ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 31, nº 1 (1995), p. 273-285

<http://www.numdam.org/item?id=AIHPB_1995__31_1_273_0>

Ann. Inst. Henri Poincaré, (In Memoriam C. KIPNIS)

Vol. 31, n° 1, 1995, p. 273-285. *Probabilités et Statistiques*

Self Diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion

by

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Dedicated to the memory of Claude Kipnis.

ABSTRACT. – We consider a tagged particle in a simple exclusion model where the probability distribution of the jump sizes has zero mean, but is not necessarily symmetric. We establish for the tagged particle, a central limit theorem under the usual scaling.

RÉSUMÉ. – Nous considérons une particule marquée dans un modèle d'exclusion simple où la loi de probabilité des sauts est de moyenne nulle mais pas nécessairement symétrique. Nous démontrons pour le déplacement de la particule marquée un théorème central limite avec le changement d'échelle usuel.

1. INTRODUCTION

We consider a random walk with simple exclusion. This means that we have a collection of particles in $Z^d, d \geq 1$ with at most one particle per site. Each particle waits for an exponential time with mean 1 and at the end of this time picks a random site to jump to. The probability that a

(*) Supported by NSF grant DMS-9201222 and ARO grant DAAL03-92-G-0317.
Classification A.M.S.: 60 K 35, 60 F 17.

particle, located at x, picks the site y to jump to is given by $p(y-x)$. However the jump can be executed only if the site y is free. Otherwise the particle remains in the original site and waits for a new exponential time. All the particles are doing this simultaneously and independently of each other. Since we are dealing with processes in continuous time ties will never occur and this describes the evolution completely.

For a more mathematical definition of the model we start with the state space Ω consisting of functions η on Z^d taking values either 0 or 1. If $\eta(x)=1$, then there is a particle at x and if $\eta(x)=0$, the site x is free.

We next define certain transformations on the space Ω.

$$
\begin{aligned}
(\sigma_{x,y}\eta)(a) &= \eta(a) \quad if \quad a \neq x \text{ or } y \\
(\sigma_{x,y}\eta)(x) &= \eta(y) \\
(\sigma_{x,y}\eta)(y) &= \eta(x)
\end{aligned}
$$

The transformation $\sigma_{x,y}$ for two sites x, y in Z^d with $x \neq y$ could be either a particle jumping from x to y or one jumping from y to x. Of course if $\eta(x)=\eta(y)$ then $\sigma_{x,y}$ does nothing and is identity. Given $p(x) \geq 0$ with $p(0)=0$ and $\Sigma p(x)=1$, *i.e.* the distribution of jumps, we can define a formal infinitesimal generator acting on suitable functions F on Ω

$$
(LF)(\eta) = \Sigma_{x,y}\eta(x)(1-\eta(y))p(y-x)[F(\sigma_{x,y}\eta)-F(\eta)]
$$

It is known [3] that this generator defines a good Markov Process on Ω . For each ρ in $0 \leq \rho \leq 1$, we have on Ω the Bernoulli product measure P_ρ with $P_\rho[\eta(x)=1]=\rho$ for all x in Z^d. Each P_ρ is an invariant measure for our evolution. Corresponding to each ρ we have a system of interacting particles with initial distribution P_ρ which is in equilibrium. Of $p(x)$ we assume that it is zero outside a finite set. The case when $p(x)=p(-x)$ is the symmetric case and then the process is reversible with respect to any one of the invariant measures. But in this article we will assume only that

$$
\Sigma < x, \theta > p(x) = 0
$$

or that the jumps have mean zero. Now the process is not necessarily reversible. We will also assume an irreducibility condition namely that $\{x : p(x) > 0\}$ generates the whole group Z^d .

If we start our evolution under the assumption that there is a particle at 0 and that at other sites we have a random Bernoulli configuration with density ρ , we can tag the particle at 0 as it moves around and denote its location at time t by z_t . We are interested in proving a Brownian scaling limit for z_t , *i.e.* to establish that $\lambda^{-\frac{1}{2}}z_{\lambda t}$ converges as $\lambda \to \infty$ to a

Brownian motion in distribution. In [2], with Claude Kipnis we proved that this was indeed so in the symmetric case. The case $p(1) = p(-1) = \frac{1}{2}$ in one dimension is special and leads to Brownian motion with zero variance. In all other cases it is nondegenerate. In this article we will extend these results to the asymmetric mean zero case.

2. EVOLUTION OF THE TAGGED PARTICLE

In order to follow the motion of a tagged particle the state space has to be changed. The state space Ω cannot distinguish between particles. We shall find it convenient to describe the current state of the system by giving the location of the tagged particle and the environment around the tagged particle. In other words the state space is $\hat{\Omega} = Z^d \times \Omega_0$. Here Ω_0 is the space of configurations on $Z^d - \{0\}$. In order to describe the evolution in the space $\hat{\Omega}$ we need to describe a transformation τ_x that acts on Ω_0 for each $x \neq 0$.

$$(\tau_x \eta)(a) = \eta(x + a) \quad if \quad a \neq 0 \;\; or \;\; -x$$
$$(\tau_x \eta)(-x) = 0$$

$\tau_x \eta$ is meaningful only if $\eta(x) = 0$ and describes the effect of the tagged particle, which by definition is always at zero, jumping to x. We relocate the origin at x the new location of the tagged particle. The transformations $\sigma_{x,y}$ for $x, y \neq 0$ are well defined on Ω_0. The evolution of the tagged particle on $\hat{\Omega}$ is governed by the following generator

$$\begin{aligned}(\hat{L}F)(z, \eta) = \Sigma_{x,y\neq 0}\eta(x)(1 - \eta(y))p(y - x)[F(z, \sigma_{x,y}\eta) - F(z, \eta)] \\ + \Sigma_x(1 - \eta(x))p(x)[F(z + x, \tau_x\eta) - F(z, \eta)].\end{aligned}$$

If F were a function of η only, then

$$\begin{aligned}(L_0F)(\eta) = \Sigma_{x,y\neq 0}\eta(x)(1 - \eta(y))p(y - x)[F(\sigma_{x,y}\eta) - F(\eta)] \\ + \Sigma_x(1 - \eta(x))p(x)[F(\tau_x\eta) - F(\eta)].\end{aligned}$$

In other words if (z_t, η_t) is the tagged system , η_t by itself is a Markov Process with generator L_0 . The Bernoulli measures P_ρ^0 on Ω_0 defined by

$$P_\rho^0[\eta(x) = 1] = \rho \quad for \; all \; x \;\in Z^d - \{0\}$$

are invariant measures for L_0. $\hat{L}$ on the other hand does not have an invariant measure because the z_t part will wander away in Z^d.

Our initial distribution on $\hat{\Omega}$ is $\delta_0 \times P_\rho^0$, *i.e.* we start from $z = 0$ and η is in equilibrium . Then under the $\hat{L}$ evolution η is always in equilibrium. z_t is the location of the tagged particle at time t and the scaling limit is to be established for it.

3. OUTLINE OF PROOF

A direct calculation yields

$$\hat{L}z = \Sigma_x (1 - \eta(x)) x p(x)$$

or equivalently for each θ in R^d

$$\begin{aligned}\hat{L} < z, \theta > &= \Sigma_x (1 - \eta(x)) < x, \theta > p(x) \\ &= < \psi(\eta), \theta > .\end{aligned}$$

Therefore,

$$< z_t, \theta > = \int_0^t < \psi(\eta_s), \theta > ds + < M(t), \theta > .$$

Here $M(t)$ is a Martingale with stationary increments. The basic idea in [2] was to replace the first term on the right in the above equation by a Martingale term and an error term.

$$\int_0^t < \psi(\eta_s), \theta > ds = < N(t), \theta > + < E(t), \theta > .$$

Our ability to do so depended on the following procedure.

Step 1.

Solve the equation

$$\lambda u_\lambda - L_0 u_\lambda = \psi$$

For simplicity let us take just one component of ψ . Based on an estimate

$$|E^{P_\rho^0}[\psi(\eta) G(\eta)]| \leq C_\rho [D_\rho(G)]^{\frac{1}{2}}$$

where $D_\rho(u) = - < L_0 u, u >_\rho$ is the Dirichlet form of u, we get for the solution u_λ

$$E[\lambda u_\lambda^2 + D_\rho(u_\lambda)] = < \psi, u_\lambda >_\rho$$
$$\leq C_\rho [D_\rho(u_\lambda)]^{\frac{1}{2}}.$$

This gives us bounds

$$\lambda E u_\lambda^2 \leq C_\rho^2$$
$$D_\rho(u_\lambda) \leq C_\rho^2$$

holding uniformly as $\lambda \to 0$.

Step 2.

Using either spectral theory or a more direct argument one establishes in fact that

$$\lambda E[u_\lambda^2] \to 0 \quad as \quad \lambda \to 0$$

and

$$D_\rho(u_{\lambda_1} - u_{\lambda_2}) \to 0 \quad as \quad \lambda_1, \lambda_2 \to 0.$$

This was enough to reduce the Brownian scaling property for z_t to the same thing for a Martingale with stationary increments which is elementary and standard.

Step 3.

We prove compactness by establishing the tightness of the distributions of

$$\int_0^t \psi(\eta_s) ds$$

under Brownian scaling.

Step 4.

Establish the nondegeneracy of the Brownian motion obtained as the scaling limit.

4. DETAILS OF PROOF

Due to the asymmetry one has to make changes in the proof along the way.

Step 1.

$$\psi(\eta) = \Sigma(1-\eta(x)) < x, \theta > p(x)$$
$$= \Sigma_{x\neq 0}\eta(x)\omega_\theta(x)$$

where $\omega_\theta(x)$ are weights with $\Sigma\omega_\theta(x) = 0$. We can therefore rewrite

$$\psi(\eta) = \Sigma_{x,y\neq 0}[\eta(x) - \eta(y)]\tilde{\omega}(x,y)$$

for some weights $\tilde{\omega}(x,y)$. Because of irreducibility we can assume that $\tilde{\omega}(x,y) \neq 0$ only if $p(y-x) > 0$. Since $\eta(x) - \eta(y) = -((\sigma_{x,y}\eta)(y) - (\sigma_{x,y}\eta)(x))$ we can write,

$$E[G(\eta)\psi(\eta)] = \frac{1}{2}E[\Sigma\tilde{\omega}(x,y)[\eta(x) - \eta(y)][G(\sigma_{x,y}\eta) - G(\eta)]$$
$$\leq C[D_1(G)]^{1/2}.$$

where

$$D_1(G) = E[\Sigma_{x,y\neq 0}p(y-x)[G(\sigma_{x,y}\eta) - G(\eta)]^2.$$

Here and in what follows E is expectation relative to P_ρ^0 for some fixed $0 < \rho < 1$. If we denote by

$$D_2(G) = E[\Sigma_x[G(\tau_x\eta) - G(\eta)]^2]$$

then the full Dirichlet form for L_0 is

$$D_0(G) = D_1(G) + D_2(G)$$

Step 2.

If now one solves the equation

$$\lambda u_\lambda - L_0 u_\lambda = \psi$$

one can get the estimates

$$\begin{aligned} \lambda E u_\lambda^2 + D_0(u_\lambda) = E\psi u_\lambda \leq C[D_1(u_\lambda)]^{\frac{1}{2}}, \\ \lambda E u_\lambda^2 \leq C, \\ D_0(u_\lambda) \leq C. \end{aligned} \tag{4.1}$$

We can as before take a subsequence of u_λ converging to a limit u in H_1. Here H_1 is the completion of nice functions with respect to the

Dirichlet norm $D_0(u)$. It contains a lot of generalized functions and consists technically of limits of equivalence classes of functions modulo constants. Let us suppose that the following estimate holds. For all functions F and G,

$$E[FL_0G] \leq C[D_0(F)]^{\frac{1}{2}}[D_0(G)]^{\frac{1}{2}} \tag{4.2}$$

then L_0 will be a bounded operator from H_1 into H_{-1} and $\lambda u_\lambda \to 0$ weakly in H_{-1}. Therefore

$$L_0 u = -\psi \quad in \quad H_{-1}$$

From this we obtain,

$$D_0(u) = < u, \psi >$$

where $<, >$ is the pairing between H_1 and H_{-1}, the latter being the formal dual of the former. Taking limits as $\lambda \to 0$ in (4.1) if $d_1 = lim\ \lambda E u_\lambda^2$ and $d_2 = lim\ D_0(u_\lambda)$ along suitable subsequences then

$$d_1 + d_2 = < u, \psi > = D_0(u)$$

By lower semicontinuity $D_0(u) \leq d_2$. Therefore $d_1 = 0$ and $d_2 = D_0(u)$. This proves the strong convergence of u_λ in H_1 and also establishes

$$lim\ \lambda E u_\lambda^2 = 0.$$

It is now routine to prove the uniqueness of the limit point. So the inequality (4.2) which is obvious with $C = 1$ in the symmetric case is the crucial step in the nonsymmetric case. We shall prove it in the last section. The rest of the details are identical to the symmetric case.

Step 3.

Whereas in [2], for the symmetric case, compactness was a consequence of the estimate in step 1, in our more general context we need to use some special properties of ψ. We start with the estimate

$$\frac{1}{t} log E \left\{ exp \left[\lambda \int_0^t \psi(\eta_s) ds \right] \right\} \leq \sup_{||G||_2 = 1} [\lambda E[\psi G^2] - D_0(G)]$$

As in step 1

$$E[G^2\psi] = \frac{1}{2} E \Sigma [\eta(x) - \eta(y)] \tilde{\omega}(x, y) [G^2(\sigma_{x,y}\eta) - G^2(\eta)]$$

and

$$\begin{aligned}
&E[(\eta(x)-\eta(y))(G^2(\sigma_{x,y}\eta)-G^2(\eta))]\\
&=E[[\eta(x)-\eta(y)][G(\sigma_{x,y}\eta)-G(\eta)][G(\sigma_{x,y}(\eta)+G(\eta)]]\\
&\le E[|G(\sigma_{x,y}\eta)-G(\eta)||G(\sigma_{x,y}\eta)+G(\eta)|]\\
&\le [E[G(\sigma_{x,y}\eta)-G(\eta)]^2]^{\frac{1}{2}}[E[G(\sigma_{x,y}\eta)+G(\eta)]^2]^{\frac{1}{2}}\\
&\le C[E[G(\sigma_{x,y}\eta)-G(\eta)]^2]^{\frac{1}{2}}.
\end{aligned}$$

We have used the facts that $|\eta(x)-\eta(y)|\le 1$ and $EG^2(\sigma_{x,y}\eta)=EG^2(\eta)=1$. Therefore

$$\begin{aligned}
\sup_{||G||_2=1}[\lambda E\psi G^2 - D_0(G)] &\le \sup_{||G||_2=1}[C\lambda(D_0(G))^{\frac{1}{2}}-D_0(G)]\\
&\le \frac{A\lambda^2}{2}
\end{aligned}$$

In other words

$$E\left[exp\left[\lambda\int_s^t \psi(\eta_s)ds\right]\right]\le exp\left[A(t-s)\frac{\lambda^2}{2}\right].$$

Compactness is now a consequence of Garsia-Rodemick-Rumsey estimate which can be found in [4].

Step 4

We now establish the nondegeneracy of the limiting Brownian motion. From the outline of the proof it is clear that what we need to establish is that the martingale produced by $\int_0^t <\psi(\eta_s),\theta> ds$ namely $<N(t),\theta>$ cannot cancel the martingale $<M(t),\theta>$. There are two classes of martingales generated by the underlying Poisson events: jumps that involve the tagged particle, identified through τ_x and those involving other particles identified through $\sigma_{x,y}$ with $x,y\neq 0$. These are orthogonal martingales because they relate to non simultaneous Poisson jumps. If $N(t)$ were to totally cancel $M(t)$, which involves only the first type of jumps, its orthogonal projection on to the second type of martingales should go to zero. This can happen only if for the corresponding u_λ, $D_1(u_\lambda)\to 0\ as\ \lambda\to 0$. But if $D_1(u_\lambda)\to 0$ then it follows from our estimates that $D_0(u_\lambda)\to 0$ as well and this forces $N(t)$ to be identically zero. With a small bit of extra calculation one can turn this into a lower bound for the variance of the limiting Brownian motion.

With only the basic estimate (4.2) remaining to be proved in the next section we have now established our main result.

THEOREM 4.1. – *If z_t is the location of the tagged particle in Z^d then as $k \to \infty$ the distribution of the stochastic process $\frac{z_{kt}}{\sqrt{k}}$ converges in Skorohod space to a Brownian motion on R^d with a nondegenerate covariance matrix.*

5. THE BASIC ESTIMATE

For the operator L_0 acting on functions defined on Ω_0

$$\begin{aligned}(L_0F)(\eta) &= \Sigma_x(1-\eta(x))[F(\tau_x\eta) - F(\eta)]p(x) \\ &\quad + \Sigma_{x,y\neq 0}\eta(x)(1-\eta(y))p(y-x)[F(\sigma_{x,y}\eta) - F(\eta)]\end{aligned}$$

we wish to establish the following estimate.

THEOREM 5.1. - *For functions $F(\eta)$ and $G(\eta)$*

$$|< F, L_0G >_\rho| \leq C_\rho[D_0(F)]^{\frac{1}{2}}[D_0(G)]^{\frac{1}{2}}$$

for some constant C_ρ depending only on ρ.

Here $<,>_\rho$ represents the inner product in $L_2(P_\rho^0)$ and $D_0(F)$ is the Dirichlet form

$$\begin{aligned}D_0(F) = \frac{1}{2}E^{P_\rho^0}\{&\Sigma_x(1-\eta(x))p(x)[F(\tau_x\eta) - F(\eta)]^2 \\ &+ \Sigma_{x,y\neq 0}\eta(x)(1-\eta(y))p(y-x)[F(\sigma_{x,y}\eta) - F(\eta)]^2\}\end{aligned}$$

Before we prove Theorem 5.1 let us make some observations.

1. If L_0 were symmetric in $L_2(P_\rho^0)$ then the inequality is valid with $C_\rho = 1$.

2. Suppose A is a Markov generator on a finite state space with an invariant measure μ and A is not symmetric. Let $\bar{A}$ be the symmetrization $\frac{A + A^*}{2}$ in $L_2(\mu)$. Since A and $\bar{A}$ have the same range, if we have $Ag = h$ we also have for some $\bar{g}$, $\bar{A}\bar{g} = h$ and the map $g \to \bar{g}$ is well defined and the two Dirichlet forms are related by

$$D(\bar{g}) \leq CD(g)$$

Let us first prove a similar estimate for the untagged system L on Ω.

$$(LF)(\eta) = \Sigma_{x,y}\eta(x)(1-\eta(y))p(y-x)[F(\sigma_{x,y}\eta) - F(\eta)]$$

LEMMA 5.2. – *For any two functions F,G on Ω and for any ρ*

$$| < F, LG >_\rho | \leq C_\rho [D(F)]^{1/2} [D(G)]^{1/2}.$$

Proof. – The proof depends on the following considerations. Suppose $a_1, a_2, \cdots a_k$ are k points in Z^d such that $a_1 + a_2 + \cdots + a_k = 0$, then the probability distribution $\pi(x)$ on Z^d defined by $\pi(a_j) = \frac{1}{k}$ for $j = 1, 2 \cdots k$ and $\pi(x) = 0$ for all other x has clearly mean 0. The sequence $y_0 = 0, y_1 = a_1, y_2 = a_1 + a_2, \cdots, y_k = a_1 + a_2 + \cdots + a_k = 0$ defines a cycle $C = \{y_0, y_1, \cdots, y_k\}$ and a π can be associated to each such cycle by taking $a_j = y_j - y_{j-1}$. Moreover one can assume that the cycle has no double points. Otherwise the cycle decomposes into two or more cycles and π_C is a convex combination of π_{C_i} corresponding to the component cycles. We can therefore limit ourselves to irreducible cycles. If $C_1, C_2, \cdots, C_l$ are l irreducible cycles and $w_1, w_2, \cdots, w_l$ are nonnegative weights adding up to 1, the convex combination $p(x) = \Sigma w_i \pi_{C_i}(x)$ is a mean zero probability distribution on Z^d. We have the converse

LEMMA 5.3. – *Any $p(x)$ with finite support and mean 0 has a representation*

$$p(x) = \Sigma w_i \pi_{C_i}(x)$$

for some weights $w_1, w_2, \cdots, w_l$ and irreducible cycles $C_1, C_2, \cdots, C_l$.

Proof. – A proof of this lemma can be found in [5].

We continue with the proof of lemma 5.2. Each cycle C introduces an operator A_C on Ω by

$$(A_C F)(\eta) = (1/k) \Sigma_{i=0}^{k-1} \eta(y_i)(1 - \eta(y_{i+1}))[F(\sigma_{y_i, y_{i+1}} \eta) - F(\eta)]$$

One can verify that A_C has P_ρ for an invariant measure and is in general nonreversible unless $k = 2$. For any cycle C we can consider the cycle $C + x$ described by $x, x + y_1, \cdots, x + y_k = x$ starting and ending at x rather than at 0 and define

$$(A_{C+x} F)(\eta) = (1/k) \Sigma_{i=0}^{k-1} \eta(x + y_i)(1 - \eta(x + y_{i+1}))$$
$$[F(\sigma_{x+y_i, x+y_{i+1}} \eta) - F(\eta)]$$

We can make a translation invariant generator on Ω by defining

$$L_C = \Sigma_{x \in Z^d} A_{C+x}$$

L_C is a typical generator for asymmetric mean zero random walk with simple exclusion. A consequence of lemma 5.3. is the representation

LEMMA 5.4. – *Our generator L for asymmetric mean zero simple exclusion has a representation*

$$L = \Sigma_{i=0}^{l} w_i L_{C_i}$$

for some $C_1, C_2, \cdots, C_l$ which are irreducible cycles.

Now we can complete the proof of lemma 5.2. The Dirichlet form for L is easily seen to be weighted sum of the forms for each C_i. Therefore there is no loss of generality in assuming $l = 1$ or $L = A_C$ for some irreducible C.

$$L = \Sigma_x A_{C+x}$$

The Dirichlet form for L_C is the sum of Dirichlet forms for each A_{C+x}. By Schwartz's inequality it is enough to prove the estimate for each A_{C+x} and this is essentially observation 2. The idea is that instead of detailed balance that produces reversibility we now have local balance that yields bounds.

Now we have to extend these considerations to the tagged system. We need another basic idea. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $T_1, T_2, \cdots, T_k$ are k measure preserving transformations. Let us assume $T_k \cdots T_1 = I$, and define

$$(LF)(\omega) = \Sigma_{i=0}^{k}[F(T_i\omega) - F(\omega)].$$

Our claim is that our basic estimate is valid in this context.

LEMMA 5.5. – *We have for all F and G,*

$$|E[FLG]| \leq C[D(F)]^{1/2}[D(F)]^{1/2}.$$

Proof.

$$\begin{aligned}\int FLGdP &= \Sigma \int F(\omega)[G(T_i\omega) - G(\omega)]dP \\ &= \Sigma \int F(T_{i-1}\cdots T_1\omega)[G(T_i\cdots T_1\omega) - G(T_{i-1}\cdots T_1\omega)]dP \\ &= \Sigma \int [F(T_{i-1}\cdots T_1\omega) - F(\omega)] \\ &\qquad [G(T_i\cdots T_1\omega) - G(T_{i-1}\cdots T_1\omega)]dP\end{aligned}$$

because

$$\begin{aligned}&\Sigma_i F(\omega)[G(T_i\cdots T_1\omega) - G(T_{i-1}\cdots T_1\omega)] \\ &\quad = F(\omega)[G(T_k\cdots T_1\omega) - G(\omega)] = 0.\end{aligned}$$

Rest is Schwartz.

Finally we return to the proof of Theorem 5.1. The operator

$$\begin{aligned}(L_0F)(\eta) &= \Sigma p(x)(1-\eta(x))[F(\tau_x\eta)-F(\eta)] \\ &\quad + \Sigma p(y-x)\eta(x)(1-\eta(y))[F(\sigma_{x,y}\eta)-F(\eta)]\end{aligned}$$

decomposes into cycles and we can assume without loss of generality that $p(x) = \pi_C(x)$ for some irreducible C. The problem is that 0 plays a special role and translates $C + x$ of the cycle C that touch 0 create trouble. We can write

$$L_0F = L_1F + L_2F$$

where L_2F involves only jumps of the untagged particles and in addition to full cycles that are estimated without any trouble there are incomplete cycles because they go through the origin. L_1 on the other hand involves only jumps of the tagged particle.

Let us remark that

$$\eta(x)(1-\eta(y))[F(\sigma_{x,y}\eta)-F(\eta)] = (1-\eta(y))[F(\sigma_{x,y}\eta)-F(\eta)]$$

so that the terms of L_1 and L_2 look similar. Let our cycle be $0 = y_0, y_1, \cdots, y_k = 0$ with $a_i = y_i - y_{i-1}$ for $i = 1, \cdots, k$. Then ignoring constants 1/k and disregarding the full cycles,

$$\begin{aligned}(L_1F)(\eta) &= \Sigma_i(1-\eta(a_i))[F(\tau_{a_i}\eta)-F(\eta)] \\ (L_2F)(\eta) &= \Sigma_{(x,y)\in\Delta}(1-\eta(y))[F(\sigma_{x,y}\eta)-F(\eta)]\end{aligned}$$

The rest of the proof can be best described in words. Suppose there is an empty site in the loop in front of the tagged particle, *i.e.* $\eta(a_1) = 0$. Then the tagged particle can move, that is we can apply τ_{a_1}. Now the empty site created at the origin is really at $-a_1 = (a_2 + \cdots + a_k)$ because the origin has shifted with the tagged particle. $-a_1$ is the last site of the loop that starts from the new origin *i.e.* the old a_1. One can effect $\sigma_{a_2+\cdots+a_{k-1}, a_2+\cdots+a_k}$ and get a free site at $a_2 + \cdots + a_{k-1}$. We can proceed in this fashion till we get a free site at a_2. Now the tagged particle can jump to a_2 and the whole process starts again. After several steps the tagged particle will return to its original starting point with an empty site in front and all other particles in exactly the same position that we started from. This takes exactly $n = k(k-1)$ steps and we use up every term of L_1 and L_2 exactly once. In other words

$$(L_1 + L_2)(F)(\eta) = \Sigma\chi_{E_j}[F(S_j\eta)-F(\eta)]$$

where $E_{j+1} = S_j S_{j-1} \cdots S_1 E_1$ and $S_n S_{n-1} \cdots S_1 = I \ on \ E_1$. We can use lemma 5.5 at this point and we are done.

6. REMARKS

The basic estimate (4.2) for the untagged system was derived and used by Lin Xu in his New York University PhD dissertation to establish a hydrodynamic scaling limit for mean zero random walks with simple exclusion. A central limit theorem for the position of a tagged particle in the case of asymmetric one dimensional simple exclusion (non zero mean) was considered in [1] by C. Kipnis.

ACKNOWLEDGEMENTS

The author wishes to thank F. Rezakhanlou for valuable discussions.

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*(Manuscript received April 9, 1994;
revised August 14, 1994.)*