 0, \\ l \equiv 0, \quad \text{or} \quad j \equiv h \equiv k \equiv l \equiv 1 \mod 2. \end{array}\right\} \qquad \text{(A.14)}$$

To secure the uniqueness of the representation (A.9) we impose the following conditions:

$$\left.\begin{array}{c} p_1,\, q_1 > 0; \qquad \gcd(k,\, l) = 1; \\ \text{either} \quad k > 0 \quad \text{or} \quad k = 0, \quad l > 0. \end{array}\right\} \qquad \text{(A.15)}$$

PROPOSITION A.2. – *If* (A.7) *holds then* (A.9) *sets a one-to-one correspondence between pairs* m, $m' \in \mathbb{Z}^2$ *with* $|m| = |m'| \neq 0$ *and the numbers* j, h, k, l, p_1, q_1 *satisfying* (A.13)-(A.15).

Proof. – Observe that (A.9) and (A.15) determine uniquely the signs of j, h, k, l so we may assume that $m_1 \pm m_1'$, $\pm m_2 \pm m_2' \geq 0$ and look for nonnegative j, h, k, l. Put

$$j = \gcd(m_1 + m_1',\, m_2 + m_2'), \qquad h = gcd(m_1 - m_1',\, -m_2 + m_2');$$
$$x = \frac{m_1 + m_1'}{j}, \quad y = \frac{m_1 - m_1'}{h}, \quad z = \frac{m_2 + m_2'}{j}, \quad w = \frac{-m_2 + m_2'}{h}.$$

Then (A.7) reduces to

$$qxy = pzw$$

with

$$\gcd(p,\, q) = \gcd(x,\, z) = \gcd(y,\, w) = 1.$$

Hence p divides xy and q divides zw so that

$$x = p_1\,k, \qquad y = p_2\,l, \qquad p_1\,p_2 = p, \qquad p_1 > 0;$$
$$z = q_1\,m, \qquad w = q_2\,n, \qquad q_1\,q_2 = q, \qquad q_1 > 0,$$

and
$$kl = mn, \qquad \gcd(k, m) = \gcd(l, n) = 1,$$

which implies that $k = n$ and $l = m$. Therefore (A.9) holds.

To prove uniqueness assume that we have another representation

$$z = q'_k\, k', \qquad w = q'_2\, l', \qquad q'_1\, q'_2 = q, \qquad q'_1 > 0.$$

Then on the one hand as we showed before, $k = n$ and $l = m$, on the other hand the same computation gives $k' = n$ and $l' = m$. Hence $k = k'$ and $l = l'$ which proves the uniqueness. Proposition A.2 is proved.

Observe that if $j,\ h,\ k,\ l$ satisfy (A.7) then $-j,\ -h,\ -k,\ -l$ satisfy (A.7) as well. So if we drop the condition on the sign of $k,\ l$ in (A.15) then we will have in (A.7) a one-to-two correspondence between $m,\ m'$ and j, $h,\ k,\ l,\ p_1,\ q_1$. Therefore,

$$S(b) = (1/2) \sum_{jhklp_1q_1} \times \exp(-(p_1\, q_1\, j^2 + p_2\, q_2\, h^2)\,(p_1\, q_2\, k^2 + p_2\, q_1\, l^2)/(4\, b)) \tag{A.16}$$

summed over all divisors p_1 of p and q_1 of q and over all $(j,\ h,\ k,\ l)$ satisfying (A.13), (A.14) and

$$\gcd(k,\ l) = 1. \tag{A.17}$$

Let us consider a partial sum $S(b;\ p_1,\ q_1)$ in (A.16) with $p_1,\ q_1$ fixed. (A.17) reduces the possibilities allowed by (A.14) to two. Therefore

$$S(b;\ p_1,\ q_1) = S_e + S_o,$$

where the terms with even $j,\ h$ are

$$S_e = (1/2) \sum_{kl} [f_1(k,\ l)\, f_2(k,\ l) - 1], \tag{A.18}$$

summed over $(k,\ l)$ satisfying (A.17), and the terms with j and h odd are

$$S_o = (1/2) \sum_{kl} g_1(k,\ l)\, g_2(k,\ l), \tag{A.19}$$

summed over over odd integers $(k,\ l)$ satisfying (A.17). The functions $(f_i,\ g_i)$ are defined by

$$\sum_x \exp(-x^2\, a_i) = f_i \quad \text{or} \quad g_i, \qquad i = 1,\ 2, \tag{A.20}$$

where the sum is over integer x for f_i and over half-odd-integer x for g_i. In (A.20) we have used the abbreviation

$$a_i = p_i\, q_i\, (p_1\, q_2\, k^2 + p_2\, q_1\, l^2)/b. \tag{A.21}$$

The (-1) in (A.18) takes account of the fact that the term $(j = h = 0)$ was omitted from (A.16). By the Poisson summation formula, (A.20) gives

$$\left.\begin{aligned} f_i &= (\pi/a_i)^{1/2} \sum_r \exp\left(-(\pi^2/a_i)\, r^2\right), \\ g_i &= (\pi/a_i)^{1/2} \sum_r (-1)^r \exp\left(-(\pi^2/a_i)\, r^2\right). \end{aligned}\right\} \tag{A.22}$$

For $a_i \geq 1$, (A.20) gives the asymptotics

$$f_i = 1 + O\left(\exp\left(-a_i\right)\right), \qquad g_i = O\left(\exp\left(-a_i\right)\right). \tag{A.23}$$

For $a_i \leq 1$, (A.22) gives

$$f_i,\, g_i = (\pi/a_i)^{1/2} \left(1 + O\left(\exp\left(-\pi^2/a_i\right)\right)\right). \tag{A.24}$$

(A.23) implies $[f_1(k,\, l)\, f_2(k,\, l) - 1] = O\left(\exp\left(-(k^2 + l^2)/b\right)\right)$, hence

$$\sum_{k^2 + l^2 > b} [f_1(k,\, l)\, f_2(k,\, l) - 1] = O(b), \qquad b \to \infty.$$

The density of $(k,\, l)$ with $\gcd(k,\, l) = 1$ is $6/\pi^2$, hence (A.24) implies

$$\begin{aligned} (1/2) &\sum_{k^2 + l^2 \leq b} [f_1(k,\, l)\, f_2(k,\, l) - 1] \\ &= \frac{3}{\pi^2} \int_{1 \leq k^2 + l^2 \leq b} \frac{\pi}{(a_1\, a_2)^{1/2}}\, dk\, ld + O(b) \\ &= \frac{3\, b}{\pi\, (pq)^{1/2}} \int_{1 \leq k^2 + l^2 \leq b} \frac{dk\, ld}{p_1\, q_2\, k^2 + p_2\, q_1\, l^2} + O(b) \\ &= \frac{3\, b}{pq} \log b + O(b). \end{aligned}$$

Therefore

$$S_e = \frac{3\, b}{pq} \log b + O(b).$$

Similar computation gives

$$S_o = \frac{b}{pq} \log b + O(b)$$

[note that the density of odd (k, l) with $\gcd(k, l) = 1$ is equal to $2/\pi^2$]. Thus

$$S(b; p_1, q_1) = \frac{4b}{pq} \log b + O(b).$$

Summing over p_1, q_1 we obtain that

$$S(b) = \frac{4\, d(p)\, d(q)}{pq}\, b \log b + O(b).$$

For p, q odd Lemma A.1 is proved. For p even and q odd the proof is similar, with a somewhat more tedious arithmetics.

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