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Exponential waiting time for filling a large interval in the symmetric simple exclusion process

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ABSTRACT. – We consider the one-dimensional nearest neighbors symmetric simple exclusion process starting with the equilibrium product distribution with density ρ. We study T_N, the first time for which the interval $\{1, \ldots, N\}$ is totally occupied. We show that there exist $0 < \alpha' \leq \alpha_N \leq \alpha'' < \infty$ such that $\alpha_N \rho^N T_N$ converges to an exponential random variable of mean 1. More precisely, we get the following uniform sharp bound: $\sup_{t \geq 0} |P\{\alpha_N \rho^N T_N > t\} - e^{-t}| \leq A \rho^{A' N}$ where A and A' are positive constants independent of N.

Key words: Symmetric simple exclusion process, occurrence time of a rare event, large deviations.

RÉSUMÉ. – Nous considérons le processus d'exclusion simple symétrique unidimensionel en équilibre avec densité de particules égale à ρ. Nous étudions T_N, le premier instant où tous les points de l'ensemble $\{1, \ldots, N\}$ sont occupés. Nous démontrons qu'il existe $0 < \alpha' \leq \alpha_N \leq \alpha'' < \infty$ tels que $\alpha_N \rho^N T_N$ converge en loi vers une distribution exponentielle de

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paramètre 1. Plus précisément, nous obtenons la borne supérieure uniforme suivante: $\sup_{t\geq 0} |P\{\alpha_N \rho^N T_N > t\} - e^{-t}| \leq A\rho^{A'N}$ où A et A' sont deux constantes positives indépendantes de N.

1. INTRODUCTION

The symmetric simple exclusion process is an infinite particle system. The process was introduced by Spitzer (1970) and its ergodic properties were discussed by Liggett (1985). In this system at most one particle is allowed in each site $x \in \mathbb{Z}$ and, at rate one, the contents of sites x and $x + 1$ are interchanged. Hence if both sites are occupied or both sites are empty, nothing happens but if one of the sites is occupied and the other is empty, the interchange is seen as a jump of the particle to the empty site. The only extremal invariant measures for this system are the product measures ν_ρ, $\rho \in [0, 1]$. Under the measure ν_ρ, the probability that a site is occupied by a particle is ρ and the occupation variables of different sites are independent random variables.

We consider this process starting from ν_ρ, the extremal equilibrium measure with density ρ. Define T_N as the first time that the sites $\{1, \dots, N\}$ get occupied. We prove that there exist $0 < \alpha' \leq \alpha_N \leq \alpha'' < \infty$ and positive A and A' such that

$$\sup_{t\geq 0} \left|P\{\alpha_N \rho^N T_N > t\} - e^{-t}\right| \leq A\rho^{A'N}. \tag{1.1}$$

The main problem in showing convergence to exponential times is to show that the process loses memory rapidly with respect to the scale being studied. We reduce this to the problem of showing that a coupled process starting with two independent configurations chosen from the invariant measure match relatively fast. This last problem can be related to the study of the decay of density in a process with two species of particles interacting by exclusion with annihilation studied recently by Belitsky (1993).

In the context of interacting particle systems the convergence to exponential times has been obtained for dissipative systems —processes for which the number of particles changes with time— by Lebowitz and Schonmann (1987) and Galves, Martinelli and Olivieri (1989). Schonmann (1991) reviews those and other results. Ferrari, Galves and Landim (1994) studied the asymmetric zero range process proving an exponential bound.

Since the zero range process and the simple exclusion process we study here conserve the number of particles, we call these systems conservative. Dissipative systems lose memory much faster than conservative systems, but somehow surprisingly, an exponential rate for the convergence has been obtained only for conservative systems.

Convergence to exponential times has also been obtained for Harris recurrent chains by Korolyuk and Sil'vestrov (1984) and Cogburn (1985). These results cannot be applied to our case because the simple exclusion process is not Harris recurrent even on the set of configurations with fixed asymptotic density. In general particle systems are not Harris recurrent. Moreover, the arguments used to show convergence to exponential distributions for Harris recurrent chains do not give a rate of convergence like that in (1.1).

A sharper result including a rate of convergence has been proved for finite state Markov chains by Aldous (1982) (1989). In view of this, a possible way to show our result would be to use a finite truncation. One considers the symmetric simple exclusion process in the box $[-\ell_N, N+\ell_N]$ with any boundary condition that leaves invariant the measure ν_ρ restricted to the box. Using our notation, Aldous and Brown (1992) proved the following bound

$$\left|P\left\{\frac{T_N}{ET_N} > t\right\} - e^{-t}\right| \leq \frac{\tau_N}{ET_N}, \tag{1.2}$$

where $1/\tau_N$ is the second eigenvalue of the continuous time Markov chain in the finite box. To apply this inequality to obtain the result for the infinite system we need to take a box of length $\ell_N = (ET_N)^{(1+\varepsilon)/2}$ for some $\varepsilon > 0$. This is necessary to guarantee that the time of occurrence of the rare event for the finite and the infinite system behave in the same way. Indeed the influence of the boundary behaves *at least* as a simple symmetric random walk, and it takes a time of the order of the square of the length of the box to arrive to the center of the box. On the other hand, Lu and Yau (1993) proved that τ_N is of the order at least of the square of the length of the box. Hence the upper bound in (1.2) diverges and the finite approximation cannot be used naively.

Another way to prove convergence to exponential times is the Chen-Stein method (see Arratia and Tavare (1993) for a review). This is a method to show convergence to a Poisson distribution for X, the number of occurrences of a rare event in a long interval of time $(0, r)$. In our case the rare event is a visit to $\mathbf{Y}_N = \{\eta : \eta(x) = 1, x \in \{1, \ldots, N\}\}$ preceded by an interval outside $\mathbf{Y}_N$ of length L with $1 << L << \rho^{-N}$.

Fixing $r = ET_N t$ we have $P\{T_N/ET_N > t\} = P\{X = 0\}$. This has been performed by Aldous and Brown (1993) for finite state Markov chains with bounds slightly worse than (1.2).

The convergence to exponential for occurrence times of rare events seems to have been studied first by Bellmann and Harris (1951) and Harris (1953). In statistical physics the question was considered in the so called "path-wise approach to metastability" by Cassandro, Galves, Olivieri and Vares (1984). Kipnis and Newman (1985) studied the case of diffusion processes. In the intermittency context the question was studied by Collet, Galves and Schmitt (1992).

2. DEFINITIONS AND RESULTS

We use Harris (1978) graphical construction to define the process. To each nearest neighbor pair of sites $\{x, y\} \subset \mathbb{Z}$ a Poisson point process of rate 1 is attached. These processes are mutually independent. Call (Ω, $\mathcal{F}$, P) the probability space where these processes are defined. Denote by $\tau_k^{\{x,y\}}$ the time at which the k-th event of the Poisson process attached to $\{x, y\}$ occurs. Let $M_t^{\{x,y\}}$ describe the number of occurrences of $\{\tau_k^{\{x,y\}}\}$ up to time t: $M_0^{\{x,y\}} = 0$ and for $t \geq 0$,

$$M_t^{\{x,y\}} = \sum_{k\geq 1} \mathbf{1}\{\tau_k^{\{x,y\}} \leq t\}. \tag{2.1}$$

For each $\omega \in \Omega$ we say that there is a path from (x, s) to (z, t), where x, z are sites and $s < t$ are times, if

(1) there exist $x_1, \ldots, x_n$ and $t_1, \ldots, t_{n+1}$ such that $x = x_1$, $|x_i - x_{i+1}| = 1$, $x_n = z$,

(2) $t_1 = s < t_2 < \cdots < t_{n+1} = t$,

(3) for $1 \leq i < n$, $t_{i+1} = \tau_k^{\{x_i, x_{i+1}\}}$ for some k and

(4) there are no events involving x_i in the time interval (t_i, t_{i+1}).

This defines a bijection map $\phi_{\omega,s,t} : \mathbb{Z} \to \mathbb{Z}$ in the following way: $\phi_{\omega,s,t} : x \mapsto y$ if and only if there is a path in ω from (x, s) to (y, t). Given an initial configuration $\zeta \in \{0, 1\}^{\mathbb{Z}}$ we define the simple exclusion process at time t starting at time $s < t$ with configuration ζ by

$$\eta_t^{\zeta,s,\omega}(x) = \zeta(\phi_{\omega,s,t}^{-1}(x)).$$

When $s = 0$ we omit it in the notation. Usually ω is also omitted. This construction implies that

$$P\{\eta_t^{\zeta,s}(x) = 1, \text{ for some } x \in F\} = P\{\zeta(y) = 1, \text{ for some } y \in \phi_{\omega,s,t}^{-1}(F)\},$$

where

$$\phi_{\omega,s,t}^{-1}(F) = \{\phi_{\omega,s,t}^{-1}(x) : x \in F\}.$$

This property is usually called *Duality*. For any $\rho \in [0,1]$, the product measure ν_ρ defined by

$$\nu_\rho(\eta(x) = 1 : x \in F) = \rho^{|F|}, \text{ for } F \subset \mathbb{Z} \text{ finite},$$

is (extremal) invariant for the simple exclusion process (Liggett (1985)). For each $N \geq 0$ let $\Lambda_N = \{1, \dots, N\}$ and

$$\mathbf{Y}_N = \{\eta : \eta(x) = 1, x \in \Lambda_N\}.$$

We are interested in the first time the system enters $\mathbf{Y}_N$:

$$T_N^\zeta = \inf\{t \geq 0 : \eta_t^\zeta \in \mathbf{Y}_N\}.$$

In case we choose the initial configuration ζ according to the law ν_ρ we will just write T_N. Therefore we use the notation

$$P\{T_N > t\} = \int d\nu_\rho(\zeta) P\{T_N^\zeta > t\}.$$

Our main result is

THEOREM. – *There exists positive constants α', α'', A and A' independent of N and a sequence $\alpha_N \in [\alpha', \alpha'']$ such that*

$$\sup_{t \geq 0} \left| P\{\alpha_N \rho^N T_N > t\} - e^{-t} \right| \leq A\rho^{A'N}.$$

This implies, in particular, that $\alpha_N \rho^N T_N$ converges in distribution to a mean one exponential random variable.

3. BOUNDS FOR T_N

For a fixed N, call $\{\tau_n\}$ the superposition of the processes $\tau_k^{\{0,1\}}$ and $\tau_k^{\{N,N+1\}}$. This is a Poisson process with intensity 2. Remark that

$$\{T_N^\zeta < \infty\} = \{T_N^\zeta = 0\} \cup \left(\cup_{k=1}^\infty \{T_N^\zeta = \tau_k\} \right) \tag{3.1}$$

Call

$$X^{\zeta}[s,t] = \mathbf{1}\{\eta_s^{\zeta} \in \mathbf{Y}_N\} + \sum_{k=1}^{\infty} \mathbf{1}\{s < \tau_k < t, \eta_{\tau_k}^{\zeta} \in \mathbf{Y}_N\} \tag{3.2}$$

where $\mathbf{1}\{.\}$ is the indicator function of the set $\{.\}$. In case ζ is chosen according to ν_ρ, we omit it in the notation. The next lemma gives an *a priori* lower bound for T_N.

LEMMA 1. – *For any positive real number r, the following holds*

$$EX[0,r] = \rho^N(1+2r). \tag{3.3}$$

As a consequence

$$P\{T_N \geq r\} \geq 1 - \rho^N(1+2r). \tag{3.4}$$

Proof. – Let $\Lambda_N = \{1, \ldots, N\}$ and define the processes

$$Z_t^{0,\zeta} = \mathbf{1}\{\eta_t^{\zeta}(x) = 1, x \in \Lambda_N \setminus \{1\} \cup \{0\}\}$$

and

$$Z_t^{N,\zeta} = \mathbf{1}\{\eta_t^{\zeta}(x) = 1, x \in \Lambda_N \setminus \{N\} \cup \{N+1\}\} .$$

We observe that

$$X^{\zeta}[0,r] = \mathbf{1}\{\zeta \in \mathbf{Y}_N\} + \sum_{x \in \{0,N\}} \int_0^r Z_{t-}^{x,\zeta} dM_t^{\{x,x+1\}}$$

Since $\left(M_t^{\{x,x+1\}} - t\right)$ is a martingale and the processes $Z_{t-}^{0,\zeta}$ and $Z_{t-}^{N,\zeta}$ are predictable with respect to the filtration defined by the family of Poisson point processes, the process

$$\sum_{x \in \{0,N\}} \int_0^r Z_{t-}^{x,\zeta} (dM_t^{\{x,x+1\}} - dt)$$

is a martingale. Therefore, since the process is in equilibrium, $P\{\zeta_0 \in \mathbf{Y}_N\} = \rho^N$ and we have

$$\begin{aligned} EX[0,r] &= \rho^N + \sum_{x \in \{0,N\}} \int_0^r EZ_{t-}^x dt \\ &= \rho^N + 2r\rho^N \end{aligned} \tag{3.5}$$

In the above expression, as usual, we have omitted the upper index ζ, as a shorthand notation to indicate that the initial configuration was chosen with the invariant distribution ν_ρ. To show (3.4) it suffices to see that

$$P\{T_N < r\} \leq EX[0,r]. \quad \square$$

PROPOSITION 2. – *There exist positive constants C, C' and C'' independent of N such that for any $t \geq 0$,*

$$P\{\rho^N T_N \geq t\} \leq 1 - \frac{t^2}{Ct^2 + C't + C''\rho^N}.$$

Proof. – Let $X = X[0, t\rho^{-N}]$. Then, calling $\beta = t\rho^{-N}$,

$$P\{T_N \leq \beta\} \geq P\{X \geq 1\}.$$

By the Schwarz inequality

$$(EX)^2 = [E(X\mathbf{1}\{X \geq 1\})]^2 \leq EX^2 P\{X \geq 1\}.$$

Therefore

$$P\{X \geq 1\} \geq (EX)^2/EX^2.$$

Using the notation of Lemma 1 we have

$$(X^\zeta[0,r])^2 = X^\zeta[0,r] + 2\int_{0<s<u<\beta} \sum_{x,y\in\{0,N\}} Z_{s-}^{x,\zeta} Z_{u-}^{y,\zeta} dM_s^{x,x+1} dM_u^{y,y+1}. \tag{3.6}$$

Then

$$E\int_{0<s<u<\beta} Z_{s-}^x Z_{u-}^y dM_s^{x,x+1} dM_u^{y,y+1} = E\int_{0<s<u<\beta} Z_{s-}^x Z_{u-}^y dM_s^{x,x+1} du$$
$$= E\int_{0<s<\beta} Z_{s-}^x F(\eta_s, s) dM_s^{x,x+1}.$$

where $F(\zeta, s) = E\left[\int_s^\beta Z_u^y du \middle| \eta_s = \zeta\right]$. Note that $F(\eta_s, s)$ is only $\mathcal{F}_s$ measurable, but when multiplied by Z_{s-}^x, it becomes $\mathcal{F}_{s-}$ measurable. Therefore, in the last expression above, the $dM_s^{x,x+1}$ can be replaced by ds, and therefore the expression becomes

$$E\int_{0<s<u<\beta} Z_s^x Z_u^y ds du.$$

We want to show that there exist positive constants c'' and c''' such that the above expression is bounded uniformly in N by $c''t^2+c'''t$. For $x \in \{0, N\}$,

$$Z_s^x \leq \mathbf{1}\{\eta_s(z) = 1, 2 \leq z \leq N-1\}$$

By translation invariance and the fact that $\int_0^\beta P\{\eta_t \in \mathbf{Y}_N\}dt = t$, it suffices to show that there exist positive constants $\tilde{c}''$ and $\tilde{c}'''$ such that

$$A_t = \int_0^\beta ds \int d\nu_\rho(\zeta)\mathbf{1}\{\zeta \in \mathbf{Y}_N\} \int_0^{\beta-s} P\{\eta_u^\zeta \in \mathbf{Y}_N\}du \leq \tilde{c}''t^2 + \tilde{c}'''t$$

uniformly in N. By duality we have

$$\begin{aligned}\int d\nu_\rho(\zeta)&\mathbf{1}\{\zeta \in \mathbf{Y}_N\}P\{\eta_s^\zeta \in \mathbf{Y}_N\}\\ &= E\big[\nu_\rho(\zeta : \zeta(x) = 1, \text{ for any } x \in \Lambda_N \cup \phi_s^{-1}(\Lambda_N))\big]\\ &= \rho^{2N} E\left[\rho^{-\#(\Lambda_N \cap \phi_s^{-1}(\Lambda_N))}\right].\end{aligned} \tag{3.7}$$

By comparison of the exclusion process with a system with independent particles (Liggett (1985), Proposition 1.7 of Chapter VIII) we have:

$$E\left[\rho^{-\#(\Lambda_N \cap \phi_u^{-1}(\Lambda_N))}\right] \leq \prod_{l=1}^N E\left[\rho^{-\mathbf{1}\{\phi_u^{-1}(l) \in \Lambda_N\}}\right]. \tag{3.8}$$

We remark that $\phi_u(l)$ is the position at time u of a continuous time simple random walk starting at time 0 from position l. Moreover $\phi_u(l)$ and $\phi_u^{-1}(l)$ have the same law. Note that

$$\begin{aligned}E\left[\rho^{-\mathbf{1}\{\phi_u(l) \in \Lambda_N\}}\right] &= \rho^{-1}P\{\phi_u(l) \in \Lambda_N\} + P\{\phi_u(l) \notin \Lambda_N\}\\ &= 1 + \frac{1-\rho}{\rho}P\{\phi_u(l) \in \Lambda_N\}.\end{aligned} \tag{3.9}$$

Putting all together we obtain

$$A_t \leq \int_0^\beta (\beta - u)\rho^{2N} \prod_{l=1}^N \left[1 + \frac{1-\rho}{\rho}P\{\phi_u(l) \in \Lambda_N\}\right] du.$$

Using the definition of β,

$$A_t \leq t \int_0^\beta \frac{\beta - u}{\beta}\rho^N \prod_{l=1}^N \left[1 + \frac{1-\rho}{\rho}P\{\phi_u(l) \in \Lambda_N\}\right] du. \tag{3.10}$$

We want to find an upper bound for (3.10). The problem is now reduced to estimates on simple symmetric random walk. On one hand,

$$\begin{aligned}\sum_{l=1}^{N} P\{\phi_u(l) \in \Lambda_N\} &= \sum_{l=1}^{N}\sum_{k=1}^{N} P\{\phi_u(l) = k\} = \sum_{l=1}^{N}\sum_{k=1}^{N} P\{\phi_u(0) = k - l\} \\ &= \sum_{l=1}^{N}\sum_{k=1-l}^{N-l} P\{\phi_u(0) = k\} = \sum_{k=-N}^{N} (N - |k|) P\{\phi_u(0) = k\} \\ &= E(N - 1 - |\phi_u(0)|)^{+}. \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}\rho^N \prod_{l=1}^{N}\left[1 + \frac{1-\rho}{\rho} P\{\phi_u(l) \in \Lambda_N\}\right] &\le \rho^N \prod_{l=1}^{N} \exp\left[\frac{1-\rho}{\rho} P\{\phi_u(l) \in \Lambda_N\}\right] \\ &= \rho^N \exp\left[\frac{1-\rho}{\rho} E(N - |\phi_u(0)|)^{+}\right]. \end{aligned} \tag{3.12}$$

On the other hand,

$$\begin{aligned}\rho\left[1 + \frac{1-\rho}{\rho} P\{\phi_u(l) \in \Lambda_N\}\right] &= 1 - (1-\rho) P\{\phi_u(l) \notin \Lambda_N\} \\ &\le \exp\left[-(1-\rho) P\{\phi_u(l) \notin \Lambda_N\}\right]. \end{aligned} \tag{3.13}$$

This implies

$$\begin{aligned}&\rho^N \prod_{l=1}^{N}\left[1 + \frac{1-\rho}{\rho} P\{\phi_u(l) \in \Lambda_N\}\right] \\ &\quad \le \exp\left[-(1-\rho)\sum_{l=1}^{N} P\{\phi_u(l) \notin \Lambda_N\}\right] \\ &\quad = \exp\left[-(1-\rho)\sum_{l=1}^{N} P\{\phi_u(l) > N\} - (1-\rho)\sum_{l=1}^{N} P\{\phi_u(l) < 1\}\right] \\ &\quad = \exp\left[-2(1-\rho)\sum_{l=1}^{N} P\{\phi_u(0) \ge l\}\right] \\ &\quad = \exp\left[-2(1-\rho) E \min\{(\phi_u(0))^{+}, N\}\right]. \end{aligned} \tag{3.14}$$

Using (3.12) and (3.14) in (3.10) we get that A_t is bounded above by

$$t\int_0^\beta \min\left\{\rho^N \exp\left[\frac{1-\rho}{\rho}E(N-|\phi_u(0)|)^+\right],\right.$$
$$\left.\exp\left[-2(1-\rho)E\min\{(\phi_u(0))^+,N\}\right]\right\}du. \tag{3.15}$$

Standard random walk estimates give

$$\begin{aligned} &E(\min\{(\phi_u(0))^+,N\}) \geq \min\{c\sqrt{u},N\},\\ &E(N-|\phi_u(0)|)^+ \leq c'(N^2/\sqrt{u}) \end{aligned} \tag{3.16}$$

for some constants c and c' that do not depend on N. Therefore using (3.16), we get

$$A_t \leq t\int_0^\beta \min\left\{\rho^N \exp\left[\frac{1-\rho}{\rho}c'(N^2/\sqrt{u})\right],\right.$$
$$\left.\exp\left[-2(1-\rho)\min\{c\sqrt{u},N\}\right]\right\}du. \tag{3.17}$$

Divide the integral in three parts and for

$$\begin{aligned} 0 \leq u \leq N^2/c^2 &\quad \text{use the bound} \quad \exp\left[-2(1-\rho)c\sqrt{u}\right]\\ N^2/c^2 < u \leq N^4 &\quad \text{use the bound} \quad \exp[-2(1-\rho)N]\\ N^4 < u \leq \beta &\quad \text{use the bound} \quad \rho^N \exp\left[\frac{1-\rho}{\rho}c'(N^2/\sqrt{u})\right] \end{aligned}$$

to show that

$$\begin{aligned} A_t \leq t\Bigg(&\int_0^{N^2/c^2} \exp\left[-2(1-\rho)c\sqrt{u}\right]du\\ &+\int_{N^2/c^2}^{N^4} \exp[-2(1-\rho)N]du\\ &+\int_{N^4}^{\beta} \rho^N \exp\left[\frac{1-\rho}{\rho}c'(N^2/\sqrt{u})\right]du\Bigg)\\ \leq t\Bigg(&\int_0^\infty \exp\left[-2(1-\rho)c\sqrt{u}\right]du\\ &+N^4\exp[-2(1-\rho)N] + t\exp\left[\frac{1-\rho}{\rho}c'\right]\Bigg) \end{aligned}$$

Since $N^4\exp[-2(1-\rho)N] <$ constant, A_t is bounded above by $t^2\tilde{c}''+t\tilde{c}'''$ uniformly in N, where $\tilde{c}''$ and $\tilde{c}'''$ are constants depending only on c and c'.

This implies that the second term in (3.6) is bounded above by $t^2c'' + tc'''$ for some constants c'' and c''' depending only on c and c'. Taking $r = t\rho^{-N}$ in Lemma 1, we get $EX = \rho^N + 2t$. Hence, $EX^2 \leq EX + t^2c'' + tc'''$ and

$$P\{X \geq 1\} \geq \frac{(EX)^2}{EX^2} \geq \frac{(\rho^N + 2t)^2}{\rho^N + 2t + c''t^2 + c'''t}.$$

Hence, for suitable positive constants C, C' and C'' independent of N,

$$P\{\rho^N T_N \geq t\} = P\{X = 0\} \leq 1 - \frac{t^2}{Ct^2 + C't + C''\rho^N}.$$

This shows the proposition □

4. THE INDEPENDENCE PROPERTY

PROPOSITION 3. – *The following upper bound holds*

$$\sup_{t>0} \left|P\{T_N \geq 2\rho^{-N}(t+s)\} - P\{T_N \geq 2\rho^{-N}t\}P\{T_N \geq 2\rho^{-N}s\}\right| \leq (1+s)C_1\rho^{C_2N}$$

where C_1 and C_2 are positive constants independent of N.

Proof. – Call $\gamma_N = 2\rho^{-N}$. We want to show that

$$\begin{aligned}|P\{X[0, \gamma_N(t+s)] = 0\}\\ -P\{X[0, \gamma_N t] = 0\}P\{X[0, \gamma_N s] = 0\}| \leq C_1\rho^{C_2N}(1+s),\end{aligned} \tag{4.1}$$

for all positive real numbers s, t. In the above formula, as usual, we have not mentioned that the initial configuration was chosen according to ν_ρ.

Take $\Delta_N < \gamma_N s$. By Lemma 1,

$$|P\{X[0, \gamma_N s] = 0\} - P\{X[\Delta_N, \gamma_N s] = 0\}| \leq (1 + 2\Delta_N)\rho^N$$

and, since the process is in equilibrium,

$$\begin{aligned}\sup_{t>0} |P\{X[0, \gamma_N(t+s)] = 0\}\\ -P\{X[0, \gamma_N t] + X[\gamma_N t + \Delta_N, \gamma_N(t+s)] = 0\}|\\ \leq (1 + 2\Delta_N)\rho^N.\end{aligned} \tag{4.2}$$

Therefore the left hand side of (4.1) is bounded above by

$$2(1+2\Delta_N)\rho^N + |P\{X[0,\gamma_N t] + X[\gamma_N t + \Delta_N, \gamma_N(t+s)] = 0\} \\ - P\{X[0,\gamma_N t] = 0\}P\{X[\Delta_N, \gamma_N s] = 0\}|. \qquad (4.3)$$

With a convenient choice of Δ_N, to prove the Proposition we need to obtain an upper bound to

$$|P\{X[0,\gamma_N t] + X[\gamma_N t + \Delta_N, \gamma_N(t+s)] = 0\} \\ -P\{X[0,\gamma_N t] = 0\}P\{X[\Delta_N, \gamma_N s] = 0\}|. \qquad (4.4)$$

Since the process is reversible with respect to ν_ρ we have

$$P\{X[0,\gamma_N t] + X[\gamma_N t + \Delta_N, \gamma_N(t+s)] = 0\} \\ = \int \nu_\rho(d\zeta) P\{X^\zeta[0,\gamma_N t] = 0\} P\{X^\zeta[\Delta_N, \gamma_N s] = 0\}. \qquad (4.5)$$

Therefore (4.4) is bounded above by

$$\int \nu_\rho(d\zeta) \int \nu_\rho(d\xi) P\{X^\xi[0,\gamma_N t] = 0\} \\ \times \left|P\{X^\xi[\Delta_N, \gamma_N s] = 0\} - P\{X^\zeta[\Delta_N, \gamma_N s] = 0\}\right|. \qquad (4.6)$$

Since the processes (η_u^ξ) and (η_u^ζ) are defined in the same probability space

$$P\{X^\xi[\Delta_N, \gamma_N s] = 0\} - P\{X^\zeta[\Delta_N, \gamma_N s] = 0\} \\ = E[\mathbf{1}\{X^\xi[\Delta_N, \gamma_N s] = 0\} - \mathbf{1}\{X^\zeta[\Delta_N, \gamma_N s] = 0\}].$$

This shows that (4.6) is bounded above by

$$2\int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) \sum_{k=1}^{\infty} \sum_{a\in\Lambda_N} \\ \times P\{\Delta_N \le \tau_k \le \gamma_N s, \eta_{\tau_k}^\xi \in \mathbf{Y}_N, \eta_{\tau_k}^\zeta(a) = 0\} + 2N\rho^{N-1}(1-\rho). \quad (4.7)$$

A martingale argument similar to the one performed in Lemma 1 and the reversibility of the process enable us to rewrite this last expression as

$$4\int_{\Delta_N}^{\gamma_N s}\int \nu_\rho(d\xi)\int \nu_\rho(d\zeta)\sum_{a\in\Lambda_N} \\ \times P\{\eta_u^\xi(x)=1,\forall x\in\Lambda_N,\eta_u^\zeta(a)=0\}du+2N\rho^{N-1}(1-\rho),$$

Let $w_N \in [0,\Delta_N]$. Its value will be fixed later. Using duality and the Markov property of the dual process,

$$P\{\eta_u^\xi(x)=1,x\in\Lambda_N,\eta_u^\zeta(a)=0\} \\ \le \sum_{F,b} P\{\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\})=F,\phi_{w_N,u}^{-1}(a)=b\} \\ \times P\{\eta_{w_N}^\xi(F)=1,\eta_{w_N}^\zeta(b)=0\}.$$

where the sum is for all subsets F of $\mathbb{Z}$ with $N-1$ elements and sites $b\in\mathbb{Z}$ such that $b\notin F$. The notation $\{\eta_{w_N}^\xi(F)=1\}$ is a shorthand for $\{\eta_{w_N}^\xi(x)=1,\forall x\in F\}$ and $\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\})$ stands for $\{\phi_{w_N,u}^{-1}(x):x\in\Lambda_N\setminus\{a\}\}$. We decompose this sum in two parts and then get upper bounds for each one of them. We first consider the pairs (F,b) for which $d(F,b)$, the minimal distance between any site $x\in F$ and b is smaller than $2l$. Using the fact that the ν_ρ is invariant with respect to the process we obtain

$$\sum_{d(F,b)\le 2l} P\{\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\})=F,\phi_{w_N,u}^{-1}(a)=b\} \\ \times\int \nu_\rho(d\xi)\int \nu_\rho(d\zeta)P\{\eta_{w_N}^\xi(F)=1,\eta_{w_N}^\zeta(b)=0\} \\ \le \sum_{d(F,b)\le 2l} P\{\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\})=F,\phi_{w_N,u}^{-1}(a)=b\}\nu_\rho\{\xi(F)=1\} \\ = P\{d(\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\}),\phi_{w_N,u}^{-1}(a))\le 2l\}\rho^{N-1}. \quad (4.8)$$

Therefore, for any value of a, the integral for u in the interval $[\Delta_N,\gamma_N s]$ of the expression appearing in (4.8) is bounded above by

$$\rho^{N-1}\int_{\Delta_N}^{\gamma_N s} P\{d(\phi_{w_N,u}^{-1}(\Lambda_N\setminus\{a\}),\phi_{w_N,u}^{-1}(a))\le 2l\}du \\ \le \rho^{N-1}\int_{\Delta_N}^{\gamma_N s}\sum_{x\in\Lambda_N\setminus\{a\}} P\{d(\phi_{w_N,u}^{-1}(x),\phi_{w_N,u}^{-1}(a))\le 2l\}du \quad (4.9)$$

Using the fact that the worst case is when x and a are nearest neighbors and translation invariance of the difference of two random walks, we can substitute x and a in each term of the right hand side of (4.9) by 0 and 1 and get that (4.9) is bounded above by

$$\begin{aligned} &2s\rho(N-1)P\{d(\phi_{w_N,\Delta_N}^{-1}(0),\phi_{w_N,\Delta_N}^{-1}(1)) \le 2l\} \\ &\quad \le s(N-1)C\frac{2l}{(\Delta_N - w_N)^{1/2}} \end{aligned} \tag{4.10}$$

for some positive constant C which does not depend on N. This is a standard upper bound for symmetric random walks. Inequality (4.10) is the first upper bound we need.

To obtain the next two upper bounds we need a new construction of process in the interval of time $[0, w_N]$. Let $\bar{\mathcal{P}}$ be an independent copy of the family of Poisson processes $\mathcal{P} = \left\{(\tau_k^{\{x,x+1\}})_{k\ge 0}, x \in \mathbb{Z}\right\}$. We use the original family $\mathcal{P}$ to construct the process $(\eta_u^\xi)_u$ starting with the configuration ξ. On the other hand, during the interval of time $[0, w_N]$, each particle which at time 0 belongs to the configuration ζ will evolve according to the independent Poisson family $\bar{\mathcal{P}}$, as long as its path does not cross the path of a particle belonging to the process $(\eta_u^\xi)_u$. If this happens both particles coalesce and from now on evolve together using the original family $\mathcal{P}$. In order to maintain the stirring property of the process $(\eta_u^\zeta)_{u\in[0,w_N]}$, each time a particle of the process starting with configuration ζ is in a neighborhood of a coalesced particle it follows only the original process $\mathcal{P}$ to jump across the corresponding bond. In other words, particles of both processes follow the original process $\mathcal{P}$ in bonds that have in one of the extremes a coalesced particle. After time w_N both processes use the original Poisson family $\mathcal{P}$ to evolve together according to Harris construction. This construction of the processes in the interval of time $[0, w_N]$ is equivalent to the *Basic Coupling* of Liggett (1985), page 382.

It is important to stress that the Harris construction keeps constant the density of discrepancies:

$$E_{\text{Harris}}\mathbf{1}\{\eta_u^\xi(0) \ne \eta_u^\zeta(0)\} = 2\rho(1-\rho)$$

where E_{Harris} is the expectation of the coupled process when only the process $\mathcal{P}$ is used. On the other hand under the Basic Coupling the density of discrepancies is non increasing in time:

$$\frac{d}{du}E_{\text{Basic}}\mathbf{1}\{\eta_u^\xi(0) \ne \eta_u^\zeta(0)\} \le 0$$

See Liggett (1985), Chapter VIII, Lemma 3.2. Belitsky (1993) gave an upper bound for the decay of density of discrepancies under the basic coupling:

THEOREM (V. Belitsky). – *For any* $\epsilon > 0$ *there exists* $u(\epsilon) < \infty$ *such that for* $u > u(\epsilon)$ *the following holds*

$$\int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P_{\text{Basic}}\{\eta_u^\xi(0) \neq \eta_u^\zeta(0)\} \leq u^{-1/4+\epsilon}.$$

Belitsky (1993) also gives an upper bound for any dimension $d \geq 1$. A sharper version of the result including a lower bound is presented in Belitsky (1994).

Now let us get back to the point in which we arrived after (4.10). We were about to consider the sum over the pairs (F, b) such that $d(F, b) > 2l$

$$\begin{aligned} &\sum_{d(F,b)>2l} P\{\phi_{w_N,u}^{-1}(\Lambda_N \backslash \{a\}) = F, \phi_{w_N,u}^{-1}(a) = b\} \\ &\qquad \times \int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P\{\eta_{w_N}^\xi(F) = 1, \eta_{w_N}^\xi(b) = 1, \eta_{w_N}^\zeta(b) = 0\} \\ &\leq \sum_{d(F,b)>2l} P\{\phi_{w_N,u}^{-1}(\Lambda_N \backslash \{a\}) = F, \phi_{w_N,u}^{-1}(a) = b, (A_{F\cup\{b\}})^c\} \\ &\qquad \times \int \nu_\rho(d\xi) P\{\eta_{w_N}^\xi(F) = 1\} \\ &\quad + \sum_{d(F,b)>2l} P\{\phi_{w_N,u}^{-1}(\Lambda_N \backslash \{a\}) = F, \phi_{w_N,u}^{-1}(a) = b, A_{F\cup\{b\}}\} \\ &\qquad \times \int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) \\ &\qquad\qquad \times P\{\eta_{w_N}^\xi(F) = 1, \eta_{w_N}^\xi(b) = 1, \eta_{w_N}^\zeta(b) = 0\}, \end{aligned} \tag{4.11}$$

where

$$A_x = \{L_{w_N}^x > x - l, R_{w_N}^x < x + l\}, \quad A_B = \bigcap_{x \in B} A_x,$$

and for any site x and time $w > 0$ the position R_w^x is the rightmost site a particle starting at x could arrive by time w using the marks of both $\mathcal{P}$ and $\bar{\mathcal{P}}$ only to jump to the right. Analogously L_w^x is defined with jumps only in the left direction. Both $-L_\cdot^x$ and $R_\cdot^x$ are Poisson processes of rate 2. The formal definition of these positions is

$$L_w^x = z \ \ (\text{respectively } R_w^x = z)$$

if

(1) there exist $x_0,\dots,x_n$ and $t_0,\dots,t_n$ such that $x_0 = x$, $x_n = z$ and $x_{i+1} = x_i - 1$, for all $i = 0,\cdots,n-1$,
(respectively

(2) there exist $x_0,\dots,x_n$ and $t_0,\dots,t_n$ such that $x_0 = x$, $x_n = z$, $x_{i+1} = x_i + 1$, for all $i = 0,\dots,n-1$)

(3) $t_n = 0 < t_{n-1} < \cdots < t_0 = w$,

(4) for any $i = 1,\dots,n-1$, $t_i = \tilde{\tau}_k^{\{x_{i+1},x_i\}}$ for some k, where the $\tilde{\tau}_k^{\{x_{i+1},x_i\}}$ is the superposition of the Poisson processes indexed by $\{x_{i+1},x_i\}$ in the families $\mathcal{P}$ and $\bar{\mathcal{P}}$
and

(5) there are no occurrences of the superposed Poisson processes indexed by $\{x_{i+1},x_i\}$ in the time interval (t_{i+1},t_i).

Using duality and the invariance of ν_ρ we rewrite the probability appearing in the first sum of the right hand side of (4.11)

$$\sum_{d(F,b)>2l} P\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\})$$
$$= F, \phi^{-1}_{w_N,u}(a) = b, (A_{F\cup\{b\}})^c\} \int \nu_\rho(d\xi) P\{\eta^\xi_{w_N}(F) = 1\}$$
$$= \sum_{d(F,b)>2l} \sum_{G,c} P\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\}) = F, \phi^{-1}_{w_N,u}(a) = b,$$
$$\phi^{-1}_{0,w_N}(F) = G, \phi^{-1}_{0,w_N}(b) = c, (A_{F\cup\{b\}})^c\}\rho^{N-1}$$
$$= \sum_{d(F,b)>2l} P\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\}) = F, \phi^{-1}_{w_N,u}(a) = b, (A_{F\cup\{b\}})^c\}\rho^{N-1}$$

The events $\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\}) = F, \phi^{-1}_{w_N,u}(a) = b\}$ and $A_{F\cup\{b\}}$ depend on disjoint parts of the space time and by the independence property of the Poisson process they are independent. This allows us to rewrite

$$P\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\}) = F, \phi^{-1}_{w_N,u}(a) = b, (A_{F\cup\{b\}})^c\}$$
$$= P\{\phi^{-1}_{w_N,u}(\Lambda_N \setminus \{a\}) = F, \phi^{-1}_{w_N,u}(a) = b\} P\big((A_{F\cup\{b\}})^c\big).$$

Now we remark that for any pair (F,b) appearing in the sum

$$P\big((A_{F\cup\{b\}})^c\big) \le NP\big(\{R^0_{w_N} \ge l\} \cup \{L^{2l}_{w_N} \le l\}\big).$$

Therefore for $w_N \le l/3$, the integral for u in the interval $[\Delta_N, \gamma_N s]$ of the first term of the right hand side of (4.11) is bounded above by

$$2\rho s N P\{R^0_{w_N} \ge l\} \le 2\rho s N \exp(-(3\log 3 - 1)w_N). \qquad (4.12)$$

This follows from the exponential Chebichev inequality applied to the Poisson distribution of mean $2w_N$. Inequality (4.12) is the second upper bound we need.

Finally let us consider the remaining terms of the right hand side of (4.11). If $d(F,b) > 2l$, in the set $A_{F\cup\{b\}}$ the independence properties of both the Poisson families $\mathcal{P}$, $\bar{\mathcal{P}}$ and ν_ρ allow us to rewrite

$$\begin{aligned}\int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P\{\eta^\xi_{w_N}(F) = 1, \eta^\xi_{w_N}(b) = 1, \eta^\zeta_{w_N}(b) = 0, A_{F\cup\{b\}}\} \\ = \int \nu_\rho(d\xi) P\{\eta^\xi_{w_N}(F) = 1, A_{F\cup\{b\}}\} \\ \times \int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P\{\eta^\xi_{w_N}(b) = 1, \eta^\zeta_{w_N}(b) = 0, A_{F\cup\{b\}}\}. \\ \le \int \nu_\rho(d\xi) P\{\eta^\xi_{w_N}(F) = 1\} \\ \times \int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P\{\eta^\xi_{w_N}(b) = 1, \eta^\zeta_{w_N}(b) = 0\}.\end{aligned}$$

Since ν_ρ is invariant,

$$\int \nu_\rho(d\xi) P\{\eta^\xi_{w_N}(F) = 1\} = \rho^{N-1}.$$

The remaining factor is bounded above by Belitsky's Theorem

$$\int \nu_\rho(d\xi) \int \nu_\rho(d\zeta) P\{\eta^\xi_{w_N}(b) = 1, \eta^\zeta_{w_N}(b) = 0\} \le {w_N}^{-1/4+\epsilon}.$$

Therefore the integral for u in the interval $[\Delta_N, \gamma_N s]$ of this last term is bounded above by

$$\rho s N {w_N}^{-1/4+\epsilon} \tag{4.13}$$

which is our last upper bound.

Collecting upper bounds (4.3), (4.7), (4.10), (4.12) and (4.13) we get that the left hand side of (4.1) is bounded above by

$$\begin{aligned}2(1+2\Delta_N)\rho^N + 2N\rho^{N-1}(1-\rho) + 4s(N-1)C\frac{2l}{(\Delta_N - w_N)^{1/2}} \\ + 8\rho s N \exp(-(3\log 3 - 1)w_N) + 4\rho s N {w_N}^{-1/4+\epsilon}.\end{aligned}$$

To conclude the proof of the Proposition we fix

$$\begin{aligned}\Delta_N &= (\gamma_N s)^{1/2} \\ l &= \Delta_N^{1/4} \\ w_N &= l/3. \ \square\end{aligned}$$

5. PROOF OF THE THEOREM

We start proving a FKG inequality.

LEMMA 5.1.

$$P(T_N > t + s) \geq P(T_N > t)P(T_N > s)$$

Proof. – As in (4.5), by reversibility,

$$P(T_N > t + s) = \int \nu_\rho(d\eta)P(T_N^\eta \geq t)P(T_N^\eta \geq s).$$

Since the functions

$$f(\eta) = P(T_N^\eta \geq t), \quad g(\eta) = P(T_N^\eta \geq s)$$

are non increasing the result follows from the FKG inequality for the Bernoulli measure (Liggett (1985) Corollary 2.12 of Chapter II). □

Define $r = r(N) = \rho^{C_2 N/2}$, where C_2 is the constant defined in Proposition 3. Let $\theta = \theta(N)$ be the solution of

$$P\{\rho^N T_N > r\} = e^{-\theta} \tag{5.1}$$

By Lemma 1 and Proposition 2,

$$1 - \rho^N - 2r \leq e^{-\theta} \leq 1 - \frac{r^2}{Cr^2 + C'r + C''\rho^N}$$

for all $r \geq 0$. Therefore, for N sufficiently large so that $1 - e^{-\theta} \geq \theta/2$,

$$\left(Cr + C' + C''\frac{\rho^N}{r}\right)^{-1} \leq \frac{\theta}{r} \leq 2\frac{\rho^N}{r} + 4.$$

Let $\alpha = \alpha_N = \theta/r$. The constant C_2 of Proposition 3 can be taken to be smaller than one. With this choice $\rho^N/r \leq 1$ and

$$(C + C' + C'')^{-1} \leq \alpha_N \leq 6 \tag{5.2}$$

Fix $t > 0$ and write $t = kr + v$ where $k = k(r) \geq 0$ is the integer part of t/r and $0 \leq v = v(r) < r$. Now,

$$\begin{aligned} &\left|P\{\rho^N T_N > t\} - e^{-\alpha t}\right| \\ &\quad \leq \left|P\{\rho^N T_N > t\} - e^{-\theta k}\right| + \left|e^{-\theta k} - e^{-\alpha t}\right|. \end{aligned} \tag{5.3}$$

To conclude the proof it suffices to show that both terms in the right hand side of (5.3) are exponentially bounded above.

Applying inductively Proposition 3 and (5.1),

$$\left|P\{\rho^N T_N > kr\} - e^{-\theta k}\right| \leq (1+r) C_1 \rho^{C_2 N}\left[1 + e^{-\theta} + \ldots + e^{-\theta(k-2)}\right].$$

Therefore, for r sufficiently small such that $\theta < 1$,

$$\sup_{k\geq 1}\left|P\{\rho^N T_N > kr\} - e^{-\theta k}\right| \leq 2C_1 \rho^{C_2 N}\frac{1}{1-e^{-\theta}}. \tag{5.4}$$

By monotonicity,

$$P\{\rho^N T_N > kr\} \geq P\{\rho^N T_N > t\}.$$

Using Lemma 5.1,

$$\begin{aligned} P\{\rho^N T_N > t\} &\geq P\{\rho^N T_N > kr\} P\{\rho^N T_N > v\} \\ &\geq P\{\rho^N T_N > kr\} e^{-\theta} \end{aligned} \tag{5.5}$$

by monotonicity and identity (5.1). Using (5.4) and (5.5) for the second inequality:

$$\begin{aligned} 2C_1\rho^{C_2 N}\frac{1}{1-e^{-\theta}} &\geq P\{\rho^N T_N > t\} - e^{-\theta k} \\ &\geq -\left(1 - e^{-\theta}\right) e^{-\theta k} - 2C_1 \rho^{C_2 N}\frac{1}{1-e^{-\theta}} e^{-\theta}. \end{aligned}$$

Therefore,

$$\left|P\{\rho^N T_N > t\} - e^{-\theta k}\right| \leq 1 - e^{-\theta} + 2C_1\rho^{C_2 N}\frac{1}{1-e^{-\theta}}. \tag{5.6}$$

Since for $\theta < 1$, $\theta/2 \leq 1 - e^{-\theta} \leq \theta$, we get the following upper bound for the first term in the right hand side of (5.3): for N sufficiently large,

$$\left|P\{\rho^N T_N > t\} - e^{-\theta k}\right| \leq \theta + \frac{4C_1\rho^{C_2 N}}{\theta}. \tag{5.7}$$

To bound the second term in the right hand side of (5.3) write

$$\begin{aligned} \left|e^{-\theta k} - e^{-\alpha t}\right| &\leq \left|1 - e^{\alpha t - \theta k}\right| \\ &\leq e\left(\frac{\theta}{r}(kr + v) - \theta k\right) = ev\frac{\theta}{r} \leq e\theta, \end{aligned} \tag{5.8}$$

for N big enough (such that $\alpha t - \theta k < 1$). Since $r = \rho^{C_2 N/2}$ and $\alpha = \theta/r$, (5.2) guarantees that both (5.7) and (5.8) are bounded above by $A\rho^{A' N}$ where $A' = C_2/2$ and A is some positive constant. □

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