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A new version of Doeblin's theorem

by

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ABSTRACT. — The aim of this note is to prove the existence of an operator-universal probability measure for infinitely divisible (i. d.) probability measures (p. m.) on a Banach space.

A famous theorem of W. Doeblin [2] asserts that there exists a p. m. belonging to the domain of partial attraction of every one-dimensional i. d. p. m. . The multi-dimensional version of this theorem is made by J. Barańska [1] in a Hilbert space and recently, by Ho Dang Phuc [5] in a Banach space.

Some techniques from Ho Dang Phuc [5] can be applied to prove a more general theorem. Namely, we replace the norming sequence in Doeblin's theorem by linear operators.

Throughout the paper we shall denote by $(X, \| \cdot \|)$ a real separable Banach space and consider only σ -additive non-negative Borel measures on X . For a bounded linear operator A and a measure μ we shall denote by $A\mu$ a measure defined by $A\mu(\varepsilon) = \mu(A^{-1}\varepsilon)$ ($\varepsilon \subset X$). In particular, if $Ax = cx$ for some $c \geq 0$ and for all $x \in X$ then $A\mu$ will be denoted by the usual symbol $T_c\mu$.

A p. m. μ on X is said to be infinitely divisible if for every $n = 1, 2, \dots$ there exists a p. m. μ_n such that $\mu = \mu_n^{*n}$, where the asterisk $*$ denotes the convolution operation of measures. It is known ([3], [9]) that every i. d. p. m. μ on X has a unique representation.

$$(1) \quad \mu = \rho * \tilde{\alpha}(M)$$

where ρ is a symmetric Gaussian p. m., $\tilde{e}(M)$ is a generalized Poisson p. m. corresponding to a Levy's measure M on $X \setminus \{0\}$. In particular, if M is a finite measure then $e(M) = \tilde{e}(M)$ is a Poisson p. m..

Recall that a p. m. P belongs to the domain of partial attraction of a p. m. q on X if there exist a subsequence $\{n_k\}$ of natural numbers and a sequence $\{a_k\}$ of real numbers such that $\{T_{a_k} P^{*n_k}\}$ is shift-convergent to q . Here and in the sequel the convergence of p. m.'s will be understood in the weak sense. Further, P is said to be universal for i. d. p. m.'s if it belongs to the domain of partial attraction of every i. d. p. m. q .

Let A be a bounded linear operator on X . A p. m. P is said to be A -universal for i. d. p. m.'s on X if for every i. d. p. m. q on X there exist subsequences $\{n_k\}$ and m_k of natural numbers such that $\{A^{n_k} P^{*m_k}\}$ is shift-convergent to q .

Now we shall prove the following generalized Doeblin theorem.

THEOREM. — For every invertible bounded linear operator A on X such that

$$(2) \quad \|A^n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

there exists an A -universal p. m. P for i. d. p. m.'s on X .

Remark. — It is easy to show that if X is finite-dimensional and A is a linear operator on X then from the existence of an A -universal p. m. P for i. d. p. m.'s on X it follows that A is invertible and the relation (2) holds.

We precede the proof of the theorem by two lemmas. The following lemma is due to Ho Dang Phuc [5].

LEMMA 1. — Let ρ be a symmetric Gaussian p. m. on X . Then there exists a sequence $\{\rho_n\}$ of Poisson p. m.'s convergent to ρ .

Proof. — Let Z be an X -valued random variable with distribution ρ . By Theorem 3 [7] it follows that there exist a sequence $\{x_n\}$ of elements of X and a sequence $\{\varphi_n\}$ of independent Gaussian random variables with distribution $N(0, 1)$ such that

$$Z = \sum_n x_n \varphi_n$$

where the series is convergent almost surely.

Let μ_n be the distribution of the random variable

$$Z_n = \sum_{k=1}^n x_k \varphi_k$$

Then, by Theorem 4.1 [6], it follows that $\{\mu_n\}$ is convergent to ρ . Moreover, μ_n is a Gaussian p. m. on the finite-dimensional space

$$E_n := \text{lin} \{ a_1, \dots, a_n \} \subset E$$

for all n . Hence for each n there exists a sequence $\mu_{n,m}$, $m = 1, 2, \dots$, of Poisson p. m.'s convergent to μ_n . Finally, define $\rho_n = \mu_{n,n}$ we get a sequence $\{\rho_n\}$ of Poisson p. m.'s convergent to ρ . The Lemma is thus proved.

LEMMA 2. — Let $\{M_n\}$ be a sequence of finite measures on X such that

$$(3) \quad \int_X \|x\| M_n(dx) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Then the sequence $e(M_n)$, $n = 1, 2, \dots$, of Poisson measures on X is convergent to δ_0 .

Proof. — For $r > 0$ let us denote $B_r = \{x \in X : \|x\| \leq r\}$. It can be easily seen that the condition (3) implies the following two conditions:

$$(4) \quad \{M_n|_{B_r^c}\} \quad \text{is weakly convergent to 0 and}$$

$$(5) \quad \lim_{r \downarrow 0} \limsup_n \int_{B_r} \|x\| M_n(dx) = 0$$

which, by Corollary 1.8 [4], implies that $e(M_n)$ is convergent to δ_0 . Thus the Lemma is proved.

Proof of the Theorem. — By the condition (2) it follows that there exist constants $c > 0$ and $\alpha > 1$ such that for each $n = 1, 2, \dots$

$$(6) \quad \|A^n\| \leq C\alpha^{-n}.$$

Let $\{q_n\}$ be a countable dense subset of the set of all i. d. p. m.'s on X . We may assume that the corresponding Levy's measure M_n to q_n in the representation (1) is concentrated at $B_n = \{x \in X : \|x\| \leq n\}$ and that $M_n(X) \leq n$ for $n = 1, 2, \dots$. By virtue of Lemma 1 we may suppose that

$$(7) \quad q_n = e(M_n) * \delta_{x_n}$$

for some sequence $\{x_n\} \subset X$.

Define

$$(8) \quad M = \sum_{n=1}^{\infty} [\alpha^{n^2}]^{-1} A^{-n^3} M_n$$

where $[\alpha^{n^2}]$ denotes the integer part of α^{n^2} and α is determined by (6). It is clear that M is a finite measure on X . Then we put

$$(9) \quad P = e(M).$$

We claim that P is A -universal for i. d. p. m.'s on X . Let q be an arbitrary i. d. p. m. on X . Then there is a subsequence $\{n_k\}$ of natural numbers such that the sequence $\{q_{n_k}\}$ is convergent to q . Let $t_k = [\alpha^{n_k^2}]$. Our further aim is to prove that the sequence

$$(10) \quad v_k := A^{n_k^3} p^{*t_k}, \quad k = 1, 2, \dots,$$

is shift-convergent to q .

Accordingly, putting

$$(11) \quad N_k^1 = \sum_{n > n_k} t_k [\alpha^{n^2}]^{-1} A^{n^3 - n^3} M_n$$

and

$$(12) \quad N_k^2 = \sum_{n < n_k} t_k [\alpha^{n^2}]^{-1} A^{n^3 - n^3} M_n$$

and taking into account the definition of v_k in (10) we have the equation

$$(13) \quad v_k = e(M_n) * e(N_k^1) * e(N_k^2)$$

We shall prove that $\lim_k N_k^1(X) = 0$ and $\lim_k \int_X \|x\| N_k^2(dx) = 0$, which, by virtue of (13) and Lemma 2, implies that $\{v_k\}$ is shift-convergent to q .

The first limit is clear, because

$$\begin{aligned} N_k^1(X) &\leq \sum_{n > n_k} n t_k [\alpha^{n^2}]^{-1} \\ &\leq \sum_{n=1}^{\infty} (n_k + n) [\alpha^{n_k^2}] [\alpha^{(n_k+n)^2}]^{-1} \\ &\leq \frac{\alpha}{\alpha - 1} \sum_{n=1}^{\infty} (n_k + n) \alpha^{-(2(n_k+n)n)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

On the other hand, we have, by (6),

$$\begin{aligned}
 \int_{\mathbf{X}} \|x\| N_k^2(dx) &= \sum_{n < n_k} \int_{\mathbf{X}} \|A^{n\hat{\alpha} - n^3} x\| t_k [\alpha^{n^2}]^{-1} M_n(dx) \\
 &\leq C \sum_{n < n_k} n^2 \alpha^{n^3 - n^3} [\alpha^{n^2}]^{-1} \\
 &\leq \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{n^3 - n^3} \alpha^{n^2 - n^2} \\
 &\leq \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{-n^2} \alpha^{(n_k - 1)^3 - n^3 + n^2} \\
 &\leq \frac{C\alpha}{\alpha - 1} \alpha^{-2n_k^3 + 3n_k - 1} \sum_{n=1}^{\infty} n^2 \alpha^{-n^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.
 \end{aligned}$$

Thus the theorem is fully proved.

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