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The lifetime of conditional Brownian motion in the plane

by

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SUMMARY. — In this note I give a short and perspicacious proof of a recent remarkable result due to Cranston and McConnell [3].

RÉSUMÉ. — Cette note est consacrée à une démonstration courte d'un résultat remarquable et récent de Cranston et McConnell [3].

Let D be a bounded domain in \( \mathbb{R}^d \), \( d \geq 1 \); \( H(D) \) the class of strictly positive harmonic functions in D; \( X = \{ X_t, t \geq 0 \} \) the standard Brownian motion in \( \mathbb{R}^d \); \( \tau_B = \inf \{ t > 0 : X_t \notin B \} \) for any Borel set B; \( m \) the Lebesgue measure in \( \mathbb{R}^d \); \( E^n_x \) the expectation associated with the \( h \)-conditioned Brownian motion starting at \( x \in D \).

THEOREM. — Let \( d = 2 \). There exists a constant \( C \) depending only on \( D \) such that

\[
\sup_{x \in D} E^n_x \{ \tau_D \} \leq Cm(D).
\]

We begin by stating explicitly the case where \( h \equiv 1 \), namely unconditioned Brownian motion, for a general Borel set B in \( \mathbb{R}^d \), \( d \geq 1 \).

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Lemma. — We have
\[
\sup_{x \in D} \mathbb{E}_x \{ \tau_B \} \leq A_d m(D)^{2/d}
\]
where
\[
A_d = \frac{1}{2\pi d^2} \left( d + 1 \right)^{\frac{2(d+1)}{d}}.
\]

This lemma can be proved by an elementary method using only the strong Markov property of $X$ and the form of its transition density. It is generalizable and adaptable to similar estimates; see [1], p. 148 ff.

As the first simplification in the proof of the theorem, we deal directly with a general $h$ in $H(D)$. This spares us some unnecessary « hard theory », such as the famous Martin representation, and the behavior of a minimal harmonic function at the boundary. Cf. Lemma 2.2 in [3], which is actually a result due to Doob. Thus for any $h \in H(D)$, we put for clarity:

\[
Y(t) = \begin{cases} 
\left( \frac{1}{h} \right)(X_t) & \text{for } 0 \leq t < \tau_D, \\
0 & \text{for } \tau \geq \tau_D.
\end{cases}
\]

It is a basic idea in $h$-conditioning that $\{ Y_t, \mathcal{F}_t, t \geq 0 \}$ is a super-martingale, where $\{ \mathcal{F}_t \}$ is the natural filtration of $\{ X_t \}$. Let $0 < a < b < \infty$; let $D'[a, b]$ and $U'[a, b]$ denote respectively the number of downcrossings and upcrossings of $[a, b]$ by $\{ Y_t, t \geq 0 \}$. Then we have for any $x \in D$:

\[
\mathbb{E}_x \{ D'[a, b] \} \leq \frac{b}{b - a}; \quad \mathbb{E}_x \{ U'[a, b] \} \leq \frac{a}{b - a}.
\]

For the first inequality (due to G. A. Hunt), see e. g. [2], p. 341; the second does not follow trivially from the first, but both follow from Dubins's inequalities (loc. cit.). Taking reciprocals, we deduce that if $D[a, b]$ and $U[a, b]$ denote the corresponding numbers for $\{ h(X_t), t \geq 0 \}$, then

\[
\mathbb{E}_x \{ U[a, b] \} \leq \frac{a}{b - a}; \quad \mathbb{E}_x \{ D[a, b] \} \leq \frac{b}{b - a}.
\]

We now define for any $x_0 \in D$:
\[
C_n = \{ x \in D : h(x) = 2^n h(x_0) \}, \quad D_n = \{ x \in D : 2^{n-1} h(x_0) < h(x) < 2^{n+1} h(x_0) \},
\]
where $n$ is an integer. Furthermore, we denote by $N_n$ the total number of times a path moves from inside $D_n$ to outside $D_n$. If it starts from $C_n$, this
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can be done either by a downcrossing of \([2^n h(x_0), 2^{n-1} h(x_0)]\), or an upcrossing of \([2^n h(x_0), 2^{n+1} h(x_0)]\). Hence we have by (5):

\[
\sup_{x \in C_n} E^x \{ N_n \} \leq \frac{2^n - 2^{n-1}}{2^n} + \frac{2^{n+1} - 2^n}{2^{n+1} - 2^n} = 3.
\]

Next, it is plain that for any \(x \in C_n\):

\[
E^x \{ \tau_{D_n} \} = \frac{1}{h(x)} E^x \left\{ \int_0^{\tau_{D_n}} h(X_t) dt \right\} \leq 2E^x \{ \tau_{D_n} \}.
\]

It remains to add up all the crossings, without ordering. But we must not forget that a path may leave \(D\) before completing the last crossing. In this case 1 must be added to \(N_n\) in the counting. Therefore our final estimate is as follows:

\[
E^x \{ \tau_D \} \leq (3 + 1) \sum_{n=-\infty}^{\infty} 2 \sup_{x \in C_n} E^x \{ \tau_{D_n} \} \leq 8 A_d \sum_{n=-\infty}^{\infty} m(D_n)^2
\]

by (6), (7) and the lemma. For \(d = 2\), this yields (1) with \(C = 8A_d\).

REFERENCES


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