

ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 17, n° 2 (1981), p. 219-227

http://www.numdam.org/item?id=AIHPB_1981__17_2_219_0

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Universal distribution for infinitely divisible distributions on Fréchet space

by

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SUMMARY. — In this paper we generalize to the case of separable Fréchet space the theorem of Doeblin [3], which asserts that there exists a distribution belonging to the domain of partial attraction of every one dimensional infinitely divisible distribution. That the distribution is called a *universal distribution in Doeblin sense*. Furthermore we show that each distribution concentrated on bounded subset belongs to the domain of attraction of $\delta(0)$ and there exist uncountably many universal distribution, which are not shift equivalent. The main theorem implies also that one distribution is infinitely divisible if f , its domain of partial attraction is nonempty.

INTRODUCTION

Let E be a separable Fréchet space. We denote by \mathcal{B} the σ -algebra generated by the topology in E and by \mathcal{P} the set of all distribution on E . We signe by \Rightarrow the weak convergence. It is known that \mathcal{P} with the weak topology is a separable metric space (see [5], Th. 6.2). For $q, q_n \in \mathcal{P}$ we say that $\{q_n\}$ is *shift convergent* to q if there is a sequence $\{a_n\} \in E$ such that $\delta(a_n) * q_n \Rightarrow q$, and say that q_1 and q_2 are *shift equivalent* if $q_1 = \delta(a) * q_2$ for some $a \in E$,

when $\delta(b)$ is the distribution concentrated at the point $b \in E$ and $*$ denotes the convolution determined for measures on \mathcal{B} by

$$\mu * \nu(B) = \int \mu(B - x)\nu(dx), \quad B \in \mathcal{B}.$$

Distribution p is called *infinitely divisible (inf. div.)* if for every natural n there exists one distribution p_n such that

$$p = p_n^{*n} = p_n * \underset{n\text{-times}}{\dots} * p_n$$

For finite measure μ on \mathcal{B} let

$$e(\mu) = e^{-\mu(E)} \left(\delta(0) + \frac{\mu}{1!} + \frac{\mu^{*2}}{2!} + \dots \right)$$

This is an *inf. div.* distribution called Poisson distribution with finite canonical measure μ . Moreover (see [2], Th. 1.9) for each *inf. div.* distribution p there exist a symmetric Gaussian distribution ω , a sequence of finite measures $\{G_n\}$ and an $a \in E$ such that :

$$(1) \quad \delta(a) * \omega * e(G_n) \Rightarrow p.$$

MAIN RESULTS

For $c > 0$ we define an operator T_c on the set of all measures on \mathcal{B} by

$$T_c \lambda(B) = \lambda(c^{-1}B), \quad B \in \mathcal{B}$$

Let $p, q \in \mathcal{P}$. We say that q belongs to the domain of partial attraction of p if there exist a subsequence $\{n_k\}$ of natural numbers and a sequence of positive numbers $\{v_k\}$ such that $\{T_{v_k} q^{*v_k}\}$ is shift-convergent to p . When $\{n_k\}$ coincides with same sequence of all natural numbers we say that q belongs to the domain of attraction of p .

Distribution q is called *universal* if it belongs to the domain of partial attraction of every *inf. div.* distribution.

LEMMA 1. — On separable Fréchet space the set of all Poisson distribution is dense in the set of all *inf. div.* distribution.

Proof. — Let ω be a symmetric Gaussian distribution. From theorems 3 [4] and 4 [4] there exists sequence of Gaussian distribution $\{p_n\}$ on finite dimensional subspaces weakly convergent to ω . Then for each n there

is a sequence of Poisson distribution $\{p_n^m, m = 1, 2, \dots\}$ weakly convergent to p_n . Since E is a metric space, we can find for each n one natural $m(n)$ such that sequence $\{p_n^{m(n)}\}$ weakly converges to ω . Then by virtue of (1) the proof is completed.

LEMMA 2. — Let $A = \{a_1, a_2, \dots\}$ be a dense subset of E . Then the set of all Poisson distribution with finite canonical measures concentrated on finite subsets of A is dense in the set of all inf. div. distribution.

Proof. — Let's fix one metric f in E . We can take the sets $V(k, n)$, $k, n = 1, 2, \dots$ such that for every n .

- 1) $f(x, a_k) < \frac{1}{n}$ for $x \in V(k, n)$
- 2) $V(k, n) \cap V(k', n) = \emptyset, k \neq k'$
- 3) $E = \bigcup_{k=n}^{\infty} V(k, n)$.

Let G be an arbitrary finite measure. For every n there is $k(n)$ such that

$$(2) \quad G\left(\bigcup_{k \geq k(n)} V(k, n)\right) < 2^{-n}$$

We define the measure μ_n supported on $\{a_1, a_2, \dots, a_{k(n)}\}$ following

$$\begin{aligned} \mu_n(\{a_k\}) &= G(V(k, n)), \quad k < k(n) \\ \mu_n(\{a_{k(n)}\}) &= G\left(\bigcup_{k \geq k(n)} V(k, n)\right) \end{aligned}$$

Then $\frac{\mu_n}{G(E)}, \frac{G}{G(E)}$ are probability measures.

Using (2) and Th. II. 6. 1 [5] we may easily verify that $\frac{\mu_n}{G(E)} \Rightarrow \frac{G}{G(E)}$, which implies

$$\mu_n^{*l} \Rightarrow G^{*l}, \quad l = 1, 2, \dots$$

Then, because for $\varepsilon > 0$ arbitrary small there exists $k(\varepsilon)$ such that

$$\sum_{k > k(\varepsilon)} \frac{\mu_n^{*k}(E)}{k!} = \sum_{k > k(\varepsilon)} \frac{G^{*k}(E)}{k!} < \varepsilon, \quad n = 1, 2, \dots,$$

we see by some estimations that $e(\mu_n) \Rightarrow e(G)$. Hence by virtue of (1) and lemma 1 this lemma is proved.

Using local convexity of E and the fact that for real α and $a \in E, \alpha'a \rightarrow 0$ when $\alpha \rightarrow 0$ we can easily show the following.

LEMMA 3. — Let $A = \{ a_1, a_2, \dots \}$ be a dense subset of E and let

$$E_k = \text{lin} \{ a_1, \dots, a_k \}, \quad k = 1, 2, \dots$$

$$E_\infty = \bigcup_{k=1}^{\infty} E_k$$

Then there exist a sequence $\{ V_n \}$ of convex neighborhoods of 0 and a base $\{ b_1, b_2, \dots \}$ of E_∞ such that

- a) $V_n = V_m, V_{n+1} \subseteq V_n, n = 1, 2, \dots,$
- b) $\bigcap_{n=1}^{\infty} V_n = \{ 0 \},$
- c) $\forall(k) \exists(m(k)) : \{ b_1, \dots, b_{m(k)} \}$ is a base of $E_k,$ and
- d) $\forall(n) \forall(j \leq n) : 2^{-n+1} b_j \in V_n.$

THEOREM 1. — *On separable Fréchet space there exists universal distribution.*

Proof. — We define $A, E_k, E_\infty, \{ V_n \}$ and $\{ b_1, b_2, \dots \}$ as in the lemma 3.

Let $\{ p_1, p_2, \dots \}$ be a sequence of inf. div. distributions dense in the set of all inf. div. distributions. By virtue of Lemma 2 we may assume that

$$p_n = \delta(d_n) * e(F_n), \quad n = 1, 2, \dots$$

with $d_n \in E, F_n$ is finite measure supported on C_{2^n} and

$$F_n(E) \leq 2^n$$

there C_r^n is defined for natural $n,$ real $r > 0$ by

$$C_r^n = \{ \beta_1 b_1 + \dots + \beta_{m(n)} b_{m(n)} \mid |\beta_1| + \dots + |\beta_{m(n)}| < r \}.$$

We define

$$G = \sum_{n=1}^{\infty} 2^{-n^2} T_{2^{n^3}} F_n$$

Then

$$G(E) \leq \sum_{n=1}^{\infty} 2^{-n^2+n} < \infty$$

Hence distribution $q = e(G)$ is well determined. We shall prove that it is universal distribution.

Let p be any inf. div. distribution. Then there exists subsequence $\{n_k\}$ of natural numbers such that

$$(3) \quad p_{n_k} \Rightarrow p$$

We show that

$$(4) \quad q_k = \delta(d_{n_k}) * T_{2^{-n_k}}(q^{*2_k}) \Rightarrow p$$

which asserts universalness of p .

For finite measure μ , natural n and real $c > 0$. We have:

$$e(\mu)^{*n} = e(n\mu) \\ T_c(\mu) = e(T_c\mu)$$

Hence

$$(5) \quad q_k = p_{n_k} * e(N_k^1) * e(N_k^2)$$

with

$$N_k^1 = \sum_{n < n_k} 2^{n_k^2 - n^2} T_{2^{n^3 - n_k^3}} F_n \\ N_k^2 = \sum_{n > n_k} 2^{n_k^2 - n^2} T_{2^{n^3 - n_k^3}} F_n$$

Because

$$N_k^2(E) \leq 2^{-n_k} \sum_{n=1}^{\infty} 2^{-n^2 + n} \xrightarrow{k} 0$$

then

$$(6) \quad e(N_k^2) \Rightarrow \delta(0)$$

We prove following that $e(N_k^1) \Rightarrow \delta(0)$ which together with (3), (5), (6) implies (4).

Let's fix one natural $m_0 > G(E)$ and let for each natural l the number $H(l) = 2^{2l+1} m_0^2$. Then

$$\frac{(2^l G(E))^{H(l)}}{H(l)!} \leq \frac{(2^l m_0)^{2^{2l+1} m_0^2}}{(2^{2l+1} m_0^2)!} = \frac{1}{(2^l m_0^2)!} - \frac{2^{2l} m_0^2}{2^{2l} m_0^2 + 1} \cdots \frac{2^{2l} m_0^2}{2 \cdot 2^{2l} m_0^2} < \frac{1}{2^l}$$

Hence

$$\sum_{n > H(l)} \frac{(2^l G(E))^n}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{2^l} \frac{1}{2^n} = 2^{-l},$$

and because for each k ,

$$N_k^1(E) \leq 2^{n_k} T_{2^{-n_k}} G(E) = 2^{n_k} G(E)$$

then

$$(7) \quad \sum_{n > H(n_k)} \frac{(N_k^1)^{*n}(E)}{n!} \leq \sum_{n > H(n_k)} \frac{(2^{n_k} G(E))^n}{n!} < 2^{-n_k}$$

On other hand, if F is any measure supported on C_r^n then for $c > 0$, $T_c F$ is supported on C_{cr}^n , F^{*m} is supported on C_{mr}^n hence for $m \leq H(n_k)$; $(N_k^1)^{*m}$ is concentrated on $C(k)$, when

$$C(k) = C_{m_0^2}^{n_k-1} 2^{-3n_k^2 + 6n_k - 1}$$

Then (7) implies

$$e(N_k^1)(C(k)) + 2^{-n_k} \geq e^{-N_k^1(E)} \left(\sum_{0 \leq m \leq H(n_k)} \frac{(N_k^1)^{*m}(E)}{m!} + \sum_{m > H(n_k)} \frac{(N_k^1)^{*m}(E)}{m!} \right) = e(N_k^1)(E) = 1$$

Hence

$$(8) \quad e(N_k^1)(C(k)) \geq 1 - 2^{-n_k}$$

Since $m_0^2 2^{-3n_k^2 + 6n_k - 1} \rightarrow 0$, there is $k(0)$ such that for $k > k(0)$

$$\beta_k = 2^{-m(n_k-1)+1} \geq m_0^2 2^{-3n_k^2 + 6n_k - 1}$$

Then from (a)-(d), since every element of $C(k)$ is linear combination of

$$\beta_k b_1, \dots, \beta_k b_{m(n_k-1)} \in V_{m(n_k-1)}$$

with coefficients which absolute values have sum less than 1, we have

$$(9) \quad C(k) \subseteq V_{m(n_k-1)} \searrow \{0\}$$

when $k \rightarrow \infty$.

For arbitrary $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$(10) \quad 2^{n_k} < \varepsilon, \quad k \geq k(\varepsilon)$$

We denote $C = \bigcup_{k \geq k(\varepsilon)} C(k)$. Let $\{x_n\} \subseteq C$, then

a) If $\{x_n\} \bigcup_{n(0) \geq k \geq k(\varepsilon)} C(k)$ for any natural $n(0)$ then because $C(k)$ are bounded

closed subsets in finite-dimensional spaces, they are compact, $\{x_n\}$ contains any convergent subsequence.

b) If, in inverse, there exist subsequence $\{k_1\}$ of natural numbers $k' > k(\varepsilon)$ and subsequence $\{x_{k'}\} \subseteq \{x_n\}$ such that $x_{k'} \in C(k')$ then (9) implies $x_{k'} \rightarrow 0$.

Hence C is relatively compact.

From Theorem II.3.2 [5] there exists a compact set K_1 such that

$$e(N_k^1)(K_1) \geq 1 - \varepsilon \quad \text{for} \quad k < k(\varepsilon)$$

Let $K = K_1 \cup \bar{C}$. Then from (8) and (10) for each k

$$e(N_k^1)(K) \geq 1 - \varepsilon$$

Hence by Prokhorov Theorem (see, for example, Th. II.6.7 [5]), the sequence $\{e(N_k^1)\}$ is relatively compact.

Simultaneously, if U is any open set containing 0 then from (9) there is natural $k(U)$ such that for $k > k(U)$

$$C_{(k)} \subseteq V_{m(n_k-1)} \subseteq U$$

Then from (8)

$$e(N_k^1)(U) \geq 1 - 2^{-n_k}$$

Hence $e(N_k^1)(U) \rightarrow 1$ and if π is any limit-distribution of some subsequence of $\{e(N_k^1)\}$ then π must be $\delta(0)$. This implies from Th. 2.3 [1] that

$$e(N_k^1) \Rightarrow \delta(0)$$

Using above Theorem we may easily prove following theorem:

THEOREM 2. — *One distribution on separable Fréchet space in inf. div. if it's domain of partial attraction is non empty.*

THEOREM 3. — *If ξ is a distribution concentrated on bounded subset of E then ξ belongs to the domain of attraction of $\delta(0)$ and hence of $\delta(a)$ for every $a \in E$.*

Proof. — We fix any sequence $\{V_n\}$ of convex neighborhood of 0 satisfying (a) and (b). Suppose B is a bounded subset on which ξ is supported. Then for each n there is, $k(n)$ such that $k^{-1/2}B \subseteq V_n$ for $k \geq k(n)$.

For every k , ξ^{*k} is supported on

$$B_k = \underbrace{B + \dots + B}_{k\text{-times}}$$

We denote

$$(11) \quad \xi_k = T_{k^{-3/2}} \xi^{*k}$$

Then ξ_k is supported on

$$k^{-3/2}B_k = \frac{1}{k}(k^{-1/2}B) + \underbrace{\dots}_{k\text{-times}} + \frac{1}{k}(k^{-1/2}B) \subseteq V_k$$

for $k > k(n)$.

Every ξ_k is tight, then for arbitrary $\varepsilon > 0$ there exists a compact set K_k such that

$$\xi_k(K_k) \geq 1 - \varepsilon$$

With next proof part we may suppose that $K_k \subseteq V_n$ if $k \geq k(n)$.

Similarly as in the proof of Th. 1 we see that $K = \bigcup_{k=1}^{\infty} K_k \cup \{0\}$ is compact and for every k

$$\xi_k(K) \geq 1 - \varepsilon$$

Hence from Prokhorop Theorem, sequence $\{\xi_k\}$ is relatively compact.

As in proof of Th. 1, because

$$\xi_k(V_n) = 1 \quad \text{for} \quad k \geq k(n)$$

then

$$(12) \quad \xi_k \Rightarrow \delta(0)$$

From here for every $a \in E$

$$\delta(a) * \xi_k \Rightarrow \delta(a)$$

This proof is completed.

We see that if q is universal distribution then $\delta(a) * q$ is also universal.

COROLLARY 1. — *Let q be universal distribution defined in proof of theorem 1, and let ξ be a distribution concentrated on bounded subset of E . Then $\xi * q$ is also universal. Hence there exist uncountably many universal distributions which are not shift-equivalent.*

Proof. — Let p be any inf. div. distribution. As same as on proof of Th. 1 there are subsequence $\{n_k\}$ of natural numbers and sequence $\{d_k\} < E$ such that

$$q_k = \delta(d_k) * T_{2^{-n_k}}(q^{*2^{n_k}}) \Rightarrow p.$$

Then from (11) and (12) we have

$$\delta(d_k) * T_{2^{-n_k}}(\xi * q)^{*2^{n_k}} = \delta(d_k) * T_{2^{-n_k}}(q^{*2^{n_k}}) * T_{2^{-n_k}} \frac{3}{2} \xi^{*n_k} = q_k * \xi_{n_k} \Rightarrow p$$

Hence from Th. 2 we have:

COROLLARY 2. — *Domain of partial attraction of one distribution on*

separable Fréchet space either is empty, if the distribution isn't inf. div., either has uncountably many elements which are not shift-equivalent, if the distribution is inf. div.

Before here the Theorem 1 has proved for the case of Banach space in my paper « Universal Distribution for inf. div. distributions on Banach space ».

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(Manuscrit reçu le 25 février 1981)