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WERNER LINDE

PETER MATHÉ

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Conditional symmetries of stable measures on $\mathbf{R}^n$

by

Werner LINDE and Peter MATHÉ

Sektion Mathematik der Friedrich-Schiller-Universität Jena, DDR

ABSTRACT. — A result concerning conditional symmetries of symmetric p-stable laws μ on $\mathbf{R}^2$ is proved. μ is said to be conditional symmetric w. r. t. a real number c, if for all Borel sets B_1, B_2

$$\mu\{(\xi_1, \xi_2), \xi_1 \in B_1, \xi_2 - c\xi_1 \in B_2\} = \mu\{(\xi_1, \xi_2), \xi_1 \in B_1, c\xi_1 - \xi_2 \in B_2\}$$

is valid.

It is given a characterization of conditional symmetric stable laws. This result is then extended to $\mathbf{R}$.

RÉSUMÉ. — Pour des lois p-stables μ dans $\mathbf{R}^2$, un résultat sur des symétries conditionnelles est prouvé. μ est dit conditionnellement symétrique par rapport à un nombre réel c, au cas où vaut

$$\mu\{(\xi_1, \xi_2); \xi_1 \in B_1, \xi_2 - c\xi_1 \in B_2\} = \mu\{(\xi_1, \xi_2); \xi_1 \in B_1, c\xi_1 - \xi_2 \in B_2\}$$

pour tous les ensembles Borel B_1, B_2.

On obtient une caractérisation de lois stables conditionnellement symétriques. On peut étendre ce résultat **a** $\mathbf{R}^n$.

1. INTRODUCTION

All measures on $\mathbf{R}^n$ are assumed to be finite and they are defined on the Borel subsets of $\mathbf{R}^n$. A measure μ is said to be *symmetric* if

$$\mu(B) = \mu(-B)$$

for all Borel subsets $B \subseteq \mathbf{R}^n$. If $\alpha > 0$ then $\tau_\alpha(\mu)$ is defined by

$$\tau_\alpha\mu(B) = \mu(\alpha^{-1}B).$$

Given $0 < p \leqslant 2$ the symmetric measure μ is said to be *p-stable* if for arbitrary α, $\beta > 0$ the equality

$$\tau_\alpha(\mu) * \tau_\beta(\mu) = \tau_\gamma(\mu)$$

holds where $\gamma > 0$ can be calculated by $\gamma^p = \alpha^p + \beta^p$.

Let us denote by $R_p(n)$ the set of all p-stable symmetric measures on $\mathbf{R}^n$.

Given $\mu \in R_2(2)$, i. e. μ is Gaussian on $\mathbf{R}^2$, there exists a real number c such that

$$(+) \quad \mu\{(\xi_1, \xi_2); \xi_1 \in B_1, c\xi_1 - \xi_2 \in B_2\} = \mu\{(\xi_1, \xi_2); \xi_1 \in B_1, \xi_2 - c\xi_1 \in B_2\}$$

for all Borel subsets B_1, $B_2 \subseteq \mathbf{R}^2$. The number c can be calculated by

$$c = \int_{\mathbf{R}^2} \xi_1\xi_2 d\mu \Big/ \int_{\mathbf{R}^2} |\xi_1|^2 d\mu$$

provided $\mu\{\xi_1 = 0\} = 0$.

One may ask now whether or not measures in $R_p(2)$, $0 < p < 2$, possess the same property $(+)$ as Gaussian ones. A first result in this direction was proved by M. Kanter ([*1*]) in 1972: If (X, Y) is a random vector whose distribution belongs to $R_p(2)$, $1 < p < 2$, then there is a constant $c \in \mathbf{R}$ with

$$\mathbf{E}(cX - Y \mid X) = \mathbf{E}(Y - cX \mid X).$$

Here $\mathbf{E}(Z \mid X)$ means the conditional expectation of Z under the condition X. Later on A. Tortrat ([*6*]) stated the following theorem: For each $\mu \in R_p(2)$ there exists a constant $c \in \mathbf{R}$ such that $(+)$ holds. Unfortunately this is false in general. Thus the two following questions remained open:

(1) Which $\mu \in R_p(2)$ satisfy $(+)$ with some $c \in \mathbf{R}$?
(2) Is every $\mu \in R_p(2)$ invariant under some reflection?

The purpose of this paper is to answer both questions. We hope that these results clarify some geometric properties of p-stable symmetric measures, $0 < p < 2$, which are completely different from those of Gaussian measures (compare also [3]).

2. AUXILIARY RESULTS

In the sequel p always denotes a real number with $0 < p < 2$. If μ is a measure on $\mathbf{R}^n$ its *characteristic function* $\hat{\mu}:\mathbf{R}^n \mapsto \mathbf{C}$ (field of complex numbers) is defined by

$$\hat{\mu}(a) = \int_{\mathbf{R}^n} \exp(i\langle x, a\rangle)d\mu(x), \qquad a \in \mathbf{R}^n.$$

Then the following are equivalent ([2]):

(1) $\mu \in \mathrm{R}_p(n)$.

(2) There are $f_1, \ldots, f_n \in \mathrm{L}_p(\Omega, \mathrm{P})$ such that

$$\hat{\mu}(a) = \exp\left(-\int_\Omega \left|\sum_{i=1}^n \alpha_i f_i\right|^p d\mathrm{P}\right)$$

for all $a = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$.

(3) If $\|\,.\,\|$ is a norm on $\mathbf{R}^n$, $\partial\mathrm{U}$ the unit sphere defined by this norm then there is a measure λ on $\partial\mathrm{U}$ such that

$$\hat{\mu}(a) = \exp\left(-\int_{\partial\mathrm{U}} |\langle x, a\rangle|^p d\lambda(x)\right), \qquad a \in \mathbf{R}^n.$$

The measure λ on $\partial\mathrm{U}$ is called the *spectral measure* of μ. It is uniquely determined in the following sense: If $\tilde{\lambda}$ also generates $\hat{\mu}$ as in (3) then

$$\lambda(\mathrm{B}) + \lambda(-\mathrm{B}) = \tilde{\lambda}(\mathrm{B}) + \tilde{\lambda}(-\mathrm{B})$$

for all Borel subsets $\mathrm{B} \subseteq \partial\mathrm{U}$. Particularly,

$$\lambda(\mathrm{B}) = \tilde{\lambda}(\mathrm{B})$$

whenever $\mathrm{B} = -\mathrm{B} \subseteq \partial\mathrm{U}$ is measurable.

As a consequence of the uniqueness we get:

LEMMA 1. — *Let f_1, f_2 be in $\mathrm{L}_p(\Omega, \mathrm{P})$ and let g_1, g_2 be in $\mathrm{L}_p(\Omega', \mathrm{P}')$ such that*

$$\int_\Omega |\alpha_1 f_1 + \alpha_2 f_2|^p d\mathrm{P} = \int_{\Omega'} |\alpha_1 g_1 + \alpha_2 g_2|^p d\mathrm{P}'$$

for all $(\alpha_1, \alpha_2) \in \mathbf{R}^2$. *Then*

$$\int_{\{f_1=0\}} |f_2|^p dP = \int_{\{g_1=0\}} |g_2|^p dP' \tag{1}$$

and

$$\int_{\{f_1\neq 0\}} |f_2|^p dP = \int_{\{g_1\neq 0\}} |g_2|^p dP'. \tag{2}$$

Proof. — Because of $\int_\Omega |f_2|^p dP = \int_{\Omega'} |g_2|^p dP'$ (choose $\alpha_1 = 0, \alpha_2 = 1$) the second equality is an easy consequence of the first one. Define $f=(f_1,f_2)$ and $g=(g_1, g_2)$ as mappings from Ω and Ω' into $\mathbf{R}^2$, respectively. If $\|\,.\,\|$ is the Euclidean norm on $\mathbf{R}^2$ we put $B = \{0\} \times \{1, -1\} \subseteq \partial U$. Then

$$\int_{\{f/\|f\|\in B\}} \|f\|^p dP = \int_{\{g/\|g\|\in B\}} \|g\|^p dP' \qquad \text{proving (1).}$$

Let μ be in $R_p(n)$. Then we denote by $X_1, \ldots, X_n$ the random variables defined by

$$X_j(x) = \xi_j, \qquad x = (\xi_1, \ldots, \xi_n).$$

PROPOSITION 1 ([5]). — *Let* μ *be in* $R_p(n)$ *with*

$$\hat{\mu}(a) = \exp\left(-\int_\Omega \left|\sum_{i=1}^n \alpha_i f_i\right|^p dP\right).$$

Then, if $1 \leqslant k, l \leqslant n$, *the random variables* X_k *and* X_l *are (stochastically) independent if and only if*

$$P(f_k . f_l = 0) = 1 .$$

Proof (¹). — Clearly, $P(f_k . f_l = 0) = 1$ implies the independence of X_k and X_l.

To prove the converse it suffices to treat the case $n = 2$. This follows by projecting $\mathbf{R}^n$ onto $\mathbf{R}^2$. Thus we assume that μ belongs to $R_p(2)$ with

$$\hat{\mu}(a) = \exp\left(-\int_\Omega |\alpha_1 f_1 + \alpha_2 f_2|^p dP\right)$$

(¹) We enclose the proof of proposition 1 because the inequality used in [5] (p. 419) is false in the case $1 < p < 2$.

and X_1 and X_2 independent. Then

$$\hat{\mu}(a) = \exp\left(- \int_{\partial U} | \langle x, a \rangle |^p d\lambda(x) \right)$$

where

$$\lambda = \int_{\Omega} | f_1 |^p dP . \delta_{e_1} + \int_{\Omega} | f_2 |^p dP . \delta_{e_2}$$

with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. By lemma 1 we get

$$\int_{\{f_1 \neq 0\}} | f_2 |^p dP = \int_{\{\xi_1 \neq 0\}} | \xi_2 |^p d\lambda = 0$$

proving $P(f_1 \cdot f_2 \neq 0) = 0$.

COROLLARY 1 ([*6*]). — *Let* $(Y_1, \ldots, Y_n)$ *be a random vector whose distribution belongs to* $R_p(n)$. *Then it is independent if and only if it is pairwise independent.*

The next proposition is a slight modification of a theorem due to Rudin ([*4*]). Originally it was formulated for complex valued random variables.

PROPOSITION 2. — *Let f and g be two* **real** *valued random variables with* $f, g \in L_p(\Omega, P)$. *If*

$$\int_{\Omega} | 1 + \alpha f |^p dP = \int_{\Omega} | 1 + \alpha g |^p dP$$

for all real numbers α *then*

$$f(P) = g(P), \qquad \text{i. e.}$$
$$P(f \in B) = P(g \in B) \qquad \textit{for all Borel sets } B \subseteq \mathbf{R}.$$

Proof. — We want to reduce the real version of Rudin's theorem to its complex one. To do so choose a complex number $z = \alpha + i\beta$. If γ_1, γ_2 are independent standard Gaussian random variables it follows

$$\int_{\Omega} | 1 + (\alpha + \beta(\gamma_2(\omega')/\gamma_1(\omega')))f |^p dP = \int_{\Omega} | 1 + (\alpha + \beta(\gamma_2(\omega')/\gamma_1(\omega')))g |^p dP$$

for all $\omega' \in \Omega'$. Multiplying both sides with $| \gamma_1(\omega') |^p$ by integrating with respect to ω' we get

$$\int_{\Omega} (| 1 + \alpha f |^2 + | \beta f |^2)^{p/2} dP = \int_{\Omega} (| 1 + \alpha g |^2 + | \beta g |^2)^{p/2} dP$$

proving $$\int_\Omega |1 + zf|^p dP = \int_\Omega |1 + zg|^p dP$$

for all complex numbers z. Now, proposition 2 follows by Rudin's theorem.

Remark. — Rudin's theorem remains true for all $p \in (0, \infty)$ with $p \neq 2, 4, 6, \ldots$.

The formulation of our main result requires a representation theorem for the characteristic function of measures in $R_p(2)$.

PROPOSITION 3. — *μ belongs to $R_p(2)$ if and only if there are a finite measure σ on $\mathbf{R}$ with $\int_{\mathbf{R}} |t|^p d\sigma(t) < \infty$ and a real number $b \geqslant 0$ such that*

$$\hat{\mu}(a) = \exp\left(-\int_{-\infty}^{\infty} |\alpha_1 + \alpha_2 t|^p d\sigma(t) - b|\alpha_2|^p\right), \qquad a = (\alpha_1, \alpha_2) \in \mathbf{R}^2.$$

Moreover, σ and b are uniquely determined.

Proof. — Of course, μ belongs to $R_p(2)$ whenever its characteristic function can be represented in this way. Now, let μ be in $R_p(2)$ with characteristic function

$$\hat{\mu}(a) = \exp\left(-\int_\Omega |\alpha_1 f_1 + \alpha_2 f_2|^p dP\right)$$
$$= \exp\left(-\int_{\{f_1 \neq 0\}} |\alpha_1 + \alpha_2 f_2/f_1|^p |f_1|^p dP - |\alpha_2|^p \int_{\{f_1 = 0\}} |f_2|^p dP\right).$$

Defining σ and b by

$$\sigma(B) = \int_{\{f_2/f_1 \in B\}} |f_1|^p dP, \qquad B \subseteq \mathbf{R} \text{ measurable},$$

and

$$b = \int_{\{f_1 = 0\}} |f_2|^p dP$$

we get a representation of $\hat{\mu}$ as stated in the proposition. It remains to prove that σ and b are uniquely determined. Because of proposition 2 it suffices to show that b is uniquely determined. But this easily follows from lemma 1 above.

In view of proposition 3 we may write $\mu \sim (\sigma, b)$ whenever

$$\hat{\mu}(a) = \exp\left(-\int_{-\infty}^{+\infty} |\alpha_1 + \alpha_2 t|^p d\sigma(t) - b|\alpha_2|^p\right)$$

for all $a = (\alpha_1, \alpha_2) \in \mathbf{R}^2$.

3. p-STABLE MEASURES INVARIANT UNDER REFLECTIONS

Let μ be in $R_p(2)$ and let c be a real number. Then μ is said to be *conditional symmetric* with respect to c if

$$\mu(\xi_1 \in B_1, \xi_2 - c\xi_1 \in B_2) = \mu(\xi_1 \in B_1, c\xi_1 - \xi_2 \in B_2)$$

for all Borel subsets $B_1, B_2 \subseteq \mathbf{R}$.

We denote in the following the matrix

$$\begin{pmatrix} 1 & 0 \\ 2c & -1 \end{pmatrix}$$

by T_c. Then μ is conditional symmetric with respect to c if and only if $T_c(\mu) = \mu$. Without loss of generality we can and do assume

$$\operatorname{supp}(\mu) = \mathbf{R}^2$$

since otherwise μ is concentrated on an 1-dimensional subspace. Those measures are conditional symmetric. We start with a formula for the calculation of c provided it exists. Besides it proves that c is uniquely determined.

PROPOSITION 4. — *Let μ be in $R_p(2)$ with $T_c(\mu) = \mu$. Then*

$$c = \int_{\mathbf{R}^2} \xi_2 \xi_1^{q-1} d\mu(x) \Big/ \int_{\mathbf{R}^2} |\xi_1|^q d\mu(x), \qquad 1 < q < p,$$

where $\xi_1^{q-1} = |\xi_1|^{q-1} \operatorname{sign} \xi_1$.

If $0 < p \leqslant 1$ then $c \in \mathbf{R}$ is the uniquely determined real number with

$$\int_{\{\xi_2/\xi_1 > c\}} |\xi_1|^q d\mu = \int_{\{\xi_2/\xi_1 < c\}} |\xi_1|^q d\mu$$

for some (each) $q < p$.

Proof. — If $q < p$ by $T_c(\mu) = \mu$ we get

$$\int_{\mathbf{R}^2} |\langle x, a \rangle|^q d\mu = \int_{\mathbf{R}^2} |\langle x, a \rangle|^q dT_c(\mu)(x) \qquad \text{for all} \qquad a \in \mathbf{R}^2.$$

If $a = (1, \alpha)$ this implies

$$\int_{\mathbf{R}^2} |1 + \alpha(\xi_2/\xi_1)|^q |\xi_1|^q d\mu(x) = \int_{\mathbf{R}^2} |1 + \alpha(2c - \xi_2/\xi_1)|^q |\xi_1|^q d\mu(x)$$

for all $\alpha \in \mathbf{R}$. Consequently, by proposition 2

$$\int_{\{\xi_2/\xi_1 \in B\}} |\xi_1|^q d\mu(x) = \int_{\{2c - \xi_2/\xi_1 \in B\}} |\xi_1|^q d\mu(x)$$

for all Borel sets $B \subseteq \mathbf{R}$. If $q > 1$ then the integral

$$\int_{\mathbf{R}^2} (\xi_2/\xi_1) |\xi_1|^q d\mu(x)$$

exists and

$$2c \int_{\mathbf{R}^2} |\xi_1|^q d\mu(x) = 2 \int_{\mathbf{R}^2} (\xi_2/\xi_1) |\xi_1|^q d\mu(x)$$

which proves the first part of proposition 4.

Now we choose $B = (-\infty, c)$. Then the second equality is satisfied. Moreover, if $\alpha < \beta$ then

$$\int_{\{\alpha < \xi_2/\xi_1 < \beta\}} |\xi_1|^q d\mu(x) > 0$$

because of supp $(\mu) = \mathbf{R}^2$. Thus c is uniquely determined by the second equality.

Now we are able to prove the main result of this section.

PROPOSITION 5. — *Let* $\mu \sim (\sigma, b)$ *in* $R_p(2)$ *be given. Then* $T_c(\mu) = \mu$ *if and only if* $h_c(\sigma) = \sigma$ *where* $h_c(t) = 2c - t$.

Proof. — The equality $h_c(\sigma) = \sigma$ implies $\hat{\mu}(T_c^* a) = \hat{\mu}(a)$, $a \in \mathbf{R}^2$, i. e. $T_c(\mu) = \mu$.

On the other hand, if $T_c(\mu) = \mu$ then $h_c(\sigma) = \sigma$ because of $T_c(\mu) \sim (h_c(\sigma), b)$ by the uniqueness of the generating measure on $\mathbf{R}$.

COROLLARY 2. — *Let* μ *be in* $R_p(2)$. *Then* $T_c(\mu) = \mu$ *if and only if* $\mu = \nu * T_c(\nu)$ *for some* $\nu \in R_p(2)$.

Proof. — Given $\mu \in R_p(2)$ with $T_c(\mu) = \mu$ such that $\mu \sim (\sigma, b)$ we define ν by $\nu \sim (\rho, b/2)$ where

$$\rho(B) = \sigma(B \cap (c, \infty)) + (\sigma\{c\}/2)\delta_c(B).$$

Then $\mu = T_c(\nu) * \nu$. The converse follows immediately.

REMARK 1. — Using proposition 5 it is rather easy to construct measures μ in $R_p(2)$ with $T_c(\mu) \neq \mu$ for all $c \in \mathbf{R}$. Thus, theorem 2 of [*6*] is false.

REMARK 2. — If $T_c(\mu) = \mu$ with $\mu \sim (\sigma, b)$ then c can be calculated by

$$c = \int_{-\infty}^{\infty} t d\sigma(t)/\sigma(-\infty, \infty)$$

provided the integral exists (for instance if $p \geqslant 1$).

REMARK 3. — Corollary 2 is a special case of a more general result proved by the second named author.

The equality $T_c(\mu) = \mu$ means that μ is invariant under a very special reflection. But as we saw not every measure in $R_p(2)$ has this property. Thus it is very natural to ask whether or not each measure in $R_p(2)$ is invariant under an appropriate reflection. It turns out that this is not true in general. We give an example of an element in $R_p(2)$ which is only invariant under some trivial linear mappings, namely under the identity map and under the transformation mapping x onto $-x$.

To construct such an example we need the following proposition:

PROPOSITION 6. — *Let* $\mu \sim (\sigma, 0)$ *be in* $R_p(2)$ *and let*

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$$

be a matrix. Then $T(\mu) = \mu$ *if and only if*

$$\sigma(\tau_{11} + \tau_{12}t = 0) = 0 \tag{1}$$

and

$$\int_{\{t \in \mathbf{R}; (\tau_{21} + \tau_{22}t)/(\tau_{11} + \tau_{12}t) \in B\}} |\tau_{11} + \tau_{12}t|^p d\sigma(t) = \sigma(B) \tag{2}$$

for all Borel subsets $B \subseteq \mathbf{R}$.

Proof. — Because of

$$\widehat{T(\mu)}(a) = \exp\left(-\int_{-\infty}^{\infty} |\alpha_1(\tau_{11} + \tau_{12}t) + \alpha_2(\tau_{21} + \tau_{22}t)|^p d\sigma(t)\right)$$ for $a = (\alpha_1, \alpha_2)$ we get

$$\int_{\{\tau_{11} + \tau_{12}t = 0\}} |\tau_{21} + \tau_{22}t|^p d\sigma(t) = 0$$

provided $T(\mu) = \mu \sim (\sigma, 0)$ (lemma 1). Then either $\sigma(\tau_{11} + \tau_{12}t = 0) = 0$ or there is a $t \in \mathbf{R}$ with $\tau_{21} + \tau_{22}t = \tau_{11} + \tau_{12}t = 0$.

Since we assumed supp $T(\mu) =$ supp $(\mu) = \mathbf{R}^2$ the mapping T must be an automorphism. Consequently, the second case cannot happen proving (1). (2) is an easy consequence of proposition 2.

The converse follows immediately.

PROPOSITION 7. — *Let* $\mu \in R_p(2)$ *be defined by*

$$\hat{\mu}(a) = \exp(-|\alpha_1+\alpha_2|^p - |\alpha_1+\alpha_2/2|^p - |\alpha_1-\alpha_2|^p), \qquad a=(\alpha_1,\alpha_2).$$

Then $T(\mu) = \mu$ *implies either*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{or} \quad T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. — Assume $T(\mu) = \mu$. Then, if $t_1 = 1$, $t_2 = 1/2$ and $t_3 = -1$, for each k, $k = 1, 2, 3$, there exists a uniquely determined t_j, $j=1, 2, 3$, such that

$$(\tau_{21} + \tau_{22}t_k)/(\tau_{11} + \tau_{12}t_k) = t_j \qquad \text{and} \qquad |\tau_{11} + \tau_{12}t_k| = 1 .$$

By some easy calculations it follows that this is possible if and only if $\tau_{21} = \tau_{12} = 0$ and $\tau_{22} = \tau_{11} = \pm 1$.

This proves proposition 7.

4. REFLECTIONS IN $\mathbf{R}^n$

The purpose of this section is to extend some results of the third section to the n-dimensional case. As in [*6*] we only investigate measures in $R_p(n)$ for which the first $n-1$ coordinate functionals $X_1, \ldots, X_{n-1}$ are independent. Let $c = (c_1, \ldots, c_{n-1}) \in \mathbf{R}^{n-1}$ be given. Then we define an $n \times n$ matrix S_c by

$$S_c = \begin{pmatrix} 1 \,. & 0 & \\ 0 & \dot{} \,.\, 1 & \\ 2c_1 \ldots & 2c_{n-1} & -1 \end{pmatrix}.$$

We want to investigate measures μ in $R_p(n)$ having $n-1$ independent coordinate functionals such that

$$S_c(\mu) = \mu .$$

They satisfy

$$\mu\left\{ \xi_1 \in B_1, \ldots, \sum_{i=1}^{n-1} c_i\xi_i - \xi_n \in B_n \right\} = \mu\left\{ \xi_1 \in B_1, \ldots, \xi_n - \sum_{i=1}^{n-1} c_i\xi_i \in B_n \right\}$$

for arbitrary Borel sets $B_1, \ldots, B_n \subseteq \mathbf{R}$.

The following was stated in [*6*], Let μ be in $R_p(n)$ with $X_1, \dots, X_{n-1}$ independent. Then there is a vector $c=(c_1, \dots, c_{n-1})$ such that $S_c(\mu)=\mu$. But this is false in general. This follows for instance by proposition 9 below.

Let us start with a representation theorem for measures having $n-1$ independent coordinate functionals.

PROPOSITION 8. — *Let μ be in $R_p(n)$ with $X_1, \dots, X_{n-1}$ independent. Let ν_i in $R_p(2)$ be the distribution of (X_i, X_n), $1 \leqslant i \leqslant n-1$, and let ν_n be the distribution of X_n on $\mathbf{R}$. Then*

$$\hat{\mu}(a) = \hat{\nu}_1(\alpha_1, \alpha_n) \dots \hat{\nu}_{n-1}(\alpha_{n-1}, \alpha_n)(\hat{\nu}_n(\alpha_n))^{2-n}, \qquad a=(\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n.$$

Proof. — Assume

$$\hat{\mu}(a) = \exp\left(-\int_\Omega \left|\sum_{i=1}^n \alpha_i f_i\right|^p dP\right).$$

If $A_i = \{\omega \in \Omega; f_i(\omega) \neq 0\}$ by proposition 1

$$P(A_i \cap A_j) = 0, \qquad 1 \leqslant i, j \leqslant n-1, \qquad i \neq j.$$

Putting

$$A = \Omega \Big\backslash \bigcup_{i=1}^{n-1} A_i$$

we get

$$\begin{aligned}
\int_\Omega \left|\sum_{i=1}^n \alpha_i f_i\right|^p dP &= \sum_{i=1}^{n-1} \int_{A_i} |\alpha_i f_i + \alpha_n f_n|^p dP + |\alpha_n|^p \int_A |f_n|^p dP \\
&= \sum_{i=1}^{n-1} \int_\Omega |\alpha_i f_i + \alpha_n f_n|^p dP - \sum_{i=1}^{n-1} |\alpha_n|^p \int_{\Omega \backslash A_i} |f_n|^p dP \\
&\qquad + |\alpha_n|^p \int_A |f_n|^p dP \\
&= \sum_{i=1}^{n-1} \int_\Omega |\alpha_i f_i + \alpha_n f_n|^p dP - (n-2)|\alpha_n|^p \int_\Omega |f_n|^p dP.
\end{aligned}$$

This proves proposition 8.

PROPOSITION 9. — *Let μ and $\nu_1, \dots, \nu_{n-1}$ be defined as above. Then $S_c(\mu) = \mu$ if and only if*

$$T_{c_i}(\nu_i) = \nu_i, \qquad 1 \leqslant i \leqslant n-1,$$

where $c = (c_1, \ldots, c_{n-1})$ *and*

$$T_{c_i} = \begin{pmatrix} 1 & 0 \\ 2c_i & -1 \end{pmatrix}.$$

Proof. — Because of

$$S_c^*(a) = (\alpha_1 + 2c_1\alpha_n, \ldots, \alpha_{n-1} + 2c_{n-1}\alpha_n, -\alpha_n)$$

by proposition 8 we get

$$\hat{\mu}(S_c^* a) = \hat{\mu}(a)$$

provided that

$$\hat{v}_i(\alpha_i, \alpha_n) = \hat{v}_i(T_{c_i}^*(\alpha_i, \alpha_n)) = \hat{v}_i(\alpha_i + 2c_i\alpha_n, -\alpha_n), \qquad 1 \leqslant i \leqslant n-1.$$

To prove the converse we fix i with $1 \leqslant i \leqslant n-1$. If

$$a = (0, \ldots, \alpha_i, 0, \ldots, 0, \alpha_n) \qquad \text{from} \qquad S_c(\mu) = \mu$$

and proposition 8 it follows

$$\hat{v}_1(2c_1\alpha_n, -\alpha_n) \ldots \hat{v}_i(\alpha + 2c_i\alpha_n, -\alpha_n) \ldots \hat{v}_{n-1}(2c_{n-1}\alpha_n, -\alpha_n)$$
$$= \hat{v}_1(0, \alpha_n) \ldots \hat{v}_i(\alpha, \alpha_n) \ldots \hat{v}_{n-1}(0, \alpha_n).$$

Then the quotient

$$d(\alpha_n) = \frac{\hat{v}_i(\alpha + 2c_i\alpha_n, -\alpha)}{\hat{v}_i(\alpha, \alpha_n)}$$

is independent of α. Choosing $\alpha = -\beta - 2c_i\alpha_n$ we get

$$d(\alpha_n) = \frac{\hat{v}_i(-\beta, -\alpha_n)}{\hat{v}_i(-\beta - 2c_i\alpha_n, \alpha_n)} = \frac{\hat{v}_i(\beta, \alpha_n)}{\hat{v}_i(\beta + 2c_i\alpha_n, -\alpha_n)} = d(\alpha_n)^{-1}.$$

Since $d(\alpha_n) > 0$ we have $d(\alpha_n) = 1$ for each $\alpha_n \in \mathbf{R}$.
Consequently,

$$\hat{v}_i(T_{c_i}^*(\alpha, \alpha_n)) = \hat{v}_i(\alpha, \alpha_n)$$

and

$$v_i = T_{c_i}(v_i), \qquad 1 \leqslant i \leqslant n-1.$$

This proves proposition 9.

REFERENCES

[1] M. KANTER, Linear sample spaces and stable processes. *J. Functional Analysis*, t. **9**, 1972, p. 441-459.

[2] P. LEVY, *Théorie de l'addition des variables aléatoires*. 2nd ed. Gauthier-Villars, Paris 1937.

[3] W. LINDE, Operators generating stable measures on Banach spaces, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **60**, 1982, p. 171-184.

[4] W. RUDIN, L_p-isometries and equimeasurability. *Indiana Univ. Math. J.*, t. **25**, 1976, p. 215-228.

[5] M. SCHILDER, Some structure theorems for the symmetric stable laws, *Ann. Math. Stat.*, t.**41**, 1970, p. 412-421.

[6] A. TORTRAT, Pseudo-martingales et lois stables. *C. R. Acad. Sc. Paris Ser. A*, t. **281**, 1975, p. 463-465.

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