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## A new version of Doeblin's theorem

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**ABSTRACT.** — The aim of this note is to prove the existence of an operator-universal probability measure for infinitely divisible (i. d.) probability measures (p. m.) on a Banach space.

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A famous theorem of W. Doeblin [2] asserts that there exists a p. m. belonging to the domain of partial attraction of every one-dimensional i. d. p. m. . The multi-dimensional version of this theorem is made by J. Barańska [1] in a Hilbert space and recently, by Ho Dang Phuc [5] in a Banach space.

Some techniques from Ho Dang Phuc [5] can be applied to prove a more general theorem. Namely, we replace the norming sequence in Doeblin's theorem by linear operators.

Throughout the paper we shall denote by  $(X, \| \cdot \|)$  a real separable Banach space and consider only  $\sigma$ -additive non-negative Borel measures on  $X$ . For a bounded linear operator  $A$  and a measure  $\mu$  we shall denote by  $A\mu$  a measure defined by  $A\mu(\varepsilon) = \mu(A^{-1}\varepsilon)$  ( $\varepsilon \subset X$ ). In particular, if  $Ax = cx$  for some  $c \geq 0$  and for all  $x \in X$  then  $A\mu$  will be denoted by the usual symbol  $T_c\mu$ .

A p. m.  $\mu$  on  $X$  is said to be infinitely divisible if for every  $n = 1, 2, \dots$  there exists a p. m.  $\mu_n$  such that  $\mu = \mu_n^{*n}$ , where the asterisk  $*$  denotes the convolution operation of measures. It is known ([3], [9]) that every i. d. p. m.  $\mu$  on  $X$  has a unique representation.

$$(1) \quad \mu = \rho * \tilde{\alpha}(M)$$

where  $\rho$  is a symmetric Gaussian p. m.,  $\tilde{e}(M)$  is a generalized Poisson p. m. corresponding to a Levy's measure  $M$  on  $X \setminus \{0\}$ . In particular, if  $M$  is a finite measure then  $e(M) = \tilde{e}(M)$  is a Poisson p. m..

Recall that a p. m.  $P$  belongs to the domain of partial attraction of a p. m.  $q$  on  $X$  if there exist a subsequence  $\{n_k\}$  of natural numbers and a sequence  $\{a_k\}$  of real numbers such that  $\{T_{a_k} P^{*n_k}\}$  is shift-convergent to  $q$ . Here and in the sequel the convergence of p. m.'s will be understood in the weak sense. Further,  $P$  is said to be universal for i. d. p. m.'s if it belongs to the domain of partial attraction of every i. d. p. m.  $q$ .

Let  $A$  be a bounded linear operator on  $X$ . A p. m.  $P$  is said to be  $A$ -universal for i. d. p. m.'s on  $X$  if for every i. d. p. m.  $q$  on  $X$  there exist subsequences  $\{n_k\}$  and  $m_k$  of natural numbers such that  $\{A^{n_k} P^{*m_k}\}$  is shift-convergent to  $q$ .

Now we shall prove the following generalized Doeblin theorem.

**THEOREM.** — For every invertible bounded linear operator  $A$  on  $X$  such that

$$(2) \quad \|A^n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

there exists an  $A$ -universal p. m.  $P$  for i. d. p. m.'s on  $X$ .

*Remark.* — It is easy to show that if  $X$  is finite-dimensional and  $A$  is a linear operator on  $X$  then from the existence of an  $A$ -universal p. m.  $P$  for i. d. p. m.'s on  $X$  it follows that  $A$  is invertible and the relation (2) holds.

We precede the proof of the theorem by two lemmas. The following lemma is due to Ho Dang Phuc [5].

**LEMMA 1.** — Let  $\rho$  be a symmetric Gaussian p. m. on  $X$ . Then there exists a sequence  $\{\rho_n\}$  of Poisson p. m.'s convergent to  $\rho$ .

*Proof.* — Let  $Z$  be an  $X$ -valued random variable with distribution  $\rho$ . By Theorem 3 [7] it follows that there exist a sequence  $\{x_n\}$  of elements of  $X$  and a sequence  $\{\varphi_n\}$  of independent Gaussian random variables with distribution  $N(0, 1)$  such that

$$Z = \sum_n x_n \varphi_n$$

where the series is convergent almost surely.

Let  $\mu_n$  be the distribution of the random variable

$$Z_n = \sum_{k=1}^n x_k \varphi_k$$

Then, by Theorem 4.1 [6], it follows that  $\{\mu_n\}$  is convergent to  $\rho$ . Moreover,  $\mu_n$  is a Gaussian p. m. on the finite-dimensional space

$$E_n := \text{lin} \{ a_1, \dots, a_n \} \subset E$$

for all  $n$ . Hence for each  $n$  there exists a sequence  $\mu_{n,m}$ ,  $m = 1, 2, \dots$ , of Poisson p. m.'s convergent to  $\mu_n$ . Finally, define  $\rho_n = \mu_{n,n}$  we get a sequence  $\{\rho_n\}$  of Poisson p. m.'s convergent to  $\rho$ . The Lemma is thus proved.

LEMMA 2. — Let  $\{M_n\}$  be a sequence of finite measures on  $X$  such that

$$(3) \quad \int_X \|x\| M_n(dx) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Then the sequence  $e(M_n)$ ,  $n = 1, 2, \dots$ , of Poisson measures on  $X$  is convergent to  $\delta_0$ .

*Proof.* — For  $r > 0$  let us denote  $B_r = \{x \in X : \|x\| \leq r\}$ . It can be easily seen that the condition (3) implies the following two conditions:

$$(4) \quad \{M_n|_{B_r^c}\} \quad \text{is weakly convergent to 0 and}$$

$$(5) \quad \lim_{r \downarrow 0} \limsup_n \int_{B_r} \|x\| M_n(dx) = 0$$

which, by Corollary 1.8 [4], implies that  $e(M_n)$  is convergent to  $\delta_0$ . Thus the Lemma is proved.

*Proof of the Theorem.* — By the condition (2) it follows that there exist constants  $c > 0$  and  $\alpha > 1$  such that for each  $n = 1, 2, \dots$

$$(6) \quad \|A^n\| \leq C\alpha^{-n}.$$

Let  $\{q_n\}$  be a countable dense subset of the set of all i. d. p. m.'s on  $X$ . We may assume that the corresponding Levy's measure  $M_n$  to  $q_n$  in the representation (1) is concentrated at  $B_n = \{x \in X : \|x\| \leq n\}$  and that  $M_n(X) \leq n$  for  $n = 1, 2, \dots$ . By virtue of Lemma 1 we may suppose that

$$(7) \quad q_n = e(M_n) * \delta_{x_n}$$

for some sequence  $\{x_n\} \subset X$ .

Define

$$(8) \quad M = \sum_{n=1}^{\infty} [\alpha^{n^2}]^{-1} A^{-n^3} M_n$$

where  $[\alpha^{n^2}]$  denotes the integer part of  $\alpha^{n^2}$  and  $\alpha$  is determined by (6). It is clear that  $M$  is a finite measure on  $X$ . Then we put

$$(9) \quad P = e(M).$$

We claim that  $P$  is  $A$ -universal for i. d. p. m.'s on  $X$ . Let  $q$  be an arbitrary i. d. p. m. on  $X$ . Then there is a subsequence  $\{n_k\}$  of natural numbers such that the sequence  $\{q_{n_k}\}$  is convergent to  $q$ . Let  $t_k = [\alpha^{n_k^2}]$ . Our further aim is to prove that the sequence

$$(10) \quad v_k := A^{n_k^3} p^{*t_k}, \quad k = 1, 2, \dots,$$

is shift-convergent to  $q$ .

Accordingly, putting

$$(11) \quad N_k^1 = \sum_{n > n_k} t_k [\alpha^{n_k^2}]^{-1} A^{n_k^3 - n^3} M_n$$

and

$$(12) \quad N_k^2 = \sum_{n < n_k} t_k [\alpha^{n^2}]^{-1} A^{n_k^3 - n^3} M_n$$

and taking into account the definition of  $v_k$  in (10) we have the equation

$$(13) \quad v_k = e(M_n) * e(N_k^1) * e(N_k^2)$$

We shall prove that  $\lim_k N_k^1(X) = 0$  and  $\lim_k \int_X \|x\| N_k^2(dx) = 0$ , which, by virtue of (13) and Lemma 2, implies that  $\{v_k\}$  is shift-convergent to  $q$ .

The first limit is clear, because

$$\begin{aligned} N_k^1(X) &\leq \sum_{n > n_k} n t_k [\alpha^{n^2}]^{-1} \\ &\leq \sum_{n=1}^{\infty} (n_k + n) [\alpha^{n_k^2}] [\alpha^{(n_k + n)^2}]^{-1} \\ &\leq \frac{\alpha}{\alpha - 1} \sum_{n=1}^{\infty} (n_k + n) \alpha^{-(2(n_k + n)n)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

On the other hand, we have, by (6),

$$\begin{aligned}
 \int_{\mathbf{X}} \|x\| N_k^2(dx) &= \sum_{n < n_k} \int_{\mathbf{X}} \|A^{n_k - n} x\| t_k [\alpha^{n^2}]^{-1} M_n(dx) \\
 &\leq C \sum_{n < n_k} n^2 \alpha^{n^3 - n_k^3} [\alpha^{n^2}]^{-1} \\
 &\leq \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{n^3 - n_k^3} \alpha^{n_k^2 - n^2} \\
 &\leq \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{-n^2} \alpha^{(n_k - 1)^3 - n_k^3 + n_k^2} \\
 &\leq \frac{C\alpha}{\alpha - 1} \alpha^{-2n_k^3 + 3n_k - 1} \sum_{n=1}^{\infty} n^2 \alpha^{-n^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.
 \end{aligned}$$

Thus the theorem is fully proved.

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