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Superprocesses and projective limits of branching Markov process

by

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ABSTRACT. — We present an a. s. limit theorem to construct Watanabe-Dawson “superprocesses” starting from a projective system of branching Markov processes.

RÉSUMÉ. — Cet article présente un théorème de limite presque sûre permettant d’obtenir des super-processus de Watanabe-Dawson à partir d’un système projectif de processus de branchement markoviens.

Branching Markov processes are generally considered as measure valued processes. However, as stressed by Neveu [N1], even in the case of the simplest Galton-Watson processes, the genealogy of the points can be also usefully described and a formalization in terms of trees of paths can be given as explained in the following. We will consider a family of such processes indexed by $\tau \in \mathbb{R}^+$. The paths start at a point x with a number of points following a Poisson law of mean $\psi\tau^{-1/\beta}$, and points are moving as independent Markov processes of semigroup P_t and branching at exponential times of expectation τ according to a reproduction law $\{q\}$

defined by the generating function :

$$\hat{q}(u) = \sum q_k u^k = \frac{1}{1+\beta} (1-u)^{1+\beta} + u,$$

β being a parameter in $]0, 1[$.

Denote $\tilde{Z}_t^{(\tau)}$ the induced multiplet-valued process and $\tilde{Z}_{t,s}^{(\tau)}$ the multiplet of the ancestors of $\tilde{Z}_t^{(\tau)}$ at time $s < t$. One can check, as in [N2] that $(\tilde{Z}_{t+h,\tau}^{(\tau)}, t \in \mathbb{R}^+)$ and $(\tilde{Z}_t^{(\tau+h)}, t \in \mathbb{R}^+)$ have the same distributions.

Taking a projective limit, we can define the processes simultaneously in such a way that the above identities hold almost surely, for all τ and $h > 0$.

By a martingale argument, it appears that the measure $\psi^{-1} \varepsilon^{1/\beta} \tilde{Z}_{t-\varepsilon}^\varepsilon \mathbf{P}_\varepsilon$ converges a. s. towards a measure μ_t as $\varepsilon \downarrow 0$, and that $(\mu_t, t > 0)$ is a Markov process.

If $U_t f(x) = -\text{Log}(E(e^{-\mu_t(f)}))$, we have

$$U_t f = P_t f - \frac{\psi^{-\beta}}{1+\beta} \int_0^t P_s ((U_{t-s}(f))^{1+\beta}) ds$$

We recover a class of the measure valued process defined by Watanabe and Dawson, called superprocesses by Dynkin and studied by many authors : see [W], [Da], [D], [F1], [I], [P], [LG], [R].

When $\beta = 1$, this construction is closely related to the one given by Le Gall in [LG1], using the Brownian excursion to generate the branching system cf. [NP], [LG2].

These results have been announced in [LJ].

Remarks : (a) Considering a general semi-group P_t instead of, for example, diffusions, has some drawbacks (measurability problems need more care) but also advantages. It allows to apply the theory to path-valued processes.

(b) The three first paragraphs introduce basic notions and technical (mostly measure-theoretical) preliminaries.

(c) Proofs of intuitively obvious properties have been only briefly sketched.

1. PATHS WITH EXPONENTIAL LIFETIME

Let E be a Polish space, with \mathcal{E} its Borel σ -field. Let $P_t(x, dy)$ be a semi-group of markovian kernels on E . Let F be the space of mappings ω from a finite interval $]S(\omega), T(\omega)]$ into E . Denote X_t its coordinates ($\omega(t) = X_t(\omega)$). Let \mathcal{F}^0 be the σ -field generated by the sets $\{T \geq t > S\} \cap \{X_t \in A\}$, ($t \in \mathbb{R}$, $\mathcal{A} \in \mathcal{E}$) and set $\mathcal{F} = \mathcal{F}^0 \vee \{X_T^{-1}(A), A \in \mathcal{E}\}$. T and S are \mathcal{F}^0 -measurable. The restriction of \mathcal{F} to

$$E^{[s, t]} = T^{-1}(t) \cap S^{-1}(s) \quad \text{is } \sigma(X_u, s < u \leq t).$$

Kolmogorov's theorem allows to construct a unique system of probabilities $\mathbb{P}_{x, s, t}$ on $E^{[s, t]}$, $\sigma(X_u, u \in]s, t])$ such that :

$$\mathbb{P}_{x, s, t}(\cap \{X_{u_i} \in A_i\}) = P_{u_i - s} 1_{A_1} \dots P_{u_n - u_{n-1}} 1_{A_n}(x)$$

for $s < u_1 \leq \dots \leq u_n \leq t$ and $A_i \in \mathcal{E}$.

Using a generating semi algebra (cf. [N3], § I-6) of \mathcal{F} , composed with sets of the form $\{S < t_1\} \cap \{t_n \leq T\} \cap \cap_i \{X_{t_i} \in A_i\} \cap \{X_T \in B\}$ with $t_1 < t_2 < \dots < t_n$, A_i and B belonging to \mathcal{E} , and the monotone class theorem, it appears that :

(1) For any $A \in \mathcal{F}$, $\mathbb{P}_{x, s, t}(A \cap E^{[s, t]})$ is a measurable function of (x, s, t) .

For any parameter $\tau > 0$, we can define a probability $\mathbb{P}_{x, s}^\tau$ on (F, \mathcal{F}) , carried by $\{S = s\}$, by setting

$$\mathbb{P}_{x, s}^\tau(A) = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} \mathbb{P}_{x, s, s+t}(A \cap E^{[s, s+t]}) dt.$$

For any $\omega \in F$ and $u \in]S(\omega), +\infty[$ ($]-\infty, T(\omega)[$), define $k_u(\omega)$, $(\theta_u \omega)$ to be the restriction of ω to $]S(\omega), T(\omega) \wedge u]$, $(S(\omega) \vee u, T(\omega)]$.

For $0 \leq h < T(\omega) - S(\omega)$, let $e_h(\omega)$ be the restriction of ω to $]S(\omega), T(\omega) - h]$, i. e. $k_{T(\omega) - h} \omega$.

The following measurability properties are easy to check :

(2) For any $A \in \mathcal{F}$ and $u \in \mathbb{R}$, $k_u^{-1}(A)$ and $\theta_u^{-1}(A)$ belong to \mathcal{F} .

(3) For any $B \in \mathcal{F}^0$ and $h \in \mathcal{R}^+$, $e_h^{-1}(B)$ belongs to \mathcal{F}^0 .

If ω and ω' belong to F and $S(\omega') = T(\omega)$, the path $\omega \omega'$ is defined on $]S(\omega), T(\omega')]$ by : $\omega \omega' = \omega$ on $]S(\omega), T(\omega)]$, $\omega \omega' = \omega'$ on $]S(\omega'), T(\omega')]$.

(4) For any $A \in \mathcal{F}$, $\{(\omega, \omega'), \omega \omega' \in \mathcal{A}\}$ is $\mathcal{F}_0 \otimes \mathcal{F}$ measurable.

The following properties can be easily checked.

P1 (Markov property).

For any $A, B \in \mathcal{F}$, $t > s \in \mathbb{R}$, $x \in E$,

$$\mathbb{P}_{x, s}^\tau(k_t^{-1}(A) \cap \theta_t^{-1}(B)) = \mathbb{E}_{x, s}^\tau(k_t^{-1}(A) \mathbb{P}_{X_t, t}^\tau(\theta_t^{-1}B))$$

P2 For any $A \in \mathcal{F}^0$ and $h > 0$, $\mathbb{P}_{x,s}^\tau(e_h^{-1}(A)) = e^{-h/\tau} \mathbb{P}_{x,s}^\tau(A)$

P3 For any $A \in \mathcal{F}$ and $\tau_1, \tau_2 > 0$,

$$\int \{ \omega \omega' \in A \} \mathbb{P}_{x,s}^{\tau_1}(d\omega) \mathbb{P}_{X_T(\omega), T(\omega)}^{\tau_2}(d\omega') = \mathbb{P}_{x,s}^{\tau_1 + \tau_2}(A).$$

2. BRANCHING PATHS

Let U be the space of empty or finite sequences of positive integers, *i. e.* “words”. There is a natural composition law on U as well as a lexicographical order induced by the order on $\mathbb{N} - \{0\}$. Define the mapping π from $U - \{\emptyset\}$ onto U by $\pi(j_1 \dots j_n) = j_1 \dots j_{n-1}$. Set $|j_1 \dots j_n| = n$, and $|\emptyset| = 0$, to define the length of a word.

Following Neveu [N1], we define trees as sets of words δ containing \emptyset and such that if $uj \in \delta$, for $j \in \mathbb{N} - \{0\}$ and $u \in U$, then $u \in \delta$ and also $ui \in \delta$ for any $i \leq j$.

Define the height $|\delta|$ of a tree as the length of the longest word it contains. If δ is a tree, and $u \in \delta$, then $T_u \delta = \{v \in U, uv \in \delta\}$ is a tree.

Set $v(\delta) = \sup\{j \in \mathbb{N} - \{0\}, j \in \delta\}$ if $\delta \neq \{\emptyset\}$, and $v(\{\emptyset\}) = 0$.

Let \hat{F} be the space of branching paths $\alpha = (\alpha_0, \xi_\alpha)$ defined by a tree α_0 and a mapping ξ_α from α_0 into F , such that $T(\xi_\alpha(\pi(u))) = S(\xi_\alpha(u))$ for any $u \in \alpha_0 - \emptyset$.

F is imbedded in \hat{F} in a canonical way. If $\omega \in F$, we identify it with the element ω' of \hat{F} such that $\omega'_0 = \{\emptyset\}$ and $\xi_{\omega'}(\emptyset) = \omega$.

We set $v(\alpha) = v(\alpha_0)$, $|\alpha| = |\alpha_0|$.

If $u \in \alpha_0$, $T_u \alpha$ is defined by $(T_u \alpha)_0 = T_u(\alpha_0)$ and $\xi_{T_u \alpha}(v) = \xi_\alpha(uv)$.

Set $T_\alpha(u) = T(\xi_\alpha(u))$, $X_\alpha(u) = X_T(\xi_\alpha(u))$ and $S_\alpha(u) = S(\xi_\alpha(u))$ for any $u \in \alpha_0$. We define also $\zeta(\alpha) = \sup_{u \in \alpha_0} T_\alpha(u)$, $\sigma(\alpha) = S_\alpha(\emptyset)$.

Let $\hat{\mathcal{F}}$ be the smallest σ -algebra on \hat{F} containing the sets

$$\Lambda_{u,A} = \{ \alpha \in \hat{F}, u \in \pi(\alpha_0) \text{ and } X_\alpha(u) \in A \},$$

for all $u \in U$, $A \in \mathcal{E}$ and

$$\Gamma_{u,B} = \{ \alpha \in \hat{F}, u \in \alpha_0 \text{ and } \xi_\alpha(u) \in B \},$$

for all $u \in U$, $B \in \mathcal{F}^0$.

(5) If $\Xi \in \hat{\mathcal{F}}$, $\Xi^{(i)} = \{ \alpha, i \in \alpha_0 \text{ and } T_i \alpha \in \Xi \} \in \hat{\mathcal{F}}$.

(If $\Xi = \Lambda_{u,A}$, $\Xi^{(i)} = \Lambda_{iu,A}$ etc.)

Set $\hat{\mathcal{F}}_0 = \{ \{ \alpha, \xi_\alpha(\emptyset) \in A \} \mid A \in \mathcal{F}^0 \}$, and define by induction the σ -algebra $\hat{\mathcal{F}}_{(n)}$ generated by subsets of \hat{F} of the form :

$$\left\{ \left\{ \alpha, v(\alpha) = 0 \text{ and } \xi_\alpha(\emptyset) \in A_0 \right\} \cup \bigcup_{i=1}^{\infty} \left\{ \alpha, v(\alpha) = i, \xi_\alpha(\emptyset) \in A_i \text{ and } T_j \alpha \in \Xi_{i,j} \text{ for } j \leq i \right\}, \right. \\ \left. A_0 \in \mathcal{F}^0, A_i \in \mathcal{F}, \Xi_{i,j} \in \hat{\mathcal{F}}_{(n+1)} \right\}.$$

($\hat{\mathcal{F}}_{(n)}$ describes the history until the n -th generation).

By (5), $\hat{\mathcal{F}}_{(n)} \subseteq \hat{\mathcal{F}}$ for every n . Moreover $\Lambda_{u, A} \in \hat{\mathcal{F}}_{(|u|+1)}$ and $\Gamma_{u, B} \in \hat{\mathcal{F}}_{(|u|)}$. Hence

(6) $\hat{\mathcal{F}}$ is the σ -algebra generated by the algebra $\cup \hat{\mathcal{F}}_{(n)}$.

Let q be a probability on \mathbb{N} with $\sum_0^\infty k q(k) < \infty$. We shall extend the probabilities $\mathbb{P}_{x,s}^\tau$ to \hat{F} as follows :

PROPOSITION 1. — *There exist a unique system of probabilities $\mathbb{P}_{x,s}^{\tau,q}$ on $(\hat{F}, \hat{\mathcal{F}})$ such that :*

- (a) *For any $\Xi \in \hat{\mathcal{F}}$, $\mathbb{P}_{x,s}^{\tau,q}(\Xi)$ is measurable in (x, s) .*
- (b) *$\mathbb{P}_{x,s}^{\tau,q}$ induces $\mathbb{P}_{x,s}^\tau$ on (F, \mathcal{F}^0) .*
- (c) *For any $A \in \mathcal{F}^0, B \in \mathcal{E}, n \in \mathbb{N} - \{0\}, \Xi_1 \dots \Xi_m \in \hat{\mathcal{F}}_{(n)}$,*

$$\mathbb{P}_{x,s}^{\tau,q}(\{ \alpha, \xi_\alpha(\emptyset) \in A, X_\alpha(\emptyset) \in B, v(\alpha) = m \text{ and } T_i \alpha \in \Xi_i \text{ for } i \leq m \}) \\ = q(m) \mathbb{E}_{x,s}^\tau \left(1_A 1_{\{X_T \in B\}} \prod_{i=1}^m \mathbb{E}_{X_T, T}^{\tau,q}(\Xi_i) \right).$$

Proof. $\mathbb{P}_{x,s}^{\tau,q}$ can be defined by induction on each $\hat{\mathcal{F}}_{(n)}$ [by (c)]. The σ -additivity of the projective limit follows from the existence of a compact class (cf. [N3], § I-6) generated by sets of the form

$$\{ u \in \pi(\alpha_0) \} \cap \{ X_\alpha(u) \in K \}, \\ \{ u \in \alpha_0 \} \cap \{ a \leq S_\alpha(u) \leq b \} \cap \{ c \leq T_\alpha(u) \leq d \} \cap \bigcap_{i=1}^n \{ X_{t_i}(\xi_\alpha(u)) \in K_i \}$$

with $K, K_i \in \mathcal{K}(E)$; where $\mathcal{K}(E)$ denotes the class of compact subsets of E .

N. B. — Set, for any $n \in \mathbb{N}, z_n = \{ u \in \alpha_0, |u| = n \}^*$:

z_n is a Galton Watson process of reproduction law q . Hence, if $\sum k q(k) \leq 1, \alpha_0$ is almost surely finite for any (x, s, τ) .

Similary set $|Z_t| = \{u \in \alpha_0, S_\alpha(u) \leq t < T_\alpha(u)\}^*$: Under $\mathbb{P}_{x,s}^{\tau, q} |Z_{t+s}|$ is a continuous time Galton Watson process with reproduction law q and time constant τ .

We extend k_t to $\{\alpha, \sigma(\alpha) < t\}$ as follows : $(k_t \alpha)_0 = \{u \in \alpha_0, S_\alpha(u) < t\}$ and $\xi_{k_t \alpha} = k_t \xi_\alpha$.

To extend θ_t , set $\hat{Z}_t(\alpha) = \{v \in \alpha_0, t \in [S_\alpha(v), T_\alpha(v)]\}$.

If $u \in \hat{Z}_t(\alpha)$, define $\theta_{t,u}(\alpha)$ as follows :

$(\theta_{t,u} \alpha)_0 = T_u \alpha_0$ and, if $\beta = \theta_{t,u} \alpha$, we define $\xi_\beta(\emptyset) = \theta_t \xi_\alpha(u)$ and $\xi_\xi(v) = \xi_\alpha(uv)$ for any $v \in T_u \alpha_0 - \emptyset$.

Measurability properties

(7) For any $\Xi \in \hat{\mathcal{F}}$, $k_t^{-1}(\Xi)$ is $\hat{\mathcal{F}}$ measurable.

Indeed : $k_t^{-1} \Lambda_{u,A} = \Lambda_{u,A} \cap \{T_\alpha(u) < t\}$ [since $u \in \pi(k_t \alpha)$ if $T_\alpha(u) < t$]

$$k_t^{-1}(\Gamma_{u,B}) = \Gamma_{u, k_t^{-1}(B)} \in \hat{\mathcal{F}}.$$

Set $\hat{\mathcal{F}}_t = \{k_t^{-1} \Xi, \Xi \in \hat{\mathcal{F}}\}$.

(8) For any $u \in U$ and $\Xi \in \hat{\mathcal{F}}$, $\{\alpha \in \hat{\mathcal{F}}, u \in \hat{Z}_t(\alpha) \text{ and } \theta_{t,u}(\alpha) \in \Xi\}$ belongs to $\hat{\mathcal{F}}$.

(Indeed for example $\{u \in \hat{Z}_t\} \cap \theta_{t,u}^{-1}(\Gamma_{v,B}) = \{u \in \hat{Z}_t\} \cap \Gamma_{uv,B}$ if $v \neq \emptyset$, and $\{u \in \hat{Z}_t\} \cap \theta_{t,u}^{-1} \Gamma_{\emptyset,B} = \{u \in \hat{Z}_t\} \cap \Gamma_{u, \theta_t^{-1}(B)}$.)

For $u \in \hat{Z}_t$, set $X(t, u) = X(\pi(u))$ if $t = S(u)$ and $X(t, u) = X_t(\xi(u))$ if $t > S(u)$. Then, we have

PROPOSITION 2. — For any $s < t, u_1, u_2 \dots u_n$ and $\Xi_1, \dots, \Xi_n \in \hat{\mathcal{F}}$,

$$\mathbb{E}_{x,s}^{\tau, q} \left(\bigcap_{i=1}^n \{u_i \in \hat{Z}_t\} \cap \theta_{t,u_i}^{-1}(\Xi_i) \mid \hat{\mathcal{F}}_t \right) = \prod_{i=1}^n 1_{\{u_i \in \hat{Z}_t\}} \mathbb{E}_{X(t, u_i), t}^{\tau, q}(\Xi_i).$$

Sketch of proof. First one shows that for any $A \in \mathcal{F}$ and $\Xi \in \hat{\mathcal{F}}$,

$$\mathbb{P}_{x,s}^{\tau, q} (\{\xi(\emptyset) \in k_t^{-1}(A)\} \cap \theta_{t,\emptyset}^{-1}(\Xi) \cap \{t < T(\emptyset)\}) = \mathbb{E}_{x,s}^{\tau, q} (k_t^{-1}(A) \cap \{t < T\} \mid \mathbb{E}_{X_t, t}^{\tau, q}(\Xi))$$

by using the Markov property P1.

Then, one proves the Proposition by induction on the “degree” $|u_1| + |u_2| + \dots + |u_n|$, using the branching property c in proposition 1 to reduce the degree after choosing test set, in $\hat{\mathcal{F}}_t$ of the form $k_t^{-1}(A)$, with $A \in \hat{\mathcal{F}}_{(n)}$ for some n .

N. B. A stronger version of this proposition has been proved in [C], when P_t is the heat semigroup.

3. GENERATING FUNCTIONAL

For any positive bounded measure μ on E and $f \in \mathcal{E}^+$, $0 < f(x) \leq 1$ for every x , set :

$$f^\mu = \exp\left(\text{Log } f \, d\mu\right).$$

Define $Z_t(\alpha) = \left(\sum_{u \in Z_t(\alpha)} \varepsilon_{X(t,u)}\right)$.

The law of Z_t is determined by the generating functional

$$\Phi_{t-s}^\tau(f)(x) = E_{x,s}^{\tau,q}(f^{Z_t}) \quad (t > s).$$

Applying proposition 1 (c), we have

$$(9) \quad \Phi_t^\tau(f) = e^{-t/\tau} P_t f + \frac{1}{\tau} \int_0^t e^{-s/\tau} P_s \hat{q}(\Phi_{t-s}^\tau(f)) ds$$

\hat{q} being the generating function of q .

(9) can be written in the equivalent form

$$(10) \quad \Phi_t^\tau(f) = P_t f + \frac{1}{\tau} \int_0^t P_s [\hat{q}(\Phi_{t-s}^\tau(f)) - \Phi_{t-s}^\tau(f)] ds$$

Indeed, assuming (9), (10) is equivalent to the identity

$$(1 - e^{-t/\tau}) P_t f + \frac{1}{\tau} \int_0^t (1 - e^{-s/\tau}) P_s \hat{q}(\Phi_{t-s}^\tau(f)) ds = \frac{1}{\tau} \int_0^t (P_{t-s}(\Phi_s^\tau(f))) ds$$

Replacing $\Phi_s^\tau(f)$ by its expression in (9) in the second member, we get the first one by performing an integration.

The semigroup property

$$(11) \quad \Phi_{t+s}^\tau = \Phi_t^\tau \circ \Phi_s^\tau$$

follows from the branching markov property (proposition 2).

For any f , $0 < f \leq 1$, equation (10) has a unique solution $\Phi_t^\tau(f)$ verifying

$$\Phi_0^\tau(f) = f, \quad 0 \leq \Phi_t^\tau(f) \leq 1.$$

[Set $a_t = \Phi_t^{(1)}(f) - \Phi_t^{(2)}(f)$. Then

$$a_t = \int_0^t (P_s (\hat{q}(\Phi_{t-s}^{(1)}) - \hat{q}(\Phi_{t-s}^{(2)})) ds$$

and

$$\|a_t\|_\infty \leq \frac{1}{\tau} \int_0^t \|a_s\|_\infty (1 + \sum k q(k)) ds,$$

hence $a_t = 0$]

The semigroup property follows also from the uniqueness.

(Set $\Phi_t^{(s)} = \Phi_t$ for $t \leq s$ and $\Phi_t^{(s)} = \Phi_{t-s} \circ \Phi_s$ for $t \geq s$, and check $\Phi_t^{(s)}$ verifies the same integral equation as Φ_t .)

Note also that for any bounded $g \in \mathcal{E}^+$, and $t > s$,

$$\mathbb{E}_{x,s}^{\tau,q} \left(\int g dZ_t \right) = - \frac{d}{d\varepsilon} \Phi_{t-s}^\tau(e^{-\varepsilon g}) \Big|_{\varepsilon=0}.$$

Differentiating (10) and solving the resulting linear integral equation yields

$$(12) \quad \mathbb{E}_{x,s}^{\tau,q} \left(\int g dZ_t \right) = \exp \left(\frac{t-s}{\tau} \hat{q}'(1) - \frac{t-s}{\tau} \right) P_t g.$$

Note also that $\mathbb{P}_{x,s}^{\tau,q}(\zeta < t) = \Phi_{t-s}^\tau(0+)$.

In the following we shall be particularly interested in the laws of reproduction $q_\beta, \beta \in]0, 1]$ defined by the generating functions

$$\hat{q}_\beta(u) = \frac{1}{1+\beta} (1-u)^{1+\beta} + u, \quad u \in [0, 1] \quad (cf. [Z], [H]).$$

Then equation (10) can be solved for $f = u$ (const.) to yield

$$\Phi_t^\tau(u) = 1 - \frac{1-u}{(1+t\tau^{-1}(1-u)^\beta)^{1/\beta}}.$$

In particular $\Phi_t^\tau(0+) = 1 - \left(\frac{\tau}{\tau+t} \right)^{1/\beta}$.

4. ERASURE

e_h extends to $\{\zeta - \sigma > h\}$ as follows :

$$(e_h \alpha)_0 = \{u \in \alpha_0, \zeta(\alpha) - h > S_\alpha(u)\}$$

and $\xi_{e_h \alpha}(u) = k_{\zeta(\alpha)-h} \xi_\alpha(u)$ for any $u \in (e_h \alpha)_0$.

(13) For any $\Xi \in \hat{\mathcal{F}}, e_h^{-1}(\Xi) \in \hat{\mathcal{F}}$.

Indeed

$$e_h^{-1} \Lambda_{u, A} = \Lambda_{u, A} \cap \{ \zeta - h > T_\alpha(u) \}$$

$$e_h^{-1} \Gamma_{u, B} = \{ \zeta - h > S_\alpha(u) \} \cap \Gamma_{u, k_{\zeta^{-1}h} B},$$

where $\Gamma_{u, k_{\zeta^{-1}h} B} = \{ \zeta - h \geq t \} \cap \Gamma_{u, B} \in \widehat{\mathcal{F}}$.

PROPOSITION 3. — For any $\Xi \in \widehat{\mathcal{F}}$,

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\Xi)) = \lambda_h \mathbb{P}_{x, s}^{\tau, q^{(h)}}(\Xi)$$

with

$$\lambda_h = \mathbb{P}_{x, s}^{\tau}(\zeta > s + h) = 1 - \Phi_h^{\tau}(0+)$$

and

$$q^{(h)}(k) = \frac{1}{1 - \lambda_h} \left(\sum_{l \geq k} q(l) \binom{l}{k} \left(1 - \lambda_h \right)^k \lambda_h^{l-k} - \lambda_h \delta_{0k} \right).$$

The proof given by Neveu in [N2], p. 106-107, can be easily generalized to this context, using the characterization of $\mathbb{P}_{x, 0}^{\tau, q}$ given in proposition 1 and property P2 to prove the identity for $\Xi \in \widehat{\mathcal{F}}_{(n)}$, by induction on n .

To start the recurrence, we need to compute $\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{\xi \in A\}))$ for $A \in \mathcal{F}_0$.

We show that for $k > 0$,

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{\xi \in A\} \cap \{v = k\})) = \sum_{l \leq k} q(l) \binom{l}{k} (1 - \lambda_h)^{l-k} \lambda_h^l \mathbb{P}_{x, s}^{\tau}(A).$$

Then, if μ denotes the law of ζ - σ under $\mathbb{P}^{\tau, q}$,

$$\begin{aligned} & \mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{\xi \in A\} \cap \{v = 0\})) \\ &= \sum q(l) \int_0^h d_\mu(z_1) \dots \int_0^h d_\mu(z_l) \mathbb{E}_{x, s}^{\tau}(e_{h - \max z_i}^{-1}(A)) \\ &= \text{some constant) } \mathbb{P}_{x, s}^{\tau}(A) \quad \text{by P}_2. \end{aligned}$$

It follows that

$$\mathbb{P}_{x, s}^{\tau, q}(e_n^{-1}(\{\xi \in A\})) = \lambda_h \mathbb{P}_{x, s}^{\tau, q}(A)$$

and

$$\mathbb{P}_{x, s}^{\tau, q}(e_h^{-1}(\{\xi \in A\} \cap \{v = 0\})) = \lambda_h q^{(h)}(0) \mathbb{P}_{x, s}^{\tau}(A).$$

Then one proves the recurrence on events of the form

$$\{\xi \in A\} \cap \{v = k\} \cap \bigcap_{i=1}^k T_i^{-1}(\Xi_i)$$

with $k > 0$, $A \in \mathcal{F}_0$ and $\Xi_i \in \widehat{\mathcal{F}}_{(n)}$.

In particular, if $q = q_\beta$,

$$1 - \lambda_h = \left(\frac{\tau}{\tau + h} \right)^{1/\beta} \quad \text{and} \quad \hat{q}_\beta^{(h)}(u) = \frac{\tau}{\tau + h} \frac{(1-u)^{1+\beta}}{1+\beta} + u.$$

Set $Z_{t,s} = Z_s \circ e_{t-s}$ for $t > s$. Then

$$Z_{t,s}(\alpha) = \sum_{u \in Z_s(\alpha)} \varepsilon_{X(s,u)} 1_{\{\zeta(\theta_{s,u}\alpha) \geq t\}}.$$

Hence for any $t_1 < t_2 < \dots < t_n, f_1, \dots, f_n \in \mathcal{E}^+$,

$$\prod_{i=1}^n f_i^{Z_{t_i,t_i}} = \prod_{u \in Z_{t_1}} \left[f_1(X(t_1, u)) \theta_{t_1, u}^{-1} \left(1_{\{\zeta \geq t_2\}} \prod_{i=2}^n f_i^{Z_{t_i, t_i}} \right) + \theta_{t_1, u}^{-1} (1_{\{\zeta \leq t_2\}}) \right].$$

By the branching Markov property, it follows that for $t_0 < t_1$

$$\mathbb{E}_{x, t_0}^{r, q} \left[\prod_{i=1}^n (f_i^{Z_{t_i, t_i}}) \right] = \Phi_{t_1 - t_0} R_{t_2 - t_1}^{f_1} \dots R_{t_n - t_{n-1}}^{f_n} (f_n)(x)$$

with $R_s^f(g) = f(\Phi_s(g) - \Phi_s(0+) + \Phi_s(0+))$.

Hence, it is easy to check that, for $h > 0$

$$\mathbb{E}_{x, 0}^{r, q} (f^{Z_t} | \sigma(Z_{t,s}, s < t-h)) = \left[\frac{\Phi_h(f) - \Phi_h(0+)}{1 - \Phi_h(0+)} \right]^{Z_t, t-h}$$

This formula says that the conditional law of Z_t given $Z_{t,s} = \mu$ is identical to the conditional law of Z_{t-s} given $Z_0 = \mu$ and that each point of μ survives until $t-s$. In particular

$$(14) \quad \mathbb{E}_{x, 0}^{r, q} \left(\int g dZ_t | \sigma(Z_{t,s}, s < t-h) \right) = \frac{1}{1 - \Phi_h^r(0+)} \int P_h g dZ_{t, t-h}$$

5. REDUCTION

We define a reduction operation R on trees, by induction on the height. It amounts to extending the lifetime of particles whose offspring is only one particle. $R(\delta)$ will be defined together with a mapping N_δ from $R(\delta)$

into \mathbb{N} , as follows :

- $R(\{\emptyset\}) = (\{\emptyset\})$, $N_{\{\emptyset\}}(\emptyset) = 0$;

- if $v \in R(\delta)$, $N_\delta(v) = \inf(m, v 1^m \in \delta \text{ and } v(T_{v 1^m} \delta) \neq 1)$ (1) and $vi \in R(\delta)$ if and only if $N(v) < \infty$ and $i \leq v(T_{v 1^{N_\delta(v)}} \delta)$.

In the following, we assume $q(1) \neq 1$ so that $N(v) < \infty$ a. s. when $v \in R(\alpha_0)$.

Set

$$\hat{F}_0 = \{ \alpha \in \hat{\mathcal{F}}, N(v) < \infty \text{ for any } v \in R(\alpha_0) \}.$$

$\hat{F}_0 \in \hat{\mathcal{F}}$, is stable under $k_t, \theta_{t,u}$ etc. and has full measure under all $P_{x,s}^{\tau,q}$. Hence all the results we have obtained are still valid on \hat{F}_0 . From now on, we shall work on \hat{F}_0 only.

R can be extended to \hat{F}_0 as follows : if $\alpha \in \hat{F}_0$, $(R\alpha)_0 = R\alpha_0$ and $\xi_{R\alpha}(v) = \prod_{i=0}^{N(v)} \xi_\alpha(v 1^i)$ for any $v \in R\alpha_0$ (where the product refers to the concatenation of paths).

PROPOSITION 4. - $R P_{x,s}^{\tau,q} = P_{x,s}^{\tau_R, q_R}$ with

$$\tau_R = \frac{\tau}{1 - q(1)} \quad \text{and} \quad q_R(k) = \frac{1}{1 - q(1)} (q(k) - q(1) \delta_1(k)).$$

Sketch of proof. Note first that for any $A \in \mathcal{F}$,

$$P_{x,s}^{\tau,q} \left(\prod_{i=0}^{N(\emptyset)} \xi(1^i) \in A \right) = P_{x,s}^{\tau_R, q_R}(A) \quad \text{by P3.}$$

Then prove that $P_{x,s}^{\tau,q}(R^{-1}(\Xi)) = P_{x,s}^{\tau_R, q_R}(\Xi)$ for Ξ in $\hat{\mathcal{F}}_{(n)}$ by induction on n .

COROLLARY. - For any $\Xi \in \hat{\mathcal{F}}$,

$$P_{x,s}^{\tau, q_\beta}(R^{-1} \circ e_h^{-1}(\Xi)) = \left(\frac{\tau}{\tau + h} \right)^{1/\beta} P_{x,s}^{h+\tau, q_\beta}(\Xi).$$

[An easy computation shows $(q_\beta^{(h)})_R = q_\beta$.]

(1) 1^m means 1 written m times.

6. SUPERPROCESSES

Let μ be a probability on (E, \mathcal{E}) , ψ and ε be positive parameters. Set $\mathbb{P}_\mu^\varepsilon = \int \mu(dx) \mathbb{P}_{x, \beta}^{\varepsilon, q_\beta}$ (β will be fixed in the following). Let π_μ^ε be a Poisson point process in \hat{F}_0 , with intensity $\psi \varepsilon^{-1/\beta} \mathbb{P}_\mu^\varepsilon$. For any $F \in \hat{\mathcal{F}}^+$,

$$(15) \quad \text{Log}(\mathbb{E}(e^{-\int F(\alpha) \pi_\mu^\varepsilon(d\alpha)})) = \psi \varepsilon^{-1/\beta} \mathbb{E}_\mu^\varepsilon(e^{-F} - 1).$$

π_μ^ε describes the evolution of a population starting at time 0 with a Poisson distribution of intensity $\psi \varepsilon^{-1/\beta} \mu$, and with reproduction law q_β . Each particle has an exponential lifetime of average ε and moves independently of the others according to the law determined by the semigroup P_t .

From the corollary to proposition 4, it follows that $R \circ e_h \pi_\mu^\varepsilon$ and $\pi_\mu^{\varepsilon+h}$ have the same distribution.

Taking a sequence $\varepsilon_n \downarrow 0$ and applying the Ionescu-Tulcea theorem, it appears one can define the processes $\pi_\mu^{\varepsilon_n}$ simultaneously, so that

$$(16) \quad R \circ e_h \pi_\mu^{\varepsilon_n} = \pi_\mu^{\varepsilon_n+h} \quad \text{for any } \varepsilon_n, h \in \mathbb{R}^+ - \{0\}.$$

Set $\tilde{Z}_t^{\varepsilon, \mu} = \int Z_t(\alpha) \pi_\mu^\varepsilon(d\alpha)$. (For fixed ε , it describes the position of the population living at time t .)

Similarly, set $\tilde{Z}_{t,s}^{\varepsilon, \mu} = \int Z_{t,s}(\alpha) \pi_\mu^\varepsilon(d\alpha)$. (It describes the population at times s whose offspring shall be alive at time t .)

By (16), we have for any $t > h > 0$,

$$(17) \quad \tilde{Z}_{t,t-h}^{\varepsilon, \mu} = \tilde{Z}_{t-h}^{\varepsilon+h, \mu}.$$

Hence, from identity (14), for any bounded function f , and $0 < \eta < \varepsilon$, $\mathbb{E}(\langle \tilde{Z}_{t-\eta}^{\eta, \mu}, P_\eta f \rangle \mid \sigma(\tilde{Z}_{t-\eta}^{\gamma, \mu}, \gamma \geq \varepsilon))$ can be computed and is equal to

$$(1 - \Phi_{\varepsilon-\eta}^\eta(0+))^{-1} \langle \tilde{Z}_{t-\varepsilon}^{\varepsilon, \mu}, P_\varepsilon f \rangle.$$

(If μ is a measure and f a function, $\langle \mu, f \rangle = \int f d\mu$.)

[One uses the fact that for any σ -field $\mathcal{B} \subseteq \hat{\mathcal{F}}$, and bounded measurable function F , $\mathbb{E}\left(\int F d\pi^\varepsilon \mid \sigma(\pi^\varepsilon(\Xi), \Xi \in \mathcal{B})\right) = \pi^\varepsilon(\mathbb{E}_\mu^\varepsilon(F \mid \mathcal{B}))$.]

Moreover $(1 - \Phi_{\varepsilon-\eta}^\eta(0+)) = \left(\frac{\eta}{\varepsilon}\right)^{1/\beta}$. Hence, we have, for f positive,

(18) For any sequence $\varepsilon_n \downarrow 0$, $Y_t^{\varepsilon_n, \mu}(f) = \varepsilon_n^{1/\beta} \Psi^{-1} \langle \tilde{Z}_{t-\varepsilon_n}^{\varepsilon_n, \mu}, P_{\varepsilon_n} f \rangle$ is a positive martingale which has to converge a. s. for all f towards a limit $Y_t^\mu(f)$, clearly independent of the sequence ε_n .

Using a dense countable \mathbb{Q} -vector subspace V of $C(E)$, we define a random measure μ_t such that

$$\langle \mu_t, f \rangle = Y_t^\mu(f) \quad \text{a. s. for any } f \text{ in } C(E).$$

Since $|\mathbb{E}(\mu_t(f))| \leq \mu P_t(|f|)$, the a. s. equality extends to any bounded measurable function f .

THEOREM. — μ_t is a Markov process whose law is determined by the semigroup Q_t verifying :

$$-\text{Log} \int Q_t(\mu, d\nu) \exp\left(-\int f d\nu\right) = \int U_t(f) d\mu$$

for any positive bounded measurable function f , where $U_t(f)$ is uniquely determined by the equation

$$U_t f = P_t f - \frac{\Psi^{-\beta}}{1+\beta} \int_0^t P_s (U_{t-s}(f))^{1+\beta} ds.$$

Proof. For any bounded f in \mathcal{E}^+ , set

$$\begin{aligned} U_t(\mu, f) &= -\text{Log} \mathbb{E}(\exp(-Y_t^\mu(f))) \\ U_t^\varepsilon(\mu, f) &= -\text{Log} \mathbb{E}(\exp(-Y_t^{\varepsilon, \mu}(f))). \end{aligned}$$

By (15), we have, $U_t^\varepsilon(\mu, f) = -\frac{1}{\varepsilon'} \mathbb{E}_\mu^\varepsilon(e^{-\varepsilon' Z_{t-\varepsilon} P_\varepsilon f} - 1)$, where $\varepsilon' = \varepsilon^{1/\beta} \Psi^{-1}$.

Hence $U_t^\varepsilon(\mu, f) \leq \|f\|_\infty$ for all t, ε and more generally

$$|U_t^\varepsilon(f) - U_t^\varepsilon(g)| \leq \|f - g\|_\infty.$$

Also, if we set $U_t^\varepsilon(f)(x) = U_t^\varepsilon(\varepsilon_x, f)$, we have

$$U_t^\varepsilon(\mu, f) = \int \mu(dx) U_t^\varepsilon(f)(x),$$

and

$$(19) \quad U_t^\varepsilon(f) = \frac{1}{\varepsilon'} (1 - \Phi_{t-\varepsilon}^\varepsilon(e^{-\varepsilon' P_\varepsilon f})).$$

By (18) since e^{-x} is convex, $U_t^\varepsilon(\mu, f)$ decreases with ε . By dominated convergence its limit has to be $U_t(\mu, f)$.

From (19) and (10) applied to $\Phi_{t-\varepsilon}^\varepsilon$ and $q = q_\beta$,

$$U_t^\varepsilon(f) = \frac{1}{\varepsilon'} P_{t-\varepsilon} (1 - e^{-\varepsilon' P_\varepsilon f}) - \frac{\Psi^{-\beta}}{1 + \beta} \int_0^{t-\varepsilon} P_s (U_{t-s}^\varepsilon(f)^{1+\beta}) ds.$$

Set $U_t(f)(x) = U_t(\varepsilon_x, f)$.

Since, for any ε_0 and $\varepsilon < \varepsilon_0$,

$$\begin{aligned} \int_0^{t-\varepsilon_0} P_s (U_{t-s}^\varepsilon(f)^{1+\beta}) ds + \|f\|_\infty^{1+\beta} \varepsilon_0 \\ \geq \int_0^{t-\varepsilon} P_s (U_{t-s}^\varepsilon(f)^{1+\beta}) ds \geq \int_0^{t-\varepsilon_0} P_s (U_{t-s}^\varepsilon(f)^{1+\beta}) ds \end{aligned}$$

we obtain $U_t(f) = P_t f - \frac{\Psi^{-\beta}}{1 + \beta} \int_0^t P_s (U_{t-s}(f)^{1+\beta}) ds$ by dominated convergence.

Uniqueness can be shown in the same way as for equation (10).

We also have the semigroup property $U_{t+s} = U_t \circ U_s$.

We still have to show that μ_t is a Markov process, which is equivalent to prove that

$$-\text{Log E} \left(\exp \left(- \sum_{i=1}^n Y_{t_i}(f_i) \right) \right) = U_{t_1}(f_1 + U_{t_2-t_1}(f_2 + \dots))$$

for $0 < t_1 < \dots < t_n$ and bounded $f_i \in \mathcal{E}^+$.

The first member equals

$$\begin{aligned} - \lim_{\varepsilon \downarrow 0} \text{Log E} \left(\exp \left(- \varepsilon' \int \langle Z_{t_i-\varepsilon}(\alpha), P_\varepsilon f_i \rangle \pi_\mu^\varepsilon(d\alpha) \right) \right) \\ = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon'} (1 - \Phi_{t-\varepsilon}^\varepsilon (e^{-\varepsilon' P_\varepsilon f_1} \Phi_{t_2-t_1}^\varepsilon (e^{-\varepsilon' P_\varepsilon f_2} \dots))) \\ = \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon (P_\varepsilon f_1 + V_{t_2-t_1}^\varepsilon (P_\varepsilon f_2 + \dots)) = (*) \end{aligned}$$

with

$$V_t^\varepsilon(f) = -\frac{1}{\varepsilon'} \log(\Phi_t^\varepsilon(e^{-\varepsilon' f}))$$

and

$$\tilde{U}_t^\varepsilon(f) = \frac{1}{\varepsilon'} \mathbb{E}^\varepsilon (1 - e^{-\varepsilon' Z_t(f)}) = \frac{1}{\varepsilon'} (1 - \Phi_t^\varepsilon(e^{-\varepsilon' f}))$$

(note that $U_t^\varepsilon = \tilde{U}_{t-\varepsilon}^\varepsilon \circ P_\varepsilon$). We can replace V_t^ε by \tilde{U}_t^ε . Indeed,

$$\|V_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(f)\|_\infty \leq \frac{1}{2\varepsilon'} \|1 - \Phi_t^\varepsilon(e^{-\varepsilon' f})\|_\infty^2 \leq \frac{\varepsilon' \|f\|_\infty^2}{2}$$

when $\varepsilon' \|f\|_\infty < \frac{1}{2}$.

Moreover, $|\tilde{U}_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(g)| \leq \|f - g\|$. Hence,

$$(*) = \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon(P_\varepsilon f_1 + \tilde{U}_{t_2-t_1}^\varepsilon(P_\varepsilon f_2 + \dots)).$$

Finally since

$$\begin{aligned} |\tilde{U}_\varepsilon^\varepsilon(f) - P_\varepsilon f| &\leq \frac{1}{\varepsilon' \varepsilon (1 + \beta)} \int_0^\varepsilon P_s (1 - \Phi_{t-s}^\varepsilon(e^{-\varepsilon' f}))^{1+\beta} ds \\ &+ P_\varepsilon \left| \frac{1 - e^{-\varepsilon' f}}{\varepsilon'} - f \right| \leq \frac{1}{\varepsilon \varepsilon' (1 + \beta)} \left| \int_0^\varepsilon P_s (\varepsilon' \tilde{U}_{t-s}^\varepsilon(f))^{1+\beta} ds \right| + \frac{\varepsilon' \|f\|_\infty^2}{2} \\ &\leq \frac{\varepsilon \psi^{-\beta}}{1 + \beta} \|f\|^{1+\beta} + \frac{\varepsilon' \|f\|_\infty^2}{2} \quad \text{and} \quad \tilde{U}_{t_1-t_2}^\varepsilon = \tilde{U}_\varepsilon^\varepsilon \tilde{U}_{t_1-t_2-\varepsilon}^\varepsilon \end{aligned}$$

$$\begin{aligned} (*) &= \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon(P_\varepsilon(f_1 + \tilde{U}_{t_2-t_1-\varepsilon}^\varepsilon(P_\varepsilon(f_2 + \dots)))) \\ &= \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1}^\varepsilon(f_1 + U_{t_2-t_1}^\varepsilon(f_2 + \dots)) \\ &= U_{t_1}(f_1 + U_{t_2-t_1}(f_2 + \dots)). \end{aligned}$$

Q.E.D.

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