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Small tails for the supremum of a Gaussian process

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ABSTRACT. — Let T be a compact metric space. Let $(X_t)_{t\in T}$ be a Gaussian process with continuous covariance. Assume that the variance has a unique maximum at some point τ and that X_t has a. s. bounded sample paths. We prove that

$$\lim_{u\to\infty} P(\operatorname{Sup} X_t > u)/P(X_\tau > u) = 1$$

if and only if

$$\lim_{h\to 0} h^{-1} E(\operatorname{Sup}_{t\in T_h}(X_t - X_\tau)) = 0$$

where $T_h = \{t\in T;\ E(X_t X_\tau) \geqq E(X_\tau^2) - h^2\}$.

Key words : Gaussian process on abstract sets, Borell's inequality.

RÉSUMÉ. — Soit T un espace compact métrique, et $(X)_{t\in T}$ un processus gaussien à convariance continue. Supposons que (X_t) soit p. s. borné et que l'écart type de X_t ait un maximum unique en un certain point τ. Nous montrons que

$$\lim_{u\to\infty} P(\sup X_t > u)/P(X_\tau > u) = 1$$

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si et seulement si

$$\lim_{h \to 0} h^{-1} \mathrm{E}(\operatorname{Sup}_{t \in \mathrm{T}_n} (\mathrm{X}_t - \mathrm{X}_\tau)) = 0$$

où $\mathrm{T}_h = \{t \in \mathrm{T};\ \mathrm{E}(\mathrm{X}_t \mathrm{X}_\tau) \geqq \mathrm{E}(\mathrm{X}_\tau^2) - h^2\}$.

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1. INTRODUCTION

Let T be a compact metric space. Let $(\mathrm{X}_t)_{t \in \mathrm{T}}$ be a (real separable) Gaussian process with mean zero and continuous covariance. Asume that (X_t) has a. s. bounded sample paths, so $\operatorname{Sup}_{t \in \mathrm{T}} \mathrm{X}_t < \infty$ a. s. We want to determine when the tail of the distribution of $\operatorname{Sup} \mathrm{X}_t$ is minimal in some sense. Let $\sigma^2(t) = \mathrm{EX}_t^2$, and $\sigma = \operatorname{Sup}_{t \in \mathrm{T}} \sigma(t)$. If Y is standard normal, we have for all u

$$\mathrm{P}(\operatorname{Sup} \mathrm{X}_t > u) \geqq \mathrm{P}(\sigma \mathrm{Y} > u).$$

We want to characterize the Gaussian processes for which the following occurs:

$$\lim_{u \to \infty} \mathrm{P}(\operatorname{Sup} \mathrm{X}_t > u)/\mathrm{P}(\sigma \mathrm{Y} > u) = 1. \qquad (\bigstar)$$

Our work is motivated by several recent papers ([1], [2], [7]) that give sufficient conditions for ($\bigstar$) in specific situations. These results rely on conditions on the covariance function, and appropriate computations. On the other hand, we will approach the problem from the abstract point of view, and our result will be an elementary consequence of (a weak form of) Borell's inequality. (*See* [8] for a previous use of Borell's inequality in the study of Sup X_t).

THEOREM. — *Condition* ($\bigstar$) *is equivalent to the following two conditions*

(I) *There exists a unique* τ *in* T *such that* $\sigma(\tau) = \sigma$.

(II) *If, for* $h > 0$, *we set*

$$\mathrm{T}_h = \{t \in \mathrm{T};\ \mathrm{E}(\mathrm{X}_t \mathrm{X}_\tau) \geqq \sigma(\tau)^2 - h^2\},$$

we have

$$\lim_{h \to 0} h^{-1} \mathrm{E} \operatorname{Sup}_{t \in \mathrm{T}_h} (\mathrm{X}_t - \mathrm{X}_\tau) = 0.$$

Both of these conditions seem rather easy to check in practice. (In particular condition (II) is completely elucidated by the theory of [9].) As an illustration, we prove the result of [1]: let $(X_t)_{t \geqq 0}$ be a Gaussian process with mean zero and stationary increments, and such that $X_0=0$. Assume that the incremental variance function $\sigma^2(t)$ is convex and that $\lim_{t \to 0} \sigma^2(t)/t=0$. Then condition (★) holds on each interval $[0, \tau]$ for $\tau > 0$.

First (I) is obviously satisfied. We now check (II). Since

$$E(X_t X_\tau)=\frac{1}{2}(\sigma^2(t)+\sigma^2(\tau)-\sigma^2(\tau-t)),$$

for $t \in T_h$, we have $\sigma^2(t) \geqq \sigma^2(\tau)-2h^2$. Since σ^2 is convex, it has a left derivative at each point, so for t in T_h we have $t \geqq \tau - K h^2$ for some constant K. Since the process has stationary incremants, to check (II) it is enough to show that

$$\lim_{h \to 0} h^{-1} E \sup_{0 \leqq t \leqq h^2} |X_t|=0.$$

This is done in a few lines of computations using, e. g., the bound of $E \sup_{0 \leqq t \leqq h^2} |X_t|$ by Dudley's entropy integral (*see* e. g. [6], p. 25).

2. PREPARATION

In this section we list some facts that our proof will use. Let $\varphi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$ be the standard normal density function. Let

$$\psi(z)=\int_z^\infty \varphi(x)\,dx=P(Y > z).$$

We denote by K_1, K_2, ... positive universal constants. The first two lemmas are elementary.

LEMMA 1. — *The following hold*

For $u \geqq 1$, $K_1^{-1}\varphi(u) \leqq u\psi(u) \leqq \varphi(u)$. (1)

For $x \geqq -1$, $\psi(x) \leqq K_2 \exp-(x^2/2)$ (2)

For any u, t, $\varphi(u-t) \leqq \varphi(u)\exp tu$. (3)

For $0 \leqq v \leqq u$, *we have* $\psi(v) \geqq \psi(u)+u(u-v)\psi(u)$. (4)

Proof. — (1) is well known, and implies the crude inequality (2). (3)

follows from the explicit form of φ. To prove (4), we note that since $\varphi(x)$ decreases for $x \geqq 0$, we have

$$\psi(v)-\psi(u)=\int_v^u \varphi(x)\,dx \geqq (u-v)\,\varphi(u) \geqq u(u-v)\,\psi(u)$$

by (1).

LEMMA 2. — *Let* $(w_n)_{n \geqq 0}$ *be a sequence with* $w_n > 0$ *and* $w_{n+1} \geqq 2w_n$ *for each n. Then*

$$\sum_n w_n \exp\left(-\frac{1}{2}\left(w_n-\frac{1}{2}\right)^2\right) \leqq \mathrm{K}_3.$$

Proof. — This is an obvious consequence of the fact that for $x > 0$;

$$x\exp\left(-\frac{1}{2}\left(x-\frac{1}{2}\right)^2\right) \leqq \mathrm{K}_4 \min(x, x^{-1}).$$

The following is well known and goes back to [4] and [5].

LEMMA 3. — *If* $\mathrm{P}(\operatorname{Sup} \mathrm{X}_t \leqq \mathrm{A}) \geqq 1/2$, *then* $\mathrm{E}\operatorname{Sup}\mathrm{X}_t \leqq \mathrm{K}_5\mathrm{A}$.

We finally state the version of Borell's inequality [3] that we need.

LEMMA 4. — *Let* (Y_t) *be a separable Gaussian process. Assume that* $\mathrm{P}(\operatorname{Sup}\mathrm{Y}_t \geqq w) \leqq 1/2$, *and let* $\theta=\sup \sigma(\mathrm{Y}_t)$. *Then for* $u \geqq w$

$$\mathrm{P}(\operatorname{Sup}\mathrm{Y}_t \geqq u) \leqq \psi((u-w)/\theta).$$

In particular, for all u

$$\mathrm{P}(\operatorname{Sup}\mathrm{Y}_t \geqq u) \leqq 2\,\psi((u-w)/\theta). \tag{5}$$

3. PROOF OF THE THEOREM

Changing X_t in $\sigma^{-1}\mathrm{X}_t$ we can assume $\sigma=1$. The necessity of condition (I) follows from the following elementary observation, whose proof we leave to the reader. If the couple (Y, Z) of random variables is jointly Gaussian, and if

$$\mathrm{EY}^2=\mathrm{EZ}^2=1, \qquad \mathrm{E(YZ)} < 1$$

then $\lim\limits_{u \to \infty} \mathrm{P}(\operatorname{Max}(\mathrm{Y}, \mathrm{Z}) > u)/\psi(u)=2$.

We now on assume (I). For simplicity of notation, we set $a(t)=\mathrm{E}(X_t X_\tau)$, $Z_t=X_t-a(t)X_\tau$. The process (Z_t) is hence independent of X_τ. We have

$$Z_t-(X_t-X_\tau)=(1-a(t))X_\tau,$$

so we have

$$\mathrm{E}\sup_{a(t)>1-h^2}|Z_t-(X_t-X_\tau)|\leqq h^2\,\mathrm{E}|X_\tau|.$$

This shows that condition (II) is equivalent to the following condition

(III) $\lim_{h\to 0} h^{-1}\,\mathrm{E}\sup_{a(t)>1-h^2} Z_t=0.$

Proof that (★) ⇒ (III). – Let $\varepsilon>0$ be small enough that $2\varepsilon^2<1$, and let $w\geqq 1$ be such that

$$\forall u\geqq w,\quad \mathrm{P}(\operatorname{Sup} X_t>u)\leqq(1+\varepsilon^2)\,\psi(u). \tag{6}$$

Fix $u\geqq w$. If we condition equation (6) with respect to X_t, we have

$$\int_{-\infty}^{\infty}\mathrm{P}(\operatorname{Sup}(Z_t+a(t)y)>u)\,\varphi(y)\,dy\leqq(1+\varepsilon^2)\,\psi(u). \tag{7}$$

We set $\mathrm{T}'=\{t\in\mathrm{T};\ a(t)\geqq 0\}$, and

$$\eta(y)=\mathrm{P}(\sup_{t\in\mathrm{T}'}(Z_t+a(t)y)>u).$$

This is obviously an increasing function of y. For $y>u$, taking $t=\tau$, we see that $\eta(y)=1$. It follows from (7) that

$$\int_{-\infty}^{u}\eta(y)\,\varphi(y)\,dy\leqq\varepsilon^2\,\psi(u).$$

Let $v=u-2\varepsilon^2/u$, so $v>0$ and $u(u-v)=2\varepsilon^2$.

Using (4), we have

$$2\varepsilon^2\,\psi(u)\,\eta(v)\leqq\eta(v)(\psi(v)-\psi(u))\leqq\int_v^u\eta(y)\,\varphi(y)\,dy\leqq\varepsilon^2\,\psi(u).$$

This shows that $\eta(v)\leqq 1/2$. So we have

$$\mathrm{P}(\sup_{t\in\mathrm{T}}(Z_t+a(t)v)<u)=\mathrm{P}(\forall t\in\mathrm{T}',\ Z_t<u-a(t)v)\geqq 1/2.$$

We observe that for $h > 0$, $a(t) \geqq 1-h^2$, we have

$$u-a(t)v \leqq u-v+h^2 v \leqq u-v+h^2 u.$$

It follows that

$$\mathrm{P}(\sup_{a(t) \geqq 1-h^2} \mathrm{Z}_t < u-v+h^2 u) \geqq 1/2. \tag{8}$$

Suppose now that $h \leqq \varepsilon/w$, and take $u=\varepsilon/h \geqq w$. *We have* $u-v=2\varepsilon^2/u=2\varepsilon h$, and $h^2 u=\varepsilon h$, so (8) becomes

$$\mathrm{P}(\sup_{a(t) \geqq 1-h^2} \mathrm{Z}_t < 3\varepsilon h) \geqq 1/2.$$

It follows from lemma 3 that $\mathrm{E}(\sup_{a(t) \geqq 1-h^2} \mathrm{Z}_t) \leqq 3\varepsilon h \mathrm{K}_5$ and this concludes the proof.

Proof that (I) and (III) imply (★). — We take $\eta > 0$ with $\eta < 2^{-4}$. Let $\alpha > 0$ be such that $\alpha \leqq 2^{-3}$, and that

$$\forall h \leqq \alpha, \quad \mathrm{E} \sup_{a(t) \geqq 1-h^2} \mathrm{Z}_t \leqq \eta h. \tag{9}$$

For $n \geqq 0$, we define the following subsets of T:

$$\mathrm{A}_n=\{t;\ a(t) \geqq 1-4^{-n}\alpha^2\}; \qquad \mathrm{B}_n=\mathrm{A}_n \setminus \mathrm{A}_{n+1}.$$

We note that

$$\forall t \in \mathrm{B}_n, \quad a(t) \leqq 1-4^{-n-1}\alpha^2. \tag{10}$$

Since the covariance is continuous and T is compact, condition (I) implies that

$$\theta = \sup_{a(t) \leqq 1-\alpha^2} \sigma(t) < 1.$$

Take now w large enough that $\mathrm{P}(\sup \mathrm{X}_t \leqq w) \geqq 1/2$. For $u \geqq w$, it follows from Lemma 4 that

$$\mathrm{P}(\sup_{a(t) \leqq 1-\alpha^2} \mathrm{X}_t \geqq u) \geqq \psi((u-w)/\theta).$$

This implies that

$$\lim_{u \to \infty} \mathrm{P}(\sup_{a(t) \leqq 1-\alpha^2} \mathrm{X}_t \geqq u)/\psi(u)=0.$$

To prove (★), it is enough to show that for some universal constant K, we have

$$\limsup_{u \to 0} \mathrm{P}(\operatorname{Sup}_{\mathrm{A}_0} \mathrm{X}_t > u)/\psi(u) \leqq 1 + \mathrm{K}\,\eta^2. \tag{11}$$

We first note that from (9) we have

$$\mathrm{E} \operatorname{Sup}_{\mathrm{A}_n} \mathrm{Z}_t \leqq \eta\alpha 2^{-n},$$

so in particular

$$\mathrm{P}(\operatorname{Sup}_{\mathrm{A}_n} \mathrm{Z}_t \leqq \eta\alpha 2^{-n+1}) \geqq 1/2.$$

Since $\mathrm{Z}_\tau = 0$, for t in A_n we have $\mathrm{EZ}_t^+ \leqq \eta\alpha 2^{-n}$, so

$$\theta_n = \operatorname{Sup}_{\mathrm{A}_n} \sigma(\mathrm{Z}_t) \leqq (2\pi)^{1/2}\,\eta\alpha 2^{-n} \leqq \eta\alpha 2^{-n+2}. \tag{12}$$

It follows from (10) that, for $y \geqq 0$, we have

$$\begin{aligned}\mathrm{P}(\operatorname{Sup}_{\mathrm{B}_n} \mathrm{X}_t > u \mid \mathrm{X}_\tau = y) &= \mathrm{P}(\operatorname{Sup}_{\mathrm{B}_n} (\mathrm{Z}_t + a(t)\,y) \geqq u) \\ &\leqq \mathrm{P}(\operatorname{Sup}_{\mathrm{B}_n} \mathrm{Z}_t \geqq u - y(1 - 4^{-n-1}\alpha^2)).\end{aligned}$$

Since $\mathrm{B}_n \subset \mathrm{A}_n$, it follows from (12) and (5) that

$$\mathrm{P}(\operatorname{Sup}_{\mathrm{B}_n} \mathrm{X}_t > u \mid \mathrm{X}_\tau = y) \leqq 2\,\psi((u - y(1 - 4^{-n-1}\alpha^2))/\eta\alpha 2^{-n+2} - 1/2). \tag{13}$$

We set

$$\mathrm{I}(u) = \int_0^u \mathrm{P}(\operatorname{Sup}_{\mathrm{A}_0} \mathrm{X}_t > u \mid \mathrm{X}_\tau = y)\,\varphi(y)\,dy.$$

$$\mathrm{J}(u) = \int_{-\infty}^0 \mathrm{P}(\operatorname{Sup}_{\mathrm{A}_0} \mathrm{X}_t > u \mid \mathrm{X}_\tau = y)\,\varphi(y)\,dy.$$

We have

$$P(\operatorname*{Sup}_{A_0} X_t > u) = \psi(u) + P(\operatorname*{Sup}_{A_0} X_t > u,\ X_\tau < u) \tag{14}$$

$$= \psi(u) + I(u) + J(u).$$

For $y \leqq 0$, we have

$$P(\operatorname*{Sup}_{A_0} X_t > u \mid X_\tau = y) = P(\operatorname*{Sup}_{A_0} (Z_t + a(t)y) \geqq u)$$

$$\leqq P(\operatorname*{Sup}_{A_0} Z_t \geqq u)$$

so we have $J(u) \leqq P(\operatorname*{Sup}_{A_0} Z_t \geqq u)$. Since $\operatorname*{Sup}_{t \in A_0} EZ_t^2 < 1$, we see by lemma 4 that $\lim_{u \to \infty} J(u)/\psi(u) = 0$.

From (13), we have, since $A_0 = \bigcup_{n \geqq 0} B_n$,

$$I(u) \leqq \sum_{n \geqq 0} \int_0^u P(\operatorname*{Sup}_{B_n} X_t > u \mid X_\tau = y)\, \varphi(y)\, dy$$

$$\leqq \sum_{n \geqq 0} \int_0^u 2\psi((u - y(1 - 4^{-n-1}\alpha^2))/\eta\alpha 2^{-n+2} - 1/2)\, \varphi(y)\, dy.$$

Making the change of variable $y = u - z$, and setting

$$\gamma_n = 2^{-n-4}\alpha/\eta; \qquad \delta_n = (1 - 4^{-n-1}\alpha^2)/\eta\alpha\, 2^{-n+2},$$

we get

$$I(u) \leqq 2 \sum_{n \geqq 0} \int_0^\infty \psi(\gamma_n u + \delta_n z - 1/2)\, \varphi(u - z)\, dz.$$

Using (1) and (3), we have

$$\varphi(u - z) \leqq \varphi(u) \exp uz \leqq K_1 u \psi(u) \exp uz,$$

so

$$I(u) \leqq 2K_1 \psi(u) \sum_{n \geqq 0} \int_0^\infty u \psi(\gamma_n u + \delta_n z - 1/2) \exp(uz)\, dz.$$

We note now the important fact that, since $\alpha^2 \leqq 2^{-3}$, $\eta < 2^{-4}$,

$$\gamma_n \delta_n = 2^{-6}(1 - 4^{-n+1}\alpha^2)/\eta^2 \geqq 2^{-7}/\eta^2 \geqq 2. \tag{15}$$

Using (2), we get after a short computation, that for some constants K_6, K_7,

$$I(u) \leqq K_6 \psi(u) \sum_{n \geqq 0} \int_0^\infty u \exp\left[-\frac{1}{2}\left(\gamma_n u - \frac{1}{2}\right)^2 - \frac{1}{2}\left(\delta_n z - \frac{1}{2}\right)^2\right] dz.$$

$$\leqq K_7 \psi(u) \sum_{n \geqq 0} \frac{u}{\delta_n} \exp\left(-\frac{1}{2}\left(\gamma_n u - \frac{1}{2}\right)^2\right).$$

And from (15) we have

$$I(u) \leqq \eta^2 2^7 K_7 \psi(u) \sum_{n \geqq 0} u \gamma_n \exp\left(-\frac{1}{2}\left(\gamma_n u - \frac{1}{2}\right)^2\right).$$

Since $u \gamma_{n+1} = u \gamma_n / 2$, it follows from lemma 2 that for some universal constant K, we have $I(u) \leqq K \eta^2 \psi(u)$. Together with (14), this proves (11), and concludes the proof.

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