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Convolutional attractors of stationary sequences of random measures on compact groups

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ABSTRACT. — We consider a stationary ergodic sequence $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$, of random probability measures on a compact group G and study the asymptotic behaviour of their convolutions

$$v_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \dots * \mu_m(\omega)$$

in the weak topology as $n \rightarrow \infty$.

Let $\mathcal{A}_m(\omega)$ be the set of all limit points of $v_m^{(n)}(\omega)$ as $n \rightarrow \infty$, $A_m(\omega) = \left(\bigcup_{n=1}^{\infty} \text{supp } v_m^{(n)}(\omega) \right)^-$ and $\lambda_m(\omega) = \lim_{n \rightarrow \infty} \tilde{v}_m^{(n)}(\omega) * v_m^{(n)}(\omega)$. There exists a compact \mathcal{A}_∞ such that a. s.

$$\mathcal{A}_\infty = A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \overline{\lim}_{m \rightarrow \infty} \mathcal{A}_m(\omega) = \left(\bigcup_{m=1}^{\infty} \mathcal{A}_m(\omega) \right)^-$$

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We call this set \mathcal{A}_∞ the convolutional attractor of $\{\mu_m\}$, since also $\mathcal{A}_\infty = (\dot{v}_m^{(n)}(\omega), m \in \mathbb{N})^-$ a.s. where the sequence $\dot{v}_m^{(n)} = v_m^{(n)}(\omega) * \lambda_m(\omega)$ is asymptotically equivalent to $v_m^{(n)}(\omega)$ as $n \rightarrow \infty$ a.s. Describing properties of \mathcal{A}_∞ we in particular find conditions under which $\lambda_m(\omega)$, $A_m(\omega)$ and $\mathcal{A}_m(\omega)$ do not depend essentially on ω and \mathcal{A}_∞ forms a group of measures as in the well known case of convolution powers $\mu^{(n)}$ of a single measure μ .

Key words : Random measures, convergence of convolutions, compact groups.

RÉSUMÉ. — Nous considérons une suite stationnaire et ergodique $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$, de mesures de probabilités sur un groupe compact G et étudions le comportement asymptotique des produits de convolution $v_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \dots * \mu_m(\omega)$ dans la topologie faible lorsque $n \rightarrow \infty$.

Soit $\mathcal{A}_m(\omega)$ l'ensemble de tous les points d'adhérence de $v_m^{(n)}(\omega)$ lorsque $n \rightarrow \infty$, $A_m(\omega) = \left(\bigcup_{n=1}^{\infty} \text{supp } v_m^{(n)}(\omega) \right)^-$ et $\lambda_m(\omega) = \lim_{n \rightarrow \infty} \tilde{v}_m^{(n)}(\omega) * v_m^{(n)}(\omega)$.

Il existe un ensemble compact \mathcal{A}_∞ tel que, p. p.,

$$\mathcal{A}_\infty = A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \overline{\lim}_{m \rightarrow \infty} \mathcal{A}_m(\omega) = \left(\bigcup_{m=1}^{\infty} \mathcal{A}_m(\omega) \right)^-$$

Nous appelons l'attracteur convolutionnel de la suite $\{\mu_m\}$, puisque

$$\mathcal{A}_\infty = (\dot{v}_m^{(n)}(\omega), n, m \in \mathbb{N})^- \text{ p. p.}$$

où la suite $\dot{v}_m^{(n)} = v_m^{(n)}(\omega) * \lambda_m(\omega)$ est p. p. asymptotiquement équivalente à la suite $v_m^{(n)}(\omega)$ lorsque $n \rightarrow \infty$ p. p.

En décrivant les propriétés de \mathcal{A}_∞ nous trouvons en particulier des conditions pour que $\lambda_m(\omega)$, $A_m(\omega)$ et $\mathcal{A}_m(\omega)$ ne dépendent pas essentiellement de ω , et pour que \mathcal{A}_∞ forme un groupe de mesures comme dans le cas bien connu des puissances de convolution $\mu^{(n)}$ d'une mesure unique μ est p. p.

1. INTRODUCTION

Let G be a compact Hausdorff group and $\mathcal{M}^1(G)$ be the convolution semigroup of Borel probability measures on G with the weak topology.

We consider a stationary random process $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$, defined on the probability space (Ω, \mathcal{F}, P) with values in $\mathcal{M}^1(G)$ and study the limit behaviour of the random measures.

$$v_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) * \dots * \mu_m(\omega), \quad m, n \in \mathbb{N}$$

for the typical realizations of the process $\mu_n(\omega)$ as $n \rightarrow \infty$.

The convergence of convolutions of probability measures on a compact group has been examined by many authors (e. g. *see* [1], [4], [6], [7], [8], [10], [11], [14]-[16] and references cited there).

Precisely, the asymptotic behaviour of the sequence of the convolution powers $v^{(n)} = \mu \star \dots \star \mu$ (n -times), $n \in \mathbb{N}$ for a fixed $\mu \in \mathcal{M}_1(G)$ is described as follows (*see* [4], ch. II).

THEOREM 1.0. — *a) The set $\mathcal{A} = \text{LmP}_{n \rightarrow \infty} v^{(n)}$ of all limit points of the sequence $\{v^{(n)}\}_{n=1}^\infty$ has the form*

$$\mathcal{A} = \lambda H = \{ \lambda x, x \in H \}$$

where $\lambda = \lambda_K$ is the normalized Haar measure of the subgroup

$$K = \left[\bigcup_{n=1}^\infty S(\tilde{v}^{(n)} \star v^{(n)}) \right]^-$$

K is a normal subgroup of

$$H = [S(\mu)]^- = \left[\bigcup_{n=1}^\infty S(v^{(n)}) \right]^- = \overline{\lim}_{n \rightarrow \infty} S(v^n)$$

and furthermore

$$\lambda = \lim_{n \rightarrow \infty} \tilde{v}^{(n)} \star v^{(n)} = \lim_{n \rightarrow \infty} v^{(n)} \star \tilde{v}^{(n)}$$

b) The sequence $v^{(n)}$ is asymptotically equivalent to the sequence $\dot{v}^{(n)} = \dot{\mu} \star \dots \star \dot{\mu}$ of the convolution powers of the measure $\dot{\mu} = \lambda \star \mu$, i. e.

$$\lim_{n \rightarrow \infty} (v^{(n)} - \dot{v}^{(n)}) = 0$$

and

$$\mathcal{A} = \text{LmP}_{n \rightarrow \infty} \dot{v}^{(n)} = (\dot{v}^{(n)}, n \in \mathbb{N})^-$$

Here and elsewhere $[A]$ denotes the group generated by the set A and A^- is its closure. $S(\mu)$ denotes the support of the measure μ and we use the notation μx and $x\mu$ instead $\mu \star \delta_x$ and $\delta_x \star \mu$ where δ_x is a Dirac measure in a point. The measure $\dot{\mu}$ is the image of μ by the involution $x \rightarrow x^{-1}$, $x \in G$. The definition of $\overline{\lim}$ and $\underline{\lim}$ see in [4], ch. 2, or in [9], § 29, and $\text{LmP}_{n \rightarrow \infty}$ means the set of all limit (accumulation) points of the corresponding sequence as $n \rightarrow \infty$.

It's natural to call the set \mathcal{A} in the above theorem 1.0 the *convolutional attractor* (CA) of the measure μ .

The main purpose of the paper is to construct the analogous (as it is possible) convolutional attractor for a stationary sequence of random measures (SSRM) $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$. To this end we shall investigate the limit points of the corresponding convolutions $v_m^{(n)}(\omega)$ as $n \rightarrow \infty$.

For a given SSRM $\{\mu_n\}_{n=1}^\infty$ on G we introduce the following notation.

Denote by $\mathcal{A}^{(n)}$ the essential image of the random element $v_m^{(n)}$, *i. e.* the support of its distribution $P^\circ(v_m^{(n)})^{-1}$ on $\mathcal{M}_1(G)$. Put also

$$\mathcal{A}^{(\infty)} = \overline{\lim}_{n \rightarrow \infty} \mathcal{A}^{(n)}, \quad \mathcal{B}^{(\infty)} = \left(\bigcup_{n=1}^{\infty} \mathcal{A}^{(n)} \right)^-,$$

$$H = [S(v), v \in \mathcal{B}^{(\infty)}]^{-}, \quad K = [S(\tilde{v} \star v), v \in \mathcal{B}^{(\infty)}]^{-}$$

We shall assume everywhere in the course of the paper that the following conditions hold.

A) The SSRM $\{\mu_n(\omega)\}_{n=1}^{\infty}$ is ergodic, *i. e.* every stationary event has the probability 0 or 1.

B) The compact set $\mathcal{B}^{(\infty)}$ (and therefore $\mathcal{A}^{(n)}$ for all n) has a countable base of its topology.

The condition B) is equivalent to the metrizability of the compact set $\mathcal{B}^{(\infty)}$ (*see* [9], § 41. II). But we do not assume any conditions of separability or metrizability on G .

The main results of the paper are the theorems 1.1-1.4 stated below

THEOREM 1.1. — *For all m and a.a. ω the following statements hold.*

a) *The set $\mathcal{A}_m(\omega) = \text{Lm P}_{n \rightarrow \infty} v_m^{(n)}(\omega)$ of all limit points of the sequence $v_m^{(n)}(\omega)$ as $n \rightarrow \infty$ has the form*

$$\mathcal{A}_m(\omega) = A_m(\omega) \lambda_m(\omega)$$

where

$$A_m(\omega) = \overline{\lim}_{n \rightarrow \infty} S(v_m^{(n)}(\omega)) = \left(\bigcup_{n=1}^{\infty} S(v_m^{(n)}(\omega)) \right)^-$$

and

$$\lambda_m(\omega) = \lim_{n \rightarrow \infty} \tilde{v}_m^{(n)}(\omega) \star v_m^{(n)}(\omega)$$

are the Haar measures of the subgroups

$$K_m(\omega) = \left[\bigcup_{n=1}^{\infty} S(\tilde{v}_m^{(n)}(\omega) \star v_m^{(n)}(\omega)) \right]^{-}$$

and

$$K = [K_m(\omega), m \in \mathbb{N}]^{-}$$

Herewith the subgroups $K_m(\omega)$ are conjugated in H and

$$\text{Lm P}_{n \rightarrow \infty} v_m^{(n)}(\omega) \star \tilde{v}_m^{(n)}(\omega) = (\lambda_m(\omega), m \in \mathbb{N})^{-}$$

b) *The equality*

$$\dot{\mu}_n(\omega) = \mu_n(\omega) \star \lambda_n(\omega)$$

defines a SSRM such that the sequence of corresponding convolutions

$$\dot{v}_m^{(n)}(\omega) = \dot{\mu}_{m+n-1}(\omega) \star \dots \star \dot{\mu}_m(\omega)$$

is asymptotically equivalent to $v_m^{(n)}(\omega)$ as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} (v_m^{(n)}(\omega) - \dot{v}_m^{(n)}(\omega)) = 0$$

for all m and a.a. ω .

c) There exists a compact subset \mathcal{A}_∞ of $\mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)}$ such that

$$\begin{aligned} \mathcal{A}_\infty &= A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \overline{\lim}_{m \rightarrow \infty} \mathcal{A}_m(\omega) \\ &= \left(\bigcup_{m=1}^{\infty} \mathcal{A}_m(\omega) \right)^- = (\dot{v}_m^{(n)}(\omega), n, m \in \mathbb{N})^- \end{aligned}$$

for a.a. ω .

We shall call the above set \mathcal{A}_∞ the *convolutional attractor* of the SSRM

$$\{\mu_n\}_{n=1}^{\infty}$$

The asymptotic behavior of the convolutions $v_m^{(n)}$ as $n \rightarrow \infty$ is completely defined by the convolutions $\dot{v}_m^{(n)}(\omega)$ of the limiting SSRM $\{\dot{\mu}_n\}_{n=1}^{\infty}$. The correspondence

$$\{\mu_n\}_{n=1}^{\infty} \rightarrow \{\dot{\mu}_n\}_{n=1}^{\infty}$$

is retractive i. e. the limiting SSRM of $\{\dot{\mu}_n\}_{n=1}^{\infty}$ is $\{\{\dot{\mu}_n\}\}$ itself

It should be mentioned that the sets $K_m(\omega)$, $A_m(\omega)$ and $\mathcal{A}_m(\omega)$ (unlike K , H , \mathcal{A}_∞ , $\mathcal{A}^{(\infty)}$ and $\mathcal{B}^{(\infty)}$) can essentially depend on ω and m . The main new phenomenon arising here is that *CA need not to be a group of measures*. In particular it can contain the Haar measures of a family of distinct conjugated subgroups $K_m(\omega)$ of the group K .

Such phenomenon appears even in the case when $\{\mu_n\}$ forms a Markov chain with a finite state space (sec. 6). But it disappears for independent random measures μ_n .

THEOREM 1.2. — *The following conditions are related by 8) \Rightarrow 7) \Leftrightarrow 6) \Rightarrow 5) and 1)-5) are equivalent among themselves.*

- 1) the mapping $\omega \rightarrow \lambda_m(\omega)$ is constant a. e.;
- 2) $\lambda_m(\omega) = \lambda_K$ a. e., where λ_K is the Haar measure of K ;
- 3) there exists $\lim_{n \rightarrow \infty} v_m^{(n)}(\omega) * \tilde{v}_m^{(n)}(\omega)$ a. e.;
- 4) $\lim_{n \rightarrow \infty} v_m^{(n)}(\omega) * \tilde{v}_m^{(n)}(\omega) = \lambda_K$ a. e.;
- 5) $\lambda_K \in \mathcal{B}^{(\infty)}$
- 6) \mathcal{A}_∞ is a subgroup of the semigroup $\mathcal{M}_1(G)$.
- 7) $\mathcal{A}_\infty = \lambda_K H$;
- 8) $\mathcal{A}^{(n)} = \mathcal{A}^{(1)} * \dots * \mathcal{A}^{(1)}$ (n -times), $n \in \mathbb{N}$.

COROLLARY 1.3. — *If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed (i. i. d.) random measures, then the condition 8) and hence the other conditions of the Theorem 1.2 hold.*

In fact the i. i. d. sequence $\{\mu_n\}$ satisfies the following condition:

$$S(P_n) = S(P_1) \times \dots \times S(P_1) \text{ (} n\text{-times)}, \quad n \in \mathbb{N},$$

where P_n be the distributions of the random vectors (μ_1, \dots, μ_n) . Thus 8) holds too.

Thus the CA of a sequence of i. i. d. random measures always has a quite similar form and properties as in the case of convolution powers $\{\mu^n\}_{n=1}^\infty$ (theorem 1.0).

As a consequence we obtain the convergence conditions for $v_m^{(n)}(\omega)$.

THEOREM 1.4. — *The following properties are equivalent.*

- 1) *One of the limits $\lim_{n \rightarrow \infty} v_m^{(n)}(\omega)$ exists a. e.;*
- 2) $\lim_{n \rightarrow \infty} v_m^{(n)}(\omega) = \lambda_H$, a. e. $\forall m \in \mathbb{N}$;
- 3) $K_m(\omega) = H$ a. e. for some (or for all) $m \in \mathbb{N}$;
- 4) $A_m(\omega) = \overline{\lim}_{n \rightarrow \infty} S(v_m^{(n)}(\omega))$ with positive probability;
- 5) $\underline{\lim}_{n \rightarrow \infty} S(v_m^{(n)}(\omega)) \neq \emptyset$ with positive probability.
- 6) $\lambda_H \in \mathcal{B}^{(\infty)}$.

This theorem generalizes the familiar Ito-Kawada theorem (see [6], [7], [8], [15] and [4], ch. 2). It is an easy consequence of the above results. The condition 2) in the above theorem means the compositional convergence of the sequence $\{\mu_n(\omega)\}_{n=1}^\infty$ in the sense of Maksimov [11].

Our method of the study of the CA is based on the notion of a normal sequence, which is introduced in sec. 2. These are sequence with a block recurrence property in the topological sense. Every Borel normal sequence (see [16]) is a normal in our sense but not conversely.

It is easily verified (see ass. 5.1) that almost all realizations of a SSRM $\{\mu_n\}$ satisfying A) and B) are normal sequences. Therefore we can consider the CA of an arbitrary normal sequence of measures and obtain the above results as a consequence of the corresponding theorems for normal sequences in the sections 2-4. Some of the results about normal sequences (th. 3.1, th. 4.1 and others) are of independent interest.

A part of the results of this paper was announced in [12], [13].

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2. NORMAL SEQUENCES

Recall that a sequence $\{a_n\}_{n=1}^\infty$ is said to be Borel normal (see e. q. [16]) if for every $l \geq 1$ there exist infinitely many numbers n such that

$$a_{n+i} = a_i, \quad i = 1, 2, \dots, l$$

DEFINITION 2.1. — A sequence $\{a_n\}_{n=1}^\infty$ of elements of a topological space E will be called normal if for every $l \geq 1$ and for any collection of neighborhoods V_1, \dots, V_l of the points a_1, \dots, a_l there exist infinitely many numbers n such that

$$a_{n+i} \in V_i, \quad i=1, 2, \dots, l \tag{2.1}$$

Every Borel normal sequence is obviously normal and these two notions coincide when E has the discrete topology.

The strictly increasing sequence $\{n_k\}_{k=1}^\infty$ which consists all n satisfying (2.1) will be called the recurrence sequence of the block (a_1, \dots, a_l) into the neighborhood $V_1 \times \dots \times V_l$.

The next theorem plays an important part in the sequel.

Let now E be a compact semigroup and for an arbitrary sequence $\{a_n\}_{n=1}^\infty$ in E consider its partial products

$$b_n = a_n \dots a_1, \quad n \in \mathbb{N}.$$

THEOREM 2.2. — Let $\{a_n\}_{n=1}^\infty$ be a normal sequence in a compact semigroup E and \mathcal{L} denotes the set of all limit points of the corresponding sequence $\{b_n\}_{n=1}^\infty$. Then \mathcal{L} contains at least one idempotent.

Proof. — Let \mathcal{U} be the totality of all sequences $\{U_n\}_{n=1}^\infty$, where U_n is an neighborhood of a_n for each n . We shall fix one such sequence $u = \{U_n\}_{n=1}^\infty \in \mathcal{U}$ and for every $l \geq 1$ consider the recurrence sequence $n_k = n_k(u, l)$, $k \geq 1$, of the block $(a_1 \dots a_l)$ into $U_1 \times \dots \times U_l$.

Let now $\mathcal{L}(u, l)$ be the set of all limit points of the sequence $\{b_{n_k}\}_{k=1}^\infty$ where $n_k = n_k(u, l)$.

The set $\mathcal{L}(u, l)$ is closed as the totality of all limits of the convergent subnets of the sequence $\{b_{n_k}\}_{k=1}^\infty$ and $\mathcal{L}(u, l) \neq \emptyset$ on account of the normality of $\{a_n\}$.

Since

$$\{n_k(u, l), k \leq l\} \supset \{n_k(u, l+1), k \leq l\}$$

we have a decreasing sequence of non-empty closed subsets $\{\mathcal{L}(u, l)\}_{l=1}^\infty$, which has the non-empty intersection $\mathcal{L}(u) = \bigcap_{l=1}^\infty \mathcal{L}(u, l)$.

We may define the intersection of a finite subset $\{u_i, i=1, \dots, s\}$ of \mathcal{U} by

$$\bigcap_{i=1}^s u_i = \left\{ \bigcap_{i=1}^s U_{n,i} \right\}_{n=1}^\infty \in \mathcal{U}$$

where $u_i = \{U_{n,i}\}_{n=1}^\infty \in \mathcal{U}$. Since $\{a_n\}$ is normal

$$\bigcap_{i=1}^s \mathcal{L}(u_i) = \bigcap_{l=1}^\infty \left(\bigcap_{i=1}^s \mathcal{L}(u_i, l) \right) \supset \bigcap_{l=1}^s \left(\mathcal{L} \left(\bigcap_{i=1}^s u_i, l \right) \right) = \mathcal{L} \left(\bigcap_{i=1}^s u_i \right) \neq \emptyset \tag{2.2}$$

We obtain the system $\{\mathcal{L}(u), u \in \mathcal{U}\}$ of nonempty closed subsets of \mathcal{L} . It is a centered system by (2.2), i.e. it has the finite intersection property. Thus its intersection $\mathcal{L}_0 = \bigcap_{u \in \mathcal{U}} \mathcal{L}(u)$ is a non-empty closed subset of \mathcal{L} .

We shall show now that

$$b_n \mathcal{L}_0 \subset \mathcal{L}, \quad n \in \mathbb{N} \tag{2.3}$$

If this inclusion is false there exist $l \in \mathbb{N}$ and $b \in \mathcal{L}_0$ such that $b_l b \notin \mathcal{L}$. One can choose $u = \{U_n\}_{n=1}^\infty \in \mathcal{U}$, which satisfies

$$(U_1 \cdot \dots \cdot U_1 b)^- \cap \mathcal{L} = \emptyset \tag{2.4}$$

and $U_n = E$ for $n > l$. Since $b \in \mathcal{L}_0 \subset \mathcal{L}(u, l)$, it is a limit point of the sequence $\{b_{n_k}\}_{k=1}^\infty$, where $n_k = n_k(u, l)$ is the recurrence sequence of the block $(a_1 \dots a_l)$ into $U_1 \times \dots \times U_l$. Taken a convergent net $b_{n_k(\alpha)} \rightarrow b$ we deduce from

$$b_{n_k+l} \in U_l \cdot \dots \cdot U_1 b_{n_k}$$

that the set $(U_l \cdot \dots \cdot U_1 b)^-$ contains limit points of the net $b_{n_k(\alpha)+l}$ and then limit points of b_n . This contradicts (2.4).

Thus (2.3) holds and hence $\mathcal{L} \mathcal{L}_0 \subset \mathcal{L}$.

By construction we have $\mathcal{L}_0 \subset \mathcal{L}$ and then \mathcal{L} contains the compact semigroup $\left(\bigcup_{n=1}^\infty \mathcal{L}_0^n \right)^-$ generated by \mathcal{L}_0 . Any compact semigroup contains an idempotent ([5], 9.18). Employing this assertion to the semigroup $\left(\bigcup_{n=1}^\infty \mathcal{L}_0^n \right)^-$ we complete the proof. ■

3. CENTERED CONVERGENCE AND ITS CONSEQUENCES

In the course of the sections 3 and 4 we shall consider a fixed normal sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{M}_1(G)$ and its convolutions

$$v_m^{(n)} = \mu_{m+n-1} \star \dots \star \mu_m, \quad m, n \in \mathbb{N} \tag{3.1}$$

Introduce the compact groups

$$K_m = \left[\bigcup_{n=1}^\infty S(\tilde{v}_m^{(n)}) S(v_m^{(n)}) \right]^-, \quad m \in \mathbb{N}$$

of the group $H = [S(\mu_n), n \in \mathbb{N}]^-$ and denote by λ_m the probability Haar measure of K_m .

The next theorem on centered convergence will be the main tool to describe limit points of $v_m^{(n)}$ as $n \rightarrow \infty$

THEOREM 3.1. — *For a normal sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{M}_1(G)$ there exist the following limits*

$$\lim_{n \rightarrow \infty} \tilde{x}_m^{(n)} v_m^{(n)} = \lambda_m, \quad m \in \mathbb{N}$$

where λ_m is the Haar measure of the subgroup K_m and $\{\tilde{x}_m^{(n)}\}_{n=1}^\infty$ is an arbitrary sequence of elements $\tilde{x}_m^{(n)} \in S(\tilde{v}_m^{(n)})$.

To prove this theorem we make use the left regular representation of G and $\mathcal{M}_1(G)$ in the Hilbert space $\mathcal{H} = L_2(G, \lambda_G)$, which are defined by

$$L(g)f = \delta_g * f, \quad L(\mu)f = \mu * f$$

for $f \in \mathcal{H}$, $g \in G$ and $\mu \in \mathcal{M}_1(G)$. The mapping L is in fact a unitary representation of G and a $*$ -representation of the convolutional semigroup $\mathcal{M}_1(G)$; $L(\tilde{\mu}) = L(\mu)^*$ and $\|L(\mu)\| \leq 1$ (see [5], § 27). Herewith, $L: \mu \rightarrow L(\mu)$ is a topological isomorphism of $\mathcal{M}_1(G)$ onto $L(\mathcal{M}_1(G))$ with the strong operator (so)-topology or with the weak operator (wo)-topology on $L(\mathcal{M}_1(G))$ on account of the compactness of $\mathcal{M}_1(G)$.

Proof of theorem 3.1. — It is enough to consider the case $m = 1$.

Denote $T_n = L(v_1^{(n)})$, $n \in \mathbb{N}$. We will use the order on $L(\mathcal{M}_1(G))$ which is induced by the cone of all non-negative defined operators on \mathcal{H} , i. e.

$$T \leq T' \Leftrightarrow ((T' - T)f, f) \geq 0, \quad \forall f \in \mathcal{H}$$

Then $0 \leq T_n^* T_n \leq I$, where $I = id_{\mathcal{H}}$, and

$$0 \leq L(\mu_n)^* L(\mu_n) \leq I$$

implies

$$0 \leq T_n^* T_n = T_{n-1}^* L(\mu_n)^* L(\mu_n) T_{n-1} \leq T_{n-1}^* T_{n-1} \leq I \quad (3.2)$$

i. e. the sequence $\{T_n^* T_n\}_{n=1}^\infty$ is a decreasing one and it is bounded below. Hence there exists the limit

$$(wo)\text{-} \lim_{n \rightarrow \infty} T_n^* T_n = E, \quad 0 \leq E \leq I, \quad E \in L(\mathcal{M}_1(G))$$

(see [3], prob. 94).

On the other hand, there is an idempotent in the set \mathcal{A}_1 of all limit points of $v_1^{(n)}$ as $n \rightarrow \infty$ by the Theorem 2.2. Then λ is a limit point of the sequence $\tilde{v}_1^{(n)} * v_1^{(n)}$. Since $L: \mu \rightarrow L(\mu)$ is a homeomorphism, there exists the limit $\lim_{n \rightarrow \infty} \tilde{v}_1^{(n)} * v_1^{(n)} = \lambda$, where $L(\lambda) = E$.

The operator $L(\lambda)$ is an orthogonal projector on \mathcal{H} and it gives the orthogonal decomposition $\mathcal{H} = X_1 \oplus Y_1$ where $X_1 = \text{Im } E$ and $Y_1 = \text{Ker } E$.

We have by (3.2)

$$\begin{aligned}
 f \in X_1 &\Leftrightarrow T_n^* T_n f \rightarrow f \Rightarrow (T_n^* T_n f, f) \rightarrow (f, f) \\
 &\Leftrightarrow \|T_n f\| \rightarrow \|f\| \Leftrightarrow \|T_n f\| = \|f\| \forall n \\
 &\Rightarrow (T_n^* T_n f, f) = (f, f) \forall n \Leftrightarrow T_n^* T_n f = f \forall n \\
 &\Rightarrow E f = f \Leftrightarrow f \in X_1
 \end{aligned}$$

and

$$\begin{aligned}
 f \in Y_1 &\Leftrightarrow T_n^* T_n f \rightarrow 0 \Rightarrow (T_n^* T_n f, f) = \|T_n f\|^2 \rightarrow 0 \\
 &\Rightarrow (E f, f) = 0 \Rightarrow E f = f \Leftrightarrow f \in Y_1
 \end{aligned}$$

Thus

$$X_1 = \{ f \in \mathcal{H} : \|T_n f\| = \|f\| \forall n \} = \{ f \in \mathcal{H} : T_n^* T_n f = f \forall n \} \quad (3.3)$$

$$Y_1 = \{ f \in \mathcal{H} : \|T_n f\| \rightarrow 0, n \rightarrow \infty \} \quad (3.4)$$

We want to show now that $\lambda = \lambda_1$.

We have $\lambda * \lambda_1 = \lambda_1$ by $\tilde{v}_1^{(n)} * v_1^{(n)} \rightarrow \lambda$ and $S(\tilde{v}_1^{(n)} * v_1^{(n)}) \subset K_1$. Conversely, if $\lambda * f = f, f \in \mathcal{H}$ (i.e. $f \in X_1$) then $\tilde{v}_1^{(n)} * v_1^{(n)} * f = f$ for all n by (3.3) and hence $\delta_x * f = f$ a.e. for all $x \in S(\tilde{v}_1^{(n)} * v_1^{(n)})$, $n \in \mathbb{N}$. Therefore $\delta_x * f = f$ a.e. for all $x \in K_1$ and $\lambda_1 * f = f$. Thus $\lambda_1 * \lambda = \lambda$ and hence $\lambda_1 = \lambda$. (It was used, that $\mu * f = f \Leftrightarrow \delta_x * f = f$ a.e. for all $x \in S(\mu)$, (See [4], 1.2.7.) Let now $\{\tilde{x}_1^{(n)}\}_{n=1}^\infty$ with $\tilde{x}_1^{(n)} \in S(\tilde{v}_1^{(n)})$. For $f \in X_1$ we have $\tilde{x}_1^{(n)} v_1^{(n)} * f = f = \lambda_1 * f$ a.e. by (3.3) since $S(\tilde{x}_1^{(n)} v_1^{(n)}) \in K_1$. For $f \in Y_1$ we have

$$\|\tilde{x}_1^{(n)} v_1^{(n)} * f\| = \|v_1^{(n)} * f\| \rightarrow 0, \quad n \rightarrow \infty$$

by (3.4). Taking into account the decomposition $\mathcal{H} = X_1 \oplus Y_1$ and $X_1 = L(\lambda_1) \mathcal{H}$ we obtain

$$\|\tilde{x}_1^{(n)} v_1^{(n)} * f - \lambda_1 * f\| \rightarrow 0, \quad n \rightarrow \infty$$

for all $f \in \mathcal{H}$ and hence $\tilde{x}_1^{(n)} * v_1^{(n)} \rightarrow \lambda_1$. ■

COROLLARY 3.2. — For all $m \in \mathbb{N}$ the following limits exist

- a) $\lim_{n \rightarrow \infty} \tilde{v}_m^{(n)} * v_m^{(n)} = \lambda_m$
- b) $\lim_{n \rightarrow \infty} (v_m^{(n)} - v_m^{(n)} * \lambda_m) = 0$
- c) $\lim_{n \rightarrow \infty} (v_m^{(n)} * \tilde{v}_m^{(n)} - x_m^{(n)} \lambda_m \tilde{x}_m^{(n)}) = 0$

for all $x_m^{(n)} \in S(v_m^{(n)})$ and $\tilde{x}_m^{(n)} \in S(\tilde{v}_m^{(n)})$.

Remark 3.3. — The choice of a centering sequence $\tilde{x}_m^{(n)}$ on the left side of $v_m^{(n)}$ is essentially connected with the order of the factors $\mu_{m+n-1} \cdot \dots \cdot \mu_m$ in $v_m^{(n)}$. The following simple example shows that the sequence $v_m^{(n)} * \tilde{v}_m^{(n)}$ need not converge as $n \rightarrow \infty$. In this case $v_m^{(n)} x^{(n)}$ does not converge under any choice of $x^{(n)}$.

Example 3.4. — Let L_1 and L_2 be a pair of conjugate subgroups of G and $L_2 = x L_1 x^{-1}$, $L_1 \neq L_2$. Consider a periodic sequence $\{\mu_n\}$, supposing

$$\mu_{3k} = \lambda_{L_1}, \quad \mu_{3k+1} = \delta_x, \quad \mu_{3k+2} = \delta_x^{-1}, \quad k = 0, 1, 2 \dots$$

For $n \geq 3$ we have $\tilde{v}_1^{(n)} * v_1^{(n)} = \lambda_{L_1}$, but $v_1^{(n)} * \tilde{v}_1^{(n)} = \lambda_{L_2}$ for $n = 3k + 1$ and $v_1^{(n)} * \tilde{v}_1^{(n)} = \lambda_{L_1}$ otherwise. Then $v_m^{(n)} * \tilde{v}_m^{(n)}$ has exactly two limit points λ_{L_1} and λ_{L_2} .

Remark 3.5 If the Second Axiom of Countability holds on G the centering sequence always exists for every (even non-normal) sequence in $\mathcal{M}_1(G)$ (see [8]). In the case of a normal sequence we need not SAC-condition and the limit of the centered sequence of measures always has the form $x\lambda$, where $x \in H$ and λ is an idempotent.

We are able to describe now the limits points of $v_m^{(n)}$ as $n \rightarrow \infty$. Introduce the following notation.

$$B_m = \left(\bigcup_{n=1}^{\infty} S(v_m^{(n)}) \right)^-, \quad A_m = \overline{\lim_{n \rightarrow \infty} S(v_m^{(n)})}$$

and C_m be the set of all limit points of all possible sequences $\{x_m^{(n)}\}_{n=1}^{\infty}$ as $n \rightarrow \infty$ where $x_m^{(n)} \in S(v_m^{(n)})$. At last let, \mathcal{A}_m be the set of all limit points of $v_m^{(n)}$ as $n \rightarrow \infty$ and fixed $m \in \mathbb{N}$. i. e. $\mathcal{A}_m = L m P_{n \rightarrow \infty} v_m^{(n)}$.

THEOREM 3.6. — For a normal sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathcal{M}_1(G)$ and $m \in \mathbb{N}$ the following assertions hold:

- a) $A_m = B_m = C_m^- \supset K_m$
- b) $\mathcal{A}_m = A_m \lambda_m \ni \lambda_m$.

Proof. — We may suppose $m = 1$.

1) $C_1^- = A_1$. It is obvious that $C_1 \subset A_1 = A_1^-$ and hence $C_1^- \subset A_1$. For every $x \in A_1$ and an arbitrary neighborhood U and of x one can choose a sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ such that $x_1^{(n)} \in S(v_1^{(n)})$, $n \in \mathbb{N}$ and $x_1^{(n)} \in U$ for infinitely many of n . By the compactness the sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ has a limit point in U^- . Hence $C_1 \cap U^- \neq \emptyset$ for every neighborhood U of x and $x \in C_1^-$. Thus $A_1 \subset C_1^-$.

2) $\mathcal{A}_1 = C_1 \lambda_1$ follows from theorem 3.1, since

$$\mathcal{A}_1 = L m P_{n \rightarrow \infty} (x_1^{(n)} \lambda_1) = (L m P_{n \rightarrow \infty} x_1^{(n)}) \lambda_1$$

for any sequence $\{x_1^{(n)}\}_{n=1}^{\infty}$ with $x_1^{(n)} \in S(v_1^{(n)})$.

3) $\mathcal{A}_1 \ni \lambda_1$. By theorem 2.2 \mathcal{A}_1 contains an idempotent λ , which has the form $\lambda = x \lambda_1$ by 2). Then $\lambda = \lambda_1$.

4) $A_1 \supset K_1$. Since $\lambda_1 \in \mathcal{A}_1$ there exists a subnet $\{v_1^{(n(\alpha))}\}$ of the sequence $\{v_1^{(n)}\}$ which converges to λ_1 . For any n_0 there exists α_0 such that $n(\alpha) > n_0$ for all $\alpha > \alpha_0$. Hence

$$K_1 = S(\lambda_1) = S(\lim_{\alpha} v_1^{(n(\alpha))}) \subset \left(\bigcup_{\alpha > \alpha_0} S(v_1^{(n(\alpha))}) \right)^- \subset \left(\bigcup_{n > n_0} S(v_1^{(n)}) \right)^-$$

On the other hand for $x \notin A_1$ one can choose a number n_0 and a neighborhood U of x such that $S(v_1^{(n)}) \cap U = \emptyset$ for all $n > n_0$ and hence $U \cap K_1 = \emptyset$, i. e. $x \notin K_1$. Thus $K_1 \subset A_1$.

5) $A_1 \supset B_1$. The equality $v_{m+1}^{(n)} * v_1^{(m)} = v_1^{(m+n)}$ implies

$$S(v_{m+1}^{(n)}) \cdot S(v_1^{(m)}) = S(v_1^{(m+n)}), \quad m, n \in \mathbb{N}.$$

Hence $C_{m+1} S(v_1^{(m)}) \subset C_1$, $m \in \mathbb{N}$. Using 1) to A_{m+1} and A_1 , we have also $A_{m+1} S(v_1^{(m)}) \subset A_1$, $m \in \mathbb{N}$.

Applying 4) to the set A_{m+1} we obtain $A_{m+1} \supset K_{m+1} \ni e$, where e is the unit element of G . Hence $S(v_1^{(m)}) \subset A_1$, $m \in \mathbb{N}$ and $B_1 \subset A_1$. The inverse inclusion is obvious. ■

THEOREM 3.7. — *For a normal sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{M}_1(G)$ the following equalities hold for all $m, n \in \mathbb{N}$ and $x_m^{(n)} \in S(v_m^{(n)})$*

$$v_m^{(n)} * \lambda_m = \lambda_{m+n} * v_m^{(n)} = x_m^{(n)} \lambda_m = \lambda_{m+n} x_m^{(n)}$$

Proof. — We again may suppose $m = 1$.

Chosen any $x_1^{(k)} \in S(v_1^{(k)})$ and $x_{k+1}^{(n)} \in S(v_{k+1}^{(n)})$ we deduce by theorem 3.1 as $n \rightarrow \infty$

$$(x_{k+1}^{(n)})^{-1} v_{k+1}^{(n)} \rightarrow \lambda_{k+1} \quad \text{and} \quad (x_{k+1}^{(n)} x_1^{(k)})^{-1} v_1^{(n+k)} \rightarrow \lambda_1$$

Then taking into account the equality

$$v_{k+1}^{(n)} * v_1^{(k)} = v_1^{(n+k)}$$

we obtain

$$(x_1^{(k)})^{-1} \lambda_{k+1} * v_1^{(k)} = \lambda_1$$

that is

$$\lambda_{k+1} * v_1^{(k)} = x_1^{(k)} \lambda_1, \quad k \in \mathbb{N}, \quad x_1^{(k)} \in S(v_1^{(k)})$$

Taking integration over $x_1^{(k)} \in S(v_1^{(k)})$ by the measures $v_1^{(k)}$ we have also

$$\lambda_{k+1} * v_1^{(k)} = v_1^{(k)} * \lambda_1$$

To prove the last equality

$$x_1^{(n)} \lambda_1 = \lambda_{n+1} x_1^{(n)}$$

we need the following lemma.

LEMMA 3.8. — *Let $\mathcal{H} = X_m \oplus Y_m$ be the decomposition of the Hilbert space \mathcal{H} defined by the orthoprojector $L(\lambda_m)$, $m \in \mathbb{N}$. Then*

$$L(v_1^{(m)}) X_1 = X_{m+1}, \quad m \in \mathbb{N}$$

Proof. — It is obvious $L(v_1^{(m)}) X_1 \subset X_{m+1}$. Since G is compact the representation L is decomposed into the direct sum of finite dimensional sub-representations $L = \bigotimes_{s \in \Xi} L^s$ acting in the subspaces \mathcal{H}^s where

$\dim \mathcal{H}^s < \infty$, and $\bigoplus_{s \in \Xi} \mathcal{H}^s = \mathcal{H}$. Herewith every operator $L(\mu)$, $\mu \in \mathcal{M}_1(G)$ admits the decomposition (see [5], § 27).

$$L(\mu) = \bigoplus_{s \in \Xi} L^s(\mu)$$

Therefore it is enough to check the equalities

$$L^s(v_1^{(m)}) X_1^s = X_{m+1}^s, \quad \text{where } X_{m+1}^s = \mathcal{H}^s \cap X_{m+1}$$

By the theorem 3.6 $\lambda_1 \in \mathcal{A}_1$ and hence $L^s(\lambda_1)$ is a limit point of the sequence $L^s(v_1^{(m)})$ as $m \rightarrow \infty$. Since $L^s(v_1^{(m)})$ are contractions and $\dim \mathcal{H}^s < \infty$ we obtain for all s

$$\dim L^s(v_1^{(m)}) X_1^s = \dim X_1^s = \dim X_{m+1}^s < \infty$$

that implies the required equality. ■

From the above lemma it is seen that

$$\lambda_{k+1} = v_1^{(k)} * \lambda_1 * \tilde{v}_1^{(k)}, \quad k \in \mathbb{N}$$

and using $v_1^{(k)} * \lambda_1 = x_1^{(k)} \lambda_1$ we conclude

$$\lambda_{k+1} x_1^{(k)} = x_1^{(k)} \lambda_1, \quad k \in \mathbb{N}, \quad x_1^{(k)} \in S(v_1^{(k)})$$

Thus the theorem 3.7 is proved. ■

COROLLARY 3.9. — For all $x_m^{(n)} \in S(v_m^{(n)})$ and $\tilde{x}_m^{(n)} \in S(\tilde{v}_m^{(n)})$ the following relations hold

- a) $K_{m+n} = x_m^{(n)} K_m \tilde{x}_m^{(n)}$
- b) $A_{m+n} x_m^{(n)} = A_m$.

4. CONVOLUTIONAL ATTRACTORS OF NORMAL SEQUENCES OF MEASURES

The aim of this section is to describe the convolutional attractors for arbitrary normal sequences in $\mathcal{M}_1(G)$.

In common with the sec. 3 let $\{\mu_n\}_{n=1}^\infty$ be a fixed normal sequence in $\mathcal{M}_1(G)$ and $v_m^{(n)}$, $m, n \in \mathbb{N}$ be its convolutions defined by (3.1). We preserve all notation of the sec. 3 and introduce also the sets:

$$\left. \begin{aligned} \mathcal{A}^{(n)} &= L m P_{m \rightarrow \infty} v_m^{(n)}, & \mathcal{B}^{(n)} &= (v_m^{(n)}, m \in \mathbb{N})^- \\ \mathcal{A}^{(\infty)} &= \overline{\lim}_{n \rightarrow \infty} \mathcal{A}^{(n)}, & \mathcal{B}^{(\infty)} &= \left(\bigcup_{n=1}^\infty \mathcal{B}^{(n)} \right)^- \\ \mathcal{A}_\infty &= \overline{\lim}_{m \rightarrow \infty} \mathcal{A}_m, & \mathcal{B}_\infty &= \left(\bigcup_{m=1}^\infty \mathcal{A}_m \right)^- \end{aligned} \right\} \quad (4.1)$$

THEOREM 4.1. — For any normal sequence $\{\mu_n\}_{n=1}^\infty$

- a) $\mathcal{A}^{(n)} = \mathcal{B}^{(n)}, n \in \mathbb{N}$
- b) $\mathcal{A}_\infty = \mathcal{B}_\infty \subset \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)}$
- c) $\mathcal{A}_\infty = A_m \lambda_m A_m^{-1}, m \in \mathbb{N}$

Proof. — By th. 3.6, 3.7 and cor. 3.9

$$\begin{aligned} \mathcal{A}_m &= A_m \lambda_m = A_1 \lambda_1 S(v_1^{(m)})^{-1}, \quad m \in \mathbb{N} \\ \mathcal{A}_\infty &= \overline{\lim}_{m \rightarrow \infty} \mathcal{A}_m = A_1 \lambda_1 (\overline{\lim}_{m \rightarrow \infty} S(v_1^{(m)})^{-1}) = A_1 \lambda_1 A_1^{-1} \\ \mathcal{B}_\infty &= \left(\bigcup_{m=1}^\infty \mathcal{A}_m \right)^- = A_1 \lambda_1 \left(\bigcup_{m=1}^\infty S(v_1^{(m)})^{-1} \right)^- = A_1 \lambda_1 B_1^{-1} = A_1 \lambda_1 A_1^{-1} \end{aligned}$$

For $m > 1$ and any $x_1^{(m-1)} \in S(v_1^{(m-1)}), \tilde{x}_1^{(m-1)} \in S(\tilde{v}_1^{(m-1)})$

$$A_m \lambda_m A_m^- = A_m x_1^{(m-1)} \lambda_1 \tilde{x}_1^{(m-1)} A_m^{-1} = A_1 \lambda_1 A_1^{-1}.$$

Further, for any fixed n the sequence $\{v_m^{(n)}\}_{m=1}^\infty$ is normal since $\{\mu_m\}_{m=1}^\infty$ is a such one. Therefore $\mathcal{A}^{(n)} = \mathcal{B}^{(n)}, n \in \mathbb{N}$

The set $\mathcal{A}^{(\infty)} = \bigcap_{k=1}^\infty (\bigcup_{n \geq k} \mathcal{B}^{(n)})^-$ contains of all limit points of all possible sequences $\{v_m^{(n)}\}_{n=1}^\infty$, where $v_m^{(n)} \in \mathcal{B}^{(n)}$. Hence $\mathcal{A}_m \subset \mathcal{A}^{(\infty)}$ for all m and $\mathcal{A}_{(\infty)} \subset \mathcal{A}^{(\infty)}$.

The inclusion $\mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)}$ is obvious. ■

We shall call the set \mathcal{A}_∞ the *convolutional attractor* (CA) of the normal sequence $\{\mu_n\}_{n=1}^\infty$. The equality

$$\dot{\mu}_n = \mu_n * \lambda_n, \quad n \in \mathbb{N}$$

defines the “limiting sequence” $\{\dot{\mu}_n\}_{n=1}^\infty$ for $\{\mu_n\}_{n=1}^\infty$ such that the sequences $v_m^{(n)}$ and

$$\dot{v}_m^{(n)} = \dot{\mu}_{m+n-1} * \dots * \dot{\mu}_m, \quad m, n \in \mathbb{N}$$

are asymptotically equivalent as $n \rightarrow \infty$, that is

$$\lim_{n \rightarrow \infty} (v_m^{(n)} - \dot{v}_m^{(n)}) = 0, \quad m \in \mathbb{N}$$

It is easy to see that the CA

$$\dot{\mathcal{A}}_\infty = \overline{\lim}_{m \rightarrow \infty} L m P_{n \rightarrow \infty} \dot{v}_m^{(n)}$$

of the sequence $\{\dot{\mu}_n\}_{n=1}^\infty$ coincides with \mathcal{A}_∞ and moreover

$$\mathcal{A}_\infty = \dot{\mathcal{A}}_\infty = (\dot{v}_m^{(n)}, m \in \mathbb{N}, n \in \mathbb{N})^- \tag{4.2}$$

Let us describe now the set \mathcal{E}_∞ of all idempotents of \mathcal{A}_∞

COROLLARY 4.2. — For all $m \in \mathbb{N}$

$$\begin{aligned} \mathcal{E}_\infty &:= \{ \alpha \in \mathcal{A}_\infty : \alpha^2 = \alpha \} = \{ \tilde{\alpha} * \alpha, \alpha \in \mathcal{A}_\infty \} \\ &= \{ \alpha * \tilde{\alpha}, \alpha \in \mathcal{A}_\infty \} = \text{LmP}_{n \rightarrow \infty} v_m^{(n)} * \tilde{v}_m^{(n)} \\ &= (\lambda_n, n \in \mathbb{N})^- = \{ x \lambda_m x^{-1}, x \in A_m \} \end{aligned}$$

This is a direct consequence of the equality $\mathcal{A}_\infty = A_m \lambda_m A_m^{-1}$, $m \in \mathbb{N}$, (see th. 4.1 c).

COROLLARY 4.3. — Let $K = [K_m, m \in \mathbb{N}]^-$ be the smallest compact subgroup containing the subgroups $K_m, m \in \mathbb{N}$. Then

$$\begin{aligned} K &= [\bigcup_{v \in \mathcal{B}^{(\infty)}} S(\tilde{v} * v)]^- = [\bigcup_{v \in \mathcal{B}^{(\infty)}} S(\tilde{v} * v)]^- \\ &= [S(\lambda), \lambda \in \mathcal{E}_\infty]^- = [x K_m x^{-1}, x \in A_m]^- , \quad m \in \mathbb{N} \end{aligned}$$

and K is a subgroup of the group $H = [\bigcup_{v \in \mathcal{B}^{(\infty)}} S(v)]^-$.

We are going to elucidate now when the CA forms a group of measures and when the sequence $v_1^{(n)} * \tilde{v}_1^{(n)}$ converges (cf. ex. 3.4).

THEOREM 4.4. — The following conditions are related by $8) \Rightarrow 7) \Leftrightarrow 6) \Rightarrow 5)$ and $1) - 5)$ are equivalent among themselves:

- 1) $\lambda_m = \lambda_1, m \in \mathbb{N}$,
- 2) $\lambda_m = \lambda_K, m \in \mathbb{N}$,
- 3) there exists $\lim_{n \rightarrow \infty} v_m^{(n)} * \tilde{v}_m^{(n)}$,
- 4) $\lim_{n \rightarrow \infty} v_m^{(n)} * \tilde{v}_m^{(n)} = \lambda_K$,
- 5) $\lambda_K \in \mathcal{B}^{(\infty)}$,
- 6) \mathcal{A}_∞ is a subgroup of the semigroup $\mathcal{M}_1(G)$,
- 7) $\mathcal{A}_\infty = \lambda_K H$,
- 8) $\mathcal{A}^{(n)} = \mathcal{A}^{(1)} * \dots * \mathcal{A}^{(1)}$ (n -times), $n \in \mathbb{N}$.

Proof. — 1), 2), 3), 4) are equivalent by cor. 4.2 and 4.3.

2) \Rightarrow 5). $\lambda_K = \lambda_1 \in \mathcal{A}_1 \subset \mathcal{B}^{(\infty)}$ by th. 3.6 b),

5) \Rightarrow 2). If $\lambda_K \in \mathcal{B}^{(\infty)} = \{ v_m^{(n)}, m, n \in \mathbb{N} \}^-$, then $\lambda_K \in \{ v_m^{(n)} * \lambda_K, m, n \in \mathbb{N} \}^-$ and $\lambda_K \in \{ \lambda_m, m \in \mathbb{N} \}^- = \mathcal{E}_\infty$.

Thus $\mathcal{E}_\infty = \{ \lambda_K \}$ and $\lambda_K = \lambda_m, m \in \mathbb{N}$.

7) \Rightarrow 5) is obvious

6) \Rightarrow 7) If \mathcal{A}_∞ is a group, the set $\mathcal{E}_\infty = \{ \lambda_m, m \in \mathbb{N} \}^-$ of all its idempotents coincides to $\{ \lambda_K \}$. Then K is a normal subgroup of H , the group \mathcal{A}_∞ has the form

$$\mathcal{A}_\infty = A_1 \lambda_K A_1^{-1} = (A_1 A_1^{-1}) \lambda_K \subset H \lambda_K$$

The group \mathcal{A}_∞ contains also the sets $(A_1 A_1^{-1})^n \lambda_K, n \in \mathbb{N}$ and hence $H \lambda_K \subset \mathcal{A}_\infty$.

7) \Rightarrow 6) since K is a normal subgroup of H in this case.

8) \Rightarrow 6). If 8) holds the set $\mathcal{B}^{(\infty)} = \left(\bigcup_{n=1}^{\infty} \mathcal{A}^{(n)} \right)^{-}$ is a semigroup and

$\mathcal{A}^{(\infty)} = \bigcap_{m=1}^{\infty} \left(\bigcup_{n \geq m} \mathcal{A}^{(n)} \right)^{-}$ is a subsemigroup of $\mathcal{B}^{(\infty)}$. Hence $\lambda_m * \lambda_n \in \mathcal{A}^{(\infty)}$ for

all $m, n \in \mathbb{N}$, and $\lambda_K \in \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)}$, since λ_K is contained in the compact semigroup generated by $\mathcal{E}_{\infty} = \{ \lambda_n, n \in \mathbb{N} \}^{-}$. Using 5) \Rightarrow 2) we see that $\mathcal{E}_{\infty} = \{ \lambda_K \}$.

Then $v_m^{(n)} * \lambda_K = \lambda_K * v_m^{(n)} \in \mathcal{A}_{\infty}$ for all $m, n \in \mathbb{N}$ and

$$\lambda_K * \mathcal{A}^{(\infty)} = \lambda_K * \mathcal{B}^{(\infty)} = \mathcal{A}_{\infty}$$

is the smallest left and in the same time right ideal of the compact semigroups $\mathcal{A}^{(\infty)}$ and $\mathcal{B}^{(\infty)}$. Thus \mathcal{A}_{∞} is a group ([5], 9.22). ■

Remark 4.5 a) The conditions 1)-5) do not imply 6) in a general case. For example, if $\mu_{2k} = \lambda x$, $\mu_{2k-1} = \lambda x^{-1}$, $k \in \mathbb{N}$, where $\lambda^2 = \lambda = x \lambda x^{-1}$ and $\lambda x^2 \neq \lambda$, one has the normal sequence $\{ \mu_n \}$ with $\mathcal{E}_{\infty} = \{ \lambda \}$ and $\mathcal{A}_{\infty} = \{ \lambda, \lambda x, \lambda x^{-1} \}$ which is not a group and even semigroup.

b) Remember that the smallest two-sided ideal of a compact semigroup is called its Sushkevich kernel. ([5], 9.21). We have proved now that provided condition 8) of th. 4.4 holds the CA \mathcal{A}_{∞} of a normal sequence $\{ \mu_n \}$ is the Sushkevich kernel of the semigroups $\mathcal{B}^{(\infty)}$ and $\mathcal{A}^{(\infty)}$ and it is a group.

It should be also noted that both inclusions $\mathcal{A}_{\infty} \subset \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)}$ may be strict (see sec. 6).

As a consequence of the above results we can prove now the convergence theorem.

Denote $D_m = \varliminf_{n \rightarrow \infty} S(v_m^{(n)})$, $m \in \mathbb{N}$.

THEOREM 4.6. — *For any normal sequence $\{ \mu_n \}_{n=1}^{\infty}$ the following conditions are equivalent*

- 1) $\lim_{n \rightarrow \infty} v_m^{(n)}$ exists,
- 2) $\lim_{n \rightarrow \infty} v_m^{(n)} = \lambda_H$ for all $m \in \mathbb{N}$,
- 3) $K_m = H$,
- 4) $A_m = D_m$,
- 5) $D_m \neq \emptyset$,
- 6) $\lambda_H \in \mathcal{B}^{(\infty)}$,

Each of the conditions 1)-5) holds for all $m \in \mathbb{N}$ if it does for some one.

Proof. — 2) \Rightarrow 1) and 4) \Rightarrow 5) are obvious

1) \Rightarrow 3) If $\mathcal{A}_m = A_m \lambda_m$ consists of the only point then $A_m \subset K_m$ and hence $K_m = H$,

3) \Rightarrow 2) If $K_m = H$ then $\mathcal{A}_m = A_m \lambda_m = A_m \lambda_H = \{\lambda_H\}$,

2) \Rightarrow 4) $H = S(\lim_{n \rightarrow \infty} v_m^{(n)}) \subset D_m \subset A_m \subset H$,

2) \Rightarrow 6) $\lambda_H = \lim_{n \rightarrow \infty} v_m^{(n)} \in \mathcal{A}_\infty \subset \mathcal{B}^{(\infty)}$,

6) \Rightarrow 2) If $\lambda_H \in \mathcal{B}^{(\infty)} = (v_m^{(n)}, m, n \in \mathbb{N})^-$,

then $\lambda_H \in (v_m^{(n)} * \lambda_m, m, n \in \mathbb{N})^- = \mathcal{A}_\infty = A_m \lambda_m A_m^{-1}$,

and $\mathcal{A}_\infty = \{\lambda_H\}$ i. e. 2) holds

5) \Rightarrow 3) If $x \in D_m$ then for every open $U \ni x$ there exists n_0 such that $U \cap S(v_m^{(n)}) \neq \emptyset$ for all $n > n_0$. Hence for $x_m^{(n)} \in S(v_m^{(n)}) \cap U$ we have

$$S(v_m^{(n)}) \subset x_m^{(n)} K_m \subset UK_m, \quad n > n_0$$

and

$$A_m = \overline{\lim_{n \rightarrow \infty} S(v_m^{(n)})} \subset UK_m$$

If U runs the filter of open neighborhoods of x the open set UK_m runs the filter of neighborhoods of $x K_m$. We have now

$$K_m \subset A_m \subset x K_m$$

Hence $A_m \subset K_m$ and $H \subset K_m$ and $H = K_m$.

We have proved now 1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1) and 2) \Leftrightarrow 6) and 2) \Rightarrow 4) \Rightarrow (5) \Rightarrow 3). ■

In the simplest case, when $v_m^{(n)} = \mu * \dots * \mu$ (n -times), the theorem proved above is the well known Ito-Kavada theorem (see [1], [2], [4], [5], [7] and [8], ch. 2). For Borel normal sequences the implications 1) \Leftrightarrow 2) \Leftrightarrow 3) have been proved by Urbanik [3]. The convergence of convolutions $v_m^{(n)}$ as $n \rightarrow \infty$ for every m to the same limit means the compositional convergence in the Maksimov sense [6].

5. THE PROOF OF THE MAIN THEOREMS 1.1-1.4

In this section we shall deduce the main results stated in the introduction from the theorems of the sec. 3 and 4.

Consider a SSRM $\{\mu_n\}_{n=1}^\infty$, $\mu_n = \mu_n(\omega)$, $\omega \in \Omega$, on G which satisfies the conditions A) and B) and let $v_m^{(n)} = v_m^{(n)}(\omega)$, $m, n \in \mathbb{N}$ be the corresponding sequences of their convolutions defined in the sec. 1. We shall use again the notations of the sec. 1.

Let $\mathcal{A}^{(n)} \subset \mathcal{M}_1(G)$ be the support of the distribution $P \circ (v_m^{(n)})^{-1}$ of the random measures $v_m^{(n)}(\omega)$. (It does not depend on m). Also $\mathcal{A}^{(n)}(\omega) = L m P_{m \rightarrow \infty} v_m^{(n)}(\omega)$, $\mathcal{B}^{(n)}(\omega) = (v_m^{(n)}(\omega), m, n \in \mathbb{N})^-$.

Assertion 5.1. — a) $\{v_m^{(n)}(\omega)\}_{m=1}^\infty$ is a normal sequence in $\mathcal{M}_1(G)$ for a.a. ω .

b) $\mathcal{A}^{(n)} = \mathcal{A}^{(n)}(\omega) = \mathcal{B}^{(n)}(\omega)$ for a.a. ω .

Proof. — One can transfer the SSRM $\{v_m^{(n)}\}_{m=1}^\infty$ onto the space $(\bar{\Omega}, \bar{P})$ of its realizations by the mapping

$$\varphi: \Omega \ni \omega \rightarrow \{v_m^{(n)}(\omega)\}_{m=1}^\infty \in \bar{\Omega}$$

where $\bar{\Omega}$ is a compact subset of the countable direct product of the copies of $\mathcal{A}^{(n)}$. The compact set $\bar{\Omega}$ has a countable base of the topology by B). Herewith the shift transformation θ on $\bar{\Omega}$ preserves the measure $\bar{P} = P \circ \varphi^{-1}$ and θ is ergodic by A).

One can now deduce a) and b) from the Poincaré recurrence theorem and ergodicity of θ , considering the countable system of open sets

$$U_{k_1} \times \dots \times U_{k_m}, \quad m \in \mathbb{N},$$

where $\{U_k\}_{k=1}^\infty$ is a base of the topology on $\mathcal{A}^{(n)}$ (see [1], ch. 1 § 1, § 2). ■

Consider now the sets $\mathcal{A}^{(\infty)}$, $\mathcal{B}^{(\infty)}$ and the subgroups K, H defined in sec. 1 and denote

$$\mathcal{A}^{(\infty)}(\omega) = \overline{\lim}_{n \rightarrow \infty} \mathcal{A}^{(n)}(\omega), \quad \mathcal{B}^{(\infty)}(\omega) = \left(\bigcup_{n=1}^\infty \mathcal{B}^{(n)}(\omega) \right)^-,$$

$$H(\omega) = [S(\mu_n(\omega)), n \in \mathbb{N}]^-,$$

$$K(\omega) = [S(\tilde{v}_m^{(n)}(\omega) \star v_m^{(n)}(\omega)), m, n \in \mathbb{N}]^-$$

Assertion 5.2. — a) $\mathcal{A}^{(\infty)}(\omega) = \mathcal{A}^{(\infty)}$, $\mathcal{B}^{(\infty)}(\omega) = \mathcal{B}^{(\infty)}$,

b) $H(\omega) = H$, $K(\omega) = K$

for a.a. $\omega \in \Omega$.

This assertion follows immediately from the above one.

Consider now the CA

$$\mathcal{A}_\infty(\omega) = \overline{\lim}_{m \rightarrow \infty} L m P_{n \rightarrow \infty} v_m^{(n)}(\omega)$$

of the sequence $\{\mu_n(\omega)\}_{n=1}^\infty$ with a fixed ω .

Assertion 5.3. — The mapping $\omega \rightarrow \mathcal{A}_\infty(\omega)$ is constant a. s.

Proof. — The sequence $\{\mu_n(\omega)\}_{n=1}^\infty$ is normal for a.a. ω . For any such ω the limit

$$\lambda_m(\omega) = \lim_{n \rightarrow \infty} \tilde{v}_m^{(n)}(\omega) \star v_m^{(n)}(\omega)$$

exists and one can consider the limiting sequence $\dot{\mu}_m(\omega) = \mu_m(\omega) \star \lambda_m(\omega)$.

For the corresponding n -th convolutions $\dot{v}_m^{(n)}(\omega) = v_m^{(n)}(\omega) \star \lambda_m(\omega)$ the equality

$$\mathcal{A}_\infty(\omega) = \dot{\mathcal{A}}_\infty(\omega) = \dot{\mathcal{A}}^{(\infty)}(\omega) = \dot{\mathcal{B}}^{(\infty)}(\omega)$$

holds on account of (4.2), where

$$\dot{\mathcal{A}}_\infty(\omega) = \overline{\lim}_{m \rightarrow \infty} L m P_{n \rightarrow \infty} \dot{v}_m^{(n)}(\omega)$$

and

$$\dot{\mathcal{B}}^{(\infty)}(\omega) = (\dot{v}_m^{(n)}(\omega), m, n \in \mathbb{N})^-.$$

On the other hand applying the above assertion for the SSRM $\{\dot{\mu}_n\}_{n=1}^\infty$, one can see that the mapping $\omega \rightarrow \dot{\mathcal{B}}^{(\infty)}(\omega)$ is constant a. s. ■

In order to prove the rest statements of the theorems 1.1-1.4 and to complete their proofs one can apply now the results of the sec. 3 and 4 for a fixed normal sequence $\{\mu_n(\omega)\}_{n=1}^\infty$.

6. EXAMPLES

We shall give here three simple examples of CA to illustrate the objects under consideration.

1) Let $x_n = x_n(\omega)$, $n \in \mathbb{N}$ be i. i. d. random elements on Ω with the values in a compact group G which has a countable base of its topology. Define the SSRM

$$\mu_n(\omega) = \delta_{x_{n+1}(\omega) \cdot x_n(\omega)^{-1}}, \quad n \in \mathbb{N}, \quad \omega \in \Omega$$

and denote by S the essential image of x_n .

In this case we obtain for a.a. ω

$$\begin{aligned} v_m^{(n)}(\omega) &= \delta_{x_{n+m}(\omega) \cdot x_m(\omega)^{-1}}, \quad m, n \in \mathbb{N} \\ S &= (x_n(\omega), n \in \mathbb{N})^- = L m P_{n \rightarrow \infty} x_n(\omega), \quad H = [S]^- \\ A_m(\omega) &= S x_n(\omega)^{-1}, \quad \mathcal{A}_m(\omega) = \{ \delta_x, x \in S \cdot x_m(\omega)^{-1} \} \\ K &= K_m(\omega) = \{ e \}, \quad \mathcal{E}_\infty = \{ \delta_e \} \\ \mathcal{A}^{(n)} &= \mathcal{A}_\infty = \mathcal{A}^{(\infty)} = \mathcal{B}^{(\infty)} = \{ \delta_x, x \in SS^{-1} \} \end{aligned}$$

One can see that $A_m(\omega)$ and $\mathcal{A}_m(\omega)$ essentially depend on m and ω and the CA \mathcal{A}_∞ need not coincide with $\lambda_K H$.

2) Let H be a subgroup of a finite group G and K_0 be a non-normal subgroup of H.

Denote

$$\Gamma = \{ x \lambda_0 y^{-1}, x, y \in H \} \subset \mathcal{M}_1(G)$$

where λ_0 denotes the Haar measure of K_0 .

Consider the SSRM $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$, which is a Markov chain with the finite state space Γ , and transition probability matrix

$$Q = \{ q_{\alpha\beta} \}_{\alpha, \beta \in \Gamma}, \text{ where } q_{\alpha, \beta} = P \{ \mu_{n+1} = \beta \mid \mu_n = \alpha \},$$

and stationary vector of probabilities

$$q_\alpha = P \{ \mu_n = \alpha \}, \quad \alpha \in \Gamma$$

We demand that the transition matrix Q satisfies the condition

$$q_{\alpha\beta} > 0 \Leftrightarrow \beta * \alpha \in \Gamma \quad (6.1)$$

(ones sees $\beta * \alpha \in \Gamma \Leftrightarrow \alpha * \tilde{\alpha} = \tilde{\beta} * \beta$).

This Markov chain is mixing and for the corresponding convolutions $v_m^{(n)}(\omega)$ we have a. s. for $n, m \in \mathbb{N}$

$$\begin{aligned} \mathcal{A}^{(n)} &= \Gamma, & A_m(\omega) &= H, & \mathcal{E}_\infty &= \{x \lambda_0 x^{-1}, x \in H\} \\ \lambda_m(\omega) &= \tilde{\mu}_m(\omega) * \mu_m(\omega), & K_m(\omega) &= S(\lambda_m(\omega)) \\ \mathcal{A}_m(\omega) &= H \lambda_m(\omega), & K &= [x K_0 x^{-1}, x \in H] \end{aligned}$$

and

$$\mathcal{A}_\infty = \mathcal{A}^{(\infty)} = \mathcal{B}^{(\infty)} = \Gamma$$

Since K_0 is not a normal subgroup of H the CA $\mathcal{A}_\infty = \Gamma$ contains a nontrivial set of idempotents $\mathcal{E}_\infty = \{x \lambda_0 x^{-1}, x \in H\}$.

As in the example 1) the SSRM $\{\mu_n\}$ coincides with its limiting sequence $\{\tilde{\mu}_n\}$ and $\mathcal{A}_\infty = \Gamma$.

3) We can change the previous example extending the state space of the considering Markov chain as follows

$$\Gamma' = \Gamma \cup \{\delta_x, x \in H\}$$

and taking the matrix $Q' = \{q'_{\alpha, \beta}\}_{\alpha, \beta \in \Gamma'}$ which satisfies the same requirement (6.1) as Q .

Then the CA \mathcal{A}_∞ of the obtained SSRM $\{\mu_n\}_{n=1}^\infty$ coincides with Γ which is not equal to Γ' and we have the strict inclusion $\mathcal{A}_\infty \subset \mathcal{A}^{(\infty)}$ in such case.

One can construct a lot of different examples of CA replacing the “ \Leftrightarrow ” in the condition (6.1) on “ \Rightarrow ” or considering generalization on the continuous state space case.

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