ANNALES DE L'I. H. P., SECTION B

DAVID M. MASON

A strong invariance theorem for the tail empirical process

Annales de l'I. H. P., section B, tome 24, nº 4 (1988), p. 491-506
<http://www.numdam.org/item?id=AIHPB_1988__24_4_491_0>

Ann. Inst. Henri Poincaré,

Vol. 24, n° 4, 1988, p. 491-506. *Probabilités et Statistiques*

A strong invariance theorem for the tail empirical process

by

David M. MASON

Department of Mathematical Sciences,
501 Ewing Hall, University of Delaware,
Newark, DE 19716, U.S.A.

ABSTRACT. — The tail empirical process is defined to be for each integer $n \geqq 1$, $a(n)^{-1/2} \alpha_n(a(n)s)$, $0 \leqq s \leqq 1$, where α_n is the uniform empirical process based on n independent uniform $(0, 1)$ random variables and $a(n)$ is a sequence of positive constants such that $a(n) \to 0$ and $na(n) \to \infty$ as $n \to \infty$. The tail empirical process converges weakly to a standard Wiener process on $[0, 1]$. A strong approximation to the tail empirical process by a sequence of Gaussian processes in obtained which, among other results, leads to a functional law of the iterated logarithm for the tail empirical process. These results have application to the study of the almost sure limiting behavior of sums of extreme values.

Key words : Uniform empirical process, strong approximation, functional law of the iterated logarithm.

RÉSUMÉ. — Le processus empirique de queue est défini, pour tout $n \geqq 1$, par $a(n)^{-1/2} \alpha_n(a(n)s)$, $0 \leqq s \leqq 1$, où α_n est le processus empirique uniforme basé sur n variables aléatoires indépendantes et uniformes sur $(0, 1)$, et $a(n)$ est une suite de constantes positives telles que $a(n) \to 0$ et $na(n) \to \infty$ quand $n \to \infty$. Le processus empirique de queue converge faiblement sur $[0, 1]$ vers un processus de Wiener standard. On obtient une approximation

Classification A.M.S. : Primary 60 F 99, 60 F 17.

(*) Research partially supported by the Alexander von Humboldt Foundation while the author was visiting the University of Munich on leave from the University of Delaware.

forte du processus empirique de queue par une suite de processus gaussiens. Ceci mène, entre autres résultats, à une loi fonctionnelle du logarithme itéré pour le processus empirique de queue. Ces résultats peuvent être appliqués à l'étude du comportement asymptotique presque sûr de sommes de valeurs extrêmes.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let U_1, U_2, ..., be a sequence of independent uniform (0, 1) random variables and for each integer $n \geqq 1$ let

$$G_n(s) = n^{-1} \sum_{i=1}^{n} 1(U_i \leqq s), \qquad 0 \leqq s \leqq 1,$$

where $1(x \leqq y)$ denotes the indicator function, be the empirical distribution function based on the first n of these uniform (0, 1) random variables. The uniform empirical process will be written

$$\alpha_n(s) = n^{1/2}\{G_n(s) - s\}, \qquad 0 \leqq s \leqq 1.$$

Throughout this paper $a(n)$ will denote a sequence of positive constants such that for all integers $n \geqq 1$, $0 < a(n) < 1$, and $a(n) \to 0$ and $k(n) := na(n) \to \infty$ as $n \to \infty$.

In this paper we shall be concerned with the tail empirical process defined in terms of a sequence $a(n)$ to be

$$w_n(s) = a(n)^{-1/2} \alpha_n(a(n)s), \qquad 0 \leqq s \leqq 1.$$

An easy application of Theorem 1 of Mason and van Zwet (1987) shows that on a rich enough probability space there exist a sequence of independent uniform (0, 1) random variables and a sequence of Brownian bridges B_1, B_2, ..., such that for any sequence $a(n)$

$$(1.1) \quad \sup_{0 \leqq s \leqq 1} |w_n(s) - a(n)^{-1/2} B_n(a(n)s)| = O_p(k(n)^{-1/2} \log k(n)) = o_p(1).$$

Writing for each integer $n \geqq 1$

$$(1.2) \qquad B_n(s) = W_n(s) - s W_n(1), \qquad 0 \leqq s \leqq 1,$$

where W_n is a standard Wiener process on [0, 1], we see that (1.1) leads to

$$(1.3) \qquad \sup_{0 \leqq s \leqq 1} |w_n(s) - a(n)^{-1/2} W_n(a(n)s)| = o_p(1).$$

Next, by observing that $a(n)^{-1/2}\,\mathrm{W}_n(a(n)s)$, $0 \leqq s \leqq 1$, is a standard Wiener process, it is clear that (1.3) implies w_n converges weakly in $\mathrm{D}[0, 1]$ to a standard Wiener process. [This weak convergence fact is implicit in the work of Cooil (1985) and M. Csörgő and Mason (1985).]

What is known about the almost sure behavior of the supremum of the tail empirical process can be summarized as follows [*cf.* Einmahl and Mason (1988 *a*) and Kiefer (1972).]

Assume that the sequence $a(n)$ satisfies

(A) $\qquad a(n) \downarrow 0 \quad$ and $\quad k(n) = na(n) \uparrow \infty \quad$ as $n \to \infty$,

(here and elsewhere $\uparrow$ denotes non-decreasing and $\downarrow$ non-increasing), and set

$$l(n) = 2\log\log(\max(n,3)).$$

(1.4.i) Whenever $o\,(k\,(n)) = \log\log n$, then

$$\limsup_{n\to\infty} \sup_{0\leqq s\leqq 1} l(n)^{-1/2}\,|w_n(s)| = 1 \quad \text{a. s.}$$

(1.4.ii) Whenever $k\,(n) \sim c \log\log n$ for some $0 < c < \infty$, then

$$\limsup_{n\to\infty} \sup_{0\leqq s\leqq 1} l(n)^{-1/2}\,|w_n(s)| = (c/2)^{1/2}(\beta_c - 1) \quad \text{a. s.,}$$

where $\beta_c > 1$ and $\beta_c\,(\log \beta_c - 1) + 1 = c^{-1}$.

(1.4.iii) Whenever $k\,(n) = o\,(\log\log n)$, then

$$\limsup_{n\to\infty} \sup_{0\leqq s\leqq 1} l(n)^{-1/2}\,|w_n(s)| = \infty \quad \text{a. s.}$$

It is proven in Komlós, Major and Tusnády [KMT] (1975), that on a rich enough probability space there exist a sequence of independent uniform (0, 1) random variables $\mathrm{U}_1, \mathrm{U}_2, \ldots$, and a sequence of *independent* Brownian bridges $\mathrm{B}_1, \mathrm{B}_2, \ldots$, such that

$$(1.5) \qquad \sup_{0\leqq s\leqq 1} \left| n^{1/2}\alpha_n(s) - \sum_{m=1}^{n} \mathrm{B}_m(s) \right| = O\,((\log n)^2) \quad \text{a. s.}$$

Writing B_m as in (1.2) for each $m \geqq 1$ and noting that by the law of the iterated logarithm

$$\sum_{m=1}^{n} \mathrm{W}_m(1) = O\,((n\log\log n)^{1/2}) \quad \text{a. s.,}$$

we obtain after some algebra that on the probability space of (1.5), with probability one,

$$(1.6) \quad \sup_{0\leqq s\leqq 1} \left| w_n(s) - k\,(n)^{-1/2} \sum_{m=1}^{n} \mathrm{W}_m(a(n)s) \right|$$
$$= O\,(k\,(n)^{-1/2}(\log n)^2 + (a\,(n)\log\log n)^{1/2}),$$

which for sequences $a(n)$ satisfying

$$(1.7) \qquad o(k(n)) = (\log n)^4 / \log\log n$$

yields

$$(1.8) \qquad \sup_{0 \leqq s \leqq 1} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = o((\log\log n)^{1/2}) \quad \text{a. s.}$$

Thus if we knew that for sequences $a(n)$ satisfying (A)

$$(1.9) \qquad \limsup_{n \to \infty} \sup_{0 \leqq s \leqq 1} (l(n)k(n))^{-1/2} \left| \sum_{m=1}^{n} W_m(a(n)s) \right| = 1 \quad \text{a. s.},$$

then part, but not all, of (1.4.i) would follow from strong invariance, that is, for sequences $a(n)$ satisfying (A) and (1.7) the lim sup statement in (1.4.i) would be a consequence of (1.8) and (1.9).

The main purpose of this paper is to completely characterize those sequences $a(n)$ satisfying (A) for which the strong approximation statement (1.8) holds on a suitably constructed probability space or for which it cannot hold on any probability space. It turns out that the condition $o(k(n)) = \log\log n$ of (1.4.i) is in a certain sense necessary and sufficient for (1.8). Theorem 2, below, implies that (1.9) is true for all sequences $a(n)$ satisfying (A). Thus (1.4.i) follows from Theorem 2 in combination with our strong approximation Theorem 1. Since the KMT (1975) strong approximation (1.5) only yields (1.8) for sequences $a(n)$ satisfying (1.7), we see that our Theorem 1 constitutes an improvement to their strong approximation *in the tails*, i.e. when $o(k(n)) = \log\log n$ but (1.7) does not hold.

We now state our results.

THEOREM 1. — *Let $a(n)$ satisfy (A). Whenever $o(k(n)) = \log\log n$, there exist a sequence of independent uniform $(0, 1)$ random variables $U_1, U_2, \ldots$, and a sequence of independent standard Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space such that with probability one*

$$(1.10) \qquad \lim_{n \to \infty} \sup_{0 \leqq s \leqq 1} l(n)^{-1/2} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = 0.$$

Our next theorem provides the functional law of the iterated logarithm version of (1.9).

Let $B[0, 1]$ denote the space of bounded real-valued functions defined on $[0, 1]$ with the usual supremum norm and $K[0, 1]$ denote the set of absolutely continuous functions $f \in B[0, 1]$ such that

$$f(0) = 0 \qquad \text{and} \qquad \int_0^1 (f'(s))^2 \, ds \leqq 1.$$

For any sequence $a(n)$ and sequence of independent standard Wiener processes $W_1, W_2, \ldots$, set for integers $n \geqq 1$

$$(1.11) \qquad X_n(s) = k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s), \qquad 0 \leqq s \leqq 1.$$

THEOREM 2. — *Let $a(n)$ satisfy* (A), *then with probability one the sequence $l(n)^{-1/2} X_n$ is relatively compact in* B[0, 1] *with set of limit points equal to* K[0, 1].

An immediate consequence of Theorem 2, (1.4.ii) and (1.4.iii) is the following corollary, which shows that a strong approximation to the tail empirical process of the type given in Theorem 1 is not possible when $k(n)/\log\log n \to c \in [0, \infty)$.

COROLLARY 1. — *Let $a(n)$ satisfy* (A) *and $k(n) = c_n \log\log n$ where $c_n \to c$ as $n \to \infty$ with $0 \leqq c < \infty$, then for any sequence of independent uniform* (0, 1) *random variables $U_1, U_2, \ldots$, and sequence of independent standard Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space with probability one*

$$(1.12) \quad \limsup_{n \to \infty} \sup_{0 \leqq s \leqq 1} l(n)^{-1/2} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = \gamma(c),$$

where $0 < \gamma(c) < \infty$ when $0 < c < \infty$ and $\gamma(0) = \infty$.

Our second corollary gives the functional law of the iterated logarithm version of (1.4.i). It follows readily from Theorem 1 and Theorem 2, and may be viewed as the Finkelstein (1971) theorem for the tail empirical process.

COROLLARY 2. — *Let $a(n)$ satisfy* (A) *and $o(k(n)) = \log\log n$, then with probability one the sequence $l(n)^{-1/2} w_n$ is relatively compact in* B[0, 1] *with set of limit points equal to* K[0, 1].

Remark 1. — From (1.4.ii) and (1.4.iii) it is easy to see that Corollary 2 no longer holds when $a(n)$ satisfies the conditions of Corollary 1 in the sense that K[0, 1] is no longer the set of limit points of $l(n)^{-1/2} w_n$.

In order to motivate our final three corollaries, consider now the following weighted versions of the tail empirical process defined for integers $n \geqq 1$ and $0 < v < 1/2$ to be

$$w_n^{(v)}(s) = w_n(s) s^{-1/2+v}, \qquad 0 \leqq s \leqq 1.$$

Using Theorem 2 of Mason and van Zwet (1987), along with (1.2), it is straight-forward to show that on a rich enough probability space there exist a sequence of independent uniform (0, 1) random variables $U_1, U_2, \ldots$, and a sequence of standard Wiener processes $W_1, W_2, \ldots$, such that

$$\sup_{0 \leq s \leq 1} \left| w_n^{(v)}(s) - a(n)^{-1/2} W_n(a(n)s)\, s^{-1/2+v} \right| = o_p(1),$$

which of course implies that $w_n^{(v)}$ converges weakly in D [0, 1] to $WI^{-1/2+v}$, where here and elsewhere in this paper W denotes a standard Wiener process and I the identity function on [0, 1]. [Again this weak convergence result is implicit in Cooil (1985) and M. Csörgő and Mason (1985).]

For any sequence $a(n)$ satisfying (A) and $0<v<1/2$ set for integers $n \geq 3$

$$e(n, v) = \{ nk(n)^{2v/(1-2v)} (\log\log n)^{1/(1-2v)} \}^{-1}.$$

As a special case of Theorem 1 of Einmahl and Mason (1988 *a*) one obtains

$$(1.13) \qquad \limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \left| w_n^{(v)}(s) \right| = 1 \quad \text{or} \quad \infty \quad \text{a. s.}$$

according as the following series is finite or infinite

$$(1.14) \qquad \sum_{n=3}^{\infty} e(n, v).$$

It is natural then to consider the strong approximation analogue of Theorem 1 for these weighted tail empirical processes. The following corollary provides this analogue.

COROLLARY 3. – *Let* $a(n)$ *satisfy* (A) *and* $0<v<1/2$.

(I) *Whenever the series in* (1.14) *is finite, then on the probability space of Theorem* 1 *with probability one*

$$(1.15) \qquad \lim_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \left| w_n^{(v)}(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s)\, s^{-1/2+v} \right| = 0.$$

(II) *Whenever the series in* (1.14) *is infinite, then for any sequence of independent uniform* (0, 1) *random variables* $U_1, U_2, \ldots$, *and sequence of independent Wiener processes* $W_1, W_2, \ldots$, *sitting on the same probability space with probability one*

$$(1.16) \qquad \limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \times \left| w_n^{(v)}(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s)\, s^{-1/2+v} \right| = \infty.$$

For any $0<v<1/2$, let

$$K^{(v)}[0, 1] = \{ f I^{-1/2+v} : f \in K[0, 1] \}$$

and for any sequence $a(n)$ and $0<v<1/2$ set for integers $n \geqq 1$

$$(1.17) \qquad X_n^{(v)}(s)=k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s)s^{-1/2+v}, \qquad 0 \leqq s \leqq 1.$$

Our next corollary provides the functional law of the iterated logarithm version of Theorem 2 for the sequence of processes $X_n^{(v)}$.

COROLLARY 4. — *Let* $a(n)$ *satisfy* (A) *and* $0<v<1/2$, *then with probability one the sequence* $l(n)^{-1/2} X_n^{(v)}$ is relatively compact in $B[0, 1]$ *with set of limit points equal to* $K^{(v)}[0, 1]$.

As our final corollary we obtain the functional law of the iterated logarithm form of (1.13), whose proof is immediate from Corollaries 3 and 4. This corollary may be viewed as the James (1975) theorem for the weighted tail empirical process.

COROLLARY 5. — *Let* $a(n)$ *satisfy* (A) *and* $0<v<1/2$, *then whenever the series in* (1.14) *is finite with probability one the sequence* $l(n)^{-1/2} w_n^{(v)}$ *is relatively compact in* $B[0, 1]$ *with set of limit points equal to* $K^{(v)}[0, 1]$.

Remark 2. — Whenever the series in (1.14) is infinite, it is obvious from (1.13) that Corollary 5 does not hold.

Remark 3. — The statements of Theorem 1 and Corollary 2 remain true when in the definition of w_n the uniform empirical process α_n is replaced by the uniform quantile process β_n. The resulting process is called the tail quantile process. These results are contained in Einmahl and Mason (1988 *b*).

For applications of these results to the derivation of functional laws of the iterated logarithm for sums of extreme value processes the interested reader is referred to Mason (1988).

2. PROOFS

Proof of Theorem 1. — First we must establish a number of lemmas, some of which may be of separate interest.

From now on we write for integers $n \geqq 1$, $b(n)=(k(n)l(n))^{-1/2}$ and $[x]$ will denote the integer part of x.

LEMMA 1. — *Let* $a(n)$ *satisfy* (A) *and* $o(k(n))=\log\log n$, *then for each* $1<\lambda<\infty$ *there exist a sequence of independent uniform* (0, 1) *random variables* $U_1, U_2, \ldots$, *and a sequence of independent standard Wiener processes* $W_1, W_2, \ldots$, *sitting on the same probability space* (Ω, A, P) *such* *that*

$$(2.1)\quad \lim_{r\to\infty} b(n_r) \sup_{0\leq s\leq 1} \left| n_r^{1/2}\alpha_{n_r}(a(n_r)s) - \sum_{m=1}^{n_r} W_m(a(n_r)s)\right| = 0 \quad a.s.,$$

where $n_r = [\lambda^r]$.

Proof. — Let $m_1 = n_1$ and $m_r = n_r - n_{r-1}$ for $r = 2, 3, \ldots$ For each integer $r \geqq 1$ construct using Theorem 1 of Mason and van Zwet (1987) independent uniform (0, 1) random variables $U_1^{(r)}, \ldots, U_{m_r}^{(r)}$ and a standard Wiener process $W^{(r)}$ sitting on the same probability space (Ω_r, A_r, P_r) such that for universal positive constants C, K and λ independent of r

$$P(\sup_{0\leqq s\leqq k/m_r} m_r^{1/2} | \alpha_{m_r}^{(r)}(s) - (W^{(r)}(s) - sW^{(r)}(1)) | > C\log k + x) \leqq K e^{-\lambda x}$$

for all $0 \leqq x < \infty$ *and* $1 \leqq k \leqq m_r$, where $\alpha_{m_r}^{(r)}$ is the empirical processes based on $U_1^{(r)}, \ldots, U_{m_r}^{(r)}$, which by an elementary bound for the tail of the standard normal distribution [*cf.* Feller (1968)] gives

$$(2.2)\quad P(\sup_{0\leqq s\leqq k/m_r} m_r^{1/2} | \alpha_{m_r}^{(r)}(s) - W^{(r)}(s) | > C\log k + x)$$
$$< K_1 e^{-\lambda_1 x} + K_2 e^{-\lambda_2 x^2 m_r/k^2}$$

for universal positive constants K_1, K_2, λ_1 and λ_2. We can assume that

$$W^{(r)} = m_r^{-1/2} \sum_{i=1}^{m_r} W_i^{(r)},$$

where $W_1^{(r)}, \ldots, W_{m_r}^{(r)}$ are independent standard Wiener processes.

Now set

$$(\Omega, A, P) = (\times_{r=1}^{\infty} \Omega_r, \times_{r=1}^{\infty} A_r, \times_{r=1}^{\infty} P_r),$$

$U_i = U_i^{(1)}$, $W_i = W_i^{(1)}$ for $i = 1, \ldots, m_1$, and $U_{n_r+i} = U_i^{(r+1)}$ and $W_{n_r+i} = W_i^{(r+1)}$ for $i = 1, \ldots, m_{r+1}$ for $r = 1, 2, \ldots$

CLAIM 1:

$$(2.3)\quad \lim_{r\to\infty} b(m_r) \sup_{0\leqq s\leqq 1} m_r^{1/2} | \alpha_{m_r}^{(r)}(a(m_r)s) - W^{(r)}(a(m_r)s) | = 0 \quad a.s.$$

Proof. — Choose any $\varepsilon > 0$. We have by (2.2)

$$P(\sup_{0\leqq s\leqq 1} m_r^{1/2} | \alpha_{m_r}^{(r)}(a(m_r)s) - W^{(r)}(a(m_r)s) | > C\log k(m_r) + \varepsilon b^{-1}(m_r))$$
$$< K_1 \exp(-\varepsilon\lambda_1 b^{-1}(m_r)) + K_2 \exp(-\varepsilon^2 \lambda_2 (2\log\log m_r)(m_r/k(m_r))),$$

which since as $r \to \infty$

$$b^{-1}(m_r)/\log\log m_r \to \infty \qquad \text{and} \qquad m_r/k(m_r) \to \infty$$

is less than r^{-2} for all large enough r. Hence by the Borel-Cantelli lemma in combination with the fact that as $r \to \infty$

$$b(m_r) \log k(m_r) = (\log k(m_r))/(2k(m_r) \log\log m_r)^{1/2} \to 0$$

we have shown (2.3).

CLAIM 2. — *For each integer* $k \geqq 1$,

$$\limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(n_r) \left| \sum_{m=1}^{n_{r-k}} W_m(a(n_r)s) \right| \leqq \lambda^{-k/2} \quad a.s. \tag{2.4}$$

and

$$\limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(n_r) \left| n_{r-k}^{1/2} \alpha_{n_{r-k}}(a(n_r)s) \right| \leqq \lambda^{-k/2} \quad a.s. \tag{2.5}$$

Proof. — First consider (2.4). Notice that

$$\sum_{m=1}^{n_{r-k}} W_m(a(n_r)s) \overset{d}{=} (n_{r-k} a(n_r))^{1/2} W(s), \qquad 0 \leqq s \leqq 1.$$

Therefore for any $\varepsilon > 0$

$$P\left(\sup_{0 \leqq s \leqq 1} b(n_r) \left| \sum_{m=1}^{n_{r-k}} W_m(a(n_r)s) \right| > ((1+\varepsilon)\lambda^{-k})^{1/2} \right)$$
$$= P\left(\sup_{0 \leqq s \leqq 1} |W(s)| > b^{-1}(n_r)(n_{r-k} a(n_r))^{-1/2}((1+\varepsilon)\lambda^{-k})^{1/2} \right),$$

which is less than or equal to

$$4P(W(1) > b^{-1}(n_r)(n_{r-k} a(n_r))^{-1/2}((1+\varepsilon)\lambda^{-k})^{1/2}).$$

This expression is for all large enough r less than or equal to

$$4P(W(1) > (2(1+\varepsilon/2)\log\log n_r)^{1/2}) < \exp(-(1+\varepsilon/2)\log\log n_r).$$

The Borel-Cantelli lemma completes the proof of (2.4).

Turning to (2.5), we have by a straightforward application of Inequality 2 on p. 444 of Shorack and Wellner (1986) that for any $\varepsilon > 0$

$$\sum_{r=1}^{\infty} P\left(\sup_{0 \leqq s \leqq 1} b(n_r) \left| n_{r-k}^{1/2} \alpha_{n_{r-k}}(a(n_r)s) \right| > ((1+\varepsilon)\lambda^{-k})^{1/2} \right) < \infty.$$

This completes the proof of Claim 2.

To finish the proof of Lemma 1, we note that for each integer $k \geqq 1$

$$\limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(n_r) \left| n_r^{1/2} \alpha_{n_r}(a(n_r)s) - \sum_{m=1}^{n_r} W_m(a(n_r)s) \right|$$
$$\leqq \limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(n_r) \left| n_{r-k}^{1/2} \alpha_{n_{r-k}}(a(n_r)s) \right|$$
$$+ \limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(n_r) \left| \sum_{m=1}^{n_{r-k}} W_m(a(n_r)s) \right|$$
$$+ \sum_{i=1}^{k} \limsup_{r \to \infty} \sup_{0 \leqq s \leqq 1} b(m_{r+1-i}) m_{r+1-i}^{1/2}$$
$$\times \left| \alpha_{m_{r+1-i}}^{(r+1-i)}(a(m_{r+1-i})s) - W^{(r+1-i)}(a(m_{r+1-i})s) \right|,$$

which by Claims 1 and 2 is almost surely less than or equal to $2\lambda^{-k/2}$. Since k can be chosen arbitrarily large we have (2.1).

For any integer $n \geqq 1$ set

$$S_n(s) = \sum_{j=1}^{n} W_j(s), \qquad 0 \leqq s \leqq 1, \tag{2.6}$$

and

$$T_n(s) = \sum_{j=1}^{n} \{1(U_j \leqq s) - s\}, \qquad 0 \leqq s \leqq 1. \tag{2.7}$$

LEMMA 2. — *For any* $0 < a < 1$, *integer* $n \geqq 1$ *and* $\varepsilon > 0$

$$P(\max_{1 \leqq j \leqq n} \sup_{0 \leqq s \leqq a} |S_j(s)| > \varepsilon) \leqq 2P(\sup_{0 \leqq s \leqq a} |S_n(s)| > \varepsilon) \tag{2.8}$$

and if, in addition, $\varepsilon > 2(2na)^{1/2}$

$$P(\max_{1 \leqq j \leqq n} \sup_{0 \leqq s \leqq a} |T_j(s)| > \varepsilon) \leqq 2P(\sup_{0 \leqq s \leqq a} |T_n(s)| > \varepsilon/2). \tag{2.9}$$

Proof. — First, (2.8) follows from the Banach space version of Lévy's inequality for independent symmetric random variables [*cf.* Theorem 2.6 of Araujo and Giné (1980)]. Next we turn to (2.9). Choose any $0 < a < 1$ and let $\{r_k : k \geqq 0\}$ with $r_0 = 0$ be a denumeration of the rationals in [0, a]. For integers $1 \leqq j \leqq n$ and $k \geqq 1$ set

$$A_{j,k} = \{\sup_{i \leqq j,\ l < k} |T_i(r_l)| \leqq \varepsilon,\ |T_j(r_k)| > \varepsilon\}$$

and

$$B_{j,k} = \{|T_n(r_k) - T_j(r_k)| \leqq \varepsilon/2\}.$$

Notice that

$$P\left(\bigcup_{j=1}^{n}\bigcup_{k=1}^{\infty}(A_{j,k}\cap B_{j,k})\right)\leqq P(\sup_{0\leqq s\leqq a}|T_n(s)|>\varepsilon/2)$$

and by Chebyshev's inequality and the choice of ε

$$P(B_{j,k})\geqq 1-4(n-j)\,a/\varepsilon^2\geqq 1/2.$$

Proceeding now as in the proof of Lemma 2.3 of James (1975) we obtain (2.9).

For any $1<\lambda<\infty$ write for integers $r\geqq 1$

$$\Delta_r=\max_{n_r<n\leqq n_{r+1}}\sup_{0\leqq s\leqq 1}\left|\sum_{j=1}^{n-n_r}W_{j+n_r}(a(n_r)s)\right|$$

and

$$\Delta_r^*=\max_{n_r<n\leqq n_{r+1}}\sup_{0\leqq s\leqq 1}\left|\sum_{j=1}^{n-n_r}\{1(U_{j+n_r}\leqq a(n_r)s)-a(n_r)s\}\right|,$$

where $n_r=[\lambda^r]$.

LEMMA 3. — *Let $a(n)$ satisfy* (A), *then for each* $1<\lambda<\infty$

(2.10) $$\limsup_{r\to\infty} b(n_r)\Delta_r\leqq(\lambda-1)^{1/2}\quad a.s.$$

and if, in addition, $o(k(n))=\log\log n$, *then*

(2.11) $$\limsup_{r\to\infty} b(n_r)\Delta_r^*\leqq 2(\lambda-1)^{1/2}\quad a.s.$$

Proof. — First consider (2.10). Choose any $\varepsilon>0$. By Lemma 2

$$\begin{aligned}P(b(n_r)\Delta_r>((1+\varepsilon)(\lambda-1))^{1/2})&\\ \leqq 2P(\sup_{0\leqq s\leqq 1}|S_{m_{r+1}}(a(n_r)s)|>((1+\varepsilon)(\lambda-1))^{1/2}/b(n_r))&\\ =2P(\sup_{0\leqq s\leqq 1}|W(s)|>(2(1+\varepsilon)(\lambda-1)n_r/m_{r+1})^{1/2}(\log\log n_r)^{1/2}),&\end{aligned}$$

which for all large enough r is less than

$$2P(\sup_{0\leqq s\leqq 1}|W(s)|>(2(1+\varepsilon/2)\log\log n_r)^{1/2}),$$

and noting that for all $x>0$,

$$P(\sup_{0\leqq s\leqq 1}|W(s)|>x)\leqq 2P(\sup_{0\leqq s\leqq 1}W(s)>x)=4P(W(1)>x),$$

we see that the previous expression is for all large r

$$\leqq 8P(W(1)\geqq(2(1+\varepsilon/2)\log\log n_r)^{1/2}<\exp(-(1+\varepsilon/2)\log\log n_r).$$

An application of the Borel-Cantelli lemma finishes the proof of (2.10).

Assertion (2.11) is proven similarly, applying at the appropriate step Inequality 2 on p. 444 of Shorack and Wellner (1986).

LEMMA 4. — *Let $a(n)$ satisfy* (A) *and $o(k(n)) = \log\log n$, then for every $\varepsilon > 0$ there exist a sequence of independent uniform* (0,1) *random variables* $U_1, U_2, \ldots$, *and a sequence of independent standard Wiener processes* $W_1, W_2, \ldots$, *sitting on the same probability space such that*

$$\text{(2.12)}\quad \limsup_{n\to\infty} \sup_{0\leqq s\leqq 1} b(n)\left| n^{1/2}\alpha_n(a(n)s) - \sum_{m=1}^{n} W_m(a(n)s)\right| \leqq \varepsilon \quad a.s.$$

Proof. — Choose any $\varepsilon > 0$ and $1 < \lambda < \infty$ such that $3(\lambda-1)^{1/2} < \varepsilon$. Let $U_1, U_2, \ldots$, and $W_1, W_2, \ldots$, be constructed as in Lemma 1. Observe that since $a(n)\downarrow 0$ and $b(n)\downarrow 0$ for any integers $r \geqq 1$ and $n_r < n \leqq n_{r+1}$

$$b(n) \sup_{0\leqq s\leqq 1}\left| n^{1/2}\alpha_n(a(n)s) - \sum_{m=1}^{n} W_m(a(n)s)\right|$$
$$\leqq b(n_r) \sup_{0\leqq s\leqq 1}\left| n_r^{1/2}\alpha_{n_r}(a(n_r)s) - \sum_{m=1}^{n_r} W_m(a(n_r)s)\right| + b(n_r)\Delta_r + b(n_r)\Delta_r^*.$$

Obviously now, (2.12) follows from (2.1), (2.10) and (2.11) combined with the above inequality.

Lemma 4 allows us to apply a trick of Major (1976) [*cf.* also p. 397-398 of Gaenssler and Stute (1977) and p. 261-262 of Philipp and Stout (1986)] to construct a sequence of independent uniform (0,1) random variables $U_1, U_2, \ldots$, and a sequence of independent standard Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space so that (1.10) holds. For the sake of brevity these details are omitted. This completes the proof of Theorem 1.

Proof of Theorem 2. — The proof of Theorem 2 is based on Theorem 1 of Lai (1974). For any $1 < \lambda < \infty$, set for integers $r \geqq 1$

$$Y_r(s) = X_{n_r}(s), \qquad 0 \leqq s \leqq 1,$$

where $n_r = [\lambda^r]$ and X_{n_r} is as in (1.11).

LEMMA 5. — *Let $a(n)$ satisfy* (A), *then for any $1 < \lambda < \infty$ with probability one the sequence $l(n_r)^{-1/2}\, Y_r$ is relatively compact in* B[0,1] *with set of limit points equal to* K[0,1].

Proof. — First note that each Y_r is a standard Wiener process on [0,1]. For each integer $r \geqq 1$ let $\mathbf{F}_r$ denote the σ-field generated by $\{Y_m(s): 0 \leqq s \leqq 1,\ 1 \leqq m \leqq r\}$. Obviously we have for each r, $k = 1, 2, \ldots$, and $0 \leqq s \leqq 1$

$$E\{E(Y_{r+k}(s)\,|\,\mathbf{F}_r)\}^2 = s\, n_r/n_{r+k}.$$

Since

$$\lim_{r,\, k \to \infty} n_r/n_{r+k} = 0,$$

the assertion of the lemma follows from Theorem 1 of of Lai (1974).

For any $1<\lambda<\infty$ set for integers $r \geqq 1$

$$D_r = \max_{n_r < n \leqq n_{r+1}} \sup_{0 \leqq s \leqq 1} \left| l(n_r)^{-1/2} X_{n_r}(s) - l(n)^{-1/2} X_n(s) \right|$$

where $n_r = [\lambda^r]$.

LEMMA 6. — *Let* $a(n)$ *satisfy* (A), *then for any* $1<\lambda<\infty$

$$\limsup_{r \to \infty} D_r \leqq \psi(\lambda) \quad a.\, s., \tag{2.13}$$

where $\psi(\lambda) = (\lambda-1)^{1/2} + 1 - \lambda^{-1/2} + (1-\lambda^{-2})^{1/2}$.

Proof. — Recall the definitions of X_n in (1.11) and S_n in (2.6). Notice that

$$\begin{aligned} D_r \leqq & \max_{n_r < m \leqq n_{r+1}} \sup_{0 \leqq s \leqq 1} b(n_r) \left| \sum_{j=1}^{m-n_r} W_{j+n_r}(a(n_r)s) \right| \\ & + \sup_{0 \leqq s \leqq 1} l(n_r)^{-1/2} \left| X_{n_r}(s) \right| \{ 1 - b(n_{r+1})/b(n_r) \} \\ & + \max_{n_r < n \leqq n_{r+1}} \sup_{0 \leqq s \leqq 1} b(n_r) \left| S_{n_r}(a(n_r)s) - S_{n_r}(a(n)s) \right| \\ & := b(n_r)\Delta_r + D_r^{(1)} + D_r^{(2)}. \end{aligned}$$

First, by Lemma 3 we have

$$\limsup_{r \to \infty} b(n_r)\Delta_r \leqq (\lambda-1)^{1/2} \quad \text{a. s.} \tag{2.14}$$

Observe that

$$b(n_{r+1})/b(n_r) = (k(n_r) \log\log n_r / (k(n_{r+1}) \log\log n_{r+1}))^{1/2},$$

and since $a(n_{r+1}) \leqq a(n_r)$, $k(n_r)/k(n_{r+1}) \geqq n_r/n_{r+1}$. Therefore

$$1 - b(n_{r+1})/b(n_r) \leqq 1 - (n_r \log\log n_r/(n_{r+1} \log\log n_{r+1}))^{1/2},$$

and hence

$$\limsup_{r \to \infty} \{ 1 - b(n_{r+1})/b(n_r) \} \leqq 1 - \lambda^{-1/2}. \tag{2.15}$$

Statement (2.15) in combination with Lemma 5 gives

$$\limsup_{r \to \infty} D_r^{(1)} \leqq 1 - \lambda^{-1/2} \quad \text{a. s.} \tag{2.16}$$

CLAIM 3:

$$\limsup_{r \to \infty} D_r^{(2)} \leqq (1-\lambda^{-2})^{1/2} \quad a.\, s. \tag{2.17}$$

Proof. — We see that

$$D_r^{(2)} \leqq \sup_{sa(n_{r+1})/a(n_r) \leqq t \leqq s \leqq 1} b(n_r) \left| S_{n_r}(a(n_r)s) - S_{n_r}(a(n_r)t) \right| := D_r^{(3)}$$

and

$$D_r^{(3)} \stackrel{d}{=} l(n_r)^{-1/2} \sup_{sa(n_{r+1})/a(n_r) \leqq t \leqq s \leqq 1} \left| W(s) - W(t) \right| := D_r^{(4)}.$$

Since for all large enough r

$$\lambda^{-2} \leqq k(n_{r+1}) n_r / (k(n_r) n_{r+1}) = a(n_{r+1})/a(n_r) \leqq 1,$$

we have for any $\varepsilon > 0$ and all large enough r

$$\begin{aligned} &P(D_r^{(4)} > ((1+\varepsilon)(1-\lambda^{-2}))^{1/2}) \\ &\quad \leqq P(\sup_{s\lambda^{-2} \leqq t \leqq s \leqq 1} \left| W(s) - W(t) \right| \\ &\quad > ((1+\varepsilon)(1-\lambda^{-2}) 2 \log\log n_r)^{1/2}) \\ &\quad \leqq P(\sup_{0 \leqq v \leqq 1} \sup_{0 \leqq u \leqq 1 - \lambda^{-2}} \left| W(u+v) - W(v) \right| \\ &\quad \geqq ((1+\varepsilon)(1-\lambda^{-2}) 2 \log\log n_r)^{1/2}), \end{aligned}$$

which by Lemma 1.2.1 on p. 29 of M. Csörgő and Révész (1981) is

$$\leqq c(\varepsilon, \lambda) \exp(-(1+\varepsilon/2) \log\log n_r),$$

for some constant $0 < c(\varepsilon, \lambda) < \infty$ depending on only $\varepsilon > 0$ and $\lambda > 1$. The Borel-Cantelli lemma completes the proof of (2.17).

From statements (2.14), (2.15) and (2.17) we conclude (2.13), finishing the proof of Lemma 6.

Since $\psi(\lambda) \to 0$ as $\lambda \downarrow 1$, the assertion of Theorem 2 follows from an easy argument based on Lemmas 5 and 6. [*See*, for instance, Section 3 of Lai (1974).]

Proofs of Corollaries 3 *and* 4. — We require the following two lemmas.

LEMMA 7. — *Let* $a(n)$ *satisfy* (A) *and* $0 < v < 1/2$.

$$\lim_{\substack{n \to \infty \\ \delta \downarrow 0}} \sup_{0 \leqq s \leqq \delta} l(n)^{-1/2} \left| X_n^{(v)}(s) \right| = 0 \quad a.s. \tag{2.18}$$

Proof. — From Orey and Pruitt (1973) or Wichura (1973) [*cf.* also Theorem 1.12.2 on p. 61 of M. Csörgő and Révész (1981)] it is easy to conclude using a change of time scale with S_n as in (26) that

$$\limsup_{\substack{n \to \infty \\ \delta \downarrow 0}} \sup_{0 \leqq s \leqq \delta} \left| S_n(a(n)s) \right| / \left(4sk(n) \log\log\left(\frac{n^2}{sk(n)} \right) \right)^{1/2} = 1 \quad \text{a. s.} \tag{2.19}$$

Now

$$(2.20)\quad \sup_{0\leq s\leq\delta} l(n)^{-1/2}\left|X_n^{(v)}(s)\right|$$

$$\leq \sup_{0\leq s\leq\delta}\left|S_n(a(n)s)\right|\Big/\left(4\,sk(n)\log\log\left(\frac{n^2}{sk(n)}\right)\right)^{1/2} g_n(\delta),$$

where

$$g_n(\delta)=2^{1/2}\sup_{0\leq s\leq\delta} s^v((\log\{\log(n^2/k(n))+\log(1/s)\})/\log\log n)^{1/2}.$$

Since $g_n(\delta)\to 0$ as $n\to\infty$ and $\delta\downarrow 0$, (2.18) follows from (2.19) and inequality (2.20).

LEMMA 8. — *Let* $a(n)$ *satisfy* (A) *and* $0<v<1/2$. *Whenever the series in* (1.14) *is finite.*

$$(2.21)\quad \lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup_{0\leq s\leq\delta} l(n)^{-1/2}\left|w_n^{(v)}(s)\right|=0\quad a.s.$$

Proof. — It is clear that the series in (1.14) being finite implies by the (1.13) statement in this case that

$$\limsup_{n\to\infty}\sup_{0\leq s\leq 1} l(n)^{-1/2}\delta^{-v}\left|w_n^{(v)}(s\delta)\right|=1\quad \text{a.s.},$$

which gives (2.21).

The first part of Corollary 3 obviously follows from Lemmas 7 and 8 and Theorem 1, i. e. here we necessarily have $o(k(n))=\log\log n$. Corollary 4 is a consequence of Lemma 7 and Theorem 2. Finally, the second part of Corollary 3 is implied by Corollary 4 and the fact that the lim sup in (1.13) is infinite almost surely when the series in (1.14) is infinite.

ACKNOWLEDGEMENTS

The author thanks the referee for a careful reading of the manuscript and for a number of very useful suggestions that improved the presentation. He is also indebted to John H. J. Einmahl for pointing out and correcting a couple of minor errata in the original version.

REFERENCES

A. ARAUJO and E. GINÉ, *The Central Limit Theorem for Real and Banach Valued Random Variables*, John Wiley & Sons, New York, 1980.

B. COOIL, Limiting Multivariate Distribution of Intermediate Order Statistics, *Ann. Probab.*, Vol. **13**, 1985, pp. 469-477.

M. CSÖRGŐ and P. RÉVÉSZ, *Strong Approximations in Probability and Statistics*, Academic Press, New York, Akadémai Kiadó, Budapest, 1981.

M. Csörgő and D. M. Mason, On the Asymptotic Distribution of Weighted Uniform Empirical and Quantile Processes in the Middle and on the Tails, *Stochastic Process. Appl.*, Vol. **21**, 1985, pp. 119-132.

J. H. J. Einmahl and D. M. Mason, Laws of the Iterated Logarithm in the Tails for Weighted Uniform Empirical Processes, *Ann. Probab.*, Vol. **16**, 1988 *a*, pp. 126-141.

J. H. J. Einmahl and D. M. Mason, Strong Limit Theorems for Weighted Quantile Processes, *Ann. Probab.*, 1988 *b* (to appear).

W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. **1**, Third Ed., Wiley, New York, 1968.

H. Finkelstein, The Law of Iterated Logarithm for Empirical Distributions, *Ann. Math. Statist.*, Vol. **42**, 1971, pp. 607-615.

P. Gaenssler and W. Stute, *Wahrscheinlichkeitstheorie*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

B. R. James, A Functional Law of the Iterated Logarithm for Weighted Empirical Distributions, *Ann. Probab.*, Vol. **3**, 1975, pp. 762-772.

J. Kiefer, Iterated Logarithm Analogues for Sample Quantiles when $p_n \downarrow 0$, *Proceedings Sixth Berkeley Symp. Math. Statist. Probab.*, Vol. **I**, 1972, pp. 227-244, Univ. of California Press, Berkeley, Los Angeles.

J. Komlós, P. Major and G. Tusnády, An Approximation of Partial Sums of Independent rv's and the Sample df, I, *Z. Wahrsch. verw. Gebiete*, Vol. **32**, 1975, pp. 111-131.

T. L. Lai, Reproducing Kernel Hilbert Spaces and the Law of the Iterated Logarithm for Gaussian Processes, *Z. Wahrsch. verw. Gebiete, Vol.* **29**, 1974, pp. 7-19.

P. Major, Approximations of Partial Sums of i.i.d.r.v.s. when the Summands Have Only Two Moments, *Z. Wahrsch. verw. Gebiete*, Vol. **35**, 1976, pp. 221-229.

D. M. Mason, *Sums of Extreme Value Processes*, Preprint, 1988.

D. M. Mason and W. R. van Zwet, A Refinement of the KMT Inequality for the Uniform Empirical Process, *Ann. Probab.*, Vol. **15**, 1987, pp. 871-884.

S. Orey and W. E. Pruitt, Sample Functions of the N-Parameter Wiener Process, *Ann. Probab.*, Vol. **1**, 1973, pp. 138-163.

W. Philipp and W. Stout, Invariance Principles for Martingales and Sums of Independent Random Variables, *Math. Z.*, Vol. **192**, 1986, pp. 253-264.

G. R. Shorack and J. A. Wellner, *Empirical Processes with Applications to Statistics*, John Wiley & Sons, New York, 1986.

M. J. Wichura, Some Strassen-Type Laws of the Iterated Logarithm for Multiparameter Stochastic Processes with Independent Increments, *Ann. Probab.*, Vol. **1**, 1973, pp. 272-296.

(*Manuscrit reçu le* 21 *mai* 1987)
(*corrigé le* 1er *mars* 1988.)