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## Ornstein-Métivier- Brunel theorem revisited

by

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**SUMMARY.** — We give a general lemma for positive operators on  $L^\infty$  and we apply it to get an easy proof of a theorem of Brunel on Harris processes. The lemma is dual to the one proved by Foguel in order to get a Zéro-two law. We also show that Brunel's result, which is the extension of a theorem of Métivier, is best possible.

**RÉSUMÉ.** — On établit un lemme général sur les opérateurs positifs sur  $L^\infty$  pour donner une démonstration assez simple d'un théorème de Brunel sur les processus de Harris. Un lemme dual avait été démontré par Foguel pour obtenir une loi Zéro ou deux. On montre aussi, que le résultat de Brunel qui est une extension d'un théorème de Métivier est le meilleur possible.

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### 0. INTRODUCTION

In [9], M. Métivier gives an extension of a result of D. Ornstein for random walks [12] to general Markov operators on  $L^\infty$ . He proves mainly that the partial sums of the iterates of a function are uniformly bounded provided it has a zero integral and that it is supported on a « small set » described later by P. A. Meyer [10] as « modeste ». In [2], A. Brunel extends

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this theorem to functions which have their supports contained in a larger class of sets denoted by J. Neveu [11] as « special ».

In this paper, we give a relatively easy proof of Brunel's result, using only elementary measure—theoretic techniques and a general lemma for positive operators on  $L^\infty$ , similar to the one proved by S. R. Foguel in [6] in order to get a zero-two law result.

We also show that the class of sets verifying the theorem is exactly the class of special sets. Finally, we give a direct proof to the fact that the existence of a special set implies that the process is Harris: this result was obtained by S. Horowitz [8] by associating to the Markov operator a transition probability on a Compact stonian space for which the result is known.

For the case where the operator is induced by a transition probability, we refer the reader to the elegant and complete papers of Neveu [11] and Brunel and Revuz [3].

## I. PRELIMINARIES

A Markov process is defined to be a quadruple  $(X, \Sigma, m, P)$  where  $(X, \Sigma, m)$  is a finite measure space with positive measure  $m$  and where  $P$  is a positive contraction on  $L^1(m)$ . In the following, we will only deal with the operator adjoint to  $P$  defined on  $L^\infty(m)$ , which will also be denoted by  $P$ .

Let  $H$  be the class  $\{h : X \rightarrow [0, 1] / h \text{ measurable and } m(h) \neq 0\}$ . We define for every  $h \in H$ , the operator  $U_h = \sum_{n \geq 0} (PI_{1-h})^n P$  and we note as in [11] that for every  $0 \leq h \leq k \leq 1$  the following resolvent equation holds:

$$U_h = U_k + U_h I_{k-h} U_k$$

Recall that  $P$  is said to be conservative and ergodic if and only if for every  $f \geq 0$  and  $m(f) > 0$ . We have  $U_0 f = \sum_{n \geq 1} P^n f = +\infty$ . It is equivalent to say that for every  $h \in H$ ,  $U_h(h) = 1$ . In the following, we will always assume that  $P$  is conservative and ergodic.

A set  $A$  is said to be special if for every  $h \in H$  with  $h \leq 1_A$ , we have  $U_h(1_A) \in L^0$ . Using the results of [1], it is easy to see that the existence of a special set  $A$  implies the existence of a  $\sigma$ -finite,  $P$  invariant measure  $\lambda$  equivalent to  $m$ , and that the restriction  $\lambda_A$  of  $\lambda$  to  $A$  is a finite measure invariant under  $U_A I_A$ .

We also recall that every Markov operator  $P$  can be decomposed into a kernel operator  $Q$  and an operator  $R$  such that  $R$  does not dominate any non-trivial positive kernel. For every  $n \in \mathbb{N}$ , let  $Q_n$  and  $R_n$  be the operators corresponding to the decomposition of  $P^n$ . We say that  $P$  is a Harris operator if there exists an integer  $n$  such that  $Q_n 1 \neq 0$  (See [5]).

Finally, for every function  $f$  in  $L^\infty$  and measure  $\mu$ , we denote by  $f \otimes \mu$  the operator which maps a function  $g$  to  $\mu(g).f$ .

In the following, we will use frequently the lattice properties of the Riesz space of operators on  $L^\infty$  and the fact that the class of kernel operators is an order ideal in that space. For more details we refer to [5] and [14].

## II. MAIN RESULTS

Before proceeding to the proof of the main result, we prove a general lemma for operators on  $L^\infty$  similar to the one proved by S. R. Foguel in [6] in order to get a zero-two law result.

LEMMA (1). — Let  $P, Q_1$  and  $Q_2$  be three commuting positive Markov operators with  $P1 = Q_1 1 = Q_2 1 = 1$ . If there exists a positive Markov operator  $R$  such that  $R1 = \delta > 0, P^r \geq Q_1 R$  and  $P^r \geq Q_2 R$  where  $\delta$  is a real number and  $r$  is some integer, then:

$$\lim_{n \rightarrow \infty} \|(Q_2 - Q_1)P^n\| = 0,$$

*Proof.* — Put  $P^r = Q_1 R + S'_1 = Q_2 R + S''_1$  then  $P^r = \frac{1}{2}(Q_1 + Q_2)R + S_1$  where  $S_1 = \frac{1}{2}(S'_1 + S''_1) \geq 0$ . Also  $1 = P^r 1 = \delta + S_1 1$ . Let us prove by induction that

$$i) \quad P^{rn} = \frac{1}{2^n}(Q_1 + Q_2)^n R^n + S_n, \quad S_n \geq 0:$$

$$\begin{aligned} P^{r(n+1)} &= P^r \frac{1}{2^n}(Q_1 + Q_2)^n R^n + P^r S_n = \frac{1}{2^n}(Q_1 + Q_2)^n P^r R^n + P^r S_n \\ &= \frac{1}{2^n}(Q_1 + Q_2)^n \left[ \frac{1}{2}(Q_1 + Q_2)R + S_1 \right] R^n + P^r S_n \\ &= \frac{1}{2^{n+1}}(Q_1 + Q_2)^{n+1} R^{n+1} + S_{n+1} \end{aligned}$$

where

$$S_{n+1} = \frac{1}{2^n}(Q_1 + Q_2)^n S_1 R^n + P^r S_n \geq 0.$$

Apply *i)* to 1:

$$1 = P^{rn} 1 = \frac{1}{2^n}(Q_1 + Q_2)^n R^n 1 + S_n 1 = \delta^n + S_n 1.$$

Thus

$$ii) \quad S_n 1 = \text{Const.} = 1 - \delta^n.$$

Let us improve *i*):

$$iii) \quad P^{rnj} = \frac{1}{2^n} (Q_1 + Q_2)^n T_j + S_n^j, \quad T_j \geq 0 \quad \text{and} \quad T_j 1 = \text{Const.} = \beta_j$$

If  $j = 1$  take  $T_1 = R^n$  and use *i*).  $T_1 1 = \delta^n$ . Now, by (1) and by induction

$$\begin{aligned} P^{rn(j+1)} &= P^{rn} P^{rnj} = P^{rn} \frac{1}{2^n} (Q_1 + Q_2)^n T_j + P^{rn} S_n^j \\ &= \frac{1}{2^n} (Q_1 + Q_2)^n P^{rn} T_j + \left| \frac{1}{2^n} (Q_1 + Q_2)^n R^n + S_n \right| S_n^j \\ &= \frac{1}{2^n} (Q_1 + Q_2)^n T_{j+1} + S_n^{j+1} \end{aligned}$$

where  $T_{j+1} = P^{rn} T_j + R^n S_n^j$ . Thus, by *ii*),  $T_{j+1} \geq 0$  and

$$T_{j+1} 1 = \beta_j + \delta^n (1 - \delta^n)^j = \text{Const.}$$

Note, by *iii*),

$$\beta_j \geq 1.$$

Let us proceed now with the proof of the Lemma. The sequence

$$\| (Q_2 - Q_1) P^n \|$$

is monotone in  $n$ . By *iii*)

$$\| (Q_2 - Q_1) P^{rnj} \| \leq \left\| \frac{1}{2^n} (Q_2 - Q_1) (Q_1 + Q_2)^n \right\| + 2 \| S_n \|^j.$$

Now

$$\left\| \frac{1}{2^n} (Q_2 - Q_1) (Q_1 + Q_2)^n \right\| \leq \frac{1}{2^{n-1}} + \frac{1}{2^n} \sum_{k=0}^{n-1} \left| \binom{n}{k} - \binom{n}{k+1} \right|.$$

Note now that  $\binom{n}{k}$  increases when  $0 \leq k \leq \frac{n}{2}$  and decreases when  $\frac{n}{2} \leq k \leq n$ .

Thus the sum is bounded by  $\frac{1}{2^n} \left[ \frac{n}{2} \right]$  which is  $O\left(\frac{1}{\sqrt{n}}\right)$  by Stirling Formula.

Once  $n$  is fixed choose  $j$  by

$$\| S_n \|^j \leq (1 - \delta^n)^j \xrightarrow{j \rightarrow \infty} 0.$$

We shall also need the following lemmas :

LEMMA (2). — Let  $0 < \theta < 1$ , then  $P$  is a Harris operator if and only if  $U_\theta$  dominates a non-trivial kernel operator.

*Proof.* — It is enough to show that  $Q_\theta = \sum_{n \geq 0} (1 - \theta)^n Q_{n+1}$  is the kernel component of the operator  $U_\theta = \sum_{n \geq 0} (1 - \theta)^n P^{n+1}$ .

For that, let  $R_\theta = \sum_{n \geq 0} (1 - \theta)^n R_{n+1}$  and suppose there exists a kernel  $K$  such that  $R_\theta \geq K \geq 0$ . For each integer  $N$  we have

$$\begin{aligned} 0 \leq K &\leq \left[ \sum_{n=0}^N (1 - \theta)^n R_{n+1} + \sum_{n=N+1}^\infty (1 - \theta)^n R_{n+1} \right] \wedge K \\ &\leq \left[ \sum_{n=0}^N (1 - \theta)^n R_{n+1} \wedge K \right] + \left[ \sum_{n=N+1}^\infty (1 - \theta)^n R_{n+1} \right] \wedge K \\ &= \left[ \sum_{n=N+1}^\infty (1 - \theta)^n R_{n+1} \right] \wedge K \end{aligned}$$

Since for each  $n$ ,  $R_n \wedge K$  is a kernel operator dominated by  $R_n$ , hence it is zero. It follows that  $0 \leq K \leq \sum_{n=N+1}^\infty (1 - \theta)^n R_{n+1}$  for each  $N$ , thus  $K$  is a trivial kernel.

LEMMA (3). — If the whole space  $X$  is special then  $P$  is a Harris operator.

*Proof.* — It is easy to see, as noted in [4], that if  $X$  is special then for every  $h \in H$ , there exists a constant  $b(h)$  depending on  $h$  such that  $U_\theta(h) \geq b(h) > 0$ . We now claim that there exists  $\varepsilon > 0$  such that whenever  $m(B) \geq 1 - \varepsilon$  we have  $U_\theta 1_B \geq \varepsilon$ .

If it is not the case, we may find a sequence of sets  $(A_j)_{j \geq 2}$  such that  $m(A_j) \geq 1 - \frac{1}{2^j}$  and  $U_\theta 1_{A_j} < \frac{1}{2^j}$  on a set  $B_j$  with  $m(B_j) > 0$ . Let now  $E = \bigcap_{j \geq 2} A_j$ . Since  $\sum_{j \geq 2} m(A_j) \leq \frac{1}{2}$  we get that  $m(E) \geq \frac{1}{2}$ . The contradiction follows from the fact that  $U_\theta 1_E \leq U_\theta 1_{A_j} < \frac{1}{2^j}$ .

To show that  $P$  is Harris, we show that  $U_\theta$  dominates a non-trivial kernel operator. For that, denote by  $T$  the one-dimensional operator  $1 \otimes m$  and note that  $U_\theta \wedge T = \inf \left\{ U_\theta g + \int (1 - g)dm \right\}$ ;  $0 \leq g \leq 1$  is a kernel operator since it is dominated by  $T$  [5].

Fix now  $g$  between 0 and 1 and let  $B = \left\{ g \geq \frac{1}{2} \right\}$ . Clearly,  $g \geq \frac{1}{2} 1_B$  and

$$U_\theta g + \int (1 - g)dm \geq \frac{1}{2} U_\theta 1_B + \int_{B^c} (1 - g)dm \geq \frac{1}{2} U_\theta 1_B + \frac{1}{2} m(B^c).$$

If  $m(B^c) > \varepsilon$  then  $U_\theta g + \int (1 - g)dm > \frac{1}{2} \varepsilon$ . On the other hand, if  $m(B^c) \leq \varepsilon$ ,  $m(B) > 1 - \varepsilon$  and  $U_\theta 1_B \geq \varepsilon$ . It follows that  $U_\theta \wedge T$  is non-trivial and lemma (2) yields that  $P$  is Harris.

LEMMA (4). — If the whole space  $X$  is special then there exists a measure  $\mu_\theta$  equivalent to  $m$  such that:

$$U_\theta \geq 1 \otimes \mu_\theta.$$

Proof. — By the preceding lemma,  $P$  is a Harris operator, hence  $m \ll Q_c$  and the same proof of lemma 2.2 in [13, p. 71] gives that there exists two strictly positive functions  $g$  and  $h$  in  $H$  such that  $U_\theta \geq h \otimes gm$ .

Since  $X$  is special we get

$$U_{\theta^2} \geq (\theta - \theta^2)U_\theta U_\theta \geq (\theta - \theta^2)U_\theta(h \otimes gm) \geq (\theta - \theta^2)agm$$

where  $a$  is a constant such that  $U_\theta(h) \geq a > 0$ .

We are now able to prove the following

THEOREM (1) (Ornstein-Metivier-Brumel). — If  $A$  is a special set for  $P$ , then there exists a constant  $M$  such that for every  $f \in L^\infty(A)$  with  $\lambda(f) = 0$  we have:

$$\left\| \sum_{n=0}^N P^n f \right\|_\infty \leq M \|f\|_\infty \quad \text{uniformly in } N.$$

Proof. — Note first that  $X$  is special for the induced operator  $I_A U_A I_A$  thus by lemma (4) there exists a measure  $m_A$  equivalent to  $I_A m$  such that:

$${}^A U_\theta = \sum_{n \geq 0} (1 - \theta)^n (I_A U_A I_A)^{n+1} \geq 1_A \otimes m_A \quad \text{on } L^\infty(A)$$

Let  $\lambda_A = \frac{1}{\lambda(A)} I_A \lambda$  be the invariant measure for the operator  $U_A I_A$ . Denote by  $R$  and  $E$  respectively the one-dimensional operators  $1_A \otimes m_A$  and  $1 \otimes \lambda_A$ . We have

$$\begin{aligned} {}^A U_\theta &\geq R = \theta {}^A U \theta R \\ {}^A U_\theta &\geq R = ER \geq \theta ER \end{aligned}$$

Clearly  ${}^A U_\theta$  and  $E$  commute and  $\theta R 1_A = \theta m_A(A) > 0$ . Lemma (1) then implies

$$\lim_{n \rightarrow \infty} \| {}^A U_\theta^n ({}^A U_\theta - E) \| = 0$$

But  ${}^A U_\theta E = E$ , so

$$\lim_{n \rightarrow \infty} \| {}^A U_\theta^n - E \| = 0.$$

It follows that  $\| {}^A U_\theta^n \| \rightarrow 0$  on  $(I_A - E)L^\infty(A)$  and  $I_A - {}^A U_\theta^n$  is invertible for some  $n$  on  $(I_A - E)L^\infty(A)$ . If  $f \in L^\infty(A)$  and  $\int f d\lambda_A = 0$  we get, the same way as in [2].

$$\begin{aligned} f \in (I_A - E)L^\infty(A) &\subseteq (I_A - {}^A U_\theta^n)L^\infty(A) \subseteq (I_A - {}^A U_\theta)L^\infty(A) \\ &\subseteq (I_A - I_A U_A I_A)L^\infty(A) \subseteq (I - P)L^\infty \end{aligned}$$

from which the theorem follows immediately.

Now we show that the special sets are the only sets which verify theorem (1).

**THEOREM (2).** — If  $A$  is a set verifying the following:

i)  $\lambda(A) < \infty$ .

ii) For every  $f \in L^\infty(A)$  with  $\lambda(f) = 0$  one have  $\left\| \sum_{n=0}^N P^n f \right\|_\infty \leq K \| f \|_\infty$  uniformly in  $N$ .

Then  $A$  is special.

*Proof.* — Without loss of generality, we may and will assume that  $\lambda(A) = 1$ . Let  $h \in H$  and  $h \leq 1_A$ , then the function  $h - \lambda(h)1_A \in L^\infty(A)$  and is of

measure zero, thus the sequence  $\phi_N = \sum_{n=0}^N P^n (h - \lambda(h)1_A)$  is bounded in  $L^\infty$ ,

hence there exists a subsequence  $\phi_{N_j}$  which is  $\sigma(L^\infty, L^1)$  convergent to  $\phi \in L^\infty$ . Since  $(I - P)$  is continuous for the topology  $\sigma(L^\infty, L^1)$ , the function  $\phi$  clearly verifies the equation

$$h - \lambda(h)1_A = (I - P)\phi.$$

By applying  $U_h$  to both sides, we get:

$$\begin{aligned} \lambda(h)U_h(1_A) &= U_h[h - (I - P)\phi] = 1 - U_h(I - I_{1-h}P)\phi - U_hI_h\phi \\ &= 1 - P\phi - U_hI_h\phi \in L^\infty, \end{aligned}$$

which is enough to show that  $A$  is a special set.

**THEOREM (3).** — If there exists a special set for  $P$  then  $P$  is a Harris operator.

*Proof.* — Let  $A$  be a special set. For every  $h \in H$  with  $h \leq 1_A$ , we apply theorem (1) to  $1_A - \frac{h}{\lambda_A(h)}$  to get

$$\left\| \sum_{n=0}^N P^n \left( 1_A - \frac{h}{\lambda_A(h)} \right) \right\| \leq K$$

Since  $P$  is ergodic and conservatif,  $\sum_0^N P^n 1_A \rightarrow +\infty$  a. e. By Egoroff's theorem there exists  $B \subseteq A$ ,  $\lambda(B) > 0$ , such that  $\sum_0^N P^n 1_A$  goes to  $+\infty$

uniformly on  $A$ . Thus, there exists  $N$  such that  $\sum_0^N P^n 1_A \geq 2K1_B$ . It follows that:

$$2K1_B \leq \sum_0^N P^n 1_A \leq K + \frac{1}{\lambda_A(h)} \sum_0^N P^n h$$

and  $\sum_0^N P^n h \geq K\lambda_A(h)1_B$ . That is

$$U_\theta h \geq (1 - \theta)^N K \lambda_A(h) 1_B = K' \lambda_A(h) 1_B$$

where  $K'$  is the constant  $(1 - \theta)^N K$ .

Let  $T$  denote the operator  $1 \otimes \lambda_A$  and consider

$$U_\theta \wedge T(1) = \inf \left\{ U_\theta(g) + \int (1 - g)d\lambda_A : 0 \leq g \leq 1 \right\}.$$

As before, fix  $0 \leq g \leq 1$  and set  $C = \left\{ g \geq \frac{1}{2} \right\}$ . Clearly,  $g \geq \frac{1}{2} 1_C$  and

$$\begin{aligned} U_\theta g + \int (1 - g)d\lambda_A &\geq \frac{1}{2} U_\theta 1_C + \int_{C^c} (1 - g)d\lambda_A \geq \frac{1}{2} U_\theta 1_{A \cap C} + \frac{1}{2} \lambda(A \cap C^c) \\ &\geq \frac{1}{2} K' \lambda(A \cap C) 1_B + \frac{1}{2} \lambda(A \cap C^c) \end{aligned}$$

It follows that  $U_\theta \wedge T$  is a non-trivial kernel operator and by lemma (2)  $P$  is a Harris operator.

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