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D. NUALART
M. SANZ

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Changing time for two-parameter strong martingales

by

D. NUALART

Departament d'Estadística Matemàtica, Facultat de Matemàtiques,
Universitat de Barcelona, Gran Via 585, Barcelona, 7 Spain

and M. SANZ

Departament d'Estadística, Secció de Matemàtiques,
Universitat Autónoma de Barcelona, Bellaterra, Barcelona, Spain

RÉSUMÉ. — Cet article aborde le problème de transformer une martingale forte de carré intégrable, adaptée à un processus de Wiener, $\{ M_z, z \in R_+^2 \}$, en un processus de Wiener à deux indices, au moyen d'un changement de temps. Concrètement on donne des conditions suffisantes sur une famille de régions d'arrêt $\{ D_z, z \in R_+^2 \}$ pour que $\{ M(D_z), z \in R_+^2 \}$ soit un processus de Wiener.

On étudie en détail les changements de temps pour le processus de Wiener à deux indices et on caractérise les familles de régions d'arrêt déterministes transformant un drap brownien en un autre. On donne aussi une extension des résultats aux processus à n indices.

SUMMARY. — This paper studies the problem of transforming a two-parameter square integrable strong martingale $\{ M_z, z \in R_+^2 \}$ adapted to a Wiener process into a two-parameter Wiener process by means of a time change. Concretely, sufficient conditions are given for a family of stopping sets $\{ D_z, z \in R_+^2 \}$ for $\{ M(D_z), z \in R_+^2 \}$ to be a Wiener process.

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We treat in detail the particular case of changing time for the Wiener process obtaining some characterizations of all increasing families of deterministic stopping sets which transform a Brownian sheet into another one. An extension of these results to n-parameter processes is also developed.

1. INTRODUCTION

This paper studies the problem of transforming a two-parameter martingale into a two-parameter Wiener process $\{ W_z, z \in R_+^2 \}$ by means of a time change.

It is well known that, given a one-parameter continuous martingale $\{ M_t, t \in R_+ \}$, there exists a family of stopping times $\{ T_t, t \in R_+ \}$ such that $\{ M_{T_t}, t \in R_+ \}$ is a Brownian motion. For multi-parameter processes there is no immediate extension of this result because we do not have a good generalization of the concept of stopping time.

In [2] R. Cairoli and J. B. Walsh showed, through an example, the limited usefulness of stopping points to treat the problem of transforming two-parameter martingales. Indeed, they exhibit a two-parameter Gaussian strong martingale $\{ M_z, z \in R_+^2 \}$ so that for any deterministic time change of the form $z \to \Gamma(z)$, where Γ is a mapping of R_+^2 onto itself, the process $\{ M_{\Gamma(z)}, z \in R_+^2 \}$ can never be a two-parameter Wiener process.

The basic aim of our work is to study this problem using the notion of stopping set, introduced in [3] and [4]. In section 2, given a square integrable strong martingale $\{ M_z, z \in R_+^2 \}$ adapted to a two-parameter Wiener process, we set up conditions for an increasing family of stopping sets $\{ D_z, z \in R_+^2 \}$ for $\{ M(D_z), z \in R_+^2 \}$ to be a Wiener process; and a method to construct these families is given.

The third part is devoted to the characterization of all increasing families $\{ D_z, z \in R_+^2 \}$ of stopping sets which transform a two-parameter Wiener process into another one. Finally, an extension of these results for n-parameter processes is given in section 4.

We thank Prof. J. B. Walsh for having suggested these problems to us.

Next we introduce the basic notation.

R_+^2 will denote the positive quadrant of the plane with the usual ordering

$$(s_1, t_1) < (s_2, t_2) \quad \text{if and only if} \quad s_1 \leqslant s_2 \quad \text{and} \quad t_1 \leqslant t_2 .$$

$(s_1, t_1) \ll (s_2, t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$, and we will write

$(s_1, t_1) \wedge (s_2, t_2)$ if $s_1 \leqslant s_2$ and $t_1 \geqslant t_2$. If $z_1 \ll z_2$, $(z_1, z_2]$ denotes the rectangle $\{ z \in \mathrm{R}_+^2 / z_1 \ll z < z_2 \}$, and R_z denotes the rectangle $[(0, 0), z]$ for every $z \in \mathrm{R}_+^2$. Let us represent the Borel σ-field of R_+^2 by $\mathscr{B}$ and the Lebesgue measure on $\mathscr{B}$ by m.

Let $\mathrm{W} = \{ \mathrm{W}_z, z \in \mathrm{R}_+^2 \}$ be a two-parameter Wiener process in a completed probability space $(\Omega, \mathscr{F}, \mathrm{P})$, that is, a Gaussian separable process with zero mean and covariance function given by

$$\mathrm{E}\{ \mathrm{W}_{s_1 t_1} \cdot \mathrm{W}_{s_2 t_2} \} = (s_1 \wedge s_2) \cdot (t_1 \wedge t_2).$$

$\{ \mathscr{F}_z, z \in \mathrm{R}_+^2 \}$ will be the increasing family of σ-fields generated by W; that means, $\mathscr{F}_z = \sigma(\mathrm{W}_\zeta, \zeta < z)$ completed by the null sets of $\mathscr{F}$. Moreover, for each $(s, t) \in \mathrm{R}_+^2$ we will consider the following increasing families of σ-fields

$$\mathscr{F}_{st}^1 = \bigvee_{\tau \geqslant 0} \mathscr{F}_{s\tau} \quad \text{and} \quad \mathscr{F}_{st}^2 = \bigvee_{\sigma \geqslant 0} \mathscr{F}_{\sigma t} .$$

Let E_0 be the set $\{ (s, t) \in \mathrm{R}_+^2 / s = 0 \text{ or } t = 0 \}$.

An $\mathscr{F}_z$-adapted, integrable process $\mathrm{M} = \{ \mathrm{M}_z, z \in \mathrm{R}_+^2 \}$ is a martingale if $\mathrm{E}(\mathrm{M}_{z_2} / \mathscr{F}_{z_1}) = \mathrm{M}_{z_1}$ for all $z_1 < z_2$, and it is a strong martingale if $\mathrm{M}_z = 0$, when $z \in \mathrm{E}_0$ and

$$\mathrm{E}(\mathrm{M}(z_1, z_2] / \mathscr{F}_{z_1}^1 \vee \mathscr{F}_{z_2}^2) = 0 ,$$

for all $z_1 < z_2$, where $\mathrm{M}(z_1, z_2] = \mathrm{M}_{z_2} - \mathrm{M}_{(s_1, t_2)} - \mathrm{M}_{(s_2, t_1)} + \mathrm{M}_{z_1}$ if $z_1 = (s_1, t_1)$ and $z_2 = (s_2, t_2)$.

2. STOPPING SETS AND CHANGING TIME FOR STRONG MARTINGALES

Following [3], we introduce the notion of stopping set.

DEFINITION 2.1. — A *stopping set* $\mathrm{D}(\omega)$ is a map from Ω to the subsets of R_+^2 satisfying:

i) The process $\{ 1_{\mathrm{D}(\omega)}(z), z \in \mathrm{R}_+^2 \}$ is progressively measurable. (In particular it is measurable and adapted).

ii) For all $\omega \in \Omega$ such that $\mathrm{D}(\omega) \neq \phi$ we have $\mathrm{D}(\omega)$ is closed, and $z \in \mathrm{D}(\omega)$ implies $\mathrm{R}_z \subset \mathrm{D}(\omega)$.

We will say that $\mathrm{D}(\omega)$ is a bounded stopping set if there exists z_0 such that $\mathrm{D}(\omega) \subset \mathrm{R}_{z_0}$, for all $\omega \in \Omega$.

If D_1 and D_2 are stopping sets, then so are $\mathrm{D}_1 \cap \mathrm{D}_2$ and $\mathrm{D}_1 \cup \mathrm{D}_2$. To each stopping set D such that $\mathrm{E}(m(\mathrm{D})) < \infty$, we can associate the random variable $W(D) = \int_{R_+^2} 1_D(z) dW_z$ and the σ-field $\mathcal{F}_D$ generated by the variables $\{ W(D \cap R_z), z \in R_+^2 \}$ together with the null sets of $\mathcal{F}$.

Let L_W^2 be the class of all $\mathcal{F}_z$-adapted and measurable processes $\phi = \{ \phi_z, z \in R_+^2 \}$ such that $\int_{R_{z_0}} E(\phi(z)^2) dz < \infty$, for all $z_0 \in R_+^2$.

Let L_{WW}^2 be the class of all processes $\psi = \{ \psi(z, z'), z, z' \in R_+^2 \}$ satisfying

i) $\psi(z, z'; \omega)$ is measurable and $\mathcal{F}_{z \vee z'}$-adapted,

ii) $\psi(z, z') = 0$ unless $z \wedge z'$,

iii) for all $z_0 \in R_+^2$ $\int_{R_{z_0}} \int_{R_{z_0}} E(\psi(z, z')^2) dz dz' < \infty$.

For processes $\phi \in L_W^2$ and $\psi \in L_{WW}^2$, the stochastic integrals $\int_{R_{z_0}} \phi(z) dW_z$ and $\int_{R_{z_0}} \int_{R_{z_0}} \psi(z, z') dW_z dW_{z'}$ can be defined (see [*1*]). We need the following results (see [*4*]) which determine conditional expectations of such stochastic integrals with respect to the σ-field $\mathcal{F}_D$.

PROPOSITION 2.1. — Let D be a stopping set and let $\phi \in L_W^2$, $\psi \in L_{WW}^2$. For all $z_0 \in R_+^2$ we have

$$E\left(\int_{R_{z_0}} \phi(z) dW_z / \mathcal{F}_D \right) = \int_{R_{z_0}} \phi(z) 1_D(z) dW_z . \tag{2.1}$$

$$E\left(\int_{R_{z_0}} \int_{R_{z_0}} \psi(z, z') dW_z dW_{z'} / \mathcal{F}_D \right) = \int_{R_{z_0}} \int_{R_{z_0}} E(\psi(z, z') / \mathcal{F}_{D \cap R_{z \vee z'}}) 1_D(z) 1_D(z') dW_z dW_{z'} . \tag{2.2}$$

These formulae are still true if we take stochastic integrals of processes integrable over all R_+^2.

Stopping sets have properties which are analogous to those of stopping times for one-parameter processes.

1. If $D_1 \subset D_2$, then $\mathcal{F}_{D_1} \subset \mathcal{F}_{D_2}$.
2. For each stopping set D with $E(m(D)) < \infty$, the random variable W(D) is $\mathcal{F}_D$-measurable.
3. $\mathcal{F}_{D_1} \cap \mathcal{F}_{D_2} = \mathcal{F}_{D_1 \cap D_2}$ and $\mathcal{F}_{D_1} \vee \mathcal{F}_{D_2} = \mathcal{F}_{D_1 \cup D_2}$.
4. If D is a stopping set and ϕ is an $\mathcal{F}_z$-measurable random variable, then $1_D(z) \cdot \phi$ is $\mathcal{F}_{D \cap R_z}$-measurable.

For the proof of these properties see [*4*].

A stopping set D will be called *simple* if there exists a partition of R_+^2 into rectangles $(z_{ij}, z_{i+1,j+1}]$ and sets $A_{ij} \in \mathscr{F}_{z_{ij}}$ such that

$$D(\omega) = \bigcup_{ij} (z_{ij}, z_{i+1,j+1}] 1_{A_{ij}}(\omega) .$$

These rectangles will be closed if $z_{ij} \in E_0$.

For each stopping set D there exists a decreasing sequence of simple stopping sets $\{ D_n, n \in N \}$ such that $D = \bigcap_n D_n$ (see [3]). Moreover, $\mathscr{F}_D = \bigcap_n \mathscr{F}_{D_n}$ (see [4]).

Let $M = \{ M_z, z \in R_+^2 \}$ be a square integrable strong martingale. There exists a process $\phi \in L_W^2$ such that $M_{st} = \int_{R_{st}} \phi(z) dW_z$ (see [1]). For each stopping set D such that $\int_{R_+^2} E(1_D(z)\phi(z)^2) dz < \infty$ we can consider the random variable $M(D) = \int_{R_+^2} 1_D(z)\phi(z) dW_z$. Our purpose is to find a family of stopping sets $\{ D_z, z \in R_+^2 \}$ such that $\{ M(D_z), z \in R_+^2 \}$ is a Wiener process.

First, we establish several lemmas in order to reach the main result.

LEMMA 2.1. — If the family of stopping sets $\{ D_z, z \in R_+^2 \}$ satisfies the property

$$D_{z_1} \cap D_{z_2} = D_{z_1 \wedge z_2} , \qquad \text{for all} \quad z_1, z_2 \in R_+^2 , \tag{2.3}$$

then, the associated family of σ-fields $\{ \mathscr{F}_{D_z}, z \in R_+^2 \}$ is increasing, and for all $(s, t) \in R_+^2$, the σ-fields $\mathscr{F}^1_{D_{st}} = \bigvee_{\tau \geq 0} \mathscr{F}_{D_{s\tau}}$ and $\mathscr{F}^2_{D_{st}} = \bigvee_{\sigma \geq 0} \mathscr{F}_{D_{\sigma t}}$ are conditionally independent, given $\mathscr{F}_{D_{st}}$.

Proof. — For each $\sigma \geq s$ and $\tau \geq t$ we know (see [4]) that the σ-fields $\mathscr{F}_{D_{s\tau}}$ and $\mathscr{F}_{D_{\sigma t}}$ are conditionally independent given $\mathscr{F}_{D_{s\tau} \cap D_{\sigma t}} = \mathscr{F}_{D_{st}}$. This fact leads immediately to the statement of the lemma. □

LEMMA 2.2. — If $M = \{ M_z, z \in R_+^2 \}$ is a square integrable strong martingale and $\{ D_z, z \in R_+^2 \}$ is a family of stopping sets satisfying (2.3), and such that $\int_{R_+^2} E(1_{D_{st}}(z)\phi^2(z)) dz < \infty$ for all $(s, t) \in R_+^2$, then $\{ M(D_z), z \in R_+^2 \}$ is a strong martingale with respect to the family of σ-fields $\{ \mathscr{F}_{D_z}, z \in R_+^2 \}$.

Proof. — If $(s, t) < (s', t')$ let us write

$$M(D)((s, t), (s', t')] = M(D_{s't'}) - M(D_{s't}) - M(D_{st'}) + M(D_{st}).$$

To show that $E(M(D)((s, t), (s', t')]/\mathcal{F}^1_{D_{st}} \vee \mathcal{F}^2_{D_{st}}) = 0$, it suffices to verify

$$E(M(D)((s, t), (s', t')]/\mathcal{F}_{D_{s\tau}} \vee \mathcal{F}_{D_{\sigma t}}) = 0 \qquad \text{for all} \qquad \sigma \geqslant s' \text{ and } \tau \geqslant t'.$$

Using (2.1) and the fact that $\mathcal{F}_{D_{s\tau}} \vee \mathcal{F}_{D_{\sigma t}} = \mathcal{F}_{D_{s\tau} \cup D_{\sigma t}}$ we have

$$\begin{aligned} E\{ M(D)((s, t), (s', t')]/\mathcal{F}_{D_{s\tau}} \vee \mathcal{F}_{D_{\sigma t}} \} &= \int_{R_+^2} \phi(z) \{ 1_{D_{s't'}}(z) - 1_{D_{s't}}(z) - 1_{D_{st'}}(z) + 1_{D_{st}}(z) \} 1_{D_{s\tau}\cup D_{\sigma t}}(z) dW_z \\ &= \int_{R_+^2} \phi(z) \{ 1_{D_{s't}\cup D_{st'}}(z) - 1_{D_{s't}}(z) - 1_{D_{st'}}(z) + 1_{D_{st}}(z) \} dW_z = 0. \quad \square \end{aligned}$$

LEMMA 2.3. — If $M_{st} = \int_{R_{st}} \phi(z) dW_z$ is a square integrable strong martingale and $\{ D_z, z \in R_+^2 \}$ is a family of stopping sets satisfying (2.3) and such that $\sup_\omega \left| \int_{R_+^2} 1_{D_{st}}(z) \phi^2(z) dz \right| < \infty$ for all $(s, t) \in R_+^2$, then

$$\left\{ M(D_{st})^2 - \int_{R_+^2} 1_{D_{st}}(z) \phi(z)^2 dz, \mathcal{F}_{D_{st}}, (s, t) \in R_+^2 \right\}$$

is a martingale.

Proof. — Notice that $M(D_{st})^2 - \int_{R_+^2} 1_{D_{st}}(z)\phi(z)^2 dz$ is an integrable, $\mathcal{F}_{D_{st}}$-adapted process due to property 4 of stopping sets.

The two-parameter Itô formula (see [*6*]) applied to

$$M(D_{st}) = \int_{R_+^2} 1_{D_{st}}(z) \phi(z) dW_z,$$

claims

$$\begin{aligned} M(D_{st})^2 &= 2 \int_{R_+^2} M(D_{st} \cap R_z) \cdot 1_{D_{st}}(z) \phi(z) dW_z \\ &+ 2 \int_{R_+^2} \int_{R_+^2} 1_{D_{st}}(z) 1_{D_{st}}(z') \phi(z) \phi(z') dW_z dW_{z'} + \int_{R_+^2} 1_{D_{st}}(z) \phi(z)^2 dz. \end{aligned} \tag{2.4}$$

The second term on the right is square integrable because of condition

$$\sup_\omega \left| \int_{R_+^2} 1_{D_{st}}(z) \phi(z)^2 dz \right| < \infty$$

and by (2.2) is a $\mathscr{F}_{D_{st}}$-martingale. Indeed, if $(s', t') < (s, t)$, we have

$$E\left\{ \int_{R_+^2} \int_{R_+^2} 1_{D_{st}}(z) 1_{D_{st}}(z') \phi(z)\phi(z') dW_z dW_{z'} / \mathscr{F}_{D_{s't'}} \right\}$$
$$= \int_{R_+^2} \int_{R_+^2} 1_{D_{s't'}}(z) 1_{D_{s't'}}(z') \phi(z)\phi(z') dW_z dW_{z'},$$

since

$$E\{ 1_{D_{st}}(z) 1_{D_{st}}(z') \phi(z)\phi(z') / \mathscr{F}_{D_{s't'} \cap R_{z \vee z'}} \} \cdot 1_{D_{s't'}}(z) 1_{D_{s't'}}(z')$$
$$= 1_{D_{s't'}}(z) 1_{D_{s't'}}(z') \phi(z)\phi(z'),$$

due to property 4 of stopping sets.

The first term of (2.4) is square integrable because

$$E\int_{R_+^2} (M(D_{st} \cap R_z) 1_{D_{st}}(z)\phi(z))^2 dz$$
$$\leqslant E \sup_z M(D_{st} \cap R_z)^2 \left\| \int_{R_+^2} 1_{D_{st}}(z)\phi(z)^2 dz \right\|_\infty < \infty .$$

By a local property of stochastic integrals we have, a. s.,

$$M(D_{st} \cap R_z) \cdot 1_{D_{st}}(z) = M_z \cdot 1_{D_{st}}(z), \tag{2.5}$$

for all $z \in R_+^2$.

Then, the first term of (2.4) also defines an $\mathscr{F}_{D_{st}}$-martingale as follows from (2.1):

$$E\left(\int_{R_+^2} M(D_{st} \cap R_z) 1_{D_{st}}(z)\phi(z) dW_z / \mathscr{F}_{D_{s't'}} \right)$$
$$= E\left(\int_{R_+^2} M_z 1_{D_{st}}(z)\phi(z) dW_z / \mathscr{F}_{D_{s't'}} \right) = \int_{R_+^2} M_z 1_{D_{s't'}}(z)\phi(z) dW_z$$
$$= \int_{R_+^2} M(D_{s't'} \cap R_z) 1_{D_{s't'}}(z)\phi(z) dW_z ,$$

using (2.5), and being $(s', t') < (s, t)$. □

THEOREM 2.1. — Let $M_{st} = \int_{R_{st}} \phi(z) dW_z$, $\phi \in L_W^2$, be a strong martingale, and $\{ D_z, z \in R_+^2 \}$ a family of stopping sets verifying properties (2.3) and

$$\int_{R_+^2} 1_{D_{st}}(z)\phi(z)^2 dz = s \cdot t , \qquad \text{for all} \qquad (s, t) \in R_+^2 . \tag{2.6}$$

Then $\{ M(D_z), z \in R_+^2 \}$ is a two-parameter Wiener process.

Proof. — Let $\{ \mathscr{F}_z, z \in R_+^2 \}$ be an arbitrary increasing family of σ-fields verifying the following condition: the σ-fields $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$, for all $z \in R_+^2$.

An extension of a well known result of P. Lévy, obtained by E. Wong for two-parameter strong martingales, states that if $\{ M_z, \mathscr{F}_z, z \in \mathbb{R}_+^2 \}$ is a continuous strong martingale and $\{ M_{st}^2 - st, (s, t) \in \mathbb{R}_+^2 \}$ is an $\mathscr{F}_{st}$-martingale, then M_z is a two-parameter Wiener process.

Thus, we only have to prove that $M(D_{st})$ is a continuous process, taking into account the preceding lemmas. From Cairoli's maximal inequality we deduce that

$$
\begin{aligned}
& E\{ \sup_{z,z' \in [(s,t),(s+h,t+k)]} | M(D_z) - M(D_{z'}) |^2 \} \\
& \quad \leqslant 4 \cdot E\{ \sup_{z \in [(s,t),(s+h,t+k)]} | M(D_z) - M(D_{st}) |^2 \} \\
& \quad \leqslant 4 \cdot 2^4 \cdot E \left\{ \int_{\mathbb{R}_+^2} (1_{D_{s+h,t+k}}(z) - 1_{D_{st}}(z))^2 \phi^2(z) dz \right\} = 4 \cdot 2^4 (hk + ht + ks).
\end{aligned}
$$

For each $\varepsilon > 0$ and $(s, t) \in \mathbb{R}_+^2$ let $B_\varepsilon(s, t) = \{ z \in \mathbb{R}_+^2 / \| (s, t) - z \| < \varepsilon \}$. We have

$$
P\{ \sup_{z,z' \in B_\varepsilon(s,t)} | M(D_z) - M(D_{z'}) | > \varepsilon^{1/4} \} \leqslant 4 \cdot 2^4 k \varepsilon^{1/2}, \quad \text{for all} \quad (s, t) \in R_{z_0},
$$

because $z, z' \in B_\varepsilon(s, t)$ implies $z, z' \in [(s - \varepsilon, t - \varepsilon), (s + \varepsilon, t + \varepsilon)]$.

Taking $\varepsilon = 1/n^4$ and using the Borel-Cantelli lemma we find that

$$
\sup_{z,z' \in B_{1/n^4}(s,t)} | M(D_z) - M(D_{z'}) | \leqslant 1/n, \qquad \text{for} \quad n \geqslant n_0, \quad \text{a. e.}
$$

Thus, a. e., for each $\eta > 0$ there exists $\delta > 0$ such that

$$
\sup_{\substack{\|z - z'\| < \delta \\ z,z' \in R_{z_0}}} | M(D_z) - M(D_{z'}) | < \eta,
$$

and, therefore, $M(D_z)$ is continuous. □

We can prove the existence of such a family of stopping sets for a particular case.

Proposition 2.2. — If $\phi^2(s, t)$ is an increasing function of s, such that $\int_0^\infty \phi^2(s, y) dy = \infty \forall s$, then there exist families of stopping sets $\{ D_z, z \in \mathbb{R}_+^2 \}$ with properties (2.3) and (2.6).

Proof. — Fix $(s, t) \in \mathbb{R}_+^2$ and consider the function

$$
f_t(x, \omega) = \inf \left\{ y' \Big/ \int_0^{y'} \phi^2(x, y; \omega) dy > t \right\},
$$

defined for $0 \leqslant x \leqslant s$.

Then, define $D_{st}(\omega)$ as the closed hull of

$$\{ (x, y)/0 \leqslant x \leqslant s, 0 \leqslant y \leqslant f_t(x, \omega) \} . \tag{2.7}$$

$\{ D_z, z \in R_+^2 \}$ is a family of stopping sets satisfying (2.3) and (2.6). Indeed, since $\phi^2(s, t ; \omega)$ is increasing in s, $f_t(x, \omega)$ is decreasing for all $\omega \in \Omega$ and $D_{st}(\omega)$ verifies *ii)* of Definition 2.1 for all $(s, t) \in R_+^2$ and $\omega \in \Omega$. If $0 < x \leqslant s$ we have

$$\{ (x, y) \in D_{st}(\omega) \} = \{ \omega/0 \leqslant y \leqslant f_t(x', \omega) ; \text{for all } x' < x \} \in \mathscr{F}_{xy} ,$$

due to the continuity of the family $\{ \mathscr{F}_z, z \in R_+^2 \}$, and this implies the progressive measurability of $\{ 1_{D_{st}}(z), z \in R_+^2 \}$ in view of the remark after Proposition 2.1 of [*3*].

Conditions (2.3) and (2.6) can be easily verified. □

We can apply this result to an example studied by R. Cairoli and J. B. Walsh in [*2*]:

$$\phi^2(s, t) = \begin{cases} 1 & \text{if } s \cdot t \leqslant 1 \\ 2 & \text{if } s \cdot t > 1 . \end{cases}$$

In this case ϕ^2 is increasing in s, and we have

$$f_t(x) = \begin{cases} t & \text{if } xt \leqslant 1 , \\ \left(t + \dfrac{1}{x}\right)/2 & \text{if } xt > 1 . \end{cases}$$

The family of the corresponding stopping sets is

$$D_{st} = \begin{cases} R_{st} & \text{if } st \leqslant 1 \\ R_{(1/t,t)} \cup \left\{ (x, y)/0 \leqslant y \leqslant \left(t + \dfrac{1}{x}\right)/2, 1/t \leqslant x \leqslant s \right\} & \text{if } st > 1 . \end{cases}$$

Then, $\{ M(D_z), z \in R_+^2 \}$ is a two-parameter Wiener process, although it is shown in [*2*] that $\{ M_{\Gamma(z)}, z \in R_+^2 \}$ cannot be a two-parameter Wiener process for any transformation $\Gamma : R_+^2 \rightarrow R_+^2$.

If condition $\int_0^\infty \phi^2(s, y)dy = \infty$ does not hold, then the stopping sets D_z given by (2.7) exist only for points $z = (s, t)$ verifying

$$P\left\{ \int_0^\infty \phi^2(s, y)dy \geqslant t \right\} = 1 .$$

The following proposition states the local existence of the family D_z when the function $\phi^2(z)$ is smooth. It is clear form the next section that these families will never be unique.

PROPOSITION 2.3. — Suppose that there exists a neighborhood V of (0, 0) in R_+^2, where $\left|\frac{\partial\phi^2}{\partial x}(z)\right| \leqslant K < \infty$, and $\phi^2(z) \geqslant a > 0$ for all $z \in V$, $\omega \in \Omega$. Then, there exists a family of stopping sets $\{ D_z, z \in R_{z_0} \}$ with properties (2.3) and (2.6).

Proof. — Let $z_0 = (s_0, t_0)$ such that $R_{z_0} \subset V$, and $s_0 t_0 \leqslant a/K$.

Generalizing the method used in proposition (2.2) we can consider a decreasing continuous function $\beta : R_+ \to R_+$, and define

$$f_t(x, \omega) = \inf \left\{ y' \Big/ \int_0^{y'} \phi^2(x, y ; \omega) dy = \beta(x) t \right\}, \qquad \inf \phi = \infty . \quad (2.8)$$

Let $\alpha(s) = \inf \left\{ x \Big/ \int_0^x \beta(x')dx' = s \right\}$, $\inf \phi = \infty$.

Define D_z equal to the closed hull of

$$\{ (x, y)/0 \leqslant x \leqslant \alpha(s), 0 \leqslant y \leqslant f_t(x, \omega) \} . \quad (2.9)$$

Then, if we can prove that $f_t(x, \omega)$ is decreasing and finite for all $t \leqslant t_0$, and $\omega \in \Omega$, the family D_z will verify the required properties.

Set $\beta(x) = a - Kx$. Then $f_t(s) \leqslant t$ for all $s \leqslant s_0$, and $t \leqslant t_0$. From $\int_0^{f_t(x)} \phi^2(x, y)dy = (a - Kx)t$, taking derivatives with respect to x, we obtain, for all $(s, t) \in R_{z_0}$,

$$\int_0^{f_t(x)} \frac{\partial\phi^2}{\partial x} dy + \phi^2(x, f_t(x)) f_t'(x) = - Kt .$$

Finally, the inequalities

$$\left| \int_0^{f_t(x)} \frac{\partial\phi^2}{\partial x} dy \right| \leqslant \int_0^t \left| \frac{\partial\phi^2}{\partial x} \right| dy \leqslant Kt ,$$

imply $f_t'(x) \leqslant 0$.

3. TIME CHANGE FOR A TWO-PARAMETER WIENER PROCESS

Let $\{ D_z, z \in R_2^+ \}$ be a family of deterministic stopping sets. That means D_{st} is a closed subset of R_+^2 containing (0, 0), and such that $z \in D_{st}$ implies $R_z \subset D_{st}$, for all $(s, t) \in R_+^2$.

In this section we study all families such that $\{ W(D_z), z \in R_+^2 \}$ is a two-parameter Wiener process whenever $\{ W_z, z \in R_+^2 \}$ is.

By Theorem 2.1 it is enough to suppose that the family $\{ D_z, z \in R_+^2 \}$ verifies

$$D_{z_1} \cap D_{z_2} = D_{z_1 \wedge z_2}, \qquad \text{for all} \qquad z_1, z_2 \in R_+^2, \tag{3.1}$$

$$m(D_{st}) = s \cdot t, \qquad \text{for all} \qquad (s, t) \in R_+^2. \tag{3.2}$$

Every family of stopping sets with these properties gives rise to an isometry

$$T : L^2(R_+^2, \mathscr{B}, m) \rightarrow L^2(R_+^2, \mathscr{B}, m)$$

defined by $T(1_{R_z}) = 1_{D_z}$.

Indeed, let us define $T(1_{(uv,st]}) = 1_{D_{st}} - 1_{D_{ut}} - 1_{D_{sv}} + 1_{D_{uv}}$ and extend T by linearity to all linear combinations of characteristic functions of rectangles. By (3.1) and (3.2) T preserves inner products, so that it has a unique extension to an isometry of $L^2(R_+^2, \mathscr{B}, m)$.

If $B \in \mathscr{B}$, and $m(B) < \infty$, $T(1_B)$ equals (m-almost everywhere) the characteristic function of a Borel set which will be denoted by $\tau(B)$. The transformation τ is an endomorphism of the σ-field $\mathscr{B}$ which preserves Lebesgue measure, that is,

$$\tau\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} \tau(B_n), \qquad \tau\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} \tau(B_n),$$

$$m(B) = m(\tau(B)), \qquad \text{for all} \qquad B_n \text{ and } B \text{ in } \mathscr{B}.$$

R. Cairoli and J. B. Walsh showed (see [2]) that whenever D_z is a rectangle $R_{g(z)}$, for all $z \in R_+^2$, then the function $g : R_+^2 \rightarrow R_+^2$ belongs to the group $\mathscr{G}$ of transformations of R_+^2 generated by $g_+(s, t) = (t, s)$ and $g_\lambda(s, t) = \left(\lambda s, \frac{1}{\lambda} t\right)$, $\lambda \in R_+$.

In section 3 we give an extension of this result for multi-parameter processes.

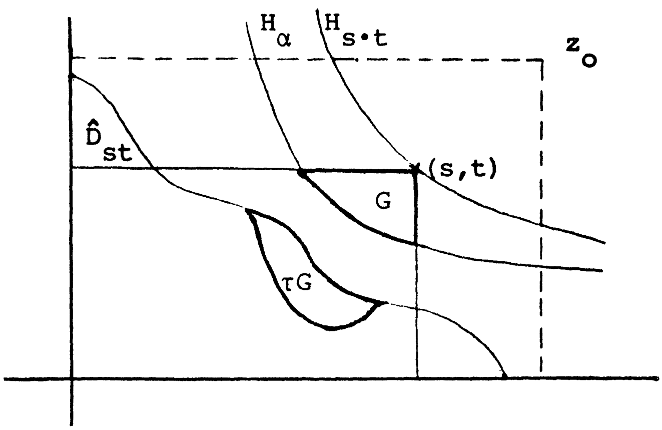
PROPOSITION 3.1. — Let $\{ D_z, z \in R_+^2 \}$ be a family of deterministic stopping sets with properties (3.1) and (3.2). If D_{z_0} is a rectangle for some $z_0 \in R_+^2$, there exists $g \in \mathscr{G}$ such that $D_z = R_{g(z)}$, for all $z < z_0$.

Proof. — Let $D_{z_0} = R_{z_1}$, and g be a transformation of $\mathscr{G}$ (which is unique if it has the form g_λ) such that $g(z_0) = z_1$. Then, $\{ g^{-1}(D_z), z \in R_+^2 \}$ is a family of stopping sets having the same properties. We will denote it by $\{ \hat{D}_z, z \in R_+^2 \}$. We have $\hat{D}_{z_0} = R_{z_0}$. The associated transformation $\tau : \mathscr{B} \rightarrow \mathscr{B}$ (defined m-almost everywhere) maps the Borel sets of R_{z_0} into Borel sets of R_{z_0}.

We will show that for all $z < z_0$, $\hat{D}_z$ is a rectangle. For all $c > 0$ consider the sets $H_c = \{ (s, t) \in R_+^2 / s \cdot t = c \}$, and $S_c = \{ (s, t) \in R_+^2 / s \cdot t \leqslant c \}$.

For all $(s, t) \in R_+^2$ we have $\hat{D}_{st} \subset S_{s\cdot t}$. For, if $z \in \hat{D}_{st}$, $R_z \subset \hat{D}_{st}$, and, consequently $m(R_z) \leqslant s\cdot t$; then $z \in S_{s\cdot t}$.

We also have $\hat{D}_{st} \cap H_{s\cdot t} \neq \phi$. Indeed, suppose that $\hat{D}_{st} \cap H_{s\cdot t} = \phi$. Then, $\alpha = \sup \{ x\cdot y/(x, y) \in \hat{D}_{st} \} < s\cdot t$, and, therefore, the set $G = R_{st} - S_\alpha$ has positive measure.



τ is an endomorphism of the σ-field of Borel sets of R_{z_0} which preserves Lebesgue measure, so it follows that

$$G \subset \limsup \tau^n(G), \qquad m\text{-a. e.}$$

We also have $(\tau^n G) \cap G = \phi$, for all $n > 0$, since $G \cap S_\alpha = \phi$, but $\tau G \subset \hat{D}_{st} \subset S_\alpha$ (which is a consequence of the definition of α), and $\tau S_\alpha \subset S_\alpha$. And that leads to a contradiction.

Therefore, there exists a point $(u, v) \in \hat{D}_{st} \cap H_{s\cdot t}$, and we must have $\hat{D}_{st} = R_{uv}$.

We define $\gamma : R_{z_0} \to R_{z_0}$ by means of the equality $\hat{D}_{st} = R_{\gamma(s,t)}$. Using methods analogous to those of [*2*] we can prove that γ is order-preserving, and, therefore, it coincides on each H_c with a transformation of $\mathcal{G}$. This can only be the identity map or g_+. Consequently, $\gamma = g_+$ or $\gamma = \text{Id.}$, and, therefore, for all $z < z_0$, $D_z = g(\hat{D}_z) = R_{g(z)}$ or $D_z = R_{gg_+(z)}$. □

COROLLARY. — Let $\{ W_z, z \in R_+^2 \}$ be a two-parameter Wiener process. If $\{ D_z, z \in R_{z_0} \}$ is a family of deterministic stopping sets contained in R_{z_0}, and such that $\{ W(D_z), z \in R_{z_0} \}$ is a two-parameter Wiener process, then for all $(s, t) \in R_+^2$ we have $D_{st} = R_{g(s,t)}$, where $g = \text{Id.}$ or

$$g(s, t) = (ts_0/t_0, st_0/s_0) .$$

If we do not restrict ourselves to a rectangle R_{z_0}, there exist, as we shall see, families of deterministic stopping sets $\{ D_z, z \in R_+^2 \}$ satisfying properties (3.1) and (3.2) which are not of the form $D_z = R_{g(z)}, g \in \mathcal{G}$.

Notice that the set $H = \{ z \in R_+^2 / D_z \text{ is a rectangle} \}$ is closed and satisfies $z \in H$ implies $R_z \subset H$. Therefore, there exists $g \in \mathcal{G}$ such that $\tau(A) = g(A)$ for all $A \subset H$, but we cannot determine the transformation τ outside H.

We will now study conditions under which there exists a continuous function $f : R_+^2 \to R_+^2$ such that $D_z = f(R_z)$, for all $z \in R_+^2$.

Notice that every deterministic stopping set can be determined by a decreasing curve: if $\overset{\circ}{D} = \{ z \in D / \exists z' \in D, z \ll z' \}$ the set $L = D - \overset{\circ}{D}$ is either the empty set, and then D is the union of two segments on the axes, or it is a continuous curve $\{ \theta(t), t \in [0, 1] \}$ such that, if $t_1 \leqslant t_2$, then $\theta(t_1) \curlywedge \theta(t_2)$. The points $\theta(0)$ and $\theta(1)$ may be infinite; $\theta(0) \in OY$ and $\theta(1) \in OX$ if $m(D)$ is finite. L will be called the stopping line associated to D.

For all $(s, t) \in R_+^2$ we define $D_s^1 = \bigcup_{\tau \geqslant 0} D_{s\tau}$, $D_t^2 = \bigcup_{\sigma \geqslant 0} D_{\sigma t}$. These sets clearly satisfy

$$z \in D_s^1 \quad \text{implies} \quad R_z \subset D_s^1 ; \qquad z \in D_t^2 \quad \text{implies} \quad R_z \subset D_t^2 .$$

Moreover, $D_s^1 \cap D_t^2 = D_{st}$ and D_s^1, D_t^2 are closed sets. In order to show this, choose a sequence z_n in D_s^1 with limit z; z_n is bounded, therefore, there exists τ such that $z_n \in D_{s\tau}$, for all n; so it follows that $z \in D_{s\tau} \subset D_s^1$.

Therefore, D_s^1 and D_t^2 are stopping sets. Let λ_s^1 and λ_t^2 be the stopping lines associated to D_s^1 and D_t^2, respectively. Notice that these curves are non disjoint, and their intersection, $\lambda_s^1 \cap \lambda_t^2$, is contained in the boundary of D_{st}.

Proposition 3.2. — Let $\{ D_z, z \in R_+^2 \}$ be a family of deterministic stopping sets verifying (3.1) and (3.2). Suppose that, for all $(s, t) \in R_+^2$, the curves λ_s^1 and λ_t^2 intersect in a unique point, then, there exists an one to one, continuous function $f : R_+^2 \to R_+^2$ which preserves Lebesgue measure and such that $D_z = f(R_z)$, for all $z \in R_+^2$.

Proof. — Define $f(s, t)$ as the intersection of λ_s^1 with λ_t^2. Let $z_n = (s_n, t_n)$ be a sequence whose limit is $z = (s, t)$. If $s_n \neq s$ and $t_n \neq t$, the curves $\lambda_{s_n}^1$, $\lambda_s^1, \lambda_{t_n}^2, \lambda_t^2$, intersect in four points and determine a compact set A_n, whose area is the same as that of the rectangle determined by z_n and z. This area goes to zero when n tends to ∞. We have $\bigcap_{n=1}^{\infty} A_n \subset \lambda_s^1 \cap \lambda_t^2$, so $\bigcap_{n=1}^{\infty} A_n = \{ f(z) \}$, and, therefore, $f(z) \underset{n \to \infty}{\longrightarrow} f(z)$.

Since τ is one to one and order-preserving, it follows immediately that f is one to one and $f(R_z) = D_z$, for all $z \in R_+^2$. It can also be shown that $\tau(B) = f(B)$ for all $B \in \mathcal{B}$ and f preserves Lebesgue measure. □

Conversely, if we have a continuous, one to one function $f : \mathrm{R}_+^2 \to \mathrm{R}_+^2$ which preserves Lebesgue measure and such that $\mathrm{R}_z \subset f(\mathrm{R}_{st})$ if $z \in f(\mathrm{R}_{st})$, then $\{ f(\mathrm{R}_z), z \in \mathrm{R}_+^2 \}$ is a deterministic stopping sets family satisfying (3.1) and (3.2). Henceforth, we restrict our study to this kind of families.

The set $\mathscr{H}$ of such functions have semigroup structure with the composition as operation and contains $\mathscr{G}$. Each function $f \in \mathscr{H}$ determines a set

$$\mathrm{H}_f = \{ z \in \mathrm{R}_+^2 / f(\mathrm{R}_z) \text{ is a rectangle} \} = \{ (s, t) \in \mathrm{R}_+^2 / f_1(s, t) \cdot f_2(s, t) = s \cdot t \} ,$$

where $f(z) = (f_1(z), f_2(z))$; and we know that the function f coincides on H_f with a transformation of the group $\mathscr{G}$, because of Proposition 3.1.

Let us consider the set

$$\mathscr{D} = \{ \mathrm{H} \subset \mathrm{R}_+^2 / \mathrm{H} \text{ is closed, } \mathrm{E}_0 \subset \mathrm{H}, \text{ and } z \in \mathrm{H} \text{ implies } \mathrm{R}_z \subset \mathrm{H} \} ,$$

and for each $\mathrm{H} \in \mathscr{D}$ let us write

$$\mathscr{H}_\mathrm{H} = \{ f \in \mathscr{H} / \mathrm{f}(z) = z, \ \forall z \in \overset{\circ}{\mathrm{H}} \text{ and } f_1(s, t) \cdot f_2(s, t) < s \cdot t, \ \forall (s, t) \notin \mathrm{H} \} .$$

We have a partition $\mathscr{H} = \bigcup_{\mathrm{H} \in \mathscr{D}} \mathscr{G} \circ \mathscr{H}_\mathrm{H}$. For, if $f \in \mathscr{H}$, there exists $g \in \mathscr{G}$ such that f coincides with g on H_f, and, therefore, $g^{-1} \circ f \in \mathscr{H}_{\mathrm{H}_f}$.

Let $\mathrm{H} \in \mathscr{D}$ be fixed and let $\alpha = \sup_{(s,t) \in \mathrm{H}} s \cdot t$; suppose $0 < \alpha < \infty$. A necessary condition for $\mathscr{H}_\mathrm{H}$ to be non empty is that $\mathrm{H} \cap \mathrm{H}_\alpha$ is a connected set. Indeed, if $f \in \mathscr{H}_\mathrm{H}$, then $\mathrm{H} \cap \mathrm{H}_\alpha$ is a connected set, since, given $z_1, z_2 \in \mathrm{H} \cap \mathrm{H}_\alpha$, we have $\mathrm{R}_{z_1} \cup \mathrm{R}_{z_2} \subset \mathrm{H}$. If G is the bounded connected component of $\mathrm{S}_\alpha - (\mathrm{R}_{z_1} \cup \mathrm{R}_{z_2})$, the properties of f imply that $f(\mathrm{G}) = \mathrm{G}$, and, therefore, f is the identity map on the segment of the hyperbola H_α determined by z_1 and z_2.

Conversely, if $\mathrm{H} \cap \mathrm{H}_\alpha$ is a connected set we may expect that $\mathscr{H}_\mathrm{H} \neq \phi$, and we will prove it when H is the rectangle R_{11}.

Besides, if $\alpha = 0$, $\mathrm{H} = \mathrm{E}_0$ and, as we shall see, $\mathscr{H}_{\mathrm{E}_0} \neq \phi$.

The C^1-class functions of the semigroup $\mathscr{H}$ are those functions $f(z) = (f_1(z), f_2(z))$ such that $f(s, t)$, or $f(t, s)$ are solutions to the following differential equation:

$$\frac{\partial f_1}{\partial s} \cdot \frac{\partial f_2}{\partial t} - \frac{\partial f_1}{\partial t} \cdot \frac{\partial f_2}{\partial s} = 1 ,$$

with the constraints

(1) $$\frac{\partial f_1}{\partial t} \leqslant 0 , \quad \frac{\partial f_2}{\partial s} \leqslant 0$$

(2) $$f_1(0, t) = f_2(s, 0) = 0 .$$

For example, if $\psi(s) : \mathrm{R}_+ \to \mathrm{R}_+$ is a C^2-class function with $\psi(0) = 0$, $\psi'(s) > 0$ and $\psi''(s) \geqslant 0$, then $f(s, t) = (\psi(s), t/\psi'(s))$ belongs to $\mathscr{H}$. In the case $\psi(s) = \lambda s$, $\lambda > 0$, we obtain functions of $\mathscr{G}$ (*).

If $\psi(s) = e^s - 1$, $f(s, t) = (e^s - 1, t \cdot e^{-s})$ belongs to $\mathscr{H}_{\mathrm{E}_0}$.

Consider the functions

$$f(s, t) = \begin{cases} (s, t) & \text{if} \quad 0 \leqslant s \leqslant 1 \\ (e^{s-1}, t \cdot e^{1-s}) & \text{if} \quad s > 1 \,, \end{cases}$$

$$g(s, t) = \begin{cases} (s, t) & \text{if} \quad 0 \leqslant t \leqslant 1 \\ (s \cdot e^{1-t}, e^{t-1}) & \text{if} \quad t > 1 \,. \end{cases}$$

It is easy to verify that $f \in \mathscr{H}_{\mathrm{H}_1}$, $g \in \mathscr{H}_{\mathrm{H}_2}$, where $\mathrm{H}_1 = \{ (s, t) \in \mathrm{R}_+^2 / 0 \leqslant s \leqslant 1 \}$ and $\mathrm{H}_2 = \{ (s, t) \in \mathrm{R}_+^2 / 0 \leqslant t \leqslant 1 \}$. Consequently, $f \circ g \in \mathscr{H}_{\mathrm{R}_{11}}$.

Let $\mathrm{M}_{st} = \displaystyle\int_{\mathrm{R}_{st}} \phi(z) d\mathrm{W}_z$, $\phi \in \mathrm{L}_{\mathrm{W}}^2$, be a strong martingale and $\{ \mathrm{D}_z, z \in \mathrm{R}_+^2 \}$ a family of stopping sets verifying properties (2.3) and (2.6). Then, by Theorem 2.1, $\{ \mathrm{M}(\mathrm{D}_z), z \in \mathrm{R}_+^2 \}$ is a two-parameter Wiener process. The results obtained in section 3 imply that this family is not unique; for, $\{ f(\mathrm{D}_z), z \in \mathrm{R}_+^2 \}$ is another family of stopping sets with the same properties, for each $f \in \mathscr{H}$.

4. EXTENSION TO *n* PARAMETERS

The partial order in R_+^n will be $(s_1, \ldots, s_n) < (t_1, \ldots, t_n)$ if $s_i \leqslant t_i$, $i = 1, \ldots, n$. Let $\{ \mathrm{W}_t, t \in \mathrm{R}_+^n \}$ be a n-parameter Wiener process, that means a separate Gaussian zero mean process with covariance function

$$\mathrm{E} \{ \mathrm{W}_s \cdot \mathrm{W}_t \} = \prod_{i=1}^{n} (s_i \wedge t_i) \,.$$

Let $\{ \mathscr{F}_t, t \in \mathrm{R}_+^n \}$ be the increasing family of σ-fields generated by W_t, that is, $\mathscr{F}_t = \sigma(\mathrm{W}_{t'}, t' < t)$.

The following result gives a different proof and generalizes to n-parameter processes the Lemma 4 of [*2*].

Proposition 4.1. — Let $f : \mathrm{R}_+^n \to \mathrm{R}_+^n$ be a mapping such that

(*) The stopping sets $\mathrm{D}_z = f(\mathrm{R}_z)$ of this kind coincide (for a fixed ω) with the stopping sets defined by (2.9), where $\phi^2 = 1$ and $\beta(x) = 1/\psi'(\psi^{-1}(x))$.

$\{ W_{f(t)}, t \in R^n_+ \}$ is an n-parameter Wiener process, then f has the form

$$f(s_1, \ldots, s_n) = f(\lambda_1 s_{\varepsilon(1)}, \ldots, \lambda_n s_{\varepsilon(n)}),$$

where $\lambda_1, \ldots, \lambda_n$ are positive real numbers with $\lambda_1 ::: \lambda_n = 1$ and ε is a permutation of $\{ 1, \ldots, n \}$. The set of these functions is a group $\mathscr{G}$.

Proof. — If Γ is the covariance function of the n-parameter Wiener process, the function f verifies

$$\Gamma(s, t) = \Gamma(f(s), f(t)), \qquad \text{for all} \qquad s, t \in R^n_+ .$$

It can be shown, as in [*2*], that f is order-preserving, that is, $s < t$ if and only if $f(s) < f(t)$.

Define $D_t = R_{f(t)}$ for all $t \in R^n_+$. Then $\{ D_t, t \in R^n_+ \}$ is a family of deterministic stopping sets (for all $t \in R^n_+$, $0 \in D_t$, D_t is closed and $s < t$ implies $R_s \subset D_t$) verifying

$$D_s \cap D_t = D_{s \wedge t}, \qquad \text{for all} \qquad s, t \in R^n_+ \tag{4.1}$$

$$m(D_s) = s_1 ::: s_n, \qquad \text{for all} \qquad s = (s_1, \ldots, s_n) \in R^n_+ \tag{4.2}$$

where m is the Lebesgue measure on the Borel σ-field $\mathscr{B}$ of R^n_+.

(4.1) holds because f is order-preserving, and (4.2) follows from $m(D_s) = \Gamma(f(s), f(s)) = \Gamma(s, s)$.

This family $\{ D_t, t \in R^n_+ \}$ gives rise, as in section 3, to a transformation $\tau : \mathscr{B} \to \mathscr{B}$ (defined m-almost everywhere) which preserves Lebesgue measure and the set operations, and such that $\tau(R_t) = D_t = R_{f(t)}$, for all $t \in R^n_+$.

We will first prove that f leaves invariant the class of lines parallel to a coordinate axis.

Given a point $s = (s_1, \ldots, s_n)$ denote by $\bar{s}_i = (s_1, \ldots, \hat{s}_i, \ldots, s_n) \in R^{n-1}_+$, and write $s = (s_i, \bar{s}_i)$. Fix the coordinates of $\bar{s}_i$ and consider the line $\{ (\sigma, \bar{s}_i), \sigma \geqslant 0 \}$, then, the points $\{ f(\sigma, \bar{s}_i), \sigma \geqslant 0 \}$ have all but one coordinates equal. Indeed, choose $\varepsilon > 0$; then $f(\bar{s}_i) < f(s_i + \varepsilon, \bar{s}_i)$, and, therefore, the regions

$$R_{f(s_i+\varepsilon,\bar{s}_i)} - R_{f(s_i,\bar{s}_i)} = \tau(R_{(s_i+\varepsilon,\bar{s}_i)}) - \tau(R_{(s_i,\bar{s}_i)}) = \tau(R_{(s_i+\varepsilon,\bar{s}_i)} - R_{(s_i,\bar{s}_i)})$$

are mutually disjoint because so are the sets $R_{(s_i+\varepsilon,\bar{s}_i)} - R_{(s_i,\bar{s}_i)}$, $i=1, \ldots, n$.

For each $i = 1, \ldots, n$, denote by $\varepsilon(i)$ the non constant component of the points $\{ f(\sigma, \bar{s}_i), \sigma \geqslant 0 \}$. We have to show that $\varepsilon(i)$ does not depend on the point s.

Let $t = (t_1, \ldots, t_n) \in R^n_+$ be another point. For a fixed i, we want to prove that the lines determined by $\{ f(\sigma, \bar{s}_i), \sigma \geqslant 0 \}$ and $f(\sigma, \bar{t}_i), \sigma \geqslant 0$

are parallel. It suffices to consider the case $\bar{s}_i < \bar{t}_i$. Then, $f(\sigma, \bar{s}_i) < f(\sigma, \bar{t}_i)$, for all $\sigma \geqslant 0$. Both lines given by $\{ f(\sigma, \bar{s}_i), \sigma \geqslant 0 \}$ and $\{ f(\sigma, \bar{t}_i), \sigma \geqslant 0 \}$ are parallel to a coordinate axis, and they contain a set of different ordered couples of points; therefore, they are parallel.

Let $f_1(s), \ldots, f_n(s)$ be the components of $f(s)$. For all $\sigma, \sigma' \geqslant 0$, we have

$$f_1(\sigma, \bar{s}_i) ::: f_n(\sigma, \bar{s}_i) = s_1 ::: s_{i-1} \cdot \sigma \cdot s_{i+1} ::: s_n ,$$
$$f_1(\sigma', \bar{s}_i) ::: f_n(\sigma', \bar{s}_i) = s_1 ::: s_{i-1} \cdot \sigma' \cdot s_{i+1} ::: s_n .$$

Consequently, if all the components are positive,

$$f_{\varepsilon(i)}(\sigma, \bar{s}_i) \cdot \sigma' = f_{\varepsilon(i)}(\sigma', \bar{s}_i) \cdot \sigma ,$$

and the quotient $f_{\varepsilon(i)}(\sigma, \bar{s}_i)/\sigma$ is independent of σ. Denote it by $\lambda_i(\bar{s}_i)$. Then, $f_{\varepsilon(i)}(s) = \lambda_i(\bar{s}_i)s_i$.

Each coordinate $\lambda_i(\bar{s}_i)s_i$ must not depend on the individual variations of s_j for $j \neq i$. Therefore, $\lambda_i(\bar{s}_i)$ is a constant, say λ_i, and the Proposition is proved. □

For a more general changing time of a n-parameter Wiener process using a family $\{ \mathrm{D}_t, t \in \mathbf{R}^n_+ \}$ of deterministic stopping sets, all results of section 3 may be properly extended. In particular, if the family satisfies (4.1) and (4.2), $\{ \mathrm{W}(\mathrm{D}_t), t \in \mathbf{R}^n_+ \}$ is a n-parameter Wiener process and we could develop an analogous characterization of such families of stopping sets.

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