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Correction to "Domains of attraction in several dimensions"

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of attraction in several dimensions »

by

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For $p > 2$ let X be a l_p-valued random variable defined as follows:

(1) $$X = \varepsilon \Sigma_{N^2 < k \leq N^2 + N} e_k$$

where $\{ e_k \}$ is the canonical basis of l_p, ε is a Bernoulli random variable with parameter $1/2$ and N is a random variable independent of ε and with distribution

$P\{ N = n \} = c/n^{1 + 2/p}$ for $n \in [2, 2^2) \cup [2^{2^2}, 2^{2^3}) \cup \ldots \cup [2^{2^{2k}}, 2^{2^{2k+1}}) \cup \ldots$
$P\{ N = n \} = 0$ otherwise .

In example 3.5 of [2] it is claimed that X is in the domain of partial attraction of only one law (one type of laws), the centered Gaussian law γ with the covariance of X, which exists, but that nevertheless X is not in its domain of attraction. To the best of our knowledge this statement is true, however the proof given in [2] of the fact that X is actually in the domain of partial attraction of γ is not correct. In this note we give a (hopefully) correct proof of this.

The notation will be as in [2]; in particular, if $x \in l_p$, $x_{(k)}$ will denote its k-th coordinate in the canonical basis.

We need the following

PROPOSITION. — Assume that X is a l_p-valued symmetric random variable.

$p \geq 2$. Let $\{X_i\}$ be independent identically distributed copies of X, and let $n_k \uparrow \infty$, $n_k \in \mathbb{N}$. Then, in order that the sequence

$$(2) \qquad \{\mathscr{L}(\Sigma_{i=1}^{n_k} X_i / n_k^{1/2})\}_{k=1}^{\infty}$$

converge weakly to a Gaussian law it is (necessary and) sufficient that the following conditions hold:

   i) X is pregaussian,
   ii) for every $\delta > 0$, $\lim_{k \to \infty} n_k P\{\|X\| > \delta n_k^{1/2}\} = 0$,
   iii) $\lim_{R \to \infty} \sup_k n_k^{1-p/2} \Sigma_{m=R+1}^{\infty} E |X_{(m)}|^p I_{[\|X\| \leq n_k^{1/2}]} = 0$.

This proposition follows directly from Theorem 3.1 in [3]. See also exercises 13 and 17, p. 205-206, in [1].

We will prove that the variable X defined in (1) satisfies conditions (i)-(iii) of the previous proposition for the sequence $\{n_k\}$ given by

$$(3) \qquad n_k = [2^{2^{2k+3/2}(2/p)}], \qquad k \in \mathbb{N}$$

where [ ] denotes « integer part of ».

It is already proved in [2] that X is pregaussian. So we need only prove that X verifies properties (ii) and (iii).

Proof of ii). — Obviously, given $\delta > 0$ there exists k_δ such that for $k > k_\delta$

$$2^{2^{2k+1}} < \delta^p n_k^{p/2}.$$

Since $\|X\| = N^{1/p}$ it follows that for $k > k_\delta$,

$$n_k P\{\|X\| > \delta n_k^{1/2}\} = n_k P\{N > \delta^p n_k^{p/2}\}$$
$$\leq c2^{2^{2k+3/2}(2/p)} \Sigma_{n=2^{2^{2k+2}}}^{\infty} n^{-1-2/p} \approx 2^{-1} cp 2^{2^{2k}(2/p)(2^{3/2}-4)}$$
$$\to 0 \qquad \text{as } k \to \infty.$$

Proof of iii). — Let us first observe that

$$(4) \quad \Sigma_{m=R+1}^{\infty} E|X_{(m)}|^p I_{[\|X\| \leq n_k^{1/2}]} = \Sigma_{m=R+1}^{\infty} P\{N^2 < m \leq N^2 + N, N \leq n_k^{p/2}\}$$

$$\leq \Sigma_{s=[R^{1/2}]}^{\infty} sP\{N = s, N \leq n_k^{p/2}\} = \begin{cases} 0 & \text{if } [R^{1/2}] > n_k^{p/2} \\ \Sigma_{s=[R^{1/2}]}^{n_k^{p/2}} sP\{N = s\} & \text{otherwise.} \end{cases}$$

Let $r \in \mathbb{N}$ be such that $2^{2^{2r}} \leq [R^{1/2}] < 2^{2^{2r+2}}$. By (4) the limit in (iii) is then bounded above by

$$\lim_{R \to \infty} \sup_{k \geq r} cn_k^{1-p/2} \int_{[2^{2^{2r}-1}, 2^{2^{2r+1}}] \cup \ldots \cup [2^{2^{2k}-1}, 2^{2^{2k+1}}]} x^{-2/p} dx$$
$$\leq \lim_{r \to \infty} \sup_{k \geq r} c(1 - 2/p)^{-1} n_k^{1-p/2} k 2^{2^{2k+1}(1-2/p)}$$
$$= \lim_{r \to \infty} c(1 - 2/p)^{-1} r 2^{2^{2r+1}(1-2/p)(1-2^{1/2})} = 0.$$

The proof that X satisfies (*ii*) and (*iii*) is thus completed. Hence, X is in the domain of partial attraction of the centered Gaussian law which has its covariance.

There is a trivial error in [2] which we also correct now. The last two lines of the proof of Theorem 3.1, part (2) should read: « ... therefore, if $\mu \{ \| x \| > 0 \} = 0$ we have $P \{ \| X \| > a_{n_i} \} / a_{n_i}^{-2} E \| X \|^2 I_{\{ \|X\| \leq a_{n_i} \}} \to 0$ as ... ».

REFERENCES

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[3] E. GINÉ, V. MANDREKAR, J. ZINN, On sums of independent random variables with values in L_p, $2 \leq p < \infty$. *Lectures Notes in Math.*, t. **709**, 1979, p. 111-124.