ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 23, nº 3 (1987), p. 499-530

<http://www.numdam.org/item?id=AIHPB_1987__23_3_499_0>

Ann. Inst. Henri Poincaré,

Vol. 23, n° 3, 1987, p.499-530. *Probabilités et Statistiques*

Potential theory for a family of several Markov processes

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ABSTRACT. — We develop some potential theory for multi-parameter processes of the form $X(t)=(X^1(t^1),\ldots, X^k(t^k))$, where $t=(t^i)\in\mathbb{R}_+^k$ are Markov processes. In particular, we investigate the "small sets" for these processes and study the properties of the fine topology.

As an application of the "small sets" results we study the existence of multiple points in the path of a Lévy process. In the symmetric case we prove a conjecture due to Hendricks and Taylor and we improve the results of previous authors for the general case. Finally, we obtain sufficient conditions for a set to contain multiples of a planar Brownian motion which are weaker than those thus far obtained.

RÉSUMÉ. — On décrit une théorie du potentiel pour des processus à indices multiples et du genre $X(t)=(X^1(t^1),\ldots, X^k(t^k))$, où $t=(t^i)\in\mathbb{R}_+^k$ et $X^1,\ldots, X^k$ sont des processus de Markov. En particulier, on étudie les « petits ensembles » de ces processus et les propriétés de la topologie fine.

Comme application des résultats concernant ces « petits ensembles » on étudie l'ensemble des points multiples sur la trajectoire d'un processus de Lévy. Dans le cas symétrique on prouve une conjecture par Hendricks et Taylor et on améliore pour le cas général les résultats des auteurs précédents. Finalement, on obtient des conditions suffisantes pour un ensemble contenant des points multiples d'un mouvement brownien plan et qui sont plus faibles que celles obtenues juqu'à présent.

Mots clés : Théorie du potentiel, processus de Markov, topologie fine, points multiples.

1. INTRODUCTION

The objects which we investigate in this paper are multi-parameter processes of the form $X(t)=(X^1(t^1), \ldots, X^k(t^k))$, where $t=(t^1, \ldots, t^k) \in \mathbb{R}_+^k$ and $X^1, \ldots, X^k$ are independent Markov processes. Some properties of these processes have already been studied in [6]. Our aim is, in the main, to extend to this setting some of the known result from the potential theory of Markov processes. We will also consider some of the applications of our results to the study of multiple points in the sample paths of one-parameter processes.

After introducing some notation in Section 2, we present a few useful preparatory results in Section 3. Chief among these are a section-type theorem and a "strong Markov" property in which the appearance of stopping-times is replaced by that of optional random measures. This formulation allows us to partially overcome the unfortunate inutility of stopping-time methods in the multi-parameter theory.

In Section 4 we employ these tools to establish a partial extension of the well-known result that if f is an excessive function for a standard Markov process Y then $f(Y_t)$ has right-continuous paths almost surely. The fine topology for X is constructed in Section 5 and the results of Section 4 are reconsidered as statements about the fine topology.

We restrict attention in Section 6 to the case where each of the X^i is a symmetric process. In this setting we discuss two notions of "small set" and develop a necessary condition for a set to be "small".

Section 7 is more or less a reprise of Section 6 for the case where each of the X^i is a Lévy process, although the methods used are markedly different.

Several applications of the "small set" results to the study of sufficient conditions for the existence of multiple points in the path of a Lévy process are given in Section 8. In particular, we verify a conjecture due to Hendricks and Taylor concerning such a condition for symmetric Lévy processes. In the general (i. e. not necessarily symmetric) case we show how our methods can improve the results of Shieh [21] for this problem.

Finally, we sharpen the condition given by Tongring [22] for a set in the plane to contain multiple points of a planar Brownian motion.

For the sake of the reader who is primarily interested in the "small set" results and their application to the study of multiple points, we remark that the results of Sections 6 to 8 are essentially independent of those contained in Sections 3 to 5.

2. NOTATIONS

Suppose that $X^i=(\Omega^i, \mathscr{M}^i, \mathscr{M}^i_{t^i}, X^i_{t^i}, \theta^i_{t^i}, P^i_{x^i})$, $1\leqq i\leqq k$, are standard processes (*see* e. g. [3], I-9) with state spaces $(E^i, \mathscr{B}^i)$ augmented by Δ^i. As usual, we set $E^i_{\Delta^i}=E^i\cup(\Delta^i)$ and let $\mathscr{B}^i_{\Delta^i}$ be the σ-field on $E^i_{\Delta^i}$ generated by $\mathscr{B}^i$.

Set $E=\prod_i E_i$ and $\mathscr{B}=\prod_i \mathscr{B}_i$. Define E_Δ and $\mathscr{B}_\Delta$ similarly. We will adopt the convention that when the domain of a function is not expressly stated it will be assumed to be E. We extend such functions to E_Δ by setting them to be 0 on $E_\Delta\setminus E$. We also adopt the analagous convention for measures.

Define a measurable space $(\Omega, \mathscr{M})$ by setting $\Omega=\prod_i \Omega^i$ and $\mathscr{M}=\prod_i \mathscr{M}^i$. We have a $\mathbb{R}^k_+$-indexed filtration on this space given by $\mathscr{M}_t=\prod_i \mathscr{M}^i_{t^i}$, where $t=(t^1,\ldots, t^k)\in\mathbb{R}^k_+$. Similarly, set $X_t(\omega)=(X^i_{t^i}(\omega^i))$, $\theta_t(\omega)=(\theta^i_{t^i}(\omega^i))$ and $P_x=\prod_i P^i_{x^i}$ for $\omega=(\omega^1,\ldots,\omega^k)\in\Omega$ and $x=(x,\ldots,x^k)\in E_\Delta$.

If μ is a σ-finite on $(E_\Delta, \mathscr{B}_\Delta)$ we may define a measure P_μ on $(\Omega, \mathscr{M})$ by $P_\mu(.)=\int\mu(dx)P_x(.)$. Let $\mathscr{M}^\mu$ be the completion of $\mathscr{M}$ with respect to P_μ. Denoting by $\mathscr{N}$ the collection of P_μ-null sets in $\mathscr{M}^\mu$, set

$$\mathscr{M}^{\mu, 1}_{s^1}=\bigcap_{T>s^1}[(\mathscr{M}^1_T\times\mathscr{M}^2\times\ldots\times\mathscr{M}^k)\vee\mathscr{N}]$$

and define $\mathscr{M}^{\mu, 1}_{s^i}$ similarly. Let $\mathscr{M}^\mu_s=\mathscr{M}^{\mu,1}_{s^1}\cap\ldots\cap\mathscr{M}^{\mu,k}_{s^k}$.

If X^i has transition function $p^i(t^i, x^i, B^i)$ set

$$p(t, x,.)=\prod_i p^i(t^i, x^i,.).$$

For $\alpha=(\alpha^1,\ldots,\alpha^k)$ with $\alpha^i>0$ put

$$G^{\alpha^i}(x^i, B^i)=\int \exp(-\alpha^i t^i)\, p^i(t^i, x^i, B^i)\, dt^i$$

and

$$G^{\alpha}(x,.)=\prod_i G^{\alpha^i}(x^i,.).$$

3. SOME GENERAL RESULTS

The following result is the analogue of the Blumenthal 0-1 Law.

(3.1) LEMMA. — *Let*

$$\begin{aligned}\mathcal{G}_t&=\sigma\{X_s: s\leqq t\}\\ &=\prod_i \sigma\{X^i_{s^i}: s^i\leqq t^i\}\\ &=\prod_i \mathcal{G}^i_{t^i}.\end{aligned}$$

Fix $x\in E_\Delta$ *and let* $\mathcal{N}$ *denote the collection of* P_x*-null sets in the* P_x*-completion of* $\mathcal{G}_\infty$. *Set*

$$\mathcal{H}=\bigcap_{t>0}(\mathcal{G}_t\vee\mathcal{N}).$$

then for each $A\in\mathcal{H}$ *we have* $P_x(A)=0$ *or* 1.

Proof. — By a monotone class argument and the one-parameter 0-1 law, the conclusion of the Lemma holds if we replace $\mathcal{H}$ by $\mathcal{G}_0$. It therefore suffices to prove that, for each $A\in\mathcal{H}$, I_A is P_x-equivalent to a $\mathcal{G}_0$-measurable random variable.

Suppose that $B=\prod_i B^i\in\mathcal{G}_\infty$, then by I-8-13 of [3]

$$\begin{aligned}P_x(B\mid\mathcal{H})&=\lim_{n\to\infty} P_x(B\mid\mathcal{G}_{(1/n,\ldots,1/n)}\vee\mathcal{N})\\ &=\lim_{n\to\infty}\prod_i P^i_{x^i}(B^i\mid\mathcal{G}^i_{1/n})\\ &=\prod_i P^i_{x^i}(B^i\mid\mathcal{G}^i_{0^+})\end{aligned}$$

$$= \prod_i \mathrm{P}^i_{x^i}(\mathrm{B}^i \mid \mathscr{G}^i_0)$$
$$= \mathrm{P}_x(\mathrm{B} \mid \mathscr{G}_0)$$

A monotone class argument then establishes that $\mathrm{P}_x(\mathrm{B} \mid \mathscr{H}) = \mathrm{P}_x(\mathrm{B} \mid \mathscr{G}_0)$ for all $\mathrm{B} \in \mathscr{G}_\infty$ and hence for all $\mathrm{B} \in [\mathscr{G}_\infty \vee \mathscr{N}]$. In particular, if $\mathrm{A} \in \mathscr{H}$ then $\mathrm{I}_\mathrm{A} = \mathrm{P}_x(\mathrm{A} \mid \mathscr{G}_0)$ and the result follows.

Suppose for the rest of this section that $k=2$. In what follows we wish to use some of the "general theory" which has been developed for two-parameter processes; in particular, those elements which appear in [2]. To this end we remark that, for a fixed probability measure μ, the probability space $(\Omega, \mathscr{M}^\mu, \mathrm{P}_\mu)$ with the two filtrations $(\mathscr{M}^{\mu_i\, i}_{s^i})$ satisfies the regularity conditions set out at the beginning of Section II in [2]. In particular we have

$$\mathrm{E}_\mu(\,.\mid \mathscr{M}^\mu_s) = \mathrm{E}_\mu(\,.\mid \mathscr{M}^{\mu_1\,1}_{s^1} \mid \mathscr{M}^{\mu_2\,2}_{s^2}) = \mathrm{E}_\mu(\,.\mid \mathscr{M}^{\mu_2\,2}_{s^2} \mid \mathscr{M}^{\mu_1\,1}_{s^1}) = \mathrm{E}_\mu(\,.\mid \mathscr{M}_s)$$

and so the F-4 commutation condition holds.

Now it is a well-known feature of the multi-parameter general theory that whilst the formal concept of a stopping-time can be extended to the multi-parameter setting the notion is nowhere near as useful as it is in the one-parameter theory. For instance, an example of Cairoli (*see* e. g. [4]) shows that there are well-behaved sets which are almost surely disjoint from the graph of any stopping time. Thus there is no hope of obtaining any sort of an analogue for the section theorem. The following result, however, indicates a means of "getting hold" of certain random sets which will be sufficient for our purposes. We refer the reader to [2] for the definitions of the optional σ-field and an integrable increasing process.

(3.2) LEMMA (*cf.* the "selection lemma" of [18]). — *For a probability measure μ on E_Δ consider the probability space $(\Omega, \mathscr{M}^\mu, \mathrm{P}_\mu)$ equipped with the filtrations $\{\mathscr{M}^{\mu_i\, i}_{s^i}\}$. If C is a non-evanescent optional subset of $\Omega \times \mathbb{R}^2_+$ then there is an optional integrable increasing process Λ such that:*

(i) *$d\Lambda_t$ is concentrated on* C;

(ii) $\mathrm{E}_\mu \Lambda_\infty > 0$.

Proof. — If we add an isolated point ∂ to $\mathbb{R}^2_+$, then the remark of III-45 [5] shows that there exists an $\mathscr{M}^\mu$-measurable mapping $\mathrm{T}: \Omega \to \mathbb{R}^2_+ \cup (\partial)$ such that $\mathrm{T}(\omega) \neq \partial \Rightarrow (\omega, \mathrm{T}(\omega)) \in \mathrm{C}$ and $\mathrm{P}_\mu\{\mathrm{T}(\omega) \neq \partial\} = \mathrm{P}_\mu\{\pi(\mathrm{C})\}$, where $\pi(\mathrm{C})$ is the projection of C onto Ω.

Let $\mathrm{A}_t = \mathrm{I}_{\{\mathrm{T} \leqq t\}}$, then A is a (non-adapted) integrable increasing process. Following [2] we may define Λ by $d\Lambda_t = \mathrm{I}_\mathrm{C}\, d\tilde{\mathrm{A}}_t$, where $\tilde{\mathrm{A}}$ is the dual optional projection of A. Then Λ is an optional increasing process satisfying (i). Moreover, by an argument similar to that which establishes Théorème 9 in [2] we have

$$\begin{aligned} E_\mu \Lambda_\infty &= E_\mu\left(\int I_C(s)\, d\tilde{A}(s)\right) \\ &= E_\mu\left(\int I_C(s)\, dA(s)\right) \\ &= E_\mu\left(\int dA(s)\right) \\ &= P_\mu(\pi(C)) > 0 \end{aligned}$$

and so (ii) also holds.

The way in which we intend to exploit Lemma 3.2 is to replace arguments which involve evaluating a process at a stopping-time by arguments which involve integrating a process with respect to the measure induced by an increasing process. This latter operation has many of the properties of the former, such as the following analogue of the strong Markov property.

(3.3) THEOREM. — *If Λ is an integrable optional increasing process on $(\Omega, \mathcal{M}^\mu, P_\mu)$ equipped with the filtrations $\{\mathcal{M}_{s}^{\mu, i}\}$ then for $B \in \mathcal{B}_\Delta$ and $t \in \mathbb{R}_+^2$ we have*

$$E_\mu\left(\int_{\mathbb{R}_+^2} E_{X(s)} I_B(X(t))\, d\Lambda(s)\right) = E_\mu\left(\int_{\mathbb{R}_+^2} I_B(X(s+t))\, d\Lambda(s)\right)$$

Proof. — Since both sides above are measures, it will suffice to show that equality holds if we replace I_B by f where $f(x^1, x^2) = f^1(x^1)\, f^2(x^2)$ with f^i a non-negative, bounded continuous function on $E_{\Delta^i}^i$.

Both sides of the resulting equation will be right continuous in t. By the unicity Laplace transforms, it thus suffices to show for $\alpha = (\alpha^1, \alpha^2)$ with $\alpha^i > 0$ that

$$E_\mu\left(\int E_{X(s)}\left[\int \exp(-\alpha . t) f(X_t)\, dt\right] d\Lambda_s\right)$$
$$= E_\mu\left(\int\left[\int \exp(-\alpha . t) f(X_{s+t})\, dt\right] d\Lambda_s\right) \quad (3.3.1)$$

For $i, j, n \in \mathbb{Z}_+$ put

$$\lambda_{n,ij} = \int_{[(i-1,j-1)2^{-n},(i,j)2^{-n}[} d\Lambda_s$$

and set $\delta_{n,ij}$ to be the point mass at $(i,j)2^{-n}$. Let

$$\Lambda_n(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{n,ij}\,\delta_{n,ij}(]0,t])$$

We will first show that (3.3.1) holds with Λ replaced by Λ_n. The right hand side of (3.3.1) is then

$$\int \exp(\alpha.\,t) \sum_i \sum_j [E_\mu(\Lambda_{n,ij} f(X((i,j)2^{-n}+t))]\,dt \qquad (3.3.2)$$

By definition $\lambda_{n,ij} \in \mathcal{M}^{\mu}_{(i,j)2^{-n}}$. An application of the Markov property of the X^i and the remark above that $E_\mu(.\,|\,\mathcal{M}^{\mu}_s) = E_\mu(.\,|\,\mathcal{M}_s)$ gives that the term in [ ] in (3.3.2) is $E_\mu(\lambda_{n,ij}\,E_{X((i,j)2^{-n})} f(X_t))$. Substituting this shows that (3.3.1) holds with Λ replaced by Λ_n.

Now note that if we set

$$\begin{aligned} g(x) &= E_x\left[\int \exp(-\alpha.\,t) f(X_t)\,dt\right] \\ &= \prod_i E^i_{x^i}\left[\int \exp(-\alpha^i t^i) f^i(X_{t^i})\,dt^i\right] \\ &= \prod_i g^i(x^i) \end{aligned}$$

then g^i is an α-excessive function for X^i and so, by II-2-12 of [3], $s \to g(X_s) = \prod_i g^i(X^i_{s^i})$ is almost surely right continuous. Clearly, the process inside [ ] of the right hand side of (3.3.1) is almost surely continuous. Hence taking limits as $n \to \infty$ shows that (3.3.1) holds.

4. COMPOSITION WITH "POTENTIALS"

An important and useful property of standard processes is, roughly speaking, that they are right processes. More specifically, if f is a λ-excessive function for some standard process Y then the paths of $f(Y_t)$ are almost surely right continuous (*see* e. g. II-2-12 of [3]).

In considering possible multi-parameter analogues for this result two difficulties present themselves. Firstly, it does not seem to be at all clear what is an appropriate definition of "excessive function" in this setting. It appears that there is no definition which shares all the pleasing properties of the one-parameter case. Secondly, the existing proofs of the above result rely on the existence of right-limited modifications of supermartingales or at least on the convergence of discrete parameter supermartingales. Unfortunately, neither of these latter two results hold in general for the natural multi-parameter analogue of the supermartingale.

We avoid these difficulties by restricting attention to the regularity properties of $g(X_t)$, where $g(x) = G^\alpha(x, f)$ with f a bounded, non-negative function. This class of functions (which must be subsumed in any reasonable multi-parameter definition of "α-excessive") has extra analytic structure which enables us to proceed without the above-mentioned convergence theorems.

We assume for the rest of this section that $k=2$.

(4.1) THEOREM. — *Let f be a non-negative, bounded measurable function. For $\alpha = (\alpha^1, \alpha^2)$ with $\alpha^i > 0$ the process $G^\alpha(X_t, f)$ is a.s. right continuous.*

The proof will proceed via a sequence of Lemmas. The first of these is a "Doob-Meyer"-type decomposition of $G^\alpha(X_t, f)$.

(4.2) LEMMA. — *Set*

$$r^1(x) = \int_{]0,\,\infty[} \exp(-\alpha^1 t^1)\, p^1(t^1, x^1, f(\,.\,, x^2))\, dt^1$$

and define r^2 analogously. Put

$$A(t; f) = \int_{]0,\,t]} \exp(-\alpha . s)\, f(X(s))\, ds$$

$$S^1(t; f) = \int_{]0,\,t^2]} \exp(-\alpha . (t^1, s^2))\, r^1(X(t^1, s^2))\, ds^2 + A(t; f)$$

$$S^2(t; \mathrm{f}) \quad \text{similarly}$$

$$M(t; f) = \exp(-\alpha . t)\, G^\alpha(X_t, f) + S^2(t; f) + S^2(t; f) - A(t; f).$$

If we fix x, then on the probability space $(\Omega, \mathscr{M}^x, P_x)$ with the filtrations $(\mathscr{M}^x_{t^i})$ we have (see [19] *for the relevant definitions of* (Δi)*-submartingale and martingale*).

(i) *$A(t; f)$ is a bounded, continuous increasing process;*

(ii) $S^i(t; f)$ *is a bounded* (Δi)-*submartingale;*
(iii) $M(t; f)$ *is a bounded martingale.*

Proof. — Suppose first that $f(y)=\prod_i f^i(y^i)$, then $G^\alpha(y, f)=\prod_i G^{\alpha^i, i}(y^i, f^i)$. Set

$$A^i(t^i; f^i)=\int_{]0, t^i]} \exp(-\alpha^i s^i) f^i(X^i(s^i))\, ds^i$$

$$M^i(t^i; f^i)=\exp(-\alpha^i t^i)\, G^{\alpha^i, i}(X^i(t^i); f^i)+A^i(t^i; f^i)$$

It is easily checked that $(M^i(t^i; f^i), \mathscr{M}^i_{t^i})$ is a one-parameter non-negative, bounded martingale on $(\Omega^i, \mathscr{M}^i, P^i_{x^i})$.

We have

$$M(t; f)=M^1(t^1; f^1)\, M^2(t^2; f^2)$$
$$S^1(t; f)=M^1(t^1; f^1)\, A^2(t^2; f^2)$$
$$S^2(t; f)=A^1(t^1; f^1)\, M^2(t^2; f^2)$$
$$A(t; f)=A^1(t^1; f^1)\, A^2(t^2; f^2).$$

It is clear that $(M(t; f), \mathscr{M}_t)$ is a martingale on $(\Omega, \mathscr{M}, P_x)$ and the examples on p. 27 of [19] show that $(S^i(t; f), \mathscr{M}_t)$ is a (Δi)-submartingale on this space.

As $E_x(\,.\,|\mathscr{M}_t)=E_x(\,.\,|\mathscr{M}^x_t)$ we see that the conclusions of the Lemma hold for f of this form.

Now note that if $\lambda_1, \lambda_2>0$ and f_1, f_2 are non-negative, bounded functions then $M(t; \lambda_1 f_1+\lambda_2 f_2)=\lambda_1 M(t; f_1)+\lambda_2 M(t; f_2)$. Also if $f_n \to f$ bounded pointwise as $n \to \infty$ then $M(t; f_n) \to M(t; f)$ bounded pointwise. Similar comments hold for the S^i. Since the classes of bounded martingales and bounded (Δi)-submartingales are each cones closed under bounded pointwise convergence, the Lemma then follows by writing f as the bounded pointwise limit of functions of the form $f_n=\sum_{1 \leq k \leq m(n)} \lambda n, k\, f_{n, k}$ where $\lambda_{n, k}>0$ and $f_{n, k}$ is of the product form considered above.

(4.3) LEMMA. — *In the notation of Lemma* 4.2 *each of the processes* $M(t; f)$, $S^i(t; f)$ *has a right-continuous modification.*

Proof. — The result for M follows from Lemma 4.2 (iii) and the Théorème of [1].

The result for S^i will follow from Lemma 4.2 (ii) and Theorem 3.4 (iii) of [19] once we have shown that $t \to E_x S^i(t; f)$ is right-continuous, but a simple calculation gives that

$$E_x S^1(t; f) = \int_{]0, (\infty, t^2)]} \exp(-\alpha . s) p(s, x, f) \, ds$$

with a similar expression holding for $E_x S^2(t; f)$.

(4.4) *Proof of Theorem* 4.1. — It will suffice to show that $Y(t; f) = \exp(-\alpha . t) G^\alpha(X_t, f)$ is P_x-a. s. right-continuous.

Let J be the class of bounded, non-negative, measurable functions f for which $Y(t; f)$ is P_x-a. s. right continuous. By II-2-12 of [3], J contains all f of the form $f(x) = I(x \in A^1 \times A^2)$ where $A^i \in \mathscr{B}^i$. In particular we have.

(i) $1_E \in J$.

We also have

(ii) If $f_1, f_2 \in J$ and $\lambda_1, \lambda_2 > 0$ then $\lambda_1 f_1 + \lambda_2 f_2 \in J$.

(iii) If $f_1, f_2 \in J$, $f_1 \geqq f_2$ then $f_1 - f_2 \in J$.

Thus if we can show

(iv) If $f_n \in J$, $f_n \to f$ bounded pointwise then $f \in J$

the Theorem will follow by a monotone class argument.

From Lemma 4.3 we know that $Y(.; f)$ has a right-continuous modification which we will denote by Z.

We have that $Y(.; f) \geqq Y(.; f_n)$ for all n and hence, using the right-continuity of Z and Y_n we have $Z(t) \geqq Y(t; f_n)$ except on an evanescent set. Thus $Z(t) \geqq Y(t; f)$ off an evanescent set.

It therefore suffices to show that the set $B = \{(\omega, t) : Z(\omega, t) > Y(\omega, t; f)\}$ is evanescent. The process $Y(; f)$ is clearly optional and Z is optional, being right-continuous (*cf.* théorème 8 of [2]). Hence B is optional. Applying Lemma 3.2 we may construct a non-trivial optional increasing process Λ such that $d\Lambda$ is concentrated on B.

Let Λ_n be defined from Λ as in the proof of Theorem 3.3. The right-continuity of Z implies

$$\int Z(s) \, d\Lambda(s) = \lim_n \int Z(s) \, d\Lambda_n(s) = \lim_n \int Y(s; f) \, d\Lambda_n(s)$$

P_x-a. s., since $Z(s) = Y(s; f) \, P_x$-a. s. for each s.

On the other hand, we have from Theorem 3.3 that

$$\begin{aligned}
&E_x\left(\int Y(s; f)\, d\Lambda_n(s)\right)\\
&\qquad = E_x\left(\int\int \exp(-\alpha.(s+t))\, f(X(s+t))\, dt\, d\Lambda_n(s)\right)\\
&\qquad \leqq E_x\left(\int\int \exp(-\alpha.(s+t))\, f(X(s+t))\, dt\, d\Lambda(s)\right)\\
&\qquad\qquad = E_x\left(\int Y(s; f)\, d\Lambda(s)\right).
\end{aligned}$$

Putting this all together, we get

$$\begin{aligned}
E_x\left(\int Z(s)\, d\Lambda(s)\right) &= E_x\left(\lim_n \int Y(s; f)\, d\Lambda_n(s)\right)\\
&\leqq \liminf_n E_x\left(\int Y(s; f)\, d\Lambda_n(s)\right)\\
&\qquad \leqq E_x\left(\int Y(s; f)\, d\Lambda(s)\right)
\end{aligned}$$

which establishes the desired contradiction.

5. THE FINE TOPOLOGY

If $B \in \mathscr{B}_\Delta$ then for each $x \in E_\Delta$ we have from Lemma 3.1 that

$$P_x(\forall t > 0,\ \exists\, 0 < s \leqq t : X_t \in B)$$

is either 0 or 1. Conforming with the one-parameter terminology, we will say that $A \subset E$ is *finely open* if for each $x \in A$ there exists a set $B \in \mathscr{B}_\Delta$ such that $(E_\Delta \setminus A) \subset B$ and the probability above is 0.

It is easy to see that the class of finely open sets form a topology on E which we will call the *fine topology*. This topology is at least as strong as the product of the fine topologies for each of the X^i.

The following result is a restatement of Theorem 4.1 in terms of these concepts.

(5.1) COROLLARY (*cf.* II-4-2 of [3]). – *Under the conditions of Theorem* 4.1, $G^\alpha(x, f)$ *is finely continuous.*

Proof. – Suppose that $U \subset \mathbb{R}$ is open. If $x \in (G^\alpha(\,.\,, f))^{-1}(U) = V$ then

$$P_x(\forall t > 0, \exists s \in [0, t] : X_s \notin V) = 0$$

by the right-continuity of $G^\alpha(X_t, f)$ established in Theorem 4.1.

6. SMALL SETS FOR SYMMETRIC PROCESSES

In this section we consider the situation where each of the transition functions $p^i(t^i, x^i, B^i)$ is symmetric with respect to a Radon measure m^i on E^i (*see* e.g. 2-2 in [9]). In this setting the classes of "small set" that we wish to investigate are:

(6.1) DEFINITION A set $B \in \mathscr{B}$ is
(i) *polar* if $P_x(\exists t > 0 : X_t \in B) = 0$ for all $x \in E$.
(ii) *exceptional* if $P_m(\exists t > 0 : X_t \in B) = 0$, where $m = \prod_i m^i$.

Clearly, every polar set is exceptional. The conditions under which the two classes coincide are the counterparts of those in the corresponding one-parameter situation (*cf.* [11], Theorems 4-2-2 and 4-3-4).

(6.2) THEOREM. – *The following are equivalent conditions on* X.
(i) *Each exceptional set is polar.*
(ii) *For each* $\alpha > 0$ *and* $x \in E$ *the measure* $G^\alpha(x, .)$ *is absolutely continuous with respect to* m.
(iii) *For each* $t > 0$ *and* $x \in E$ *the measure* $p(t, x, .)$ *is absolutely continuous with respect to* m.

Proof (i) $\Rightarrow$ (ii). – If $A^1 \in \mathscr{B}^1$ is exceptional for X^1 then $A^1 \times E^2 \times \ldots \times E^k$ is exceptional for X. By assumption, $A^1 \times E^2 \times \ldots \times E^k$ is polar for X and hence A^1 is polar for X^1. Thus every set exceptional for X^1 is also polar for X^1 and Theorem 4-2-2 of [11] gives that $G^{\alpha^1, 1}(x^1, .)$ is absolutely continuous with respect to m^1 for each $\alpha^1 > 0$ and $x^1 \in E^1$. A similar argument holds for each of the X^i and recalling that $G^\alpha(x, .) = \prod_i G^{\alpha^i, i}(x^i, .)$ completes the proof.

(ii) $\Rightarrow$ (i). — If $A \in \mathscr{B}$ is exceptional, then $u(x) = P_x(\exists t > 0 : X_t \in A) = 0$ for m — a. e. x. Note that $p(t, x, u) \leqq u(x)$ and $p(t, x, u) \to u(x)$ as $t \to 0$. Thus

$$0 = \lim_{n \to \infty} n^k G^{(n, \ldots, n)}(x, u)$$

$$= \lim_{n \to \infty} n^k \int \exp(-(n, \ldots, n).t) p(t, x, u)\, dt$$

$$= \lim_{n \to \infty} \int \exp(-(1, \ldots, 1).t) p(t/n, x, u)\, dt$$

$$= u(x), \qquad x \in E$$

and so A is polar.

(ii) $\Leftrightarrow$ (iii). — This follows from Theorem 4-3-4 of [11] once we note that $p(t, x, .) \ll m$ is equivalent to $p^i(t^i, x^i, .) \ll m^i$ for each i and that a similar equivalence holds for $G^\alpha(x, .)$.

We will now investigate analytic conditions which ensure that a given set does not belong to one of these classes.

One procedure for obtaining this sort of result in the one-parameter case (see e. g. [11]) is to associate a Hilbert space (the associated Dirichlet space) with the process and use the structure in this space to define a capacity on the subsets of E. The desired conditions are then expressed in terms of this capacity.

However in the multi-parameter case there are difficulties in setting up a relevant notion of capacity. Analytically, this seems to be intimately bound up with the fact that the Hilbert space which we associate with the process is no longer a Dirichlet space (although, typically, it is the tensor product of the Dirichlet spaces associated with the component processes [7]). In particular, "normal contractions" [9] no longer operate on the space. Seen from a probabilistic point of view, the notion of capacity in the one-parameter setting is related to the concept of first hitting times, which have no counterpart for the (partially ordered) multi-parameter problem.

One way around this impasse is to develop a theory wholly based on an extension of the classical notion of energy. Dynkin has shown in [8] that it is possible to establish many of the known one-parameter results in this way without recourse to the use of capacities. Moreover, in [6] Dynkin carries through, albeit implicity, the multi-parameter programme in the case where $p^i(t^i, x^i, .)$ is absolutely continuous with respect to m^i.

Our approach will be to use these methods to treat the general multi-parameter case.

The idea behind this technique is to demonstrate that a set A is not small by constructing a non-trivial random measure that is concentrated on the set $\{t : X_t \in A\}$. It is thus conceptually related to the Fourier-analytic "local time" technique used in [12] for treating confluences of several Brownian motions (*see* also [21]).

We begin by "embedding" the structure of our process X into the structure of a stationary process indexed by the whole of $\mathbb{R}^k$ and governed by a single σ-finite measure.

(6.3) DISCUSSION. – Consider an arbitrary symmetric standard process $(\Omega', \mathcal{M}', \mathcal{M}'_t, X'_t, \theta'_t, P'_x)$ with state space $(E', \mathcal{B}')$ and symmetrising measure m'. Then, in the sense of 0.1 in [6], there is a *canonical standard Markov process* with the same transition function (in the notation of [6], if W is the space of cadiag paths, w, in E' over some interval $[0, \zeta(w)[$ then it suffices to let P_x, $x \in E'$, be the image measure of P'_x under the mapping $\omega' \to X'.(\omega')$ on the interval $\{t : X'_t(\omega') \in E'\}$).

Given this canonical standard Markov process, we can apply the construction in 0.2 of [6] to obtain a *canonical standard time-reversible process*

$$(\Delta, x_{t+}, x_{t-}, P, \mathcal{S}(I), P^{\epsilon}_{t,x}, \rho_u)$$

in the sense of [6] Section 2. Let $Y_t = x_{t+}$ so that Y has cadlag paths on $]\alpha, \beta[$.

The salient feature of this construction which make it useful in studying our original process X′ is that $P^{+}_{0,x}(\alpha \leq 0 < \beta) = 1$ and under $P^{+}_{0,x}$ the law of Y on $]0, \beta[$ is the law of $X'(.)$ on $\{t > 0 : X'_t \in E'\}$ under P'_x.

If we now return to our family $\{X^i\}$ of standard processes, we can perform this construction to get k canonical standard time reversible processes $(\Delta^i, x^i_{t+}, x^i_{t-}, P^i, \mathcal{S}^i(I^i), P^{\epsilon, i}_{t,x}, \rho^i_u)$ and thence k processes $\{Y^i\}$. As usual set

$$P = \prod_i P^i, \qquad P^{+}_{t,x} = \prod_i P^{+,i}_{t^i,x^i}$$

$$\mathcal{F}(\prod_i I^i) = \prod_i \mathcal{F}^i(I^i)$$

and

$$Y(t) = (Y^i(t^i), \ldots, Y^k(t^k)) \quad \text{for} \quad t = (t^i) \in \prod_i \Delta^i.$$

In what follows, we adopt the convention that expressions such as $f(Y_t)$ are to be evaluated as 0 if $t \notin \prod_i \Delta^i$.

We will depart slightly from the notation of [6] and denote by Ξ_i the sample spaces for the canonical time reversible processes corresponding to the X^i. Setting $\Xi = \prod_i \Xi^i$, denote a representative of this space by $\xi = (\xi^i)$.

Also, we will denote the translation operators in Ξ^i by $\eta^i_{t^i}$ and define $\eta_t(\xi) = (\eta^i_{t^i}(\xi^i))$.

We use E to denote expectations under the measure P.

The preceding discussion sets up the "probabilistic" (typically, P is not a finite measure) structure with which we will work. The following definitions provide the complementary analytic framework which corresponds to the associated Green (or, equivalently, Dirichlet) space of the one-parameter theory.

(6.4) DEFINITION. — (i) For $f^i \in L^2(m^i)$, $t^i > 0$, $x^i \in E^i$ set $T^i_{t^i} f^i(x^i) = p^i(t^i, x^i, f^i)$.

(ii) For $f \in L^2(m)$, $t > 0$, $x \in E$ set $T_t f(x) = p(t, x, f)$. As in Proposition 2-1 [9] it is easy to check that these operators form a $]0, \infty[^k$-indexed semigroup of self-adjoint contractions on $L^2(m)$.

(iii) Denote the norm and inner-product on $L^2(m)$ by $\| \quad \|$ and $(\ , \ )$ respectively.

(iv) Denote by K the space of functions $\varphi_t(x)$, $t > 0$, $x \in E$ such that

(a) $\varphi_t(.) \in L^2(m)$, $\forall\, t$;

(b) $T_s \varphi_t = \varphi_{s+t}$, $\forall\, s, t$;

(c) $\int \| \varphi_t \|^2 \, dt < \infty$.

In much the same manner as Proposition 3-2 [9] it follows that K is a Hilbert space with inner-product $(\varphi, \varphi')_K = \int (\varphi_t, \varphi'_t)\, dt$. Set $K^+ = \{ \varphi \in K : \forall\, t, \ \varphi_t \geqq 0 \}$.

The next few results are devoted to showing that there is an "additive functional" corresponding to each element of K^+. For the rest of this section we will freely quote elements of the one-parameter theory from [6] with the implicit understanding that we are referring to the multi-parameter analogues of these results which follow by simple monotone class arguments.

(6.5) THEOREM. — *For every* $\varphi \in \mathrm{K}$ *and* $s<u$,

(i) $a_\varphi(]s, u]) = \lim_{\delta \to 0} \int_{]s, u]} \varphi_\delta(\mathrm{Y}(t)\, dt$ *exists in* $\mathrm{L}^2(\mathrm{P})$.

(ii) $$\mathrm{E}(a_\varphi(]s, u])^2) = 2^k \int_{]s, u]} \left(\int_{]0, t-s]} \| \varphi_{w/2} \|^2 dw \right) dt$$

$$\leqq 2^k \,|\,]s, u] \,|\, \int_{]0, u-s]} \| \varphi_{t/2} \|^2 dt$$

Proof. — Set

$$\mathrm{Z}_\delta = \int_{]s, u]} \varphi_\delta(\mathrm{Y}(t))\, dt.$$

Then

$$\mathrm{EZ}_\delta \mathrm{Z}_\varepsilon = \sum_{d \in \mathrm{D}} \iint_{\mathrm{R}(d)} \mathrm{E}\, \varphi_\delta(\mathrm{Y}(t))\, \varphi_\varepsilon(\mathrm{Y}(v))\, dt\, dv$$

where $\mathrm{D} = \{-1, 1\}^k$ and $\mathrm{R}(d) = \{(t, v) \in]s, u]^k : (\operatorname{sgn}(t^i - v^i)) = d\}$.
From (2-17) of [6] it follows that

$$\mathrm{E}\, \varphi_\delta(\mathrm{Y}(t))\, \varphi_\varepsilon(\mathrm{Y}(v)) = (\varphi_\delta, \mathrm{T}_{((t \vee v) - (t \vee v))}\, \varphi_\varepsilon)$$

and hence

$$\mathrm{EZ}_\delta \mathrm{Z}_\varepsilon = 2^k \iint_{\mathrm{R}((1, \ldots, 1))} \| \varphi_{(t-v+\delta+\varepsilon)/2} \|^2 dt\, dv$$

$$= 2^k \int_{]s, u]} \int_{]\delta+\varepsilon, t-s+\delta+\varepsilon]} \| \varphi_{w/2} \|^2 dw\, dt.$$

Thus

$$\mathrm{EZ}_\delta \mathrm{Z}_\varepsilon \to 2^k \int_{]s, u]} \int_{]0, t-s]} \| \varphi_{w/2} \|^2 dx\, dt$$

as $\delta, \varepsilon \to 0$ and this sufficient to establish the Theorem.

(6.6) DISCUSSION. — Let $\mathscr{A}$ be the class of intervals $]s, u] \subset \mathbb{R}_+^k$ with $0 \leqq s < u$ and let $\mathscr{A}^0$ be the subclass of $\mathscr{A}$ consisting of those intervals $]s, u]$ with $s, u \in \mathbb{Q}_+^k$.

Suppose that $\varphi \in \mathrm{K}^+$. It follows from Theorem 6.5 (i) that we may choose a sequence $\delta(n) \to 0$ and a set $\Xi' \subset \Xi$ with $\mathrm{P}(\Xi \setminus \Xi') = 0$ such that

$$a_\varphi(\xi, \mathrm{I}) = \lim_{n \to \infty} \int_{\mathrm{I}} \varphi_{\delta(n)}(\mathrm{Y}_t(\xi))\, dt$$

for all $\xi \in \Xi'$ and $\mathrm{I} \in \mathscr{A}^0$. Using Lemma 5-1 of [6] and Theorem 6.5 (ii) we may then choose $\Xi'' \subset \Xi'$ with $\mathrm{P}(\Xi' \setminus \Xi'') = 0$ such that, for $\xi \subset \Xi''$, $\mathrm{a}_\varphi(\xi, .)$ is uniquely extendable to a measure $\mathrm{A}_\varphi(\xi, .)$ on $\mathscr{B}(\mathbb{R}^k$ If we set $\mathrm{A}_\varphi(\xi, .) \equiv 0$ for $\xi \notin \Xi''$ then it is clear that A_φ is a ra n $\mathscr{B}(\mathbb{R}^k_+)$ with $\mathrm{A}_\varphi(]s, u[)$ P-equivalent to a $\mathscr{F}(]s, u[)$-n variable for all $s < u$.

(6.7) THEOREM. — *There exists a set $\Xi^* \subset \Xi$ with* $\mathrm{P}(\Xi \setminus \Xi^*)$ *for all $\xi \in \Xi^*$, $u \in \mathbb{R}^k_+$ and $\mathrm{I} \in \mathscr{A}$ we have*

$$A_\varphi(\xi, u + \mathrm{I}) = \mathrm{A}_\varphi(\eta_u \xi, \mathrm{I}).$$

Proof. — For $\mathrm{N} \in \mathbb{Z}_+$ and $\tau = \{0 = t_0 < t_1 < \ldots < t_j = \mathrm{N}\}$ put

$$s_i = (t_i, 0, \ldots, 0) \in \mathbb{R}^k_+, \qquad u_i = (t_i, \mathrm{N}, \ldots, \mathrm{N}) \in \mathbb{R}^k_+$$

and

$$\mathrm{V}_\varphi(\tau) = \sum_{i=1}^{j} \mathrm{A}_\varphi(]s_{i-1}, u_i])^2.$$

If $|\tau| = \max_{1 \leq i \leq j} (t_i - t_{i-1})$ then

$$\mathrm{EV}_\varphi(\mathrm{T}) \leq 2^k \mathrm{N}^k \int_{]0,\ (|\mathrm{T}|,\ \mathrm{N},\ \ldots,\ \mathrm{N})]} \| \varphi_{t/2} \|^2\, dt$$

by Theorem 6.5 (ii).

Thus if τ_n is a sequence of such partitions with $|\tau_n| \to 0$ we see that $\lim_n \mathrm{EV}_\varphi(\tau_n) = 0$. This and complementary arguments for the other axes show that there is a set $\Xi^{**} \subset \Xi$ with $\mathrm{P}(\Xi \setminus \Xi^{**}) = 0$ such that, for $\xi \in \Xi^{**}$, $\mathrm{A}_\varphi(\xi, .)$ has no mass in any of the hyperplanes $\{t \in \mathbb{R}^k_+ = t^i = a\}$, $a \in \mathbb{R}_+$, $1 \leq i \leq k$.

Suppose that $\xi \in \Xi^{**} \cap \Xi'' = \Xi^*$. If $\mathrm{I} \in \mathscr{A}$ then for $\delta > 0$ there exist I_1, $\mathrm{I}_2 \in \mathscr{A}^0$ with $\mathrm{I}_1 \subset \mathrm{I} \subset \mathrm{I}_2$ such that $\mathrm{A}_\varphi(\xi, \mathrm{I}_2 \setminus \mathrm{I}_1) < \delta$. Now

$$\mathrm{A}_\varphi(\xi, \mathrm{I}_1) = \lim_n \int_{\mathrm{I}_1} \varphi_{\delta(n)}(\mathrm{Y}_t(\xi))\, dt$$

$$\leqq \liminf_n \int_I \varphi_{\delta(n)}(Y_t(\xi))\,dt$$

$$\leqq \limsup_n \int_I \varphi_{\varphi(n)}(Y_t(\xi))\,dt$$

$$\leqq \lim_n \int_{I_2} \varphi_{\delta(n)}(Y_t(\xi))\,dt$$

$$= A_\varphi(\xi, I_2).$$

Thus $A_\varphi(\xi, I) = \lim_n \int_I \varphi_{\delta(n)}(Y_t(\xi))\,dt$ for all I. Since

$$\int_{u+I} \varphi_\delta(Y_t(\xi))\,dt = \int_I \varphi_\delta(Y_t(\eta_u \xi))\,dt$$

this establishes the result.

(6.8) Discussion. – For each random measure A_φ, $\varphi \in K^+$, we may define a measure on $(\mathbb{R}_+^k \times E, \mathscr{B}(\mathbb{R}_+^k) \times \mathscr{B})$ by $\nu(C) = E \int 1_C(t, Y_t)\,A_\varphi(dt)$. From Theorem 6.7 we know that $v(dt, dy) = dt\,\mu(dy)$ for some measure μ on $\mathscr{B}$. We call μ the characteristic measure of A_φ.

We wish to show that $B \in \mathscr{B}$ is not "small" for X by constructing an appropriate random measure concentrated on $\{t : X_t \in B\}$. It would therefore be useful to have some explicit means of obtaining μ from φ.

The following assumption on the transition functions $p^i(.,.,.)$ will enable us to make such a calculation. The imposition of this assumption is an intermediary technical step, and a simple device will allow us to use the calculations for this restricted case to obtain results which are applicable to general transition functions.

(6.9) Assumption. – For $f^i \in L^2(m^i)$, $c^i \in \mathbb{R}_+$ set $G^i_{c^i} f^i = \int_{]0, c^i]} T^i_{t^i} f^i\,dt^i$. Similarly, for $f \in L^2(m)$, $c \in \mathbb{R}_+^k$ set $G_c f = \int_{]0, c]} T_t f\,dt$. Denote by D_{G^i} the set of functions $f^i \in L^2(m^i)$ such that $G^i f^i = \int_{\mathbb{R}} T^i_{t^i} f^i\,dt^i$ exists in $L^2(m^i)$. Define $\tilde{D}_G$ and G similarly.

Assume that, for $i = 1, \ldots, k$, we have

(i) If $e^i \in \tilde{D}_{G^i}$ is bounded then $G^i e^i$ is also bounded.

(ii) There exists a bounded, strictly positive function $f^i \in \tilde{\mathrm{D}}_{\mathrm{G}^i}$.

(6.10) PROPOSITION. — *Under Assumption 6.9 there exists a bounded function $h>0$ such that*

(i) $h \in \mathrm{L}^2(m)$.

(ii) $h(\mathrm{Y}_t)$ *is* P-*a. s. right-continuous on* $]-\infty, \infty[\setminus\{\alpha\}$ *and* $\mathrm{P}^+_{0,\,\nu}$ *—a. s. right-continuous on* $[0, \infty[$ *for each σ-finite measure ν on* $(\mathrm{E}, \mathscr{B})$.

(iii) *For every* $\varphi \in \mathrm{K}^+$, $\int \mu(dy)\, h(y) < \infty$, *where* μ *is the characteristic measure of* A_φ.

We first prove three Lemmas.

(6.11) LEMMA. — *Under Assumption 6.9, if* $d \in \tilde{\mathrm{D}}_{\mathrm{G}^i}$ *is bounded then* $\mathrm{G}^i_b\, d(\mathrm{Y}^i(s))$ *is* P^i*—a. s. right-continuous on* $]-\infty, \infty[\setminus\{\alpha^i\}$ *and* $\mathrm{P}^{+,\,i}_{0,\,y}$*— a. s. right-continuous on* $[0, \infty[$ *for each* $b \in \mathbb{R}_+$ *and each* $y \in \mathrm{E}^i$.

Proof. — Note that $\mathrm{G}^i_b\, d = \mathrm{G}^i d - \mathrm{G}^i(\mathrm{T}^i_b\, d)$ and so, as the difference of two bounded functions which are excessive for $p^i(.,\,.,\,.)$, $\mathrm{G}^i_b\, d$ is such that $\mathrm{G}^i_b\, d(\mathrm{X}^i(s))$ is P^i_x—a. s. cadlag on $[0, \infty[$ for each $x \in \mathrm{E}^i$ (*see* II-2-12 in [3]). The result then follows from the construction of Y^i (*cf.* 0.4 of [6] for a similar argument).

(6.12) LEMMA. — *Under Assumption 6.9 set* $g = \mathrm{G}_c f$ *and* $\mathrm{F}(t, y) = \mathrm{T}_{u-t}\, g(y)$. *Then* $\mathrm{F}(t, \mathrm{Y}_t)$ *is* P-*a. s. right-continuous on* $]-\infty, u[\setminus\{\alpha\}$.

Proof. — As

$$\mathrm{F}(t, y) = \prod_i \mathrm{F}^i(t^i, y^i) = \prod_i T^i_{u^i - t^i}\, \mathrm{G}^i_{c^i} f^i(y^i)$$

it suffices to prove that $\mathrm{F}^i(t^i, \mathrm{Y}^i(t^i))$ is P^i—a. s. right-continuous on $]-\infty, u^i[\setminus\{\alpha^i\}$ for $1 \leqq i \leqq k$.

Fix i and set $\tau = \{u^i = s_0 > s_1 > \dots\}$, $s_n \to -\infty$.

Put

$$\mathrm{F}^i_\tau(s, y^i) = \mathrm{F}^i(s_k, y^i) = \mathrm{G}^i_{c^i}(\mathrm{T}^i_{u^i - s_k} f^i)(y^i)$$

for $s \in [s_{k+1}, s_k[$.

From Lemma 6.11 we have for each k that $\mathrm{F}^i(s_k, \mathrm{Y}^i(s))$ is P^i-a. s. right-continuous on $]-\infty, u[\setminus\{\alpha^i\}$ and so the same is true of $\mathrm{F}^i_\tau(s, \mathrm{Y}^i(s))$.

Observe that $|\mathrm{F}^i_\tau(s, y^i) - \mathrm{F}^i(s, y^i)| \leqq 2 \sup_z |f^i(z)|\,.\,|\tau|$, so letting $|\tau| \to 0$ gives the result.

(6.13) LEMMA. — *Under Assumption* 6.9 *and in the notation of Lemma* 6.12

$$\int \mu(dy)\, G_u g(y) \leqq \| g \| \, [E A_\varphi(]0, u])^2]^{1/2}$$

Proof. — From (2-16) of [6] we have $E g(X_u)^2 = \| g \|^2$, so it will suffice by the Cauchy-Schwartz inequality to show that

$$\int \mu(dy)\, G_u g(y) \leqq E A_\varphi(]0, u])\, g(X_u).$$

Let $\Lambda = \{\Lambda_1, \ldots, \Lambda_j\}$ be a partition of $]0, u]$ into rectangles $\Lambda_k =]s_k, t_k]$. Set $b_\Lambda(t) = t_k$ for $t \in \Lambda_k$.

Recall from the Discussion 6.6 that $A_\varphi(]s, t]) = A_\varphi(]s, t[)$ is P-equivalent to an element of $\mathcal{F}(]s, t[)$. Using (2.12) and (2.14) of [6] it then follows that

$$E A_\varphi(\Lambda_k)\, g(X_u) = E A_\varphi(\Lambda_k)\, F(t_k, Y(t_k)).$$

Thus

$$E \int_{]0,\, u]} F(b_\Lambda(t), X(b_\Lambda(t))\, A_\varphi(dt) = E A_\varphi(]0, u])\, g(X_u)$$

By considering a sequence $\Lambda_n = \{\Lambda_{n,\,k}\}$ *with* $max_{i,\,k} | t^i_{n,\,k} - s^i_{n,\,k} | \to 0$, and applying Lemma 6.12 and Fatou's Lemma we see that

$$E A_\varphi(]0, u])\, g(X_u) \geqq E \int_{]0,u]} F(t, X_t)\, A_\varphi(\mathrm{dt})$$
$$= \int_{]0,\, u]} dt \int_E \mu(dy)\, T_{u-t}\, g(y) = \int \mu(dy)\, G_u g(y)$$

(6.14) *Proof of Proposition* 6.10. — Consider $h = G_u g$. Clearly, h is bounded and (i) holds. Since $p(t, x, E) \to 1$ as $t \to 0$ for each $x \in E$ it is also clear that g and thence h is strictly positive.

Fubíni's theorem gives that $g(y) = \prod_i g^i(y^i)$ with $g^i \in \tilde{D}_{G^i}$ bounded. Lemma 6.11 then establishes statement (ii).

The proof is completed by recalling from Theorem 6.5 that $A_\varphi(]0, u]) \in L^2(P)$ and then applying Lemma 6.13.

(6.15) DEFINITION. — Let M be the class of σ-finite measures, μ, on (E, $\mathscr{B}$) such that $\int \mu(dx)\, p(t, x, dy) = \varphi_t(y)\, m(dy)$ with $\varphi_t \in K^+$.

(6.16) THEOREM. — *Under Assumption* 6.9, *if* $\nu \in M$ *then* ν *is the characteristic measure of* A_φ, *where* φ *is the element of* K^+ *corresponding to* ν.

Proof. — If f and h are as in Assumption 6.9 and Proposition 6.10 respectively and h' is a non-negative, bounded continuous function on E set $f_n = 1 \wedge nf$ and $H = hh'$.

Now, if $s<u<v$ then from (2.12) and (2.14) of [6] and the fact that $A_\varphi(]s, t])$ is P-equivalent to a $\mathscr{F}(]s, t[)$-measurable random variable we have

$$EA_\varphi(]s, u])\, H(Y_u)\, f_n(Y_v) = EA_\varphi(]s, u])\, F_n(Y_u) < \infty$$

where $F_n(y) = H(y)\, T_{v-u} f_n(y)$.

By (2-17) of [6] and Theorem 6.5

$$EA_\varphi(]s, u])\, F_n(Y_u)$$

$$= \lim_{\delta \to 0} \int_{]s,\, u]} E\, \varphi_\delta(Y_t)\, F_n(Y_u)\, dt$$

$$= \lim_{\delta \to 0} \int_{]s,\, u]} (F_n.\ T_{u-t}\, \varphi_\delta)\, dt$$

$$= \lim_{\delta \to 0} \int_{]\delta,\, u-s+\delta]} (F_n,\ \varphi_t)\, dt$$

$$= \int_{]0,\, u-s]} (F_n,\ \varphi_t)\, dt$$

Also, denoting expectations with respect to the measure $P^+_{0,\,\nu}$ by $E_{0,\,\nu}$, we have from (2.13) and (2.14) of [6] that

$$E_{0,\,\nu} \int_{]s,\, u]} H(Y_{u-t})\, f_n(Y_{v-t})\, dt$$

$$= \int_{]s,\, u]} E_{0,\,\nu}\, F_n(Y_{u-t})\, dt$$

$$= \int_{]s,\, u]} (\varphi_{u-t},\ F_n)\, dt$$

$$= \int_{]0,\, u-s]} (F_n, \varphi_t)\, dt.$$

Thus

$$EA_\varphi(]s, u])\, H(Y_u)\, f_n(Y_v) = E_{0,\, v} \int_{]s,\, u]} H(Y_{u-t})\, f_n(Y_{v-t})\, dt$$

Now suppose that $0 < c < v$ and $\Lambda = \{\Lambda_i\}$ is a partition of $]0, c]$ into rectangles. If we define $b_\Lambda(t)$ as in the proof of Lemma 6.11 then the foregoing shows that

$$E \int_{]0,\, c]} H(Y(b_\Lambda(t)))\, A_\varphi(dt)\, f_n(Y_v)$$

$$= E_{0,\, v} \int_{]0,\, c]} H(Y(b_\Lambda(t) - t))\, f_n(Y_{v-t})\, dt.$$

From Lemma 6.11 $H(Y_t)$ is P-a. s. right-continuous on $]-\infty, \infty[\setminus\{\alpha\}$ and $P^+_{0,\, v}$- a. s. right-continuous on $[0, \infty[$. By construction, A_φ concentrates its mass on $\Delta \cap [0, \infty[$. Thus, by dominated convergence

$$E \int_{]0,\, c]} H(Y_t)\, A_\varphi(dt)\, f_n(Y_v) = E_{0,\, v} \int_{]0,\, c]} H(Y_0)\, f_n(Y_{v-t})\, dt.$$

Letting $n \to \infty$, we see that if we define two measures on $D = \{(t, u) \in (\mathbb{R}^k_+)^2 : 0 < t < 1,\ t < u\}$ by

$$\gamma_1(A) = E \int_{]0,\, 1]} I_A(t, \beta)\, H(Y_t)\, A_\varphi(dt)$$

$$\gamma_2(A) = E_{0,\, v} \int_{]0,1]} I_A(t, t+\beta)\, H(Y_0)\, dt$$

then they coincide for each rectangle $]0, c] \times]v, \infty[$, $0 < c < 1$, $c < v$. Since $\int \mu(dy)\, h(y) < \infty$ we have that γ_1 is finite on each of these rectangles and so $\gamma_1(D) = \gamma_2(D)$.

That is, using 2-4 A of [6].

$$\int \mu(dy)\, h(y)\, h'(y) = E \int_{]0,1]} H(Y_t)\, A_\varphi(dt)$$

$$= \gamma_1(D)$$

$$
\begin{aligned}
&= \gamma_2(\mathrm{D}) \\
&= \mathrm{E}_{0,\,\nu} \int_{]0,1]} \mathrm{H}(\mathrm{Y}_0)\,dt \\
&= \mathrm{E}_{0,\nu}\mathrm{H}(\mathrm{Y}_0) = \int \nu(dy)\,h(y)\,h'(y).
\end{aligned}
$$

The equality between the extreme members of this chain extends to the case where h' is an arbitrary non-negative measurable function and this is sufficient to give $\mu=\nu$.

(6.17) DEFINITION. — Define transition functions by $\tilde{p}^i(t^i, x^i, \mathrm{B}^i) = \exp(-t^i)\,p^i(t^i, x^i, \mathrm{B}^i)$. It is clear that $\tilde{p}^i(.,\,.,\,.)$ is symmetric with respect to m^i. Let $\tilde{\mathrm{K}}$ and $\tilde{\mathrm{M}}$ be the spaces defined in Definitions 6.4 (iv) and 6.15, respectively, with p^i replaced by $\tilde{p}^i$.

(6.18) COROLLARY. — *If* $\mathrm{B}\in\mathscr{B}$ *and* $\mu(\mathrm{B})>0$ *for some* $\mu\in\tilde{\mathrm{M}}$ *then* B *is not exceptional for* X.

Proof. — By III-3 of [3] there exists, for each i, a standard process $\tilde{\mathrm{X}}^i$ with the transition function $\tilde{p}^i$. It follows from the construction of $\tilde{\mathrm{X}}^i$ that B will be exceptional for X if and only if it is exceptional for $\tilde{\mathrm{X}}$, where $\tilde{\mathrm{X}}$ is the process formed from the $\tilde{\mathrm{X}}^i$ in the same manner as X is formed from the X^i.

Suppose that there exists $\mu\in\tilde{\mathrm{M}}$ with $\mu(\mathrm{B})>0$. Straightforward calculations show that Assumption 6.9 holds for $\tilde{p}^i$, $i=1, \ldots, k$. If $\tilde{\mathrm{Y}}^i$ is the canonical standard time-reversible process constructed from $\tilde{\mathrm{X}}^i$ then Theorem 6.16 implies (in an obvious notation) that $\tilde{\mathrm{P}}(\exists\, t>0 : \tilde{\mathrm{Y}}(t)\in\mathrm{B})>0$. Using (2.12) and (2.15) of [6] this gives $\tilde{\mathrm{P}}^+_{0,\,m}(\exists\, t>0 : \tilde{\mathrm{Y}}(t)\in\mathrm{B})>0$. Hence, by the relationship between $\tilde{\mathrm{Y}}$ and $\tilde{\mathrm{X}}$, we have $\tilde{\mathrm{P}}_m(\exists\, t>0 : \tilde{\mathrm{X}}(t)\in\mathrm{B})>0$. Thus B is not exceptional for $\tilde{\mathrm{X}}$ and, as we remarked above, this is sufficient to show that B is not exceptional for X.

7. SMALL SETS FOR LEVY PROCESSES

In this section we consider the special case of our general set-up that obtains when each of the X^i are Levy processes on $\mathrm{E}^i=\mathbb{R}^{d^i}$. The notion of a polar set in this setting is still meaningful and our definition is unchanged from that of Definition 6.1 (i). As $p(t, x, B) = p(t, y+x, y+B)$ for $y \in E$, Lebesgue measure on E plays a distinguished role in the study of X which is similar to that played by the symmetrising measure in the study of symmetric processes. For instance, the counterpart for exceptional sets is:

(7.1) DEFINITION. — A set $B \in \mathscr{B}$ is *essentially polar* if $P_\lambda(\exists t>0 : X_t \in B) = 0$, where $\lambda = \prod_i \lambda^i$ is Lebesgue measure on E.

Every polar set is essentially polar and the conditions under which the two classes coincide are again analogues of those for the one-parameter case (*see* [14], Theorem 2-1).

(7.2) THEOREM. — *The following are equivalent conditions on* X.

(i) *Each essentially polar set is polar.*

(ii) *For each* $\alpha > 0$ *and* $x \in E$ *the measure* $G^\alpha(x, .)$ *is absolutely continuous with respect to* λ.

Proof (i) $\Rightarrow$ (ii). — This is essentially just a reprise of the proof of the similar result in Theorem 6.2 with Theorem 2.1 of [14] providing the necessary result from the one-parameter theory.

(ii) $\Rightarrow$ i). — That m is symmetrising for $p(.,.,.)$ plays no part in the proof of the corresponding result in Theorem 6.2. That proof shows generally that if $P_\mu(\exists t>0 : X_t \in B) = 0$ for some measure μ and $G^\alpha(x, .)$ is absolutely continuous with respect to μ for all α, x then B is polar.

We now investigate criteria which will ensure that a given set is not essentially polar.

(7.3) NOTATION. — If μ is a measure we denote its Fourier-Stieltjes transform by $\hat{\mu}$. As usual, we define the exponent of X^i to be the function ψ^i such that $\exp(-t^i \psi^i(z^i)) = (p^i(t^i, 0, .))\hat{}(z^i)$. If μ is a measure on E we set

$$I(\psi; \mu) = \int [\prod_i \mathrm{Re}((1+\psi^i(z^i))^{-1})] \, |\hat{\mu}(z)|^2 \, dz$$

(7.4) THEOREM. — *If* K *is compact subset of* E *then a sufficient condition for* K *to be not essentially polar is that there exists a finite measure* μ *supported on* K *such that* $I(\psi; \mu) < \infty$.

Proof. — Consider the measures

$$\tau(A) = \int \exp(-1.t) I_A(X(t)) \, dt, \qquad A \in \mathscr{B}$$

$$\tau^i(A^i)=\int \exp(-t^i)\, I_{A^i}(X^i(t^i))\, dt^i, \qquad A^i\in\mathscr{B}^i.$$

Then $\tau=\prod_i \tau^i$ and so $\hat{\tau}(z)=\prod_i \hat{\tau}^i(z^i)$. Following (10) of [15] we therefore have

$$E_0\,|\hat{\tau}(z)|^2=\prod_i E_0^i\,|\hat{\tau}^i(z^i)|^2=\prod_i \operatorname{Re}((1+\psi^i(z^i))^{-1}).$$

Thus, as in the proof of Theorem 2 in [15], we may use Theorem 1 of [15] to show that

$$(P_0\times\lambda)(\{(\omega, x):(x+X(\omega, \mathbb{R}_+^k))\cap K\neq\varnothing\})>0.$$

Therefore

$$\lambda(\{x: P_x(X(\mathbb{R}_+^k)\cap K\neq\varnothing)>0\})>0$$

and K is not essentially polar for X.

If X^i is symmetric in the sense that $p^i(t^i, 0, B^i)=p^i(t^i, 0, -B^i)$ then it is also symmetric in the sense of Section 6, with λ^i acting as the symmetrising measure. Thus, if this condition holds for each of the X^i then the considerations of Section 6 apply. In this case, the notion of exceptional set and essentially polar set coincide and Corollary 6.18 and Theorem 7.4 give two seemingly different sufficient conditions for a set not to belong to this class. As we shall see, however, in the presence of transition densities there is no essential difference between the two criteria.

(7.5) Notation. – Suppose that $p^i(t^i, x^i, .)\ll\lambda^i$, $i=1, \ldots, k$, then by Theorem 2-2 of [14] there are canonical families of probability densities $\{p_{t^i}^i\}$, $i=1, \ldots, k$, *such that:*

(i) $(t^i, x^i)\to p_{t^i}^i(x^i)$ is jointly measurable;

(ii) $x^i\to p_{t^i}^i(x^i)$ is lower semicontinuous;

(iii) $p_{t^i}^i * p_{s^i}^i=p_{t^i+s^i}^i$ everywhere;

(iv) $\int_{B^i} p_{t^i}^i(y^i-x^i)\,dy^i=p^i(t^i, x^i, B^i)$, $B^i\in\mathscr{B}^i$.

We set $u^i(x^i)=\int \exp(-t^i)\,p_{t^i}^i(x^i)\,dt^i$ and $u(x)=\prod_i u^i(x^i)$.

(7.6) Theorem. – *If $p^i(t^i, x^i, B^i)$, $i=1, \ldots, k$, is symmetric with canonical densities $p_{t^i}^i(x^i)$ then a finite measure μ is in $\tilde{M}$ if and only if $I(\psi; \mu)<\infty$.*

Prrof. — Define a pairing on the measures on E by

$$[\mu, \nu]=\iint \mu(dx)\,\nu(dy)\,u(x-y).$$

It is easy to see that $\mu\in\tilde{M}$ if and only if $[\mu, \mu]<\infty$ and if $\mu, \nu\in\tilde{M}$ then $[\mu, \nu]=(\varphi, \eta)_{\tilde{K}}$ where φ, η are the corresponding elements of $\tilde{K}$. Note also that if we define a measure $\tilde{\nu}$ by $\tilde{\nu}(A)=\nu(-A)$, then $[\mu, \nu]=(u*\mu*\tilde{\nu})(0)$.

Suppose first that $\mu\in\tilde{M}$. If, for $x\in E$, we set $\mu_x(.)=\mu(.-x)$ then $\mu_x\in\tilde{M}$ and $[\mu_x, \mu_x]=[\mu, \mu]$. The pairing [,] obeys a Cauchy-Schwartz inequlity on $\tilde{M}$, since the same is true of $(\ ,\ )_{\tilde{K}}$. Thus

$$\begin{aligned}(u*\mu*\tilde{\mu})(x)&=(u*\mu_x*\tilde{\mu})(0)\\&=[\mu_x, \mu]\\&\leqq[\mu_x, \mu_x]^{1/2}[\mu, \mu]^{1/2}\\&=[\mu, \mu]<\infty.\end{aligned}$$

Thus $u*\mu*\tilde{\mu}$ is the bounded density of a finite measure on E. By the Corollary to Theorem 3.XV-3 of [10] $(u*\mu*\tilde{\mu})\hat{}\in L^1(\lambda)$. The conclusion $I(\psi;\mu)<\infty$ follows upon noting that $\hat{u}(z)=\prod_i((1+\psi^i(z^i))^{-1})$ (*see* e. g. [14]) and ψ^i is real-valued when $p^i(., ., .)$ is symmetric.

Conversely, suppose that $I(\psi;\mu)<\infty$. Then, by Theorem 3, XV-3 of [10] the measure with density $u*\mu*\tilde{\mu}$ has a bounded continuous density. The lower semicontinuity of p_t^i and Fatou's Lemma give that $u*\mu*\tilde{\mu}$ is lower-semicontinuous. Thus

$$[\mu, \mu]=(u*\mu*\tilde{\mu})(0)<\infty$$

and $\mu\in\tilde{M}$.

8. APPLICATIONS

As a first application of our results, we verify the sufficiency of the Hendricks-Taylor conjectured condition (*see* e. g. [17]) for the existence of multiple points for a symmetric Lévy process.

(8.1) THEOREM. — *Let* Y *be a symmetric Lévy process on* $\mathbb{R}^d$ *with canonical densities* $q_s(z)$. *Set* $v(z)=\int \exp(-s)\, q_s(z)\, ds$. *If* $\int_{|z|\leq 1} [v(z)]^k\, dz<\infty$ *for some* $k\in\{2, 3 \ldots\}$ *then the sample paths of* Y *have* k*-tuple points almost surely.*

Proof. — Let $X^1, \ldots, X^k$ be k copies of Y. As we have remarked, each X^i has Lebesgue measure on $\mathbb{R}^d$ as a symmetrising measure so the considerations of Section 6 hold. In the notation of 7.5, the integrability condition ensures that $\iint \mu(dx)\, \mu(dy)\, u(x-y)<\infty$, where μ is the restriction of Lebesgue measure on $\partial=\{x\in(\mathbb{R}^d)^k : x^1=\ldots=x^k\}$ to the set $\{x\in\partial : |x^i|\leq 1/2\}$. As in the proof of Theorem 7.6 we see that $\mu\in\tilde{\mathrm{M}}$ and so, by Corollary 6.18, we have that ∂ is not exceptional (or, equivalently, not essentially polar) for X.

Set $h(x)=\mathrm{P}_x(\exists\, t>0 : \mathrm{X}_t\in\partial)$, $x\in(\mathbb{R}^d)^k$. Note that

$$0\leq \int \lambda(dy)\, p_t(y-x)\, h(y)\leq h(x),$$

where λ is Lebesgue measure on $(\mathbb{R}^d)^k$. If $h(x)=0$ for some $x\in(\mathbb{R}^d)^k$ then

$$0=\int \exp(-1.t)\left[\int \lambda(dy)\, p_t(y-x)\, h(y)\right] dt=\int \lambda(dy)\, u(y-x)\, h(y).$$

It is shown in [16] that v is strictly positive on the interior of the support of $v(z)\,dz$. Since Y is symmetric the support of $v(z)\,dz$ is the whole of $\mathbb{R}^d$. Thus $u>0$, and so this last equation implies that $h=0$ $\lambda-$a.e. which contradicts our conclusion that ∂ is not exceptional for X.

Thus $\mathrm{P}_x(\exists\, t>0 : \mathrm{X}_t\in\partial)>0$ for all $x\in(\mathbb{R}^d)^k$. Using the observation on *p.* 85 of [13] concerning the reduction on multiple point problems to range intersection problems we see that Y has k-multiple points with positive probability and a simple 0-1 argument using the independent increments property of Y completes the proof.

Hendricks and Taylor conjectured that the integrability condition in Theorem 8.1 is also necessary for the sample paths of Y to possess k-multiple points. Although we have nothing to add on this point. It is perhaps worth pointing out that Theorem 8.1 represents the best result that we can obtain using the techniques of Section 6.

More precisely, suppose that we define a pairing on the measures on $\mathbb{R}^d$ by $\langle \mu, \nu \rangle = \int \mu(dw)\,\nu(dz)\,[v(w-z)]^k$ so that, in the notation of the proof of Theorem 7.6, $\langle \mu, \nu \rangle = [\mu^*, \nu^*]$, where μ^*, ν^* are the canonical liftings of μ, ν to the family of measures on ∂. Then it might appear that we would be using the full force of Corollary 6.18 in the proof if we replaced the integrability condition by a requirement that $\langle \mu, \mu \rangle < \infty$ for some σ-finite measure μ on $\mathbb{R}^d$.

To see that this apparent gain in generality is illusory, suppose that $\langle \mu, \mu \rangle < \infty$ and let μ' be the convolution of μ with the standard normal distribution on $\mathbb{R}^d$ i.e. $\mu'(A) = \int \mu_w(A)\,\varphi(w)\,dw$ where $\mu_w(.) = \mu(. - w)$ for $w \in \mathbb{R}^d$ and φ is the standard normal density on $\mathbb{R}^d$. Referring to the proof of Theorem 7.6 we see that $\langle \mu_w, \mu_w \rangle = \langle \mu, \mu \rangle$ and that $\langle\ ,\ \rangle$ obeys a Cauchy-Schwartz inequality. Thus

$$\begin{aligned}\langle \mu', \mu' \rangle &= \iint \langle \mu_w, \mu_z \rangle\,\varphi(w)\,\varphi(z)\,dw\,dz \\ &\leqq \iint \langle \mu_w, \mu_w \rangle^{1/2} \langle \mu_z, \mu_z \rangle^{1/2}\,\varphi(w)\,\varphi(z)\,dw\,dz \\ &= \langle \mu, \mu \rangle < \infty\end{aligned}$$

and since μ' has a strictly positive continuous density the integrability condition of Theorem 8.1 holds.

Let us now remove the assumption that our Lévy process is symmetric. It is clear that the technique used in the proof of Theorem 8.1 can still be used to obatin various criteria for the existence of multiple points if we replace the use of Corollary 6.18 by an appeal to the results of Section 7. Rather than carry this programme through in the fullest generality possible, we will content ourselves with presenting a fairly simple criterion which is nevertheless powerful enough to include that obtained in [21] using "local time" methods.

(8.2) Notation. — If $a \in \mathbb{R}^d$ and f is a measurable function on $\mathbb{R}^d$ set

$$\|f\|_{k,a} = \left(\int_{|z-a|<1} |f(z)|^k\,dz \right)^{1/k}$$

$$\|f\|_k = \left(\int |f(z)|^k\,dz \right)^{1/k}$$

(8.3) THEOREM. — *Let* Y *be a Lévy process on* $\mathbb{R}^d$ *with canonical densities* $q_s(z)$. *Set* $v(z)=\int \exp(-s)\, q_s(z)\, ds$. *If* $\operatorname{supp} v(z)\, dz=\mathbb{R}^d$, *then* $\sup_{a \in \mathbb{R}^d} \| v \|_{k, a} < \infty$ *for* $k \in \{2, 3, \ldots\}$ *implies that the sample of* Y *have k-tuple points almost surely.*

Proof. — Let X, ∂ and μ be as in the proof of Theorem 8.1. In the notation of the proof of Theorem 7.6 we have, by Holder's inequality.

$$[\mu_x, \mu] \leqq \text{const.} \times \int_{|z| \leqq 1} v(z-x^1) \ldots v(z-x^k)\, dz$$
$$\leqq \text{const.} \times \prod_{1 \leqq j \leqq k} \| v \|_{k, (-x_j)}$$
$$\leqq \text{cont.} \times (\sup_{a \in \mathbb{R}^d} \| v \|_{k, a})^k.$$

Thus $\sup_{x \in (\mathbb{R}^d)^k} [\mu_x, \mu] < \infty$. As in the proof of Theorem 7.6 we see that this implies $\mathrm{I}(\psi; \mu) < \infty$ and hence, by Theorem 7.4, ∂ is not essentially polar for X. The proof is now completed in exactly the same manner as the proof of Theorem 8.1.

The following corollary is essentially Theorem 1 of [21].

(8.4) COROLLARY. — *Let* Y *be a Lévy process on* $\mathbb{R}^d$ *with canonical densities* $q_s(z)$. *Assume that q satisfies the following conditions*

(i) $q_s(z)$ *is continuous in z for each* $s>0$;

(ii) $\sup_{z \in \mathbb{R}^d, s \geqq 1} q_s(z)=\mathrm{K}<\infty$;

(iii) $q_s(0)>0$ *for each* $s>0$;

(iv) $\int_0^1 \| q_s \|_k\, ds<\infty$.

Then the sample paths of Y *have k-tuple points almost surely.*

Proof. — As $q_s(0)>0$ by assumption (ii), assumption (i) implies that there is a $\delta>0$ such that $q_s(z)>0$ for all $|z| \leqq \delta$. From 7.5 (iii) we see that $q_{ns}(z)>0$ for all $|z| \leqq n\delta$, $n \in \mathbb{Z}_+$. A similar argument shows that if $q_s(z)>0$ then $q_t(z)>0$ for all $t \geqq s$. Thus, in the notation of Theorem 8.3, $v>0$ everywhere.

Now

$$\| v \|_{k, a} \leqq \int_0^\infty \exp(-s) \| q_s \|_{k, a}\, ds$$

$$\leqq \int_0^1 \| q_s \|_{k,a} ds + \int_1^\infty \exp(-s) \mathrm{K}\, 2^d ds$$

$$\leqq \int_0^1 \| q_s \|_k ds + \mathrm{K}\, 2^d.$$

Thus $\sup_a \| v \|_{k,a} < \infty$ and the result follows from Theorem 8.3.

As a final example we will show how our methods may be used to improve the results in [22] on the characterisation of those sets in $\mathbb{R}^2$ which can contain the multiple points of a planar Brownian motion.

Set $\Phi_k(s) = (\log(s^{-1}))^k$ and let C_k denote the capacity associated with this kernel. Theorem 1 of [22] states that if $\mathrm{K} \subset \mathbb{R}^2$ is compact and $C_{2k-1}(\mathrm{K}) > 0$ then they are almost surely k-tuple points of a planar Brownian motion contained in K. We can sharpen this result as follows.

(8.5) THEOREM. – *If* $\mathrm{K} \subset \mathbb{R}^2$ *is compact and* $C_k(\mathrm{K}) > 0$ *then there are almost surely k-tuple points of a planar Brownian motion contained in* K.

Proof. – Suppose that $X^1, \ldots, X^k$ are planar Brownian motions. Fix an open ball $\mathrm{D} \supset \mathrm{K}$ and let $\bar{X}^i$ be the process obtained by killing X^i at the boundary of D. The results of III-3 in [3] show that we may construct a standard process conforming to this intuitive notion.

Theorem 4-3 of Chapter 2 [20] shows that the $\bar{X}^i$ are symmetric with respect to the restriction of Lebesgue measure on D and have transition densities $\bar{p}^i(t^i, x^i, y^i)$ with respect to this measure which we will denote by m^i. Set $\bar{g}^i(x^i, y^i) = \int \bar{p}^i(t^i, x^i, y^i)\, dt^i$ and $\bar{g}(x, y) = \prod_i \bar{g}^i(x^i, y^i)$.

Since $C_k(\mathrm{K}) > 0$, there exists a non-trivial finite measure μ supported by K such that

$$\iint \mu(dw)\, \mu(dz) (\log | w - z |^{-1})^k < \infty.$$

By the same argument as in Theorem 7-19 of Chapter 6 in [20], this implies that

$$\iint \mu(dw)\, \mu(dz) (\bar{g}^1(w, z))^k < \infty$$

and hence

$$\iint \mu^*(dx)\,\mu^*(dy)\,\bar{g}(x, y) < \infty,$$

where μ^* is the canonical lifting of μ to the family of measures on $\partial_K = \{x \in K^k : x^1 = \ldots = x^k\}$. Therefore, if $\bar{M}$ is definied as in Definition 6.15 with p^i replaced by $\bar{p}^i$, we have that $\mu^* \in \bar{M}$.

It follows from Proposition 2-7 of Chapter 2 in [20] that Assumption 6.9 holds for the $\bar{X}^i$. In the same manner as Corollary 6.18, ∂_K is not exceptional for $\bar{X}$. Certainly then, ∂_K is not exceptional for X. Since X^i has everywhere positive transition density it follows as in the proof of Theorem 8.1 that $P_x(\exists\, t > 0 : X_t \in \partial_K) > 0$ for all $x \in (\mathbb{R}^2)^k$.

The observation at the end of the proof of Theorem 8.1 and a 0-1 argument using the recurrence properties of planar Brownian motion complete the proof.

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(*Manuscrit reçu le* 6 *mai* 1986.)