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Abstract. - We consider a stationary ergodic sequence $\mu_n = \mu_n(\omega)$, $n \in \mathbb{N}$, of random probability measures on a compact group $G$ and study the asymptotic behaviour of their convolutions

$$v_m^{(n)}(\omega) = \mu_{m+n-1}(\omega) \ast \ldots \ast \mu_m(\omega)$$

in the weak topology as $n \to \infty$.

Let $A_m(\omega)$ be the set of all limit points of $v_m^{(n)}(\omega)$ as $n \to \infty$, $A_m(\omega) = \left( \bigcup_{n=1}^{\infty} \text{supp} v_m^{(n)}(\omega) \right)^{-}$ and $\lambda_m(\omega) = \lim_{n \to \infty} v_m^{(n)}(\omega) \ast v_m^{(n)}(\omega)$. There exists a compact $A_\infty$ such that a.s.

$$A_\infty = A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \lim_{m \to \infty} A_m(\omega) = \left( \bigcup_{m=1}^{\infty} A_m(\omega) \right)^{-}$$

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We call this set $\mathcal{A}_\infty$ the convolutional attractor of $\{\mu_m\}$, since also $\mathcal{A}_\infty = (\nu^{(m)}_m(\omega), m \in \mathbb{N})$ a.s. where the sequence $\nu^{(m)}_m = \nu^{(m)}_m(\omega) \ast \lambda_m(\omega)$ is asymptotically equivalent to $\nu^{(m)}_m(\omega)$ as $n \to \infty$ a.s. Describing properties of $\mathcal{A}_\infty$ we in particular find conditions under which $\lambda_m(\omega), A_m(\omega)$ and $\mathcal{A}_m(\omega)$ do not depend essentially on $\omega$ and $\mathcal{A}_\infty$ forms a group of measures as in the well known case of convolution powers $\mu^{(n)}$ of a single measure $\mu$.

Key words : Random measures, convergence of convolutions, compact groups.

RÉSUMÉ. — Nous considérons une suite stationnaire et ergodique $\mu_n = \mu_n(\omega), n \in \mathbb{N}$, de mesures de probabilités sur un groupe compact $G$ et étudions le comportement asymptotique des produits de convolution $\nu^{(m)}_m(\omega) = \mu_{m+n-1}(\omega) \ast \ldots \ast \mu_m(\omega)$ dans la topologie faible lorsque $n \to \infty$.

Soit $\mathcal{A}_m(\omega)$ l'ensemble de tous les points d'adhérence de $\nu^{(m)}_m(\omega)$ lorsque $n \to \infty$, $A_m(\omega) = \left( \bigcup_{n=1}^{\infty} \text{supp} \nu^{(m)}_m(\omega) \right)^{-1}$ et $\lambda_m(\omega) = \lim_{n \to \infty} \nu^{(m)}_m(\omega) \ast \nu^{(m)}_m(\omega)$.

Il existe un ensemble compact $\mathcal{A}_\infty$ tel que, p. p.,

$$\mathcal{A}_\infty = A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \lim_{m \to \infty} \mathcal{A}_m(\omega) = \left( \bigcup_{m=1}^{\infty} \mathcal{A}_m(\omega) \right)^{-1}$$

Nous appelons l'attracteur convolutionnel de la suite $\{\mu_m\}$, puisque $\mathcal{A}_\infty = (\nu^{(m)}_m(\omega), n, m \in \mathbb{N})$ p. p.

où la suite $\nu^{(m)}_m = \nu^{(m)}_m(\omega) \ast \lambda_m(\omega)$ est p. p. asymptotiquement équivalente à la suite $\nu^{(m)}_m(\omega)$ lorsque $n \to \infty$ p. p.

En décrivant les propriétés de $\mathcal{A}_\infty$ nous trouvons en particulier des conditions pour que $\lambda_m(\omega), A_m(\omega)$ et $\mathcal{A}_m(\omega)$ ne dépendent pas essentiellement de $\omega$, et pour que $\mathcal{A}_\infty$ forme un groupe de mesures comme dans le cas bien connu des puissances de convolution $\mu^{(n)}$ d'une mesure unique $\mu$ est p. p.

1. INTRODUCTION

Let $G$ be a compact Hausdorff group and $\mathcal{M}^1(G)$ be the convolution semigroup of Borel probability measures on $G$ with the weak topology.

We consider a stationary random process $\mu_n = \mu_n(\omega), n \in \mathbb{N}$, defined on the probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{M}^1(G)$ and study the limit behaviour of the random measures.

$$\nu^{(m)}_m(\omega) = \mu_{m+n-1}(\omega) \ast \ldots \ast \mu_m(\omega), \quad m, n \in \mathbb{N}$$

for the typical realizations of the process $\mu_n(\omega)$ as $n \to \infty$.
The convergence of convolutions of probability measures on a compact group has been examined by many authors (e.g. see [1], [4], [6], [7], [8], [10], [11], [14]-[16] and references cited there).

Precisely, the asymptotic behaviour of the sequence of the convolution powers \( v^{(n)} = \mu \ast \ldots \ast \mu \) (\( n \)-times), \( n \in \mathbb{N} \) for a fixed \( \mu \in M_1(G) \) is described as follows (see [4], ch. II).

**Theorem 1.0.** - a) The set \( \mathcal{A} = \lambda \cdot \mathcal{H} = \{ \lambda \cdot x, \ x \in H \} \) where \( \lambda = \lambda_K \) is the normalized Haar measure of the subgroup

\[
K = \left[ \bigcup_{n=1}^{\infty} S\left( \tilde{v}^{(n)} \ast v^{(n)} \right) \right]^{-}
\]

\( K \) is a normal subgroup of

\[
H = [S(\mu)]^- = \left[ \bigcup_{n=1}^{\infty} S(v^{(n)}) \right]^- = \lim_{n \to \infty} S(v^n)
\]

and furthermore

\[
\lambda = \lim_{n \to \infty} \tilde{v}^{(n)} \ast v^{(n)} = \lim_{n \to \infty} v^{(n)} \ast \tilde{v}^{(n)}
\]

b) The sequence \( v^{(n)} \) is asymptotically equivalent to the sequence \( \hat{v}^{(n)} = \hat{\mu} \ast \ldots \ast \hat{\mu} \) of the convolution powers of the measure \( \hat{\mu} = \hat{\lambda} \ast \mu \), i.e.

\[
\lim_{n \to \infty} (v^{(n)} - \hat{v}^{(n)}) = 0
\]

and

\( \mathcal{A} = \lambda \cdot m_P \to \infty \hat{v}^{(n)} = (\hat{v}^{(n)}, n \in \mathbb{N})^- \)

Here and elsewhere \([A] \) denotes the group generated by the set \( A \) and \( A^- \) is its closure. \( S(\mu) \) denotes the support of the measure \( \mu \) and we use the notation \( \mu \ast x \) and \( x \ast \mu \) instead \( \mu \ast \delta_x \) and \( \delta_x \ast \mu \) where \( \delta_x \) is a Dirac measure in a point. The measure \( \hat{\mu} \) is the image of \( \mu \) by the involution \( x \to x^{-1}, x \in G \). The definition of \( \lim \) and \( \lim \) see in [4], ch. 2, or in [9], § 29, and \( \lambda \cdot m_P \to \infty \) means the set of all limit (accumulation) points of the corresponding sequence as \( n \to \infty \).

It's natural to call the set \( \mathcal{A} \) in the above theorem 1.0 the *convolutional attractor* (CA) of the measure \( \mu \).

The main purpose of the paper is to construct the analogous (as it is possible) convolutional attractor for a stationary sequence of random measures (SSRM) \( \mu_n = \mu_n(\omega), n \in \mathbb{N} \). To this end we shall investigate the limit points of the corresponding convolutions \( \tilde{v}^{(n)}(\omega) \) as \( n \to \infty \).

For a given SSRM \( \{ \mu_n \}_{n=1}^{\infty} \) on \( G \) we introduce the following notation.
Denote by $\mathcal{A}^{(n)}$ the essential image of the random element $\nu_m^{(n)}$, i.e. the support of its distribution $P \ast (\nu_m^{(n)})^{-1}$ on $\mathcal{M}_1 (G)$. Put also

$$\mathcal{A}^{(\infty)} = \lim_{n \to \infty} \mathcal{A}^{(n)}, \quad \mathcal{B}^{(\infty)} = \left( \bigcup_{n=1}^{\infty} \mathcal{A}^{(n)} \right)^{-},$$

$$H = [S (v), v \in \mathcal{B}^{(\infty)}]^{-}, \quad K = [S (v \ast v), v \in \mathcal{B}^{(\infty)}]^{-}$$

We shall assume everywhere in the course of the paper that the following conditions hold.

A) The $\mathcal{M}$ is ergodic, i.e. every stationary event has the probability 0 or 1.

B) The compact set $\mathcal{B}^{(\infty)}$ (and therefore $\mathcal{A}^{(n)}$ for all $n$) has a countable base of its topology.

The condition B) is equivalent to the metrizability of the compact set $\mathcal{B}^{(\infty)}$ (see [9], § 41. II). But we do not assume any conditions of separability or metrizability on $G$.

The main results of the paper are the theorems 1.1-1.4 stated below.

**Theorem 1.1.** For all $m$ and $a.a. \omega$ the following statements hold.

a) The set $\mathcal{A}_m (\omega) = \lim_{n \to \infty} S (\nu_m^{(n)} (\omega))$ of all limit points of the sequence $\nu_m^{(n)} (\omega)$ as $n \to \infty$ has the form

$$\mathcal{A}_m (\omega) = A_m (\omega) \lambda_m (\omega)$$

where

$$A_m (\omega) = \lim_{n \to \infty} S (\nu_m^{(n)} (\omega)) = \left( \bigcup_{n=1}^{\infty} S (\nu_m^{(n)} (\omega)) \right)^{-}$$

and

$$\lambda_m (\omega) = \lim_{n \to \infty} \nu_m^{(n)} (\omega) \ast \nu_m^{(n)} (\omega)$$

are the Haar measures of the subgroups

$$K_m (\omega) = \left[ \bigcup_{n=1}^{\infty} S (\nu_m^{(n)} (\omega) \ast \nu_m^{(n)} (\omega)) \right]^{-}$$

and

$$K = [K_m (\omega), m \in \mathbb{N}]^{-}$$

Herewith the subgroups $K_m (\omega)$ are conjugated in $H$ and

$$L m P_n \to \nu_m^{(n)} (\omega) \ast \nu_m^{(n)} (\omega) = (\lambda_m (\omega), m \in \mathbb{N})^{-}$$

b) The equality

$$\hat{\mu}_n (\omega) = \mu_n (\omega) \ast \lambda_n (\omega)$$

defines a SSRM such that the sequence of corresponding convolutions

$$\nu_m^{(n)} (\omega) = \hat{\mu}_m + \ldots \ast \hat{\mu}_m (\omega)$$
is asymptotically equivalent to \( v_m^{(n)}(\omega) \) as \( n \to \infty \)
\[
\lim_{n \to \infty} (v_m^{(n)}(\omega) - v_m^{(n)}(\omega)) = 0
\]
for all \( m \) and \( \omega \).

c) There exists a compact subset \( \mathcal{A}_\infty \) of \( \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)} \) such that

\[
\mathcal{A}_\infty = A_m(\omega) \lambda_m(\omega) A_m(\omega)^{-1} = \lim_{m \to \infty} \mathcal{A}_m(\omega) = \left( \bigcup_{m=1}^{\infty} \mathcal{A}_m(\omega) \right)^{\infty} = (v_m^{(n)}(\omega), n, m \in \mathbb{N})
\]

for \( \omega \).

We shall call the above set \( \mathcal{A}_\infty \) the convolutional attractor of the SSRM \( \{\mu_n\}_{n=1}^{\infty} \).

The asymptotic behavior of the convolutions \( v_m^{(n)} \) as \( n \to \infty \) is completely defined by the convolutions \( v_m^{(n)}(\omega) \) of the limiting SSRM \( \{\mu_n\}_{n=1}^{\infty} \). The correspondence

\[
\{\mu_n\}_{n=1}^{\infty} \to \{\hat{\mu}_n\}_{n=1}^{\infty}
\]

is retractive i.e. the limiting SSRM of \( \{\mu_n\}_{n=1}^{\infty} \) is \( \{\hat{\mu}_n\}_{n=1}^{\infty} \) itself.

It should be mentioned that the sets \( K_m(\omega), A_m(\omega) \) and \( \mathcal{A}_m(\omega) \) (unlike \( K, H, \mathcal{A}_\infty, \mathcal{B}_\infty \)) can essentially depend on \( \omega \) and \( m \). The main new phenomenon arising here is that \( \mathcal{A} \) need not to be a group of measures. In particular it can contain the Haar measures of a family of distinct conjugated subgroups \( K_m(\omega) \) of the group \( K \).

Such phenomenon appears even in the case forms a Markov chain with a finite state space (sec. 6). But it disappears for independent random measures \( \mu_n \).

\textbf{Theorem 1.2.} - The following conditions are related by

8) \( \iff \) 7) \( \iff \) 6) \( \iff \) 5) and 1)-5) are equivalent among themselves.

1) the mapping \( \omega \to \lambda_m(\omega) \) is constant a.e.;
2) \( \lambda_m(\omega) = \lambda_K \) a.e., where \( \lambda_K \) is the Haar measure of \( K \);
3) there exists \( \lim_{n \to \infty} v_m^{(n)}(\omega) \ast \tilde{v}_m^{(n)}(\omega) \) a.e.;
4) \( \lim_{n \to \infty} v_m^{(n)}(\omega) \ast \tilde{v}_m^{(n)}(\omega) = \lambda_K \) a.e.;
5) \( \lambda_K \in \mathcal{B}(\infty) \)
6) \( \mathcal{A}_\infty \) is a subgroup of the semigroup \( \mathcal{M}_1(G) \);
7) \( \mathcal{A}_\infty = \lambda_K H \);
8) \( \mathcal{A}(n) = \mathcal{A}(1) \ast \ldots \ast \mathcal{A}(1) \) (n-times), \( n \in \mathbb{N} \).

\textbf{Corollary 1.3.} - If \( \{\mu_n\}_{n=1}^{\infty} \) is a sequence of independent identically distributed (i.i.d.) random measures, then the condition 8) and hence the other conditions of the Theorem 1.2 hold.
In fact the i. i. d. sequence $\{\mu_n\}$ satisfies the following condition:

$$S(P_n) = S(P_1) \times \ldots \times S(P_1) \text{ (n-times)}, \quad n \in \mathbb{N},$$

where $P_n$ be the distributions of the random vectors $(\mu_1, \ldots, \mu_n)$. Thus 8 holds too.

Thus the CA of a sequence of i. i. d. random measures always has a quite similar form and properties as in the case of convolution powers $\{\mu^n\}_{n=1}^{\infty}$ (theorem 1.0).

As a consequence we obtain the convergence conditions for $v^{(n)}_m(\omega)$.

**Theorem 1.4.** — The following properties are equivalent.

1) One of the limits $\lim_{n \to \infty} v^{(n)}_m(\omega)$ exists a.e.;

2) $\lim_{n \to \infty} v^{(n)}_m(\omega) = \lambda_h$, a.e. $\forall m \in \mathbb{N}$;

3) $K_m(\omega) = H$ a.e. for some (or for all) $m \in \mathbb{N}$;

4) $A_m(\omega) = \lim_{n \to \infty} S(v^{(n)}_m(\omega))$ with positive probability;

5) $\lim_{n \to \infty} S(v^{(n)}_m(\omega)) \neq \emptyset$ with positive probability.

6) $\lambda_h \in \mathcal{B}^{(\infty)}$.

This theorem generalizes the familiar Ito-Kawada theorem (see [6], [7], [8], [15] and [4], ch. 2). It is an easy consequence of the above results. The condition 2) in the above theorem means the compositional convergence of the sequence $\{\mu_n(\omega)\}_{n=1}^{\infty}$ in the sense of Maksimov [11].

Our method of the study of the CA is based on the notion of a normal sequence, which is introduced in sec. 2. These are sequence with a block recurrence property in the topological sense. Every Borel normal sequence (see [16]) is a normal in our sense but not conversely.

It is easily verified (see ass. 5.1) that almost all realizations of a SSRM $\{\mu_n\}$ satisfying A) and B) are normal sequences. Therefore we can consider the CA of an arbitrary normal sequence of measures and obtain the above results as a consequence of the corresponding theorems for normal sequences in the sections 2-4. Some of the results about normal sequences (th. 3.1, th. 4.1 and others) are of independent interest.

A part of the results of this paper was announced in [12], [13].

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**2. NORMAL SEQUENCES**

Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ is said to be Borel normal (see e. q. [16]) if for every $l \geq 1$ there exist infinitely many numbers $n$ such that

$$a_{n+i} = a_i, \quad i = 1, 2, \ldots, l$$

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**Definition 2.1.** A sequence \( \{a_n\}_{n=1}^{\infty} \) of elements of a topological space \( E \) will be called normal if for every \( l \geq 1 \) and for any collection of neighborhoods \( V_1, \ldots, V_l \) of the points \( a_1, \ldots, a_l \) there exist infinitely many numbers \( n \) such that

\[
a_{n+i} \in V_i, \quad i = 1, 2, \ldots, l \tag{2.1}
\]

Every Borel normal sequence is obviously normal and these two notions coincide when \( E \) has the discrete topology.

The strictly increasing sequence \( \{n_k\}_{k=1}^{\infty} \) which consists all \( n \) satisfying (2.1) will be called the recurrence sequence of the block \( (a_1, \ldots, a_l) \) into the neighborhood \( V_1 \times \ldots \times V_l \).

The next theorem plays an important part in the sequel.

Let now \( E \) be a compact semigroup and for an arbitrary sequence \( \{a_n\}_{n=1}^{\infty} \) in \( E \) consider its partial products

\[
b_n = a_n \cdots a_1, \quad n \in \mathbb{N}.
\]

**Theorem 2.2.** Let \( \{a_n\}_{n=1}^{\infty} \) be a normal sequence in a compact semigroup \( E \) and \( \mathcal{L} \) denotes the set of all limit points of the corresponding sequence \( \{b_n\}_{n=1}^{\infty} \). Then \( \mathcal{L} \) contains at least one idempotent.

**Proof.** Let \( \mathcal{U} \) be the totality of all sequences \( \{U_n\}_{n=1}^{\infty} \), where \( U_n \) is an neighborhood of \( a_n \) for each \( n \). We shall fix one such sequence \( u = \{U_n\}_{n=1}^{\infty} \in \mathcal{U} \) and for every \( l \geq 1 \) consider the recurrence sequence \( n_k = n_k(u, l), k \geq 1 \), of the block \( (a_1 \ldots a_l) \) into \( U_1 \times \ldots \times U_l \).

Let now \( \mathcal{L}(u, l) \) be the set of all limit points of the sequence \( \{b_{nk}\}_{k=1}^{\infty} \) where \( n_k = n_k(u, l) \).

The set \( \mathcal{L}(u, l) \) is closed as the totality of all limits of the convergent subnets of the sequence \( \{b_{nk}\}_{k=1}^{\infty} \) and \( \mathcal{L}(u, l) \neq \emptyset \) on account of the normality of \( \{a_n\} \).

Since

\[
\{n_k(u, l), k \leq 1\} \supseteq \{n_k(u, l+1), k \geq 1\}
\]

we have a decreasing sequence of non-empty closed subsets

\[
\{\mathcal{L}(u, l)\}_{l=1}^{\infty}, \text{ which has the non-empty intersection } \mathcal{L}(u) = \bigcap_{l=1}^{\infty} \mathcal{L}(u, l).
\]

We may define the intersection of a finite subset \( \{u_i, i=1, \ldots, s\} \) of \( \mathcal{U} \) by

\[
\bigcap_{i=1}^{s} u_i = \left( \bigcap_{i=1}^{s} U_{n,i} \right)_{n=1}^{\infty} \in \mathcal{U}
\]
where \( u_i = \{ U_{n,i} \}_{n=1}^{\infty} \in \mathcal{U} \). Since \( \{ a_n \} \) is normal
\[
\bigcap_{i=1}^{s} \mathcal{L}(u_i) = \bigcap_{i=1}^{s} \left( \bigcap_{l=1}^{s} \mathcal{L}(u_l, l) \right) = \bigcap_{i=1}^{s} \left( \mathcal{L}\left( \bigcap_{l=1}^{s} u_l, l \right) \right) = \mathcal{L}\left( \bigcap_{i=1}^{s} u_i \right) \neq \emptyset
\]
(2.2)

We obtain the system \( \{ \mathcal{L}(u), u \in \mathcal{U} \} \) of nonempty closed subsets of \( \mathcal{L} \). It is a centered system by (2.2), i.e. it has the finite intersection property. Thus its intersection \( \mathcal{L}_0 = \bigcap_{u \in \mathcal{U}} \mathcal{L}(u) \) is a non-empty closed subset of \( \mathcal{L} \).

We shall show now that
\[
b_n \mathcal{L}_0 \subset \mathcal{L}, \quad n \in \mathbb{N}
\]
(2.3)

If this inclusion is false there exist \( l \in \mathbb{N} \) and \( b \in \mathcal{L}_0 \) such that \( b_l b \notin \mathcal{L} \). One can choose \( u = \{ U_n \}_{n=1}^{\infty} \in \mathcal{U} \), which satisfies
\[
(U_i \cdot \ldots \cdot U_k b) - \cap \mathcal{L} = \emptyset
\]
(2.4)

and \( U_n = E \) for \( n > l \). Since \( b \in \mathcal{L}_0 \subset \mathcal{L}(u, l) \), it is a limit point of the sequence \( \{ b_{n_k} \}_{k=1}^{\infty} \), where \( n_k = n_k(u, l) \) is the recurrence sequence of the block \( (a_1 \ldots a_l) \) into \( U_1 \times \ldots \times U_l \). Taken a convergent net \( b_{n_k(a)} \to b \) we deduce from
\[
b_{n_k+l} \in U_i \cdot \ldots \cdot U_k b_{n_k}
\]
that the set \( (U_i \cdot \ldots \cdot U_k b) - \) contains limit points of the net \( b_{n_k(a)+l} \) and then limit points of \( b_n \). This contradicts (2.4).

Thus (2.3) holds and hence \( \mathcal{L} \mathcal{L}_0 \subset \mathcal{L} \).

By construction we have \( \mathcal{L}_0 \subset \mathcal{L} \) and then \( \mathcal{L} \) contains the compact semigroup \( \left( \bigcup_{n=1}^{\infty} \mathcal{L}_0^n \right)^- \) generated by \( \mathcal{L}_0 \). Any compact semigroup contains an idempotent ([5], 9.18). Employing this assertion to the semigroup \( \left( \bigcup_{n=1}^{\infty} \mathcal{L}_0^n \right)^- \) we complete the proof. ■

3. CENTERED CONVERGENCE AND ITS CONSEQUENCES

In the course of the sections 3 and 4 we shall consider a fixed normal sequence \( \{ \mu_n \}_{n=1}^{\infty} \) in \( \mathcal{M}_1(G) \) and its convolutions
\[
\nu_m^{(n)} = \mu_{m+n-1} \ast \ldots \ast \mu_m, \quad m, n \in \mathbb{N}
\]
(3.1)

Introduce the compact groups
\[
K_m = \left[ \bigcup_{n=1}^{\infty} S(\nu_m^{(n)}) S(\nu_m^{(n)}) \right]^- , \quad m \in \mathbb{N}
\]
Theorem 3.1. — For a normal sequence \( \{ \mu_n \} \) in \( \mathcal{M}_1(G) \) there exist the following limits

\[
\lim_{n \to \infty} \tilde{x}_{m}^{(n)} v_{m}^{(n)} = \lambda_{m}, \quad m \in \mathbb{N}
\]

where \( \lambda_{m} \) is the Haar measure of the subgroup \( K_m \) and \( \{ \tilde{x}_{m}^{(n)} \} \) is an arbitrary sequence of elements \( \tilde{x}_{m}^{(n)} \in S(\tilde{v}_{m}^{(n)}) \).

To prove this theorem we make use the left regular representation of \( G \) and \( \mathcal{M}_1(G) \) in the Hilbert space \( \mathcal{H} = L_2(G, \mu_G) \), which are defined by

\[
L(g)f = \delta_g * f, \quad L(\mu)f = \mu * f
\]

for \( f \in \mathcal{H}, \ g \in G \) and \( \mu \in \mathcal{M}_1(G) \). The mapping \( L \) is in fact a unitary representation of \( G \) and a \(*\)-representation of the convolutional semigroup \( \mathcal{M}_1(G) \); \( L(\tilde{\mu}) = L(\mu)^* \) and \( \| L(\mu) \| \leq 1 \) (see [5], § 27). Herewith, \( L : \mu \to L(\mu) \) is a topological isomorphism of \( \mathcal{M}_1(G) \) onto \( L(\mathcal{M}_1(G)) \) with the strong operator (so)-topology or with the weak operator (wo)-topology on \( L(\mathcal{M}_1(G)) \) on account of the compactness of \( \mathcal{M}_1(G) \).

Proof of theorem 3.1. — It is enough to consider the case \( m = 1 \). Denote \( T_n = L(v_{1}^{(n)}) \), \( n \in \mathbb{N} \). We will use the order on \( L(\mathcal{M}_1(G)) \) which is induced by the cone of all non-negative defined operators on \( \mathcal{H} \), i.e.

\[
T \leq T' \iff ((T' - T)f, f) \geq 0, \quad \forall f \in \mathcal{H}
\]

Then \( 0 \leq T_n^* T_n \leq I \), where \( I = id_{\mathcal{H}} \), and

\[
0 \leq L(\mu_n)^* L(\mu_n) \leq I
\]

implies

\[
0 \leq T_n^* T_n = T_{n-1}^* L(\mu_n) L(\mu_n) T_{n-1} \leq T_{n-1}^* T_{n-1} \leq I
\]

(3.2)

i.e. the sequence \( \{ T_n^* T_n \} \) is a decreasing one and it is bounded below. Hence there exists the limit

\[
(\text{wo}) - \lim_{n \to \infty} T_n^* T_n = E, \quad 0 \leq E \leq I, \quad E \in L(\mathcal{M}_1(G))
\]

(see [3], prob. 94).

On the other hand, there is an idempotent in the set \( \mathcal{A}_1 \) of all limit points of \( v_{1}^{(n)} \) as \( n \to \infty \) by the Theorem 2.2. Then \( \lambda \) is a limit point of the sequence \( \tilde{v}_{1}^{(n)} * v_{1}^{(n)} \). Since \( L : \mu \to L(\mu) \) is a homeomorphism, there exists the limit \( \lim_{n \to \infty} v_{1}^{(n)} * v_{1}^{(n)} = \lambda \), where \( L(\lambda) = E \).

The operator \( L(\lambda) \) is an orthogonal projector on \( \mathcal{H} \) and it gives the orthogonal decomposition \( \mathcal{H} = X_1 \oplus Y_1 \) where \( X_1 = \text{Im} E \) and \( Y_1 = \text{Ker} E \).

We have by (3.2)
\[ f \in X_1 \iff T_n^* T_n f \to f \iff (T_n^* T_n f, f) \to (f, f) \]
\[ \iff \| T_n f \| \to \| f \| \iff \| T_n f \| = \| f \| \forall n \]
\[ \Rightarrow (T_n^* T_n f, f) = (f, f) \forall n \iff T_n^* T_n f = f \forall n \]
\[ \Rightarrow E f = f \iff f \in X_1 \]
and
\[ f \in Y_1 \iff T_n^* T_n f \to 0 \iff (T_n^* T_n f, f) = \| T_n f \|^2 \to 0 \]
\[ \Rightarrow (E f, f) = 0 \Rightarrow E f = f \iff f \in Y_1 \]

Thus
\[ X_1 = \{ f \in \mathcal{H} : \| T_n f \| = \| f \| \forall n \} = \{ f \in \mathcal{H} : T_n^* T_n f = f \forall n \} \]
(3.3)
\[ Y_1 = \{ f \in \mathcal{H} : \| T_n f \| \to 0, n \to \infty \} \]
(3.4)

We want to show now that \( \lambda = \lambda_1 \).

We have \( \lambda \star \lambda_1 = \lambda_1 \) by \( \widetilde{v}^{(n)} \star v^{(n)}_1 \to \lambda \) and \( S(\widetilde{v}^{(n)} \star v^{(n)}_1) \subseteq K_1 \). Conversely, if \( \lambda \star f = f \), \( f \in \mathcal{H} \) (i.e. \( f \in X_1 \)) then \( \widetilde{v}^{(n)}_1 \star v^{(n)}_1 \star f = f \) for all \( n \) by (3.3) and hence \( \delta_x \star f = f \) a.e. for all \( x \in S(\widetilde{v}^{(n)}_1 \star v^{(n)}_1) \), \( n \in \mathbb{N} \). Therefore \( \delta_x \star f = f \) a.e. for all \( x \in K_1 \) and \( \lambda \star f = f \). Thus \( \lambda_1 \star f = f \). It was used, that \( \mu \star f = f \iff \delta_x \star f = f \) a.e. for all \( x \in S(\mu) \). (See [4], 1.2.7.) Let now \( \{ x^{(n)}_1 \}_{n=1}^{\infty} \) with \( \widetilde{v}^{(n)}_1 \in S(\widetilde{v}^{(n)}_1) \). For \( f \in X_1 \) we have \( \widetilde{x}^{(n)}_1 \star v^{(n)}_1 \star f = f = \lambda_1 \star f \) a.e. by (3.3) since \( S(x^{(n)}_1 v^{(n)}_1) \subseteq K_1 \). For \( f \in Y_1 \) we have
\[ \| \widetilde{x}^{(n)}_1 \star v^{(n)}_1 \star f \| = \| v^{(n)}_1 \star f \| \to 0, \quad n \to \infty \]
by (3.4). Taking into account the decomposition \( \mathcal{H} = X_1 \oplus Y_1 \) and \( X_1 = L(\lambda_1, \mathcal{H}) \) we obtain
\[ \| x^{(n)}_1 \star v^{(n)}_1 \star f - \lambda_1 \star f \| \to 0, \quad n \to 0 \]
for all \( f \in \mathcal{H} \) and hence \( x^{(n)}_1 \star v^{(n)}_1 \to \lambda_1 \).

**Corollary 3.2.** – For all \( m \in \mathbb{N} \) the following limits exist
\[ a) \quad \lim_{n \to \infty} \widetilde{v}^{(n)}_m \star v^{(n)}_m = \lambda_m \]
\[ b) \quad \lim_{n \to \infty} (v^{(n)}_m - \lambda_m) = 0 \]
\[ c) \quad \lim_{n \to \infty} (v^{(n)}_m \star \widetilde{v}^{(n)}_m - \lambda_v \star x^{(n)}_m \star \widetilde{v}^{(n)}_m) = 0 \]
for all \( x^{(n)}_v \in S(v^{(n)}_m) \) and \( x^{(n)}_m \in S(\widetilde{v}^{(n)}_m) \).

**Remark 3.3.** – The choice of a centering sequence \( x^{(n)}_m \) on the left side of \( v^{(n)}_m \) is essentially connected with the order of the factors \( \mu_m, \mu_{m+n-1}, \ldots, \mu_m \) in \( v^{(n)}_m \). The following simple example shows that the sequence \( v^{(n)}_m \star \widetilde{v}^{(n)}_m \) need not converge as \( n \to \infty \). In this case \( v^{(n)}_m x^{(n)}_v \) does not converge under any choice of \( x^{(n)}_v \).
Example 3.4. Let $L_1$ and $L_2$ be a pair of conjugate subgroups of $G$ and $L_2 = x L_1 x^{-1}$, $L_1 \neq L_2$. Consider a periodic sequence $\{\mu_n\}$, supposing $\mu_{3k} = \lambda_{L_1}$, $\mu_{3k+1} = \delta_x$, $\mu_{3k+2} = \delta_{x^{-1}}$, $k = 0, 1, 2 \ldots$

For $n \geq 3$ we have $\tilde{v}_1^{(n)} \ast v_1^{(n)} = \lambda_{L_1}$, but $\mu_n \ast \tilde{v}_1^{(n)} = \lambda_{L_2}$ for $n = 3k + 1$ and $\mu_n \ast v_1^{(n)} = \lambda_{L_1}$ otherwise. Then $\mu_n \ast v_1^{(n)}$ has exactly two limit points $\lambda_{L_1}$ and $\lambda_{L_2}$.

Remark 3.5 If the Second Axiom of Countability holds on $G$ the centering sequence always exists for every (even non-normal) sequence in $\mathcal{M}_1(G)$ (see [8]). In the case of a normal sequence we need not SAC-condition and the limit of the centered sequence of measures always has the form $x \lambda$, where $x \in H$ and $\lambda$ is an idempotent.

We are able to describe now the limits points of $\mu^{(n)}_m$ as $n \rightarrow \infty$

Introduce the following notation.

$$B_m = \left( \bigcup_{n=1}^{\infty} S(v^{(n)}_m) \right)^{-}, \quad A_m = \lim_{n \to \infty} S(v^{(n)}_m)$$

and $C_m$ be the set of all limit points of all possible sequences $\{x^{(n)}_m\}_{n=1}^{\infty}$ as $n \rightarrow \infty$ where $x^{(n)}_m \in S(v^{(n)}_m)$. At last let, $\mathcal{A}_m$ be the set of all limit points of $\mu^{(n)}_m$ as $n \rightarrow \infty$ and fixed $m \in \mathbb{N}$. i.e. $\mathcal{A}_m = \lim_{n \rightarrow \infty} v^{(n)}_m$.

Theorem 3.6. For a normal sequence $\{\mu^{(n)}_m\}_{n=1}^{\infty}$ in $\mathcal{M}_1(G)$ and $m \in \mathbb{N}$ the following assertions hold:

1) $C_1 = A_1$. It is obvious that $C_1 \subseteq A_1 = A_1^-$ and hence $C_1 \subseteq A_1$. For every $x \in A_1$ and an arbitrary neighborhood $U$ and of $x$ one can choose a sequence $\{x^{(n)}_1\}_{n=1}^{\infty}$ such that $x^{(n)}_1 \in S(v^{(n)}_1)$, $n \in \mathbb{N}$ and $x^{(n)}_1 \in U$ for infinitely many of $n$. By the compactness the sequence $\{x^{(n)}_1\}_{n=1}^{\infty}$ has a limit point in $U^-$. Hence $C_1 \cap U^- \neq \emptyset$ for every neighborhood $U$ of $x$ and $x \in C_1^-$. Thus $A_1 \subseteq C_1$.

2) $\mathcal{A}_1 = C_1 \lambda_1$ follows from theorem 3.1, since

$$\mathcal{A}_1 = L \cdot m \cdot P_n \rightarrow \infty \left( x_1^{(n)} \lambda_1 \right) = (L \cdot m \cdot P_n \rightarrow \infty x_1^{(n)}) \lambda_1$$

for any sequence $\{x^{(n)}_1\}_{n=1}^{\infty}$ with $x^{(n)}_1 \in S(v^{(n)}_1)$.

3) $\mathcal{A}_1 \ni \lambda_1$. By theorem 2.2 $\mathcal{A}_1$ contains an idempotent $\lambda$, which has the form $\lambda = x \lambda_1$ by 2). Then $\lambda = \lambda_1$.

4) $A_1 \ni K_1$. Since $\lambda_1 \in \mathcal{A}_1$ there exists a subnet $\left\{v_1^{(n(a))}\right\}$ of the sequence $\left\{v_1^{(n)}\right\}$ which converges to $\lambda_1$. For any $n_0$ there exists $a_0$ such that $n(a) > n_0$ for all $a > a_0$. Hence

$$K_1 = S(\lambda_1) = S(\lim_{a \rightarrow a_0} v_1^{(n(a))}) \subseteq \left( \bigcup_{a > a_0} S(v_1^{(n(a))}) \right)^{-} \subseteq \left( \bigcup_{n > n_0} S(v_1^{(n)}) \right)^{-}$$

On the other hand for \( x \notin A_1 \) one can choose a number \( n_0 \) and a neighborhood \( U \) of \( x \) such that \( S(v^{(n)}_1) \cap U = \emptyset \) for all \( n > n_0 \) and hence \( U \cap K_1 = \emptyset \), i.e. \( x \notin K_1 \). Thus \( K_1 \subseteq A_1 \).

5) \( A_1 \supseteq B_1 \). The equality \( v^{(n)}_{m+1} \ast v^{(m)}_1 = v^{(m+n)}_1 \) implies

\[
S(v^{(n)}_{m+1}) \cdot S(v^{(m)}_1) = S(v^{(m+n)}_1), \quad m, n \in \mathbb{N}.
\]

Hence \( C_{m+1} S(v^{(m)}_1) \subseteq C_1, \quad m \in \mathbb{N} \). Using 1) to \( A_{m+1} \) and \( A_1 \), we have also \( A_{m+1} S(v^{(m)}_1) \subseteq A_1, \quad m \in \mathbb{N} \).

Applying 4) to the set \( A_{m+1} \) we obtain \( A_{m+1} \supseteq K_{m+1} \supseteq e \), where \( e \) is the unit element of \( G \). Hence \( S(v^{(m)}_1) \subseteq A_1, \quad m \in \mathbb{N} \) and \( B_1 \subseteq A_1 \). The inverse inclusion is obvious.

**Theorem 3.7.** — *For a normal sequence \( \{ \mu_n \}_{n=1}^{\infty} \) in \( M_1(G) \) the following equalities hold for all \( m, n \in \mathbb{N} \) and \( S(v^{(m)}_1) \).

\[
\mu_{m+n} \ast \lambda_m = \mu_{m+n} = \mu_{m} \ast \lambda_{m+n} = \mu_{m+n}.
\]

**Proof.** — We again may suppose \( m = 1 \).

Choose any \( x^{(k)}_1 \in S(v^{(k)}_1) \) and \( x^{(n)}_{k+1} \in S(v^{(n)}_k) \) we deduce by theorem 3.1 as \( n \to \infty \)

\[
(x^{(k)}_{k+1})^{-1} v^{(n)}_{k+1} \to \lambda_{k+1} \quad \text{and} \quad (x^{(n)}_{k+1} x^{(k)}_1)^{-1} v^{(n+k)}_1 \to \lambda_1
\]

Then taking into account the equality

\[
v^{(n+k)}_{k+1} \ast v^{(k)}_1 = v^{(n+k)}_1
\]

we obtain

\[
(x^{(k)}_1)^{-1} \lambda_{k+1} \ast v^{(k)}_1 = \lambda_1
\]

that is

\[
\lambda_{k+1} \ast v^{(k)}_1 = x^{(k)}_1 \lambda_1, \quad k \in \mathbb{N}, \quad x^{(k)}_1 \in S(v^{(k)}_1)
\]

Taking integration over \( x^{(k)}_1 \in S(v^{(k)}_1) \) by the measures \( v^{(k)}_1 \) we have also

\[
\lambda_{k+1} \ast v^{(k)}_1 = v^{(k)}_1 \ast \lambda_1
\]

To prove the last equality

\[
x^{(n)}_1 \lambda_1 = \lambda_{n+1} x^{(n)}_1
\]

we need the following lemma.

**Lemma 3.8.** — *Let \( \mathcal{H} = X_m \oplus Y_m \) be the decomposition of the Hilbert space \( \mathcal{H} \) defined by the orthoprojector \( L(\lambda_m), \quad m \in \mathbb{N} \). Then

\[
L(v^{(m)}_1) X_1 = X_{m+1}, \quad m \in \mathbb{N}
\]

**Proof.** — It is obvious \( L(v^{(m)}_1) X_1 \subseteq X_{m+1} \). Since \( G \) is compact the representation \( L \) is decomposed into the direct sum of finite dimensional sub-representations \( L = \bigotimes_{s \in S} L^s \) acting in the subspaces \( \mathcal{H}^s \) where

\[
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\]
dim $\mathcal{H}^s < \infty$, and $\bigoplus_{s \in S} \mathcal{H}^s = \mathcal{H}$. Herewith every operator $L(\mu), \mu \in \mathcal{M}_1(G)$ admits the decomposition (see [5], § 27).

$$L(\mu) = \bigoplus_{s \in S} L^s(\mu)$$

Therefore it is enough to check the equalities

$$L^s(v_1^{(m)}) X^s_1 = X^s_{m+1}, \quad \text{where} \quad X^s_{m+1} = \mathcal{H}^s \cap X_{m+1}$$

By the theorem 3.6 $\lambda_1 \in \mathcal{A}_1$ and hence $L^s(\lambda_1)$ is a limit point of the sequence $L^s(v_1^{(m)})$ as $m \to \infty$. Since $L^s(v_1^{(m)})$ are contractions and $\dim \mathcal{H}^s < \infty$ we obtain for all $s$

$$\dim L^s(v_1^{(m)}) X^s_1 = \dim X^s_1 = \dim X^s_{m+1} < \infty$$

that implies the required equality. ☐

From the above lemma it is seen that

$$\lambda_{k+1} = v_1^{(k)} \ast \lambda_1 \ast \overline{v_1^{(k)}}, \quad k \in \mathbb{N}$$

and using $v_1^{(k)} \ast \lambda_1 = x_1^{(k)} \lambda_1$ we conclude

$$\lambda_{k+1} = x_1^{(k)} \lambda_1, \quad k \in \mathbb{N}, \quad x_1^{(k)} \in S(v_1^{(k)})$$

Thus the theorem 3.7 is proved. ☐

**Corollary 3.9.** - For all $x_m^{(n)} \in S(v_m^{(n)})$ and $\overline{x_m^{(n)}} \in S(\overline{v_m^{(n)}})$ the following relations hold

a) $K_{m+n} = x_m^{(n)} K_m \overline{x_m^{(n)}}$

b) $A_{m+n} \overline{x_m^{(n)}} = A_m$.

4. **CONVOLUTIONAL ATTRACTORS OF NORMAL SEQUENCES OF MEASURES**

The aim of this section is to describe the convolutional attractors for arbitrary normal sequences in $\mathcal{M}_1(G)$.

In common with the sec. 3 let $\{\mu_n\}_{n=1}^{\infty}$ be a fixed normal sequence in $\mathcal{M}_1(G)$ and $v_m, n \in \mathbb{N}$ be its convolutions defined by (3.1). We preserve all notation of the sec. 3 and introduce also the sets:

$$\mathcal{A}^{(n)} = \lim_{m \to \infty} p_{m} v_m^{(n)}, \quad \mathcal{B}^{(n)} = (v_m^{(n)}, m \in \mathbb{N})^{-}$$

$$\mathcal{A}^{(\infty)} = \lim_{n \to \infty} \mathcal{A}^{(n)}, \quad \mathcal{B}^{(\infty)} = \left( \bigcup_{n=1}^{\infty} \mathcal{B}^{(n)} \right)^{-}$$

$$\mathcal{A}_\infty = \lim_{m \to \infty} \mathcal{A}_m, \quad \mathcal{B}_\infty = \left( \bigcup_{m=1}^{\infty} \mathcal{A}_m \right)^{-}$$

(4.1)
Theorem 4.1. - For any normal sequence \( \{ \mu_n \}_{n=1}^{\infty} \)

a) \( \mathcal{A}^{(n)} = \mathcal{B}^{(n)}, \ n \in \mathbb{N} \)

b) \( \mathcal{A}_\infty = \mathcal{B}_\infty \subset \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)} \)

c) \( \mathcal{A}_\infty = A_m \lambda_m A_m^{-1}, \ m \in \mathbb{N} \)

Proof. - By th. 3.6, 3.7 and cor. 3.9

\[ \mathcal{A}_m = A_m \lambda_m = A_1 \lambda_1 S(v_1^{(m)})^{-1}, \quad m \in \mathbb{N} \]

\[ \mathcal{A}_\infty = \lim_{m \to \infty} \mathcal{A}_m = A_1 \lambda_1 \left( \lim_{m \to \infty} S(v_1^{(m)})^{-1} \right) = A_1 \lambda_1 A_1^{-1} \]

\[ \mathcal{B}_\infty = \left( \bigcup_{m=1}^{\infty} \mathcal{A}_m \right)^- = A_1 \lambda_1 \left( \bigcup_{m=1}^{\infty} S(v_1^{(m)})^{-1} \right)^- = A_1 \lambda_1 B_1^{-1} = A_1 \lambda_1 A_1^{-1} \]

For \( m > 1 \) and any \( x_1^{(m-1)} \in S(v_1^{(m-1)}), \ x_1^{(m-1)} \in S(v_1^{(m-1)}) \)

\[ A_m \lambda_m A_m^{-1} = A_1 \lambda_1 \left( x_1^{(m-1)} \right) A_m^{-1} = A_1 \lambda_1 A_1^{-1} \]

Further, for any fixed \( n \) the sequence \( \{ v_1^{(m)} \}_{m=1}^{\infty} \) is normal since \( \{ \mu_m \}_{m=1}^{\infty} \)

The set \( \mathcal{A}^{(\infty)} = \bigcap \left( \bigcup (\mathcal{B}^{(n)})^- \right) \)

contains of all limit points of all possible sequences \( \{ v_1^{(m)} \}_{m=1}^{\infty} \), \( v_1^{(m)} \in \mathcal{B}^{(n)} \). Hence \( \mathcal{A}_m \subset \mathcal{A}^{(\infty)} \) for all \( m \) and \( \mathcal{A}^{(\infty)} \subset \mathcal{A}^{(\infty)} \).

The inclusion \( \mathcal{A}^{(\infty)} \subset \mathcal{B}^{(\infty)} \) is obvious. ■

We shall call the set \( \mathcal{A}_\infty \) the convolutional attractor (CA) of the normal sequence \( \{ \mu_n \}_{n=1}^{\infty} \). The equality

\[ \hat{\mu}_n = \mu_n \ast \lambda_n, \quad n \in \mathbb{N} \]

defines the “limiting sequence” \( \{ \hat{\mu}_n \}_{n=1}^{\infty} \) for \( \{ \mu_n \}_{n=1}^{\infty} \) such that the sequences \( v_1^{(m)} \) and

\[ \hat{v}_m^{(m)} = \hat{\mu}_{m+n-1} \ast \cdots \ast \hat{\mu}_m, \quad m, n \in \mathbb{N} \]

are asymptotically equivalent as \( n \to \infty \), that is

\[ \lim_{n \to \infty} (v_1^{(m)} - \hat{v}_m^{(m)}) = 0, \quad m \in \mathbb{N} \]

It is easy to see that the CA

\[ \mathcal{A}_\infty = \lim_{m \to \infty} L m P_n \to \infty \hat{v}_m^{(n)} \]

of the sequence \( \{ \hat{\mu}_n \}_{n=1}^{\infty} \) coincides with \( \mathcal{A}_\infty \) and moreover

\[ \mathcal{A}_\infty = \mathcal{A}_\infty = (\hat{v}_m^{(n)}, \ m \in \mathbb{N}, n \in \mathbb{N})^- \quad (4.2) \]

Let us describe now the set \( \mathcal{B}_\infty \) of all idempotents of \( \mathcal{A}_\infty \).
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Corollary 4.2. For all \( m \in \mathbb{N} \)

\[
\mathcal{A}_\infty = \{ \alpha \in \mathcal{A}_\infty : \alpha^2 = \alpha \} = \{ \tilde{\alpha} * \alpha, \alpha \in \mathcal{A}_\infty \}
\]

\[
= \{ \alpha * \tilde{\alpha}, \alpha \in \mathcal{A}_\infty \} = L_m P_n \rightarrow \gamma_m(n) * \gamma_m(n)
\]

\[
= (\lambda_m, n \in \mathbb{N})^{-1} = \{ x \lambda_m x^{-1}, x \in A_m \}
\]

This is a direct consequence of the equality \( \mathcal{A}_\infty = A_m \lambda_m A_m^{-1}, m \in \mathbb{N} \), (see th. 4.1c).

Corollary 4.3. Let \( K = [K_m, m \in \mathbb{N}]^- \) be the smallest compact subgroup containing the subgroups \( K_m, m \in \mathbb{N} \). Then

\[
K = [ \bigcup_{v \in \mathcal{A}(\infty)} S(\tilde{\gamma} * v)]^- = [ \bigcup_{v \in \mathcal{A}(\infty)} S(\tilde{\gamma} * v)]^{-1}
\]

\[
= [S(\lambda), \lambda \in \mathcal{E}_\infty]^{-1} = [x K_m x^{-1}, x \in A_m]^{-1}, m \in \mathbb{N}
\]

and \( K \) is a subgroup of the group \( H = [ \bigcup_{v \in \mathcal{A}(\infty)} S(v)]^- \).

We are going to elucidate now when the CA forms a group of measures and when the sequence \( \gamma_1(n) * \gamma_1(n) \) converges (cf. ex. 3.4).

Theorem 4.4. The following conditions are related by

1) \( \lambda_m = \lambda_1, m \in \mathbb{N} \),
2) \( \lambda_m = \lambda_K, m \in \mathbb{N} \),
3) there exists \( \lim_{n \to \infty} \gamma_m(n) * \gamma_m(n) \),
4) \( \lim_{n \to \infty} \gamma_m(n) * \gamma_m(n) = \lambda_K \),
5) \( \lambda_K \in \mathcal{B}(\infty) \),
6) \( \mathcal{A}_\infty \) is a subgroup of the semigroup \( \mathcal{M}_1(G) \),
7) \( \mathcal{A}_\infty = \lambda_K H \),
8) \( \mathcal{A}(n) = \mathcal{A}(1)^* \ldots \mathcal{A}(1)^{(n)} \) (n-times), \( n \in \mathbb{N} \).

Proof. 1), 2), 3), 4) are equivalent by cor. 4.2 and 4.3.

2) \( \Rightarrow 5) \lambda_K = \lambda_1 \in \mathcal{A}_1 \subset \mathcal{B}(\infty) \) by th. 3.6 b).

5) \( \Rightarrow 2) \) If \( \lambda_K \in \mathcal{B}(\infty) = \{ \gamma_m(n), m, n \in \mathbb{N} \} \), then \( \lambda_K \in \{ \gamma_m(n) * \gamma_k, m, n \in \mathbb{N} \} \) and \( \lambda_K \in \{ \lambda_m, m \in \mathbb{N} \} \).

Thus \( \mathcal{E}_\infty = \{ \lambda_K \} \) and \( \lambda_K = \lambda_m, m \in \mathbb{N} \).

7) \( \Rightarrow 5) \) is obvious.

6) \( \Rightarrow 7) \) If \( \mathcal{A}_\infty \) is a group, the set \( \mathcal{E}_\infty = \{ \lambda_m, m \in \mathbb{N} \} \) of all its idempotents coincides to \( \{ \lambda_K \} \). Then K is a normal subgroup of H, the group \( \mathcal{A}_\infty \) has the form

\[
\mathcal{A}_\infty = A_1 \lambda_K A_1^{-1} = (A_1 A_1^{-1}) \lambda_K \subset H \lambda_K
\]

The group \( \mathcal{A}_\infty \) contains also the sets \( (A_1 A_1^{-1})^n \lambda_K, n \in \mathbb{N} \) and hence \( H \lambda_K \subset \mathcal{A}_\infty \).

7) \( \Rightarrow 6) \) since K is a normal subgroup of H in this case.

8) \implies 6). If 8) holds the set \( \mathcal{A}^{(\infty)} = \left( \bigcup_{n=1}^{\infty} \mathcal{A}^{(n)} \right)^{-} \) is a semigroup and \\
\( \mathcal{A}^{(\infty)} = \bigcap_{m=1}^{\infty} \left( \bigcup_{n=m}^{\infty} \mathcal{A}^{(n)} \right)^{-} \) is a subsemigroup of \( B^{(\infty)} \). Hence \( \lambda_m \ast \lambda_n \in \mathcal{A}^{(\infty)} \) for all \( m, n \in \mathbb{N} \), and \( \lambda_K \in \mathcal{A}^{(\infty)} \subseteq B^{(\infty)} \), since \( \lambda_K \) is contained in the compact semigroup generated by \( \mathcal{E}_{\infty} = \{ \lambda_n, n \in \mathbb{N} \}^{-} \). Using 5) \( \Rightarrow 2) \) we see that \\
\( \mathcal{E}_{\infty} = \{ \lambda_K \} \).

Then \( \nu_m^{(n)} \ast \lambda_K = \lambda_K \ast \nu_n^{(m)} \in \mathcal{A}_{\infty} \) for all \( m, n \in \mathbb{N} \) and \\
is the smallest left and in the same time right ideal of the compact semigroups \( \mathcal{A}^{(\infty)} \) and \( B^{(\infty)} \). Thus \( \mathcal{A}_{\infty} \) is a group ([5], 9.22). 

Remark 4.5 a) The conditions 1)-5) do not imply 6) in a general case. For example, if \( \mu_{2k} = \lambda x \), \( \mu_{2k-1} = \lambda x^{-1} \), \( k \in \mathbb{N} \), where \( \lambda^2 = \lambda = x \lambda x^{-1} \) and \( \lambda x^2 \neq \lambda \), one has the normal sequence \( \{ \mu_n \} \) with \( \mathcal{E}_{\infty} = \{ \lambda \} \) and \\
\( \mathcal{A}_{\infty} = \{ \lambda, \lambda x, \lambda x^{-1} \} \) which is not a group and even semigroup.

b) Remember that the smallest two-sided ideal of a compact semigroup is called its Sushkevich kernel. ([5], 9.21). We have proved now that provided condition 8) of th. 4.4 holds the CA \( \mathcal{A}_{\infty} \) of a normal sequence \( \{ \mu_n \} \) is the Sushkevich kernel of the semigroups \( B^{(\infty)} \) and \( \mathcal{A}^{(\infty)} \) and it is a group.

It should be also noted that both inclusions \( \mathcal{A}_{\infty} \subseteq \mathcal{A}^{(\infty)} \subseteq B^{(\infty)} \) may be strict (see sec. 6).

As a consequence of the above results we can prove now the convergence theorem.

Denote \( D_m = \lim_{n \to \infty} S(\nu_m^{(n)}) \), \( m \in \mathbb{N} \).

**Theorem 4.6.** For any normal sequence \( \{ \mu_n \}_{n=1}^{\infty} \) the following conditions are equivalent

1) \( \lim_{n \to \infty} \nu_m^{(n)} \) exists,
2) \( \lim_{n \to \infty} \nu_m^{(n)} = \lambda_H \) for all \( m \in \mathbb{N} \),
3) \( K_m = H \),
4) \( A_m = D_m \),
5) \( D_m \neq \emptyset \),
6) \( \lambda_H \in B^{(\infty)} \),

Each of the conditions 1)-5) holds for all \( m \in \mathbb{N} \) if it does for some one.

**Proof.** 2) \( \Rightarrow 1) \) and 4) \( \Rightarrow 5) \) are obvious

1) \( \Rightarrow 3) \) If \( \mathcal{A}_m = \lambda_m \mathcal{K}_m \) consists of the only point then \( \mathcal{A}_m \subseteq \mathcal{K}_m \) and hence \( \mathcal{K}_m = H \).
3) \(\Rightarrow\) 2) If \(K_m = H\) then \(\mathcal{A}_m = A_m \lambda_m = A_m \lambda_H = \{\lambda_H\}\),

2) \(\Rightarrow\) 4) \(H = S(\lim_{n \to \infty} v_m^{(n)}) \subseteq D_m \subseteq A_m \subseteq H\),

2) \(\Rightarrow\) 6) \(\lambda_H = \lim_{n \to \infty} v_m^{(n)} \in \mathcal{A}_\infty \subseteq \mathcal{B}(\infty)\),

6) \(\Rightarrow\) 2) If \(\lambda_H \in \mathcal{B}(\infty) = (v_m^{(n)}, m, n \in \mathbb{N})^-\),

then \(\lambda_H \in (v_m^{(n)} \ast \lambda_m, m, n \in \mathbb{N})^- = \mathcal{A}_\infty = A_m \lambda_m A_m^{-1}\),

and \(\mathcal{A}_\infty = \{\lambda_H\}\) i.e. 2) holds

5) \(\Rightarrow\) 3) If \(x \in D_m\) then for every open \(U \ni x\) there exists \(n_0\) such that \(U \cap S(v_m^{(n)}) \neq \emptyset\) for all \(n > n_0\). Hence for \(x_m^{(n)} \in S(v_m^{(n)}) \cap U\) we have

\[S(v_m^{(n)}) \subseteq x_m^{(n)} K_m \subseteq UK_m, \quad n > n_0\]

and

\[A_m = \overline{\lim_{n \to \infty} S(v_m^{(n)})} \subseteq UK_m\]

If \(U\) runs the filter of open neighborhoods of \(x\) the open set \(UK_m\) runs the filter of neighborhoods of \(x K_m\). We have now

\[K_m \subseteq A_m \subseteq x K_m\]

Hence \(A_m \subseteq K_m\) and \(H \subseteq K_m\) and \(H = K_m\).

We have proved now 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3) \(\Rightarrow\) 1) and 2) \(\Rightarrow\) 6) and 2) \(\Rightarrow\) 4) \(\Rightarrow\) 5) \(\Rightarrow\) 3). \(\blacksquare\)

In the simplest case, when \(v_m^{(n)} = \mu \ast \ldots \ast \mu\) \((n\text{-times})\), the theorem proved above is the well known Ito-Kavada theorem (see [1], [2], [4], [5], [7] and [8], ch. 2). For Borel normal sequences the implications 1) \(\Leftrightarrow\) 2) \(\Leftrightarrow\) 3) have been proved by Urbanik [3]. The convergence of convolutions \(v_m^{(n)}\) as \(n \to \infty\) for every \(m\) to the same limit means the compositional convergence in the Maksimov sense [6].

5. THE PROOF OF THE MAIN THEOREMS 1.1-1.4

In this section we shall deduce the main results stated in the introduction from the theorems of the sec. 3 and 4.

Consider a SSRM \(\{\mu_n\}_{n=1}^{\infty}, \mu_n = \mu_n(\omega), \omega \in \Omega\), on \(G\) which satisfies the conditions A) and B) and let \(v_m^{(n)} = v_m^{(n)}(\omega), m, n \in \mathbb{N}\) be the corresponding sequences of their convolutions defined in the sec. 1. We shall use again the notations of the sec. 1.

Let \(\mathcal{A}^{(n)} \subseteq \mathcal{M}_1(G)\) be the support of the distribution \(P \ast (v_m^{(n)})^{-1}\) of the random measures \(v_m^{(n)}(\omega)\). (It does not depend on \(m\)). Also \(\mathcal{A}^{(m)}(\omega) = L m P_m \to \infty v_m^{(n)}(\omega), \mathcal{B}^{(n)}(\omega) = (v_m^{(n)}(\omega), m, n \in \mathbb{N})^-\).

Assertion 5. 1. - a) \(\{v_m^{(n)}(\omega)\}_{m=1}^{\infty}\) is a normal sequence in \(\mathcal{M}_1(G)\) for a.a. \(\omega\).
b) \( \mathcal{A}^{(n)} = \mathcal{A}^{(n)}(\omega) = \mathcal{B}^{(n)}(\omega) \) for a.a.\( \omega \).

Proof. – One can transfer the SSRM \( \{ v^{(n)}_m \}_{m=1}^{\infty} \) onto the space \((\Omega, \mathcal{P})\) of its realizations by the mapping

\[
\varphi: \quad \Omega \ni \omega \mapsto \{ v^{(n)}_m(\omega) \}_{m=1}^{\infty} \in \bar{\Omega}
\]

where \( \bar{\Omega} \) is a compact subset of the countable direct product of the copies of \( \mathcal{A}^{(n)} \). The compact set \( \bar{\Omega} \) has a countable base of the topology by B). Herewith the shift transformation \( \theta \) on \( \bar{\Omega} \) preserves the measure \( \mathcal{P} = \mathcal{P} \circ \varphi^{-1} \) and \( \theta \) is ergodic by A).

One can now deduce a) and b) from the Poincare recurrence theorem and ergodicity of \( \theta \), considering the countable system of open sets

\[
U_{k_1} \times \ldots \times U_{k_m}, \quad m \in \mathbb{N},
\]

where \( \{ U_k \}_{k=1}^{\infty} \) is a base of the topology on \( \mathcal{A}^{(n)} \) (see [1], ch. 1 § 1, § 2).

Consider now the sets \( \mathcal{A}^{(\infty)}, \mathcal{B}^{(\infty)} \) and the subgroups \( K, H \) defined in sec. 1 and denote

\[
\mathcal{A}^{(\infty)}(\omega) = \lim_{n \to \infty} \mathcal{A}^{(n)}(\omega), \quad \mathcal{B}^{(\infty)}(\omega) = \left( \bigcup_{n=1}^{\infty} \mathcal{B}^{(n)}(\omega) \right)^{-},
\]

\[
H(\omega) = [S(\mu_n(\omega)), \ n \in \mathbb{N}]^{-}, \quad K(\omega) = [S(\mathcal{V}^{(n)}_m(\omega) \ast \mathcal{V}^{(m)}_n(\omega)), \ m, n \in \mathbb{N}]^{-}
\]

Assertion 5.2. – a) \( \mathcal{A}^{(\infty)}(\omega) = \mathcal{A}^{(\infty)}(\omega) = \mathcal{B}^{(\infty)}(\omega) \)

b) \( H(\omega) = H, \ k(\omega) = K \)

for a.a.\( \omega \in \Omega \).

This assertion follows immediately from the above one.

Consider now the CA

\[
\mathcal{A}^{(\infty)}(\omega) = \lim_{m \to \infty} \lim_{n \to \infty} \mathcal{P}_n \mathcal{V}^{(n)}_m(\omega)
\]

of the sequence \( \{ \mu_n(\omega) \}_{n=1}^{\infty} \) with a fixed \( \omega \).

Assertion 5.3. – The mapping \( \omega \to \mathcal{A}^{(\infty)}(\omega) \) is constant a.s.

Proof. – The sequence \( \{ \mu_n(\omega) \}_{n=1}^{\infty} \) is normal for a.a.\( \omega \). For any such \( \omega \) the limit

\[
\lambda_m(\omega) = \lim_{n \to \infty} \mathcal{V}^{(n)}_m(\omega) \ast \mathcal{V}^{(n)}_m(\omega)
\]

exists and one can consider the limiting sequence \( \tilde{\mu}_m(\omega) = \mu_m(\omega) \ast \lambda_m(\omega) \).

For the corresponding \( n \)-th convolutions \( \mathcal{V}^{(n)}_m(\omega) = \mathcal{V}^{(n)}_m(\omega) \ast \lambda_m(\omega) \) the equality

\[
\mathcal{A}^{(\infty)}(\omega) = \mathcal{A}^{(\infty)}(\omega) = \mathcal{B}^{(\infty)}(\omega)
\]

holds on account of (4.2), where

\[
\mathcal{A}^{(\infty)}(\omega) = \lim_{m \to \infty} \lim_{n \to \infty} \mathcal{P}_n \mathcal{V}^{(n)}_m(\omega)
\]
and
\[ \hat{\mathcal{B}}^{(\infty)}(\omega) = (\hat{v}_m^{(\omega)}(\omega), m, n \in \mathbb{N})^- . \]

On the other hand applying the above assertion for the SSRM \( \{ \mu_n \}_{n=1}^{\infty} \), one can see that the mapping \( \omega \rightarrow \hat{\mathcal{B}}^{(\infty)}(\omega) \) is constant a.s. ■

In order to prove the rest statements of the theorems 1.1-1.4 and to complete their proofs one can apply now the results of the sec. 3 and 4 for a fixed normal sequence \( \{ \mu_n(\omega) \}_{n=1}^{\infty} \).

6. EXAMPLES

We shall give here three simple examples of CA to illustrate the objects under consideration.

1) Let \( x_n = x_n(\omega), n \in \mathbb{N} \) be i.i.d. random elements on \( \Omega \) with the values in a compact group \( G \) which has a countable base of its topology. Define the SSRM
\[ \mu_n(\omega) = \delta_{x_{n+1}(\omega)} . x_n(\omega)^{-1}, \quad n \in \mathbb{N}, \quad \omega \in \Omega \]
and denote by \( S \) the essential image of \( x_n \).

In this case we obtain for a.a.\( \omega \)
\[ \hat{v}_m^{(\omega)}(\omega) = \delta_{x_{n+m}(\omega)} . x_m(\omega)^{-1}, \quad m, n \in \mathbb{N} \]
\[ S = (x_n(\omega), n \in \mathbb{N})^- = L_{m \rightarrow \infty} x_n(\omega), \quad H = [S]^- \]
\[ A_m(\omega) = S x_m(\omega)^{-1}, \quad \mathcal{A}_m(\omega) = \{ \delta_x, x \in S . x_m(\omega)^{-1} \} \]
\[ K = K_m(\omega) = \{ e \}, \quad \mathcal{E}_\infty = \{ \delta_e \} \]
\[ \mathcal{A}(\omega) = \mathcal{A}(\omega) = \mathcal{A}(\omega) = \{ \delta_x, x \in \mathbb{SS}^{-1} \} \]

One can see that \( A_m(\omega) \) and \( \mathcal{A}_m(\omega) \) essentially depend on \( m \) and \( \omega \) and the CA \( \mathcal{A}_\infty \) need not coincide with \( \lambda_K H \).

2) Let \( H \) be a subgroup of a finite group \( G \) and \( K_0 \) be a non-normal subgroup of \( H \).

Denote
\[ \Gamma = \{ x \lambda_0 y^{-1}, x, y \in H \} \subset \mathbb{B}_1(G) \]
where \( \lambda_0 \) denotes the Haar measure of \( K_0 \).

Consider the SSRM \( \mu_n = \mu_n(\omega), n \in \mathbb{N} \), which is a Markov chain with the finite state space \( \Gamma \), and transition probability matrix
\[ Q = \{ q_{\alpha \beta} \}_{\alpha, \beta \in \Gamma}, \text{ where } q_{\alpha \beta} = P \{ \mu_{n+1} = \beta | \mu_n = \alpha \}, \]
and stationary vector of probabilities
\[ q_\alpha = P \{ \mu_n = \alpha \}, \quad \alpha \in \Gamma \]
We demand that the transition matrix $Q$ satisfies the condition

$$q_{a0} > 0 \iff \beta \ast \alpha \in \Gamma$$

(ones sees $\beta \ast \alpha \in \Gamma \iff \alpha \ast \tilde{\alpha} = \tilde{\beta} \ast \beta$).

This Markov chain is mixing and for the corresponding convolutions $\nu_m^{(n)}(\omega)$ we have a.s. for $n, m \in \mathbb{N}$

$$A_m(\omega) = H, \quad \mathcal{E}_\omega = \{x \lambda_0 x^{-1}, x \in H\}$$

$$\lambda_m(\omega) = \mu_m(\omega) \ast \mu_m(\omega), \quad K_m(\omega) = S(\lambda_m(\omega))$$

$$\mathcal{A}_m(\omega) = H \lambda_m(\omega), \quad K = [x K_0 x^{-1}, x \in H]$$

and

$$\mathcal{A}_\omega = \mathcal{A}^{(\omega)} = \mathcal{B}^{(\omega)} = \Gamma$$

Since $K_0$ is not a normal subgroup of $H$ the CA $\mathcal{A}_\omega = \Gamma$ contains a nontrivial set of idempotents $\mathcal{E}_\omega = \{x \lambda_0 x^{-1}, x \in H\}$.

As in the example 1) the SSRM $\{\mu_n\}$ coincides with its limiting sequence $\{\hat{\mu}_n\}$ and $\mathcal{A}_\omega = \Gamma$.

3) We can change the previous example extending the state space of the considering Markov chain as follows

$$\Gamma' = \Gamma \cup \{\delta_x, x \in H\}$$

and taking the matrix $Q' = \{q'_a, \beta\}_{a, \beta \in \Gamma}$ which satisfies the same requirement (6.1) as $Q$.

Then the CA $\mathcal{A}_\omega$ of the obtained SSRM $\{\mu_n\}_{n=1}^\infty$ coincides with $\Gamma$ which is not equal to $\Gamma'$ and we have the strict inclusion $\mathcal{A}_\omega \subset \mathcal{A}^{(\omega)}$ in such case.

One can construct a lot of different examples of CA replacing the "$\iff$" in the condition (6.1) on "$\Rightarrow$" or considering generalization on the continuous state space case.

REFERENCES

CONVOLUTIONAL ATTRACTORS


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