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G-convergence of generators and weak convergence of diffusions

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ABSTRACT. — Let (L_n), L, be generators of diffusions in R^d. Let us assume uniform ellipticity and uniform boundness of the coefficients of the variational forms of the operators. Let be given a locally bounded initial density p_0. If (P_n), P are the solutions of the martingale problems with the given generators and initial law, then the sequence (P_n) in weakly convergent to P if and only if the sequence (L_n) is convergent to L in the sense defined by DeGiorgi and Spagnolo.

RÉSUMÉ. — Soient (L_n), L, générateurs de diffusion dans R^d, uniformément elliptiques et avec les coefficients de la forme variationelle uniformément bornés. Soit p_0 une densité initiale sur R^d localement bornée. Si (P_n), P, sont les solutions des problèmes des martingales avec les données précédentes, alors la suite (P_n) est convergente vers P dans la topologie étroite si et seulement si la suite (L_n) est convergente vers L dans le sens défini par DeGiorgi et Spagnolo.

1. INTRODUCTION

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of second order elliptic partial differential operators on the real d-dimensional space $\mathbb{R}^d$. Let us assume that the

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sequence $(L_n)_{n\in\mathbb{N}}$ is uniformly elliptic and that the coefficients of the variational form are uniformly bounded. Then the G-convergence of Spagnolo [*18*] makes sense for the sequence $(L_n)_{n\in\mathbb{N}}$; let L be the G-limit.

Let us assume that $(L_n)_{n\in\mathbb{N}}$, L are such that there exist solutions $(P_n)_{n\in\mathbb{N}}$ of the martingale problems (Strook-Varadhan [*19*]). Using a result in Bensoussan-Lions-Papanicolau [*4*], one can readily show that (see prop. 2.1 below)

$$\forall x \in \mathbb{R}^d \qquad P_{x,n} \xrightarrow{w} P_x \Rightarrow L_n \xrightarrow{G} L \tag{1.1}$$

(here $\xrightarrow{w}$ denotes the weak convergence of probability measures and $\xrightarrow{G}$ denotes the G-convergence). The following question was asked to the authors by prof. J. Cecconi: which sort of converse to (1.1) can be proved in general, i. e. without any reference to a particular form of the operators?

The present article gives in sec. 2 a set of assumption on the differential operators under which the implication

$$L_n \xrightarrow{G} L \Rightarrow P_{\mu_0,n} \xrightarrow{w} P_{\mu_0} \tag{1.2}$$

can be proved for absolutely continuous and with locally bounded density initial distribution μ_0. The proof is given in sec. 4; sec. 3 contains some results on the univariate laws of the diffusions.

Our method of proof is inspired by the idea of transformation of the state space by Zvonkin [*21*] and uses strong results on elliptic and parabolic partial differential equations to show the tightness of the sequence $P_{\mu_0,n}$.

2. NOTATIONS, HYPOTHESIS, AND PRELIMINARIES

2.1. Elliptic equations.

For each couple of reals $0 < \lambda < \Lambda$ we denote by $\mathscr{L}(\lambda, \Lambda)$ the set of differential operators defined by

$$L \in \mathscr{L}(\lambda, \Lambda) \Leftrightarrow \begin{cases} L = \sum_{i=1}^{d} b_i D_i + \sum_{i,j=1}^{d} D_i a_{ij} D_j \\ a = a^t \in C_B^0(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d) \\ \lambda \,|\, \xi \,|^2 \leqslant \xi^t a(x) \xi \leqslant \Lambda \,|\, \xi \,|^2 \qquad x, \xi \in \mathbb{R}^d \\ \|a\|_\infty, \|b\|_\infty \leqslant \Lambda . \end{cases} \tag{2.1}$$

(Notations as in Adams [*2*]).

Each operator $L \in \mathscr{L}(\lambda, \Lambda)$ can be written in non-variational form:

$$L = \sum_{i=1}^{d} \theta_i D_i + \sum_{i,j=1}^{d} a_{ij} D_i D_j \tag{2.2}$$

where

$$\begin{cases} \theta_i = b_i + \tilde{b}_i \\ \tilde{b}_1 = \sum_{j=1}^{d} D_j a_{ij} . \end{cases} \tag{2.3}$$

For each radius $r > 0$ we denote by $B(r)$ the open ball with center 0 and radius r; function spaces on $B(r)$ will be denoted briefly $W_0^{1,2}(r)$, $W^{2,q}(r)$... and so on.

Let be given

$$\begin{cases} f \in L^\infty(\mathbb{R}^d) \\ L \in \mathscr{L}(\lambda, \Lambda) \end{cases} \tag{2.4}$$

and consider the elliptic variational problem

$$\begin{cases} - Lu + \lambda_0 = f \\ u \in W_0^{1,2}(B(r)) \end{cases} \tag{2.5}$$

where $\lambda_0 > 0$ (Gilbart-Trudinger [*9*]).

For ease of later reference we list some regularity results for the problem (2.5) which hold under the assumption (2.4) for given r, $\lambda_0 > 0$.

$W^{2,q}$-*regularity* (Chicco [*6*]).

$$\begin{cases} \text{for all } q > 2 \\ u \in W_0^{1,2}(B(r)) \cap W^{2,q}(B(r)) . \end{cases} \tag{2.6}$$

De Giorgi-Nash bound (Gilbart-Trudinger [*9*]).

$$\begin{cases} u \in C(\overline{B(r)}) \text{ and for all } r' < r \text{ there exist constants } C_1 \text{ and } \alpha \in]0, 1[\\ \text{such that} \\ \| u \|_{C^\alpha(\overline{B(r')})} \leqslant C_1 \| f \|_{\infty, B(r)} . \end{cases} \tag{2.7}$$

Meyers bound (Meyers [*15*]).

$$\begin{cases} \text{there exist constants } C_2 > 0 \text{ and } \eta > 0 \text{ such that} \\ \| u' \|_{2+\eta, B(r)} \leqslant C_2 \| f \|_{\infty, B(r)} . \end{cases} \tag{2.8}$$

Maximum principle (Gilbart-Trudinger [*9*]).

$$(2.9)\qquad \begin{cases} \text{there exists a constant } C_3 \text{ such that} \\ \| u \|_{\infty,B(r)} \leqslant C_3 \| f \|_{\infty,B(r)} . \end{cases}$$

The stochastic representation of the solution of eq. (2.6) will be discussed in sec. 2.3 below. Let us remark that the property (2.7) does not follow from (2.6) and the Sobolev's imbedding theorems. In fact we will need that the constant C_1 in (2.7) depends only on the uniform norm of b and a, and not on the norm of $\tilde{b}$.

2.2. **Parabolic equations.**

Let be given

$$(2.10)\qquad \begin{cases} p_0 \in L^{\infty}_{loc}(\mathbb{R}^d) \\ L \in \mathscr{L}(\lambda, \Lambda) \end{cases}$$

and let us denote by L* the formal transpose of L, i. e.

$$(2.11)\qquad L^* = -\sum_{i=1}^{d} D_i b_i + \sum_{i,j=1}^{d} D_j a_{ji} D_i .$$

We will consider the following parabolic variational problem (Bensoussan-Lions [*3*], Ladisenskaya-Solonnikov-Uralceva [*11*])

$$(2.12)\qquad \begin{cases} D_t p - L^* p = 0 \\ p \in L^2(0, N ; W_0^{1,2}(B(r))) \cap C(0, N ; L^2(B(r))) \\ D_t p \in L^2(0, N ; W^{-1,2}(B(r)) \\ p(0) = p_0 \end{cases}$$

We will need later the following result.

Maximum principle (Ladiseskaya-Solonnikov-Uralceva [*11*]).

$$(2.13)\qquad \begin{cases} \text{there exist a constant } C_4 \text{ such that} \\ \| p \|_{\infty,[0,N]\times B(r)} \leqslant C_4 \| p_0 \|_{\infty,B(r)} . \end{cases}$$

The stochastic representation of the solution of eq. (2.12) will be discussed in sec. 3.

2.3. **Stochastic differential equations.**

Let $\mathbb{R}_+ = [0, +\infty[$ be a time interval and let us denote by Ω the sample space of continuous trajectories with values in $\mathbb{R}^d$. Ω has the usual polish structure and $\mathscr{B}$ denotes its Borel σ-algebra. X denotes the canonical process, X_t the coordinate projection, $\mathscr{B}_t$ the σ-algebra generated by X_s, $s < t$.

Let be given measurable real functions

$$\begin{cases} b, \tilde{b} : \mathbb{R}^d \to \mathbb{R}^d \\ \sigma : \mathbb{R}^d \to \text{Matrices } (d \times d) \end{cases} \tag{2.14}$$

We denote

$$a = \frac{1}{2}\sigma\sigma^t \tag{2.15}$$

and assume that the differential operator L defined in (2.2), (2.3) belongs to the class $\mathscr{L}(\lambda, \Lambda)$ (2.1).

The martingale problem for the operator L has a unique solution starting at $(\mu_0, 0)$, where μ_0 is an initial probability measure on $\mathbb{R}^d$, given by

$$P_{\mu_0} = \int_{\mathbb{R}^d} P_x \mu_0(dx) \tag{2.16}$$

where $(P_x)_{x \in \mathbb{R}^d}$ is the solution of the martingale problem starting at $(x, 0)$ (Strook-Varadhan [*19*]). Moreover there exists a Wiener process W on $(\Omega, \mathscr{B}, (\mathscr{B}_t)_{t \in \mathbb{R}_+}, P_{\mu_0})$ such that the canonical process X satisfies the following Ito's equation

$$dX_t = \theta(X_t)dt + \sigma(X_t)dW_t\,. \tag{2.17}$$

The stochastic representation (Friedman [*8*]) of the solution of the elliptic equation (2.5) follow under the assumption $L \in \mathscr{L}(\lambda, \Lambda)$ from the $W^{2,q}$-regularity (2.6) and the generalized Ito's formula (Zvonkin [*21*] or sec. 3 below): the solution u of eq. (2.5) satisfying (2.6), (2.7), (2.8) is the μ_0-unique continuous function on $\mathbb{R}^d$ such that

$$u(X_0) = \mathbb{E}_{\mu_0}\left(\int_0^{\tau_r} f(X_s) e^{-\lambda_0 s} ds \mid \mathscr{B}_0 \right) \tag{2.18}$$

where $\mathbb{E}_{\mu_0}$ denote the expectation with respect to P_{μ_0} and τ_r is the exit time from the ball $B(r)$.

2.4. **Convergence.**

Let be given a sequence of stochastic differential equations of the type (2.17).

$$\begin{cases} dX_t = \theta_n(X_t)dt + \sigma_n(X_t)dW_t^n & n \in \mathbb{N} \\ dX_t = \theta(X_t)dt + \sigma(X_t)dW_t\,. \end{cases} \tag{2.19}$$

Let us denote by L_n, Ln, $\in \mathbb{N}$, the associated differential operators with notations as in 2.1 and assume

$$L_n,\ L \in \mathscr{L}(\lambda, \Lambda). \tag{2.20}$$

Let us consider an initial distribution μ_0 such that

$$\mu_0 \text{ is uniform on } B(r) \tag{2.21}$$

and let us denote by P_n, $n \in \mathbb{N}$, the laws of the solutions of eq.s (2.19) with initial law μ_0.

For each given

$$f \in C^0(\mathbb{R}^d) \tag{2.22}$$

we can solve

$$\begin{cases} (-L_n + \lambda_0)u_n = f & n \in \mathbb{N} \\ (-L + \lambda_0)u = f \\ u_n,\ u \in H_0^1(B(r)) \cap C^0(\mathbb{R}^d). \end{cases} \tag{2.23}$$

PROPOSITION 2.1. — *If*

$$P_n \overset{w}{\to} P \tag{2.24}$$

then

$$u_n \to u \qquad \text{in } L^2(r). \tag{2.25}$$

Proof. — From the maximum principle (2.9) and the De Giorgi-Nash bound (2.8) it follows that

$$\{ u_n \}_{n \in \mathbb{N}} \text{ is pre-compact in } L^2(r) \text{ (in fact, in } C^0(\overline{B(r)})). \tag{2.26}$$

The functionals

$$\omega \to \varphi(\omega(0)) \int_0^{\tau_r(\omega)} f(\omega(s)) e^{-\lambda_0 s} ds \tag{2.27}$$

are bounded and P-continuous for each $\varphi \in C_B^0(\mathbb{R}^d)$, see Bensoussan-Lions-Papanicolaou [*4*], so that (2.18), (2.24), (2.21) give

$$\int_{B(r)} u_n(x)\varphi(x)dx \to \int_{B(r)} u(x)\varphi(x)dx. \tag{2.28}$$

Finally (2.26) and (2.28) give (2.25).

As announced in the introduction, the type of convergence just proved turns out to be equivalent to the G-convergence of Spagnolo [17]. For ease of reference we adopt the following definition

$$(2.29)\quad \begin{cases} L_n,\ L \in \mathscr{L}(\lambda, \Lambda) \\ L_n \overset{G}{\to} L \Leftrightarrow \forall r > 0,\ \lambda_0 > 0 \qquad (2.25) \text{ holds true}. \end{cases}$$

Remark. — The restriction given by a_n, $a \in W^{1,\infty}$ in (2.1) is imposed by the stochastic interpretation of eq.s (2.23) and by the method of proof of the converse theorem in sec. 4. This situation is not satisfactory because this restriction is not natural in the setting of the G-convergence.

3. PROBABILITY DENSITY AND GENERALIZED ITO'S FORMULA

Let be given

$$(3.1)\qquad p_0 \in L^2_{loc}(R^d)$$

and assume that P solves eq. (2.17) with initial distribution with density P_0. Let

$$(3.2)\qquad L \in \mathscr{L}(\lambda, \Lambda)$$

be the differential operator of eq. (2.17).

PROPOSITION 3.1. — *Let p be the solution of the parabolic problem* (2.12). *Then for all $t \in [0, N]$, $p(t)$ is the unnormalized conditional density of X_t given $\{ t < \tau_r \}$; i. e.*

$$(3.3)\qquad \forall f \in L^\infty(\mathbb{R}^d) \qquad \mathbb{E}(f(X_t)\mathbb{1}_{\{t<\tau_r\}}) = \int_{B(r)} p(t, x) f(x) dx .$$

Proof. — For each test function

$$(3.4)\qquad \varphi \in C_0^\infty([0, N[\times B(r))$$

we write

$$(3.5)\qquad f = (D_t + L)\varphi$$

and observe that the set of all such functions is dense in

$$(3.6)\qquad L^2(0, N ; W^{-1,2}(B(r))$$

and there exists a constant C such that (Lions [12])

$$(3.7)\qquad \| \varphi(0,.) \|_{2,B(r)} \leqslant C \| f \|_{L^2(0,N;W^{-1,2}(B(r)))} .$$

Let us consider the functional $\mathscr{E}$ defined on a dense subset of by (3.5) and

$$\mathscr{E}(f) = \mathbb{E}\left(\int_0^{\tau_r \wedge N} f(X_s, s)ds\right) = - \mathbb{E}(\varphi(X_0)). \tag{3.8}$$

This functional is bounded because from (3.7) and (3.8) it follows

$$\begin{cases} \left| E\left(\int_0^{\tau_r \wedge N} f(X_s, s)ds\right|\right. \leqslant \| \varphi \|_{2,B(r)} \| p_0 \|_{2,B(r)} \\ \leqslant C \| p_0 \|_{2,B(r)} \| f \|_{L^2(0,N;W^{-1,2}(B(r)))} . \end{cases} \tag{3.9}$$

Then there exist by Riesz theorem

$$p \in L^2(0, N ; W_0^{1,2}(B(r))) \simeq L^2(0, N ; W^{-1,2}(B(r))) \tag{3.10}$$

such that

$$\mathscr{E}(f) = \int_0^N \langle f(t), p(t) \rangle_{W^{-1,2}(B(r)),W_0^{1,2}(B(r))} . \tag{3.11}$$

In particular, using (3.5), (3.8), (3.11), it follows

$$\begin{cases} \forall \varphi \in C_0^\infty([0, N[\times B(r)) \\ \int_0^N dt \int_{B(r)} dx(D_t + L)\varphi(t, x)p(t, x) = \int_{B(r)} dx\varphi(0, x)p_0(x) \\ p \in L^2(0, N ; W^{-1,2}(B(r))) . \end{cases} \tag{3.12}$$

It can be shown (Ladizenskaja-Solonnikov-Uralceva [*11*] and Lions [*12*]) that (3.12) is equivalent to the parabolic problem in eq. (2.12). Finally (3.3) follows from the continuity of p from $[0, N]$ to $L^2(r)$ and (3.8), (3.11).

The following two proposition, are simple consequence of prop. 3.1.

PROPOSITION 2. — *For each* $q \in [2, +\infty]$ *there exist a constant* c_q *such that*

$$\sup_{t \in [0,N]} \| p(t) \|_{q,B(r)} \leqslant c_q \| p_0 \|_{q,B(r)} . \tag{3.13}$$

Proof. — From prop. 3.1, the maximum principle (2.13) and an interpolation argument.

PROPOSITION 3.2 (*Generalized Ito's formula*). — *If for* $q \in [2, +\infty]$

$$p_0 \in L^q_{loc}(\mathbb{R}^d) \tag{3.14}$$

then the Ito's formula holds true for each

$$\varphi \in W^{1,2q'}_{loc} \cap W^{2,q'}_{loc} \qquad (q^{-1} + q'^{-1} = 1) . \tag{3.15}$$

Proof. — If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of regular functions such that

$$(3.16)\qquad \begin{array}{lll} \varphi_n \to \varphi & \text{a. s.} & \\ \varphi_n' \to \varphi & \text{in} & L^{2q'}(r) \\ L\varphi_n \to L\varphi & \text{in} & L^{q'}(r) \end{array}$$

each term of the Ito's formula is convergent.

4. G-CONVERGENCE IMPLIES WEAK CONVERGENCE

Let be given stochastic differential equations of the type (2.10).

$$(4.1)\qquad dX_t = \theta_n(X_t)dt + \sigma_n(X_t)dW_t^n \qquad n \in \mathbb{N}$$

$$(4.2)\qquad dX_t = \theta(X_t)dt + \sigma(X_t)dW_t .$$

Using the same notations as in sec. 2 we assume

$$(4.3)\qquad L_n, L \in \mathscr{L}(\lambda, \Lambda)$$

and

$$(4.4)\qquad \mu_0(dx) = p_0(x)dx \qquad p_0 \in L^\infty_{loc}(\mathbb{R}^d) .$$

The probability measures that solve the problems (4.1), (4.2) are denoted P_n, P.

PROPOSITION 4.1. — *If* (4.1), (4.2), (4.3), (4.4) *hold true and*

$$(4.5)\qquad L_n \overset{G}{\to} L \qquad (\text{def. } in\ 2.29)$$

then

$$(4.6)\qquad P_n \overset{w}{\to} P .$$

Remark. — We could prove along the same lines a theorem with the hypothesis (4.4) replaced by

$$(4.4)'\qquad p_0 \in L^{\tilde q}_{loc}(\mathbb{R}^d)$$

where $\tilde q$ depends on the Meyers constant η in (2.8). The actual computation of this constant do not seem to be practical, so we have turned to the hypothesis of local boundness (4.4).

Proof of prop. 4.1. — Let $0 < r < r'$ be a couple of rad *ii* and let v be a test function $v \in \mathscr{D}(r')$. For each $n \in \mathbb{N}$ let v_n be defined as the solution of

$$(4.7)\qquad \begin{cases} (-L_n + \lambda_0)v_n = (-L + \lambda_0)v \\ v_n \in W_0^{1,2}(r') \cap W^{2,q}(r') \cap C_B^0(\mathbb{R}^d) \end{cases}$$

as in sec. 2 (the constant term in the equation belongs to $L^\infty(\mathbb{R}^d)$). Our hypothesis is

$$v_n \to v \qquad \text{in } L^2(r'). \tag{4.8}$$

From De Giorgi-Nash bound (2.6) and Ascoli-Arzela compactness criterium, it follows

$$\begin{cases} v_n \to v & \text{pointwise} \\ v_n \to v & \text{uniformly on } \overline{B(r)}. \end{cases} \tag{4.9}$$

Let τ be the exit time from $B(r)$. The generalized Ito's formula (Krilov [10], Zvonkin [21] or sec. 3) applies to the P_n-semimartingale

$$(v_n \circ X_{\tau \wedge t})_{t \in \mathbb{R}_+} \tag{4.10}$$

and gives, using also (4.7),

$$\begin{aligned} v(X_{\tau \wedge t}) &= v(X_0) + \int_0^{\tau \wedge t} (Lv)(X_s)ds + \int_0^{\tau \wedge t} v_n'(X_s)\sigma_n(X_s)dW_s^n \\ &+ [(v - v_n)(X_{\tau \wedge t}) - (v - v_n)(X_0)] - \lambda_0 \int_0^{\tau \wedge t} (v - v_n)(X_s)ds. \end{aligned} \tag{4.11}$$

Now (4.9) implies that the sequence of processes in square brachets in eq. (4.11) is uniformly bounded and uniformly convergent to zero as $n \to \infty$.

A consequence is that the limit laws of the sequence

$$(P_n \circ (v \circ X_{\tau \wedge .})^{-1})_{n \in \mathbb{N}} \tag{4.12}$$

do not depend on these terms.

Let us take a particular test function v such that

$$v(x) = x \qquad \text{on } B(r) \tag{4.13}$$

then

$$Lv = \theta \qquad \text{on } B(r) \tag{4.14}$$

the laws (4.12) are the stopped laws

$$(P_n \circ (X_{\tau \wedge .})^{-1})_{n \in \mathbb{N}} \tag{4.15}$$

and the limit coincides with the limits of the laws of the sequence

$$Y_{n,t} = v(X_0) + \int_0^{\tau \wedge t} \theta(X_s)ds + \int_0^{\tau \wedge t} v_n'(X_s)\sigma_n(X_s)dW_s^n \qquad n \in \mathbb{N}. \tag{4.16}$$

The tightness of the sequence can be shown using the Abdons [1]-

Rebolledo [16] condition. From our hypothesis the law of X_0 is constant and θ is bounded, so we have only to consider the sequence of quadratic variations.

If N is a positive constant, and stopping times $\tau_1 \leqslant \tau_2 \leqslant N$ are given such that $\tau_2 - \tau_1 = k$ is constant then from (4.16) and (4.3) it follows:

$$
(4.17)\quad \begin{cases} \langle y_n \rangle_{\tau_2} - \langle y_n \rangle_{\tau_1} = \displaystyle\int_{\tau_1}^{\tau_2} \mathbb{1}_{\{s<\tau_r\}} \sum_{i,j=1}^{d} a_{ij}^n D_i v_n(X_s) D_j v_n(X_s) ds \\ \leqslant \Lambda^2 \displaystyle\int_{\tau_1}^{\tau_2} \mathbb{1}_{\{1<\tau_r\}} \,|\, v_n'(X_s) \,|^2 ds . \end{cases}
$$

If η is the constant in Meyers bound (2.8) and $\varepsilon > 0$, from (4.17), Cebicov and Hölder inequality

$$
(4.18)\quad \begin{cases} P_n(\langle y_n \rangle_{\tau_2} - \langle y_n \rangle_{\tau_1} > \varepsilon) \\ \leqslant \left(\dfrac{\Lambda^2}{\varepsilon}\right)^{1+\frac{\eta}{2}} \mathbb{E}_n\left(\displaystyle\int_{\tau_1}^{\tau_2} 1_{\{1<\tau_r\}} \,|\, v_n'(X_s) \,|^2 ds\right)^{1+\frac{\eta}{2}} \\ \leqslant \left(\dfrac{\Lambda^2}{\varepsilon}\right)^{1+\frac{\eta}{2}} k^{\eta/2} \mathbb{E}_n\left(\displaystyle\int_0^N 1_{\{1<\tau_r\}} \,|\, v_n'(X_s) \,|^{2+\eta} ds\right). \end{cases}
$$

The last integral can be evaluated with props. 3.1 and 3.2, giving

$$
(4.19)\quad \begin{cases} P_n(\langle y_n \rangle_{\tau_2} - \langle y_n \rangle_{\tau_1} > \varepsilon) \\ \leqslant \left(\dfrac{\Lambda^2}{\varepsilon}\right)^{1+\frac{\eta}{2}} k^{\frac{\eta}{2}} \displaystyle\int_{B(r)} dx \,|\, v_n'(x) \,|^{2+\eta} \int_0^N dsp^n(s, x) \\ \leqslant \left(\dfrac{\Lambda^2}{\varepsilon}\right)^{1+\frac{\eta}{2}} C_\infty \,\|\, p_0 \,\|_{\infty, B(r)} \,\|\, v_n' \,\|_{2+\eta, B(r)}^{2+\eta} . \end{cases}
$$

Now the tightness follows the bound (2.8).

The convergence (4.6) will now follow from the uniqueness of the solution of the martingale problem for the operator L with initial distribution p_0, see Strook-Varadhan [19]. In fact, if $\tilde{P}_n$ is the law of y_n and $\tilde{P}$ is a limit point of the sequence $(\tilde{P}_n)_{n\in\mathbb{N}}$ then $\tilde{P}$ solves the martingale problem for L because

$$
(4.20)\quad \int_0^{\tau_r \wedge \cdot} \theta(X_s)\sigma(X_s) dW_s
$$

is the uniform limit of a sequence of P_n-martingales, as shown in eq. (4.11) and the subsequent remark.

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