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The lifetime of conditional Brownian motion in the plane

by

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SUMMARY. — In this note I give a short and perspicacious proof of a recent remarkable result due to Cranston and McConnell [*3*].

RÉSUMÉ. — Cette note est consacrée à une démonstration courte d'un résultat remarquable et récent de Cranston et McConnell [*3*].

Let D be a bounded domain in R^d, $d \geq 1$; H(D) the class of strictly positive harmonic functions in D; $X = \{ X_t, t \geq 0 \}$ the standard Brownian motion in R^d; $\tau_B = \inf \{ t > 0 : X_t \notin B \}$ for any Borel set B; m the Lebesgue measure in R^d; E_h^x the expectation associated with the h-conditioned Brownian motion starting at $x \in D$.

THEOREM. — *Let $d = 2$. There exists a constant* C *depending only on* D *such that*

$$\sup_{\substack{x \in D \\ h \in H(D)}} E_h^x \{ \tau_D \} \leq Cm(D). \tag{1}$$

We begin by stating explicitly the case where $h \equiv 1$, namely unconditioned Brownian motion, for a general Borel set B in R^d, $d \geq 1$.

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LEMMA. — *We have*

$$\sup_{x \in D} E^x\{\tau_B\} \leq A_d m(D)^{2/d}$$

where

$$A_d = \frac{1}{2\pi d^2}(d+1)^{\frac{2(d+1)}{d}}.$$

This lemma can be proved by an elementary method using only the strong Markov property of X and the form of its transition density. It is generalizable and adaptable to similar estimates; see [*1*], p. 148 ff.

As the first simplification in the proof of the theorem, we deal directly with a general h in H(D). This spares us some unnecessary « hard theory », such as the famous Martin representation, and the behavior of a minimal harmonic function at the boundary. Cf. Lemma 2.2 in [*3*], which is actually a result due to Doob. Thus for any $h \in H(D)$, we put for clarity:

$$\text{(2)} \qquad Y(t) = \begin{cases} \left(\frac{1}{h}\right)(X_t) & \text{for} \quad 0 \leq t < \tau_D, \\ 0 & \text{for} \quad \tau \geq \tau_D. \end{cases}$$

It is a basic idea in h-conditioning that $\{Y_t, \mathcal{F}_t, t \geq 0\}$ is a super-martingale, where $\{\mathcal{F}_t\}$ is the natural filtration of $\{X_t\}$. Let $0 < a < b < \infty$; let $D'[a, b]$ and $U'[a, b]$ denote respectively the number of downcrossings and upcrossings of $[a, b]$ by $\{Y_t, t \geq 0\}$. Then we have for any $x \in D$:

$$\text{(3)} \qquad E^x\{D'[a, b]\} \leq \frac{b}{b-a}; \qquad E^x\{U'[a, b]\} \leq \frac{a}{b-a}.$$

For the first inequality (due to G. A. Hunt), see e. g. [*2*], p. 341; the second does not follow trivially from the first, but both follow from Dubins's inequalities (*loc. cit.*). Taking reciprocals, we deduce that if $D[a, b]$ and $U[a, b]$ denote the corresponding numbers for $\{h(X_t), t \geq 0\}$, then

$$\text{(4)} \qquad E^x\{U[a, b]\} \leq \frac{a}{b-a}; \qquad E^x\{D[a, b]\} \leq \frac{b}{b-a}.$$

We now define for any $x_0 \in D$:

$$C_n = \{x \in D: h(x) = 2^n h(x_0)\},$$
$$D_n = \{x \in D: 2^{n-1}h(x_0) < h(x) < 2^{n+1}h(x_0)\},$$

where n is an integer. Furthermore, we denote by N_n the total number of times a path moves from inside D_n to outside D_n. If it starts from C_n, this can be done either by a downcrossing of $[2^n h(x_0), 2^{n-1}h(x_0)]$, or an upcrossing of $[2^n h(x_0), 2^{n+1}h(x_0)]$. Hence we have by (5):

$$\sup_{x\in C_n} E_h^x\{N_n\} \le \frac{2^{n-1}}{2^n - 2^{n-1}} + \frac{2^{n+1}}{2^{n+1} - 2^n} = 3\,. \tag{6}$$

Next, it is plain that for any $x \in C_n$:

$$E_h^x\{\tau_{D_n}\} = \frac{1}{h(x)} E^x\left\{\int_0^{\tau_{D_n}} h(X_t)dt\right\} \le 2E^x\{\tau_{D_n}\}\,. \tag{7}$$

It remains to add up all the crossings, *without ordering*. But we must not forget that a path may leave D before completing the last crossing. In this case 1 must be added to N_n in the counting. Therefore our final estimate is as follows:

$$E_h\{\tau_D\} \le (3+1)\sum_{n=-\infty}^{\infty} 2 \sup_{x\in C_n} E^x\{\tau_{D_n}\} \le 8\,A_d \sum_{n=-\infty}^{\infty} m(D_n)^{\frac{2}{d}} \tag{8}$$

by (6), (7) and the lemma. For $d = 2$, this yields (1) with $C = 8A_d$.

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