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**The class of Banach spaces,
which do not have c_0 as a spreading model,
is not L^2 -hereditary**

by

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1. INTRODUCTION

In [4] the problem was raised whether the fact, that a Banach space E does not have c_0 as a spreading model, implies that $L^2([0, 1]; E)$ has the same property. It was conjectured that the answer is no, as the property of not having c_0 as a spreading model is somewhat dual to the Banach-Saks property (see [2]) and for this latter property J. Bourgain has constructed a counterexample ([3]).

The present author has constructed independently of J. Bourgain another space E with the Banach-Saks property and $L^2(E)$ failing it ([6]) and it turns out that the dual E' gives a counterexample to the problem raised in the title.

2. THE EXAMPLE

Let $\gamma = \{n_1, n_2, \dots, n_k\}$ an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $v_i < 2^{u_i}$. As in [6] we associate to every n_i the real number $t(n_i) = v_i/2^{u_i} \in [0, 1[$ and call γ *admissible* if

$$(1) \quad k \leq n_1.$$

(2) For every $0 \leq j < 2^{u_1+1}$ there is only one i such that $t(n_i) \in [j/2^{u_1+1}, (j+1)/2^{u_1+1}[$.

For an admissible $\gamma = (n_1, \dots, n_k)$ and $x \in \mathbb{R}^{(\mathbb{N})}$, the space of finite sequences, we define

$$\|x\|_\gamma = \sum_{i=1}^k |x_{n_i}|.$$

For our purposes it will this time be convenient, not to use interpolation but to follow Baernstein's original definition ([I]): For $x \in \mathbb{R}^{(\mathbb{N})}$ define

$$\|x\|_E = \sup \left\{ \left(\sum_{l=1}^{\infty} \|x\|_{\gamma_l}^2 \right)^{1/2} \right\}$$

where the sup is taken over all increasing sequences $\{\gamma_l\}_{l=1}^{\infty}$ of admissible sets (i. e. the last member of γ_l is smaller than the first member of γ_{l+1}).

Let $(E, \|\cdot\|_E)$ be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. In an analogous way as in [6] we shall show that E has the Banach-Saks property, i. e. that it does not have a spreading model isomorphic to l^1 ; we shall also show that E' does not have a spreading model isomorphic to c_0 . In fact we shall prove a much stronger result.

PROPOSITION 1. — *a)* Every spreading model based on a normalized weak null sequence $(x_n)_{n=1}^{\infty}$ of E is isomorphic to l^2 .

b) Every spreading model based on a normalized weak null sequence $(y_n)_{n=1}^{\infty}$ of E' is isomorphic to l^2 .

Proof. — *a)* Let $(x_n)_{n=1}^{\infty}$ be a normalized weak null sequence in E . As $(x_n)_{n=1}^{\infty}$ converges to zero coordinatewise, we may assume (by a standard perturbation argument) that the x_n 's are supported by disjoint blocks, i. e. there is an increasing sequence $(r(n))_{n=1}^{\infty}$ of natural numbers, such that with $r(0) = 0$

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i.$$

Then for every sequence $\alpha_1, \dots, \alpha_k$ of scalars and $n_1 < \dots < n_k$, the estimate

$$\left\| \sum_{i=1}^k \alpha_i x_{n_i} \right\|_E \geq \left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2} \quad (1)$$

holds trivially in view of the definition of $\|\cdot\|_E$.

For the converse let $1 > \varepsilon > 0$ and choose inductively an increasing sequence $(n_k)_{k=1}^\infty$ in \mathbb{N} and infinite subsets M_k of \mathbb{N} : Let $M_0 = \mathbb{N}$ and $n_1 = 1$ and suppose M_{k-1} and n_k are defined. Let p_k be such that $2^{p_k-1} \leq r(n_k) < 2^{p_k}$ and consider the partition of $[0, 1[$ into the intervals $[j/2^{p_k}, (j+1)/2^{p_k}[$, $j = 0, \dots, 2^{p_k-1}$. For $n \geq n_k$ define

$$\mu_j^{(n)} = \max \{ |\lambda_i^{(n)}| : t(i) \in [j/2^{p_k}, (j+1)/2^{p_k}[\}.$$

Note that, for every n ,

$$\sum_{j=0}^{2^{p_k-1}} \mu_j^{(n)} \leq 1$$

in view of the definition of the norm of E and the fact that $\|x_n\|_E = 1$. Find an infinite subset \overline{M}_k of $M_{k-1} \cap [n_k + 1, n_k + 2, \dots, \infty[$ such that for every $j = 0, \dots, 2^{p_k} - 1$ the sequence $(\mu_j^{(n)})_{n \in \overline{M}_k}$ converges, to μ_j say. Clearly

$$\sum_{j=0}^{2^{p_k-1}} \mu_j \leq 1.$$

Finally let M_k be the subset of \overline{M}_k consisting of those n for which for every $j = 0, \dots, 2^{p_k} - 1$

$$\mu_j^{(n)} \leq \mu_j + 2^{-p_k} \cdot \varepsilon/3$$

and let n_{k+1} be the first element of M_k . This completes the induction.

Note that for an admissible $\gamma = (m_1, \dots, m_q)$ and $k \in \mathbb{N}$ such that $\text{inf}(\gamma) = m_1 \leq r(n_k)$ and for every choice of scalars $\alpha_{k+1}, \dots, \alpha_{k+l}$

$$\left\| \sum_{i=1}^l \alpha_{k+i} x_{n_{k+i}} \right\|_\gamma \leq (1 + \varepsilon/3) \sup_{1 \leq i \leq l} \{ |\alpha_{k+i}| \} \leq \sqrt{1 + \varepsilon} \left(\sum_{i=1}^l |\alpha_{k+i}|^2 \right)^{1/2} \quad (2)$$

Indeed, as $m_1 \leq r(n_k) < 2^{p_k}$, we see that γ may contain for every $j = 0, \dots, 2^{p_k-1}$ at most one index m_r ($1 \leq r \leq q$) with $t(m_r) \in [j/2^{p_k}, (j+1)/2^{p_k}[$; by construction the m_r 'th entry of each $x_{n_{k+i}}$ ($1 \leq i \leq l$) is bounded in absolute value by $\mu_j + 2^{-p_k} \cdot \varepsilon/3$. As the x_{n_i} are disjointly supported we get

$$\left| \sum_{i=1}^l \alpha_{k+i} x_{n_{k+i}}(m_r) \right| \leq \sup_{1 \leq i \leq l} \{ |\alpha_{k+i}| \} (\mu_j + 2^{-p_k} \cdot \varepsilon/3)$$

Summing over j and recalling the definition of $\|\cdot\|_\gamma$ we get the first inequality of (2), while the second is trivial.

We now shall pass to the general case. Fix a sequence $\alpha_1, \dots, \alpha_k$ of scalars. We shall show

$$\left\| \sum_{i=1}^k \alpha_i x_{n_i} \right\|_{\mathbb{E}} \leq \sqrt{6 + 3\varepsilon} \cdot \left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2} \quad (3)$$

which (in view of (1) and the arbitrariness of $\varepsilon > 0$) will readily prove (a). So fix an increasing sequence $\gamma_1 < \gamma_2 < \dots < \gamma_l$ of admissible sets. For brevity we write

$$x = \sum_{i=1}^k \alpha_i x_{n_i}.$$

For $i = 1, \dots, k$ let $J(i)$ be the set of $j \in \{1, \dots, l\}$ such that the last element of γ_j lies in $]r(n_{i-1}), r(n_i)]$. If $J(i)$ is not empty denote $j(i)$ the first element of $J(i)$ and let $s(j(i))$ be the element $s \in \{1, \dots, k\}$, such that the first element of $\gamma_{j(i)}$ lies $]r(n_{s-1}), r(n_s)]$. Note that for the $j \in J(i), j > j(i)$ the first and the last element of γ_j lie in $]r(n_{i-1}), r(n_i)]$, while for $j(i)$ in general only the last element lies in $]r(n_{i-1}), r(n_i)]$. So we may estimate

$$\begin{aligned} \left(\sum_{j=1}^l \|x\|_{\gamma_j}^2 \right)^{1/2} &= \left(\sum_{i=1}^k \sum_{\substack{j \in J(i) \\ J(i) \neq \emptyset}} \|x\|_{\gamma_j}^2 \right)^{1/2} \\ &= \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^k \left(\|x\|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \|x\|_{\gamma_j}^2 \right) \right)^{1/2} \\ &= \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^k \left(\left\| \sum_{s=s(j(i))}^i \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \|\alpha_i x_{n_i}\|_{\gamma_{j(i)}}^2 \right) \right)^{1/2} \\ &\leq \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^k \left(\left(\|\alpha_{s(j(i))} x_{n_{s(j(i))}}\|_{\gamma_{j(i)}} + \left\| \sum_{s=s(j(i))+1}^{i-1} \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}} + \|\alpha_i x_{n_i}\|_{\gamma_{j(i)}} \right)^2 \right. \right. \\ &\quad \left. \left. + \sum_{\substack{j \in J(i) \\ j > j(i)}} \|\alpha_i x_{n_i}\|_{\gamma_{j(i)}}^2 \right) \right)^{1/2}. \end{aligned}$$

Using (2) and the fact that $(a^{1/2} + b^{1/2} + c^{1/2})^2 \leq 3(|a| + |b| + |c|)$ we get

$$\begin{aligned} &\leq \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^k \left(\left(\left(\|\alpha_{s(j(i))} x_{n_{s(j(i))}}\|_{\gamma_{j(i)}}^2 \right)^{\frac{1}{2}} + \sqrt{1 + \varepsilon} \left(\sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 \right)^{\frac{1}{2}} \right. \right. \right. \\ &\qquad \qquad \qquad \left. \left. \left. + \left(\|\alpha_i x_{n_i}\|_{\gamma_{j(i)}}^2 \right)^{\frac{1}{2}} \right)^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \|\alpha_i x_{n_i}\|_{\gamma_{j(i)}}^2 \right) \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^k 3 \left(|\alpha_{s(j(i))}|^2 + (1 + \varepsilon) \sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 + \sum_{j \in J(i)} \|\alpha_i x_{n_i}\|_{\gamma_{j(i)}}^2 \right) \right)^{\frac{1}{2}} \\ &\leq \left(3(1 + 1 + \varepsilon) \sum_{i=1}^k |\alpha_i|^2 \right)^{\frac{1}{2}} \\ &= \sqrt{6 + 3\varepsilon} \cdot \left(\sum_{i=1}^k |\alpha_i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence we have proved (3) and thus part (a) of proposition 1.

Proof of (b). — It is easily seen using (a) that the unit vector basis $(e_i)_{i=1}^\infty$ of E is shrinking and boundedly complete (see [1] or [6]), hence the dual unit vectors $(e'_i)_{i=1}^\infty$ form a basis of E'. So let $(y_n)_{n=1}^\infty$ be a normalized sequence tending weakly (and therefore coordinatewise) to zero. Similarly as in (a) we may suppose that there is an increasing sequence $(r(n))_{n=1}^\infty$ such that

$$y_n = \sum_{i=r(n-1)+1}^{r(n)} \rho_i^{(n)} e'_i.$$

Now choose a sequence $(x_n)_{n=1}^\infty$ in E, $\|x_n\| = \langle x_n, y_n \rangle = 1$, which clearly implies that x_n is of the form

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i.$$

As in the prove of (a) find a subsequence $(n_k)_{k=1}^\infty$ such that $(x_{n_k})_{k=1}^\infty$ spans a space $\sqrt{6 + \varepsilon}$ isomorphic to l^2 .

Now fix a sequence β_1, \dots, β_k of scalars and find a sequence $\alpha_1, \dots, \alpha_k$ such that

$$\sum_{i=1}^k |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_{i=1}^k \alpha_i \beta_i = \left(\sum_{i=1}^k |\beta_i|^2 \right)^{\frac{1}{2}}.$$

Denote

$$x = \sum_{i=1}^k \alpha_i x_{n_i}$$

$$y = \sum_{i=1}^k \beta_i y_{n_i}.$$

By (a) we know that

$$\|x\|_E \leq \sqrt{6 + \varepsilon}.$$

Hence

$$\begin{aligned} \|y\|_{E'} &= \sup \{ |\langle \zeta, y \rangle| : \zeta \in E, \|\zeta\| \leq 1 \} \\ &\geq (6 + \varepsilon)^{-1/2} |\langle x, y \rangle| \\ &= (6 + \varepsilon)^{-1/2} \cdot \sum_{i=1}^k \alpha_i \beta_i \langle x_{n_i}, y_{n_i} \rangle \\ &= (6 + \varepsilon)^{-1/2} \left(\sum_{i=1}^k |\beta_i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand the reverse inequality

$$\|y\|_{E'} \leq \left(\sum_{i=1}^k |\beta_i|^2 \right)^{\frac{1}{2}}$$

is again easily checked directly from the definition of $\|\cdot\|_{E'}$. This proves (b) and therefore proposition 1. \square

Remark. — Consider the sequence of unit-vectors $(e_{2^n-1})_{n=1}^\infty$ in E (resp. $(e'_{2^n-1})_{n=1}^\infty$ in E').

It may be checked that every spreading model based on $(e_{2^n-1})_{n=1}^\infty$ (resp. $(e'_{2^n-1})_{n=1}^\infty$) is isometric to a countable l^2 -sum of 2-dimensional l^1 's, hence in this case the Banach-Mazur distance equals precisely $\sqrt{2}$.

To show that $L^2(E')$ does have c_0 as spreading model we need a trivial probabilistic lemma, whose proof is left to the reader.

LEMMA. — Let $k \in \mathbb{N}$ and $\varepsilon > 0$; there is $N(k, \varepsilon)$ such that for $M > N(k, \varepsilon)$

and for independent randomvariables X_1, \dots, X_k taking their values in $\{1, \dots, M\}$ in a uniformly distributed way, we have

$$P \left\{ \begin{array}{l} \omega : \text{there is } 1 \leq i < j \leq k \text{ with} \\ X_i(\omega) = X_j(\omega) \end{array} \right\} < \varepsilon$$

PROPOSITION 2. — $L^2_{(0,1)}(E')$ has c_0 isometrically as spreading model.

Proof. — Similarly as in [6] we let $\{\vec{f}_u\}_{u=1}^\infty$ be an independent sequence in $L^2(E')$ such that \vec{f}_u takes the value e_{2^u+v} with probability 2^{-u} (for $v = 0, \dots, 2^u - 1$). This times the e_{2^u+v} are unit-vectors in E' .

Clearly $\|\vec{f}_u\|_{L^2(E')} = 1$ and for every sequence $u_1 < u_2 < \dots < u_k$ and $\varepsilon_i = \pm 1$

$$\left\| \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')} \geq 1$$

Hence the following claim will prove the proposition.

CLAIM. — For every $k \in \mathbb{N}$

$$\limsup_{u \rightarrow \infty} \left\{ \left\| \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i} \right\|_{\varepsilon_i = \pm 1} : u \leq u_1 < \dots < u_k \right\} = 1$$

To prove the claim fix k and $\varepsilon > 0$ and let u be such that $2^u > \max(k, N(k, \varepsilon))$, where the $N(k, \varepsilon)$ is defined in the preceding lemma. Now fix $u \leq u_1 < u_2 < \dots < u_k$ and a sequence of signs $\varepsilon_1, \dots, \varepsilon_k$.

To apply the above lemma let X_1, \dots, X_k be the random variables with values in $\{1, \dots, 2^{u_1+1}\}$ defined by

$$X_i(\omega) = m \quad \text{if} \quad \vec{f}_{u_i}(\omega) = e_{2^{u_i}+v}$$

and

$$t(2^{u_i} + v) = v/2^{u_i} \in [(m - 1)/2^{u_1+1}; \quad m/2^{u_1+1}[$$

It follows from the above lemma and the definition of admissible sets γ that there is a subset $A \subseteq [0, 1[$ of measure greater than $1 - \varepsilon$ such that for $\omega \in A$ the set $\gamma_\omega = \{n_1, \dots, n_k\}$ corresponding to the indices of the unit vectors $\{\vec{f}_{u_1}(\omega), \dots, \vec{f}_{u_k}(\omega)\}$ is admissible. Hence for $\omega \in A$ we have

$$\begin{aligned} \left\| \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i}(\omega) \right\|_{E'} &= \sup \left\{ \left\langle \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \|x\|_E \leq 1 \right\} \\ &\leq \sup \left\{ \left\langle \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \|x\|_{\gamma_\omega} \leq 1 \right\} \\ &= 1. \end{aligned}$$

Integrating we obtain

$$\begin{aligned} \left\| \sum_{i=1}^k \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')}^2 &\leq \int_A \left\| \sum_{i=1}^k \vec{f}_{u_i}(\omega) \right\|_{E'}^2 d\omega + \int_{[0,1] \setminus A} \left(\sum_{i=1}^k \|\vec{f}_{u_i}\|_{E'} \right)^2 d\omega \\ &\leq 1 + k^2 \varepsilon. \end{aligned}$$

This proves the claim and therefore proposition 2. \square

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