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Brownian motion and stereographic projection

by

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ABSTRACT. — Stereographic projection from \mathbb{R}^N to S^N maps Brownian paths in \mathbb{R}^N to the paths of Brownian motion on S^N conditioned to be at the centre of the projection at a negative exponential time.

Key-words: Stereographic projection; Conditioned Brownian motion; Conformal transformations.

RÉSUMÉ. — La projection stéréographique de \mathbb{R}^N à S^N applique les trajectoires Browniennes de \mathbb{R}^N sur les trajectoires Browniennes de S^N conditionnées par le fait d'être au centre de projection à un instant de loi exponentielle.

In this brief note we shall discuss how Brownian motion in \mathbb{R}^N , for $N \geq 3$, can be interpreted as a Brownian bridge conditioned to go to the « ideal point at infinity ». This question was posed by Prof. L. Schwartz [2]. Prof. M. Yor [3] presents an alternative, more probabilistic, approach.

1. STEREOGRAPHIC PROJECTION

Consider the unit sphere S^N in \mathbb{R}^{N+1} and the hyperplane

$$\mathbb{R}^N = \{ y = (y_1, \dots, y_{N+1}) : y_{N+1} = 0 \}.$$

Stereographic projection from the point $P = (0, \dots, 0, 1)$ of S^N maps $y \in \mathbb{R}^N$

to the point $x \in S^N \setminus \{P\}$ which lies on the straight line from P through y ; see the diagram. This is a diffeomorphism between $S^N \setminus \{P\}$ and R^N , so we regard P as being the point of S^N which corresponds to the « ideal point at infinity of R^N ».

PROPOSITION 1. — *Brownian motion on R^N is mapped by stereographic projection onto a time changed version of the Brownian motion on S^N together with a drift towards P at speed $\frac{1}{2}(N-2) \tan \frac{1}{2} \theta$ on the sphere.*

Proof. — Brownian motion on a Riemannian manifold with metric $g_{ab} dx_a dx_b$ has as its infinitesimal generator one half of the Laplacian, viz.

$$\frac{1}{2} \Delta = \frac{1}{2\sqrt{g}} \sum \frac{\partial}{\partial x_a} \left(\sqrt{g} g^{ab} \frac{\partial}{\partial x_b} \right)$$

where $g = \det(g_{ab})$ and $(g^{ab}) = (g_{ab})^{-1}$. On S^N take co-ordinates (θ, z) for $x \in S^N$ where $0 \leq \theta \leq \pi$ is the angle shown in the diagram and $z = y/\|y\| \in S^{N-1} = S^N \cap R^N$.

Then

$$\|dx\|^2 = |d\theta|^2 + \sin^2 \theta \cdot \|dz\|^2$$

so the Laplacian on S^N is

$$\Delta_{S^N} = \frac{1}{\sin^{N-1} \theta} \frac{\partial}{\partial \theta} \left(\sin^{N-1} \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \Delta_{S^{N-1}}.$$

Similarly, if we take co-ordinates (r, z) for $y \in R^N$, where $r = \|y\|$, then

$$\|dy\|^2 = |dr|^2 + r^2 \|dz\|^2$$

so the usual Laplacian on R^N is

$$\Delta_{R^N} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^{N-1}}.$$

The infinitesimal generator for the deterministic motion given by a drift towards P at speed $\frac{1}{2}(N-2) \tan \frac{1}{2} \theta$ is clearly

$$\frac{1}{2} (N-2) \tan \frac{1}{2} \theta \cdot \frac{\partial}{\partial \theta}$$

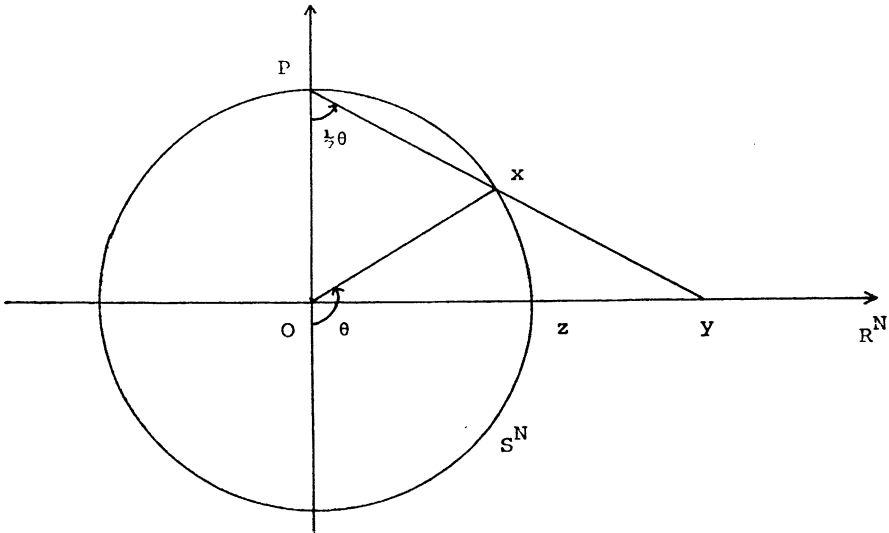
Hence, to prove the proposition we need to show that, under stereo-

graphic projection $\frac{1}{2} \Delta_{R^N}$ corresponds to some strictly positive function times

$$\mathcal{G}_P = \frac{1}{2} \Delta_{S^N} + \frac{1}{2} (N - 2) \tan \frac{1}{2} \theta \cdot \frac{\partial}{\partial \theta}.$$

Under stereographic projection we have $r = \tan \frac{1}{2} \theta$ so

$$\begin{aligned} \Delta_{S^N} &= \left(\frac{2r}{1+r^2} \right)^{1-N} \left(\frac{1+r^2}{2} \right) \frac{\partial}{\partial r} \left[\left(\frac{2r}{1+r^2} \right)^{N-1} \left(\frac{1+r^2}{2} \right) \frac{\partial}{\partial r} \right] + \left(\frac{1+r^2}{2r} \right)^2 \Delta_{S^{N-1}} \\ &= \left(\frac{1+r^2}{2} \right)^2 \left\{ \left(\frac{2}{1+r^2} \right)^{2-N} \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left[\left(\frac{2}{1+r^2} \right)^{N-2} r^{N-1} \frac{\partial}{\partial r} \right] + \frac{1}{r^2} \Delta_{S^{N-1}} \right\} \\ &= \left(\frac{1+r^2}{2} \right)^2 \left\{ \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left[r^{N-1} \frac{\partial}{\partial r} \right] - (N-2) \left(\frac{2r}{1+r^2} \right) \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}} \right\} \\ &= \left(\frac{1+r^2}{2} \right)^2 \left\{ \Delta_{R^N} - (N-2) \left(\frac{2r}{1+r^2} \right) \frac{\partial}{\partial r} \right\}. \end{aligned}$$



Equivalently,

$$\begin{aligned} \frac{1}{2} \Delta_{R^N} &= \left(\frac{2}{1+r^2} \right)^2 \left\{ \frac{1}{2} \Delta_{S^N} + \frac{1}{2} (N-2) r \left(\frac{1+r^2}{2} \right) \frac{\partial}{\partial r} \right\} \\ &= (1 + \cos \theta)^2 \left\{ \frac{1}{2} \Delta_{S^N} + \frac{1}{2} (N-2) \tan \frac{1}{2} \theta \frac{\partial}{\partial \theta} \right\}. \end{aligned}$$

This completes the proof. \square

We now wish to obtain the random process with infinitesimal generator \mathcal{G}_P by conditioning the standard Brownian motion $\text{BM}(\mathbb{S}^N)$ on the sphere to be at P at an appropriate time. To do this we will follow the analysis of conditioning given by J. L. Doob [1, Chapter 10]. Note that we are seeking a time-homogeneous process, so that conditioning $\text{BM}(\mathbb{S}^N)$ to be at P at a fixed time will not do. Furthermore, we cannot simply condition $\text{BM}(\mathbb{S}^N)$ to hit P at some time since, to do so, we would require a positive harmonic function on $\mathbb{S}^N \setminus \{P\}$ with a singularity at P . No such function exists. However, we do obtain time homogeneous processes by conditioning $\text{BM}(\mathbb{S}^N)$ to be at P at a random time T which is independent of $\text{BM}(\mathbb{S}^N)$ and has a negative exponential distribution.

PROPOSITION 2. — *Let T be a random time which is independent of $\text{BM}(\mathbb{S}^N)$ and has a negative exponential distribution with parameter $\lambda = N(N - 2)/8$. Then $\text{BM}(\mathbb{S}^N)$ conditioned to be at P at time T has infinitesimal generator*

$$\mathcal{G}_P = \frac{1}{2} \Delta_{\mathbb{S}^N} + \frac{1}{2} (N - 2) \tan \frac{1}{2} \theta \frac{\partial}{\partial \theta}$$

on $\mathbb{S}^N \setminus \{P\}$. Hence, $\text{BM}(\mathbb{R}^N)$ is mapped by stereographic projection to a time-changed version of $\text{BM}(\mathbb{S}^N)$ conditioned to be at P at the time T .

Proof. — To condition $\text{BM}(\mathbb{S}^N)$ to be at P at time T we need to find a positive function h on $\mathbb{S}^N \setminus \{P\}$ with a singularity at P and

$$\left(\frac{1}{2} \Delta_{\mathbb{S}^N} - \lambda I \right) h = 0$$

Then the conditioned process will have the h -transform:

$$u \rightarrow h^{-1} \left(\frac{1}{2} \Delta_{\mathbb{S}^N} - \lambda I \right) (h \cdot u)$$

as its infinitesimal generator. Such a function h must be a multiple of the Green's function for $\frac{1}{2} \Delta_{\mathbb{S}^N} - \lambda I$ with a pole at P and hence it must be a function of θ only. Thus we wish to solve

$$\frac{1}{2 \sin^{N-1} \theta} \frac{\partial}{\partial \theta} \left[\sin^{N-1} \theta \frac{\partial h}{\partial \theta} \right] - \lambda h = 0.$$

When $\lambda = N(N - 2)/8$ the required function h is given by $h = \left(\cos \frac{1}{2}\theta\right)^{-N+2}$

Consequently, the conditioned process has infinitesimal generator

$$\begin{aligned} u &\rightarrow h^{-1}\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)(h \cdot u) \\ &= h^{-1}\left(\frac{1}{2}h\Delta_{S^N}u + \nabla h \cdot \nabla u + \frac{1}{2}u\Delta_{S^N}h - \lambda u \cdot h\right) \\ &= \frac{1}{2}\Delta_{S^N}u + h^{-1}\nabla h \cdot \nabla u \\ &= \frac{1}{2}\Delta_{S^N}u + \frac{1}{2}(N - 2)\tan \frac{1}{2}\theta \frac{\partial}{\partial \theta} \end{aligned}$$

where ∇ is the gradient for the Euclidean metric on S^N . This proves the first assertion and the second follows from Proposition 1.

(Note that the conditioning described above does correspond to the naïve idea of conditioning a process by its position at time T . For suppose that U is a subset of S^N with a smooth boundary. If (x_t) is the Brownian motion on S^N , then we may form a new process

$$\begin{aligned} x_t^* &= x_t \quad \text{for } t < T \\ &= \partial \quad \text{for } t \geq T \end{aligned}$$

which jumps to a coffin state ∂ at the random time T . If we condition (x_t^*) so that $x_{T-}^* \in U$ then we obtain the transition semigroup P_t given by

$$\begin{aligned} P_t f(x) &= E^x(f(x_t^*) \mid x_{T-}^* \in U) \\ &= E^x(f(x_t)1_{(t < T)} \mid x_T \in U) \\ &= \frac{E^x(f(x_t)1_{(t < T)}1_U(x_T))}{E^x(1_U(x_T))} \end{aligned}$$

Setting

$$h(x) = E^x(1_U(x_T))$$

we find that

$$\begin{aligned} P_t f(x) &= h(x)^{-1}E^x(f(x_t)1_{(t < T)}h(x_T)) \\ &= h(x)^{-1}\int_t^\infty E^x(f(x_t)h(x_s))\lambda e^{-\lambda s}ds \\ &= h(x)^{-1}e^{-\lambda t}E^x(f(x_t)h(x_t)) \end{aligned}$$

by using the Markov property of the Brownian motion. Thus the condi-

tioned process is the h -transform of the Brownian motion for h the distributional solution of

$$\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)h = 1_U.$$

We can now decompose this process into an average of the processes conditioned to be at a point $X \in U$ at the time T . See J. L. Doob [1] for further details.) \square

For each $Y \in S^N$ let $h(Y, \cdot)$ be the Green's function of $\frac{1}{2}\Delta_{S^N} - \frac{N(N-2)}{8}I$ with a pole at Y . Then the Brownian motion conditioned to be at Y at the negative exponential time T has infinitesimal generator

$$u \rightarrow h(Y, x)^{-1}\left(\frac{1}{2}\Delta_{S^N} - \frac{N(N-2)}{8}I\right)(h(Y, x)u(x))$$

on $S^N \setminus \{Y\}$. As in Proposition 2 we find that this is

$$u \rightarrow \frac{1}{2}\Delta_{S^N}u(x) - (N-2)\|x - Y\|^{-1}\nabla\|x - Y\| \cdot \nabla u(x).$$

Call this generator \mathcal{G}_Y .

COROLLARY. — Let $(x_t : 0 \leq t \leq S)$ be the process with generator \mathcal{G}_P which starts from Y at time $t = 0$ and stops at the time S when it first hits P . Then the time reversed process $(\tilde{x}_\tau : 0 \leq \tau \leq S)$ given by

$$\tilde{x}_\tau = x_{S-\tau}$$

has infinitesimal generator \mathcal{G}_Y , starts from P at $\tau = 0$ and stops at the time S when it first hits Y .

Proof. — Since stereographic projection maps (x_t) onto Brownian motion in \mathbb{R}^N it is clear that $(x_t : t > 0)$ almost surely never hits Y . Thus the reversed process certainly starts from P at $\tau = 0$ and stops at the time S when it first hits Y . It remains to find its infinitesimal generator.

Let $g(Y, \cdot)$ be the Green's function for \mathcal{G}_P with pole at Y , then, for any smooth function f which is compactly supported within $S^N \setminus \{P, Y\}$, we have

$$E \int_0^S f(x_t)dt = \int g(x, Y)f(x)dV(x) = E \int_0^S f(\tilde{x}_\tau)d\tau$$

where dV is the N -dimensional Lebesgue measure on S^N .

Consequently, if we denote by $\mathcal{G}_P, (P_t)$ the generator and transition

semigroup for (x_t) and by $\tilde{\mathcal{G}}_P, (\tilde{P}_\tau)$ the corresponding operators for (\tilde{x}_τ) , then we obtain

$$\begin{aligned} \int g(x, X)f(x)P_r k(x)dV(x) &= E \int_0^s f(x_t)P_r k(x_t)dt \\ &= E \int_0^s f(x_t)k(x_{t+r})dt \\ &= E \int_0^s f(\tilde{x}_{\tau+r})k(\tilde{x}_\tau)d\tau \\ &= \int g(x, Y)k(x)\tilde{P}_r f(x)dV(x). \end{aligned}$$

So

$$\tilde{P}_r k(x) = g(x, Y)^{-1}P_r^*(g(x, Y)k(x))$$

and

$$\tilde{\mathcal{G}}_P k(x) = g(x, Y)^{-1}\mathcal{G}_P^*(g(x, Y)k(x)).$$

Now recall that $\mathcal{G}_P = h(P, \cdot)^{-1}\left(\frac{1}{2}\Delta - \lambda I\right)h(P, \cdot)$ so

$$g(x, Y) = \frac{h(Y, x)h(P, x)}{h(P, Y)}$$

and consequently

$$\tilde{\mathcal{G}}_P k(x) = h(Y, x)^{-1}\left(\frac{1}{2}\Delta - \lambda I\right)^*(h(Y, x)k(x)).$$

Since the Laplacian is self-adjoint, this gives the desired result. □

2. CONFORMAL TRANSFORMATIONS

In this section we wish to set the results of § 1 in a more general context.

For any $\lambda > 0$ we can condition $BM(S^N)$ to be at P at the independent random time T which has negative exponential distribution with parameter λ . Indeed, to do so we need only find a positive function h of θ with

$$\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)h = 0 \quad \text{on } S^N \setminus \{P\}$$

and a singularity at P . If we make the change of variables $q = \frac{1}{2}(1 - \cos \theta)$ this becomes

$$q(1 - q)\frac{d^2h}{dq^2} + \frac{1}{2}N(1 - 2q)\frac{dh}{dq} - 2\lambda h = 0$$

for $0 \leq q < 1$. This is in the standard hypergeometric form and may be solved by a power series

$$h = \sum_{n=0}^{\infty} a_n q^n.$$

This series has radius of convergence 1 and each a_n is positive, so h is certainly positive on $0 \leq q < 1$. For $\lambda \neq N(N-2)/8$ this formula does not define an elementary function. Although the conditioned process may be studied as in the previous section, it does not correspond to a simple process on \mathbb{R}^N .

The key property of stereographic projection is that it is *conformal* so it alters the metric at any point only by a scale factor. We can develop the arguments above for any such conformal transformation.

PROPOSITION 3. — *Let M be an N -manifold ($N \geq 3$) with a Riemannian metric g_{ab} and a conformally equivalent metric*

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad \text{with} \quad \Omega > 0.$$

Let R and \tilde{R} be the scalar curvature for g and \tilde{g} respectively. Then the Brownian motion relative to \tilde{g} can be obtained, up to a time change, by conditioning the Brownian motion relative to g according to its behaviour at a negative exponential time if, and only if, $R - \Omega^2 \tilde{R}$ is constant on M .

Proof. — In terms of the infinitesimal generators $\frac{1}{2}\Delta$ and $\frac{1}{2}\tilde{\Delta}$ for the Brownian motions, the Proposition states that there exists $\lambda > 0$ and strictly positive functions h and c on M with

$$\frac{1}{2}\tilde{\Delta}u = c^2 h^{-1} \left(\frac{1}{2}\Delta - \lambda I \right) (h \cdot u) \quad (1)$$

if, and only if, $R - \Omega^2 \tilde{R}$ is constant. (If we consider the second degree terms of (1) we see that the condition can only be satisfied if g and \tilde{g} are conformal. So there was no loss of generality in restricting ourselves to this case.)

The proof is simply a standard calculation of the scalar curvature for conformal metrics. We shall use the usual index notation for vectors and tensors on M . Let $\nabla_a, \tilde{\nabla}_a$ be the covariant derivatives relative to g and \tilde{g} .

Then a straightforward but tedious calculation yields the formulae:

$$\begin{aligned} \tilde{\nabla}_a v_b &= \nabla_a v_b - \Omega^{-1}(v_a \nabla_b \Omega + v_b \nabla_a \Omega - g_{ab} g^{cd} v_c \nabla_d \Omega) \\ \tilde{\Delta} u &= \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b u = \tilde{g}^{ab} \tilde{\nabla}_a (\nabla_b u) \\ &= \Omega^2 (\Delta u + (N - 2) \Omega^{-1} g^{ab} \nabla_a \Omega \nabla_b u) \\ \Omega^2 \tilde{R} &= R - 2(N - 1) \Omega^{-1} \Delta \Omega - (N - 1)(N - 4) \Omega^{-2} g^{ab} \nabla_a \Omega \nabla_b \Omega. \end{aligned}$$

Thus, for (1) to hold, we must have $c = \Omega$ and

$$h^{-1} \left(\frac{1}{2} \Delta - \lambda I \right) (h \cdot u) = \frac{1}{2} \Delta u + \frac{1}{2} (N - 2) \Omega^{-1} g^{ab} \nabla_a \Omega \nabla_b u.$$

Now

$$\Delta(h \cdot u) = g^{ab} \nabla_a \nabla_b (h \cdot u) = h \Delta u + 2g^{ab} \nabla_a h \nabla_b u + u \Delta h$$

so we obtain the two conditions:

$$h^{-1} \nabla_a h = \frac{1}{2} (N - 2) \Omega^{-1} \nabla_a \Omega$$

and

$$\left(\frac{1}{2} \Delta - \lambda I \right) h = 0.$$

The first of these is satisfied if, and only if, $h = K \cdot \Omega^{\frac{1}{2}(N-2)}$ for some constant K . In this case, the second condition becomes

$$\begin{aligned} 0 &= \left(\frac{1}{2} \Delta - \lambda I \right) (\Omega^{\frac{1}{2}(N-2)}) \\ &= \frac{1}{4} (N - 2) \Omega^{\frac{1}{2}N-2} \Delta \Omega + \frac{1}{8} (N - 2)(N - 4) \Omega^{\frac{1}{2}N-3} g^{ab} \nabla_a \Omega \nabla_b \Omega - \lambda \Omega^{\frac{1}{2}N-1}. \\ \Leftrightarrow \lambda &= \frac{1}{4} (N - 2) \Omega^{-2} \Delta \Omega + \frac{1}{8} (N - 2)(N - 4) \Omega^{-2} g^{ab} \nabla_a \Omega \nabla_b \Omega \\ &= \frac{N - 2}{8(N - 1)} \cdot (R - \Omega^2 \tilde{R}). \quad \square \end{aligned}$$

If we take g to be the Euclidean metric on S^N and \tilde{g} the metric on S^N which corresponds under stereographic projection to the Euclidean metric on R^N , then

$$\Omega = \frac{1}{1 + \cos \theta}, \quad R = N(N - 1), \quad \tilde{R} = 0$$

and we recover Proposition 2. The above formula may also be usefully applied to conformal mappings from S^N to itself.

PROPOSITION 4. — *Let $(x_t: 0 \leq t < S)$ be the process on $S^N \setminus \{P\}$ with infinitesimal generator \mathcal{G}_P and let $T: S^N \rightarrow S^N$ be a conformal automorphism of S^N . Then $(Tx_t: 0 \leq t < S)$ is a time-changed version of the process on $S^N \setminus \{TP\}$ with infinitesimal generator \mathcal{G}_{TP} .*

Proof. — Recall that the group of conformal automorphisms of S^N is generated by the inversions in spheres orthogonal to S^N . We could prove the result by direct calculation, as in § 1, of the effect of such an inversion. However, it is simpler to argue indirectly.

Let $U: \mathbb{R}^N \rightarrow S^N$ be stereographic projection with centre P and let $V: \mathbb{R}^N \rightarrow S^N$ be the stereographic projection with centre TP from the N -dimensional subspace of \mathbb{R}^{N+1} orthogonal to TP . Both of these maps are conformal, so the composite

$$Q = V^{-1}TU: \mathbb{R}^N \rightarrow \mathbb{R}^N$$

is conformal. Since $N \geq 3$, the only such conformal maps are the Euclidean similarities of \mathbb{R}^N . These similarities obviously preserve Brownian motion on \mathbb{R}^N to within alteration of the time scale by a constant factor. Now Proposition 1 shows that, to within a time change, U maps $BM(\mathbb{R}^N)$ to the process with generator \mathcal{G}_P and V maps $BM(\mathbb{R}^N)$ to the process with generator \mathcal{G}_{TP} . Therefore, $T = VQU^{-1}$ does indeed transform the process with generator \mathcal{G}_P to a time-changed version of the process with generator \mathcal{G}_{TP} . \square

If we combine Proposition 4 with the earlier Corollary, we see that time-reversal of the process starting at Y with generator \mathcal{G}_P corresponds to the image of the process under any inversion which maps S^N onto itself and interchanges Y and P . This should be compared with the results of M. Yor [3].

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