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Section B :
Calcul des Probabilités et Statistique.

Uniform semiamarts (*)

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SUMMARY. — Let $(\Omega, \mathscr{F}, P)$ be a probability space, $(\mathscr{F}_n)_{n\in\mathbb{N}}$ a stochastic basis, and write T for the set of bounded stopping times of $(\mathscr{F}_n)$. If E is a separable dual Banach space, and $(X_n)_{n\in\mathbb{N}}$ is an adapted L^1-bounded sequence of E-valued Bochner integrable random variables, then

$$\int \limsup_{n,m\in\mathbb{N}} \| X_n - X_m \| \, dP \leq 2 \limsup_{\substack{\sigma,\tau\in T \\ \sigma \geq \tau}} \int \| \mathbf{E}[X_\sigma \,|\, \mathscr{F}_\tau] - X_\tau \| \, dP .$$

This is a strengthened form of Bellow's convergence theorem for uniform amarts. A recent example of McCartney and O'Brien shows that the inequality fails in some spaces with the Radon-Nikodym property. The real-valued case of the inequality should be compared to Chacon's inequality.

RÉSUMÉ. — Soient $(\Omega, \mathscr{F}, P)$ un espace de probabilité, $(\mathscr{F}_n)_{n\in\mathbb{N}}$ une base stochastique, T l'ensemble des temps d'arrêt bornés des $(\mathscr{F}_n)$. Si E est un espace de Banach dual séparable, et (X_n) une suite adaptée bornée dans L^1, alors

$$\int \limsup_{n,m\in\mathbb{N}} \| X_n - X_m \| \, dP \leq 2 \limsup_{\substack{\sigma,\tau\in T \\ \sigma \geq \tau}} \int \| \mathbf{E}[X_\sigma \,|\, \mathscr{F}_\tau] - X_\tau \| \, dP .$$

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Cette inégalité maximale implique le théorème de A. Bellow sur la convergence p. p. des amarts uniformes. Un exemple récent de McCartney et O'Brien montre que l'inégalité est en défaut pour certains espaces de Banach qui possèdent la propriété de Radon-Nikodym. Le cas réel doit être comparé à une inégalité récente de Chacon.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $(\mathscr{F}_n)_{n=1}^{\infty}$ be an increasing sequence of sub-σ-algebras of $\mathscr{F}$, and write T for the set of bounded stopping times of the sequence $(\mathscr{F}_n)$. This data will be fixed throughout the paper.

Let $(E, \|\cdot\|)$ be a Banach space, and let $X_n : \Omega \to E$ be Bochner integrable and $\mathscr{F}_n$-measurable. In the case where E has dimension 1, it is known that the following three conditions are equivalent:

$$\text{(1)} \qquad \text{the net } \left(\int X_\tau dP\right)_{\tau \in T} \text{ converges strongly;}$$

$$\text{(2)} \qquad \lim_{m \in \mathbb{N}} \sup_{\substack{\sigma \geq m \\ \sigma \in T}} \int \| E[X_\sigma \mid \mathscr{F}_m] - X_m \| \, dP = 0;$$

$$\text{(3)} \qquad \lim_{\tau \in T} \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP = 0.$$

(See Edgar-Sucheston (1976 *b*) and Astbury (1978).)

When E is infinite-dimensional, the three conditions are not equivalent. A sequence satisfying (1) is called an *amart* (Chacon-Sucheston (1975), Edgar-Sucheston (1976 *a*)); a sequence satisfying (3) is called a *uniform amart* (see Bellow (1977) and Goussoub-Sucheston (1978)). An example given below shows that (2) and (3) are not equivalent.

We will be concerned below with sequences (X_n) which satisfy

$$\text{(3')} \qquad \limsup_{\tau \in T} \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP < \infty.$$

By analogy with [*3*] and [*9*], such a sequence will be called a *uniform semi-amart*.

The proofs of the following two elementary lemmas are left to the reader.

Lemma 1. — Let (X_n) be an adapted sequence of non-negative random variables. Then

$$\int \limsup_n X_n dP \leq \limsup_{\tau \in T} \int X_\tau dP.$$

LEMMA 2. — Let X_n be an adapted sequence of E-valued random variables, and let $\tau \in T$. Then

$$\int \sup_{n \geq \tau} \| \mathbf{E}[X_n \mid \mathscr{F}_\tau] \| \, dP \leq \sup_{\sigma \geq \tau} \int \| \mathbf{E}[X_\sigma \mid \mathscr{F}_\tau] \| \, dP .$$

LEMMA 3. — Let (X_n) be a uniformly integrable sequence of random variables with values in a separable dual space $E = F^*$. Then there is a subsequence (X_{n_k}) such that the weak* limit of $\int_A X_{n_k} dP$ exists for all $A \in \mathscr{F}$.

Proof. — Let $\mathscr{J} = \sigma(X_1, X_2, X_3, \ldots)$, a separable σ-algebra. Let $A_1, A_2, \ldots$ be dense in $\mathscr{J}$. For given A_i, the sequence $\left(\int_{A_i} X_n dP\right)_{n=1}^\infty$ is bounded in E, so repeatedly applying the Alaoglu theorem (and separability of F) there is subsequence (X_{n_k}) such that $\left(\int_{A_i} X_{n_k} dP\right)_{k=1}^\infty$ converges for all i. Let $A \in \mathscr{J}$, and $\varepsilon > 0$. By density of $\{ A_i \}$ and uniform integrability, there exists A_i with $\int_{A \triangle A_i} \| X_n \| \, dP < \varepsilon$ for all n, so

$$\left\| \int_A X_{n_k} dP - \int_A X_{n_{k'}} dP \right\| \leq \left\| \int_{A_i} X_{n_k} dP - \int_{A_i} X_{n_{k'}} dP \right\| + 2\varepsilon < 3\varepsilon$$

for large k, k'. Hence $\left(\int_A X_{n_k} dP\right)_{k=1}^\infty$ converges. Next, $\left(\int h X_{n_k} dP\right)_{k=1}^\infty$ converges for $h \in L^\infty(\mathscr{J}, P)$, since such an h is approximable uniformly by $\mathscr{J}$-simple functions. If $A \in \mathscr{F}$, let $h = P[A \mid \mathscr{J}]$. Thus $\int_A X_{n_k} dP = \int h X_{n_k} dP$ converges. $\square$

THEOREM 4. — Let E be a separable dual Banach space. Let $(X_n)_{n=1}^\infty$ be an adapted, L^1-bounded, sequence of random variables in E. Then

$$\int \limsup_{n,m} \| X_n - X_m \| \, dP \leq 2 \limsup_\tau \sup_{\sigma \geq \tau} \int \| \mathbf{E}[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP .$$

Proof. — If the right-hand side of the inequality is infinite, there is nothing to prove; we assume, therefore, that it is finite, i. e. that $(X_n)_{n\in\mathbb{N}}$ is a uniform semiamart. We have, by Lemma 2,

$$\limsup_{m\to\infty} \int \sup_{n\geq m} \| \mathbf{E}[X_n \mid \mathcal{F}_m] - X_m \| \, dP < \infty .$$

For fixed $m \in \mathbb{N}$ (sufficiently large), $\sup_{n\geq m} \| \mathbf{E}[X_n \mid \mathcal{F}_m] - X_m \| \in L^1(\Omega)$, and $\| \mathbf{E}[X_n \mid \mathcal{F}_m] \| \leq \| \mathbf{E}[X_n \mid \mathcal{F}_m] - X_m \| + \| X_m \|$, so $\sup_{n\geq m} \| \mathbf{E}[X_n \mid \mathcal{F}_m] \| \in L^1(\Omega)$. By Lemma 3, there is a subsequence (X_{n_k}) such that

$$\mu(A) = w^* \lim_{k\to\infty} \int_A X_{n_k} dP$$

exists for all $A \in \mathcal{F}_m$. (Applying a diagonal argument, we may assume the same subsequence (X_{n_k}) yields a limit $\mu(A)$ for all $A \in \bigcup_{m\in\mathbb{N}} \mathcal{F}_m$.) For $A \in \mathcal{F}_m$, we have

$$\begin{aligned} \| \mu(A) \| &\leq \limsup_k \left\| \int_A X_{n_k} dP \right\| \\ &\leq \int_A \sup_{n\geq m} \| \mathbf{E}[X_n \mid \mathcal{F}_m] \| \, dP . \end{aligned}$$

But $\sup_{n\geq m} \| \mathbf{E}[X_n \mid \mathcal{F}_m] \| \in L^1(\Omega)$, so the vector measure μ is countably additive, absolutely continuous, and has bounded variation on $\mathcal{F}_m$. Let Y_m be its Radon-Nikodym derivative.

Trivially, $(Y_m)_{m\in\mathbb{N}}$ is a martingale. If $\tau \in T$, then $\mu(A) = \int_A Y_\tau d\mu$ for all $A \in \mathcal{F}_\tau$, so

$$\begin{aligned} \left\| \int_A (Y_\tau - X_\tau) dP \right\| &= \left\| \mu(A) - \int_A X_\tau dP \right\| \\ &\leq \limsup_k \left\| \int_A X_{n_k} dP - \int_A X_\tau dP \right\| \\ &\leq \sup_{n\geq\tau} \int_A \| \mathbf{E}[X_n \mid \mathcal{F}_\tau] - X_\tau \| \, dP \\ &\leq \int_A \sup_{n\geq\tau} \| \mathbf{E}[X_n \mid \mathcal{F}_\tau] - X_\tau \| \, dP . \end{aligned}$$

Hence

$$(4) \qquad \| Y_\tau - X_\tau \| \leq \sup_{n\geq\tau} \| \mathbf{E}[X_n \mid \mathcal{F}_\tau] - X_\tau \| \qquad \text{a. s.}$$

Note $\int \| Y_n \| \, dP \leq \int \| X_n \| \, dP + \int \| Y_n - X_n \| \, dP$, so $(Y_n)_{n\in\mathbb{N}}$ is L^1-bounded. The Banach space E has the Radon-Nikodym property, so Y_n converges a. s. Using the Lemmas and inequality (4), we have:

$$\begin{aligned}
&\int \limsup_{n,m} \| X_m - X_n \| \, dP \\
&\quad \leq \int \limsup_{n,m} (\| X_m - Y_m \| + \| Y_m - Y_n \| + \| Y_n - X_n \|) dP \\
&\quad \leq 2 \int \limsup_n \| X_n - Y_n \| \, dP \\
&\quad \leq 2 \limsup_\tau \int \| X_\tau - Y_\tau \| \, dP \\
&\quad \leq 2 \limsup_\tau \int \sup_{n \geq \tau} \| E[X_n \mid \mathscr{F}_\tau] - X_\tau \| \, dP \\
&\quad \leq 2 \limsup_\tau \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP . \qquad \square
\end{aligned}$$

The reader of [*6*], [*7*] and [*9*] will recognize a « Riesz decomposition » in the above proof. The sequence X_n is decomposed as $Y_n + Z_n$ where (Y_n) is a martingale and (Z_n) satisfies

$$\int \limsup_n \| Z_n \| \, dP \leq \limsup_\tau \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP .$$

I have stated Theorem 4 in the form given so that it will remain true in a closed subspace of the separable dual. Combining this with Stegall's theorem (1975), we obtain the following.

COROLLARY 5. — Let F be a dual Banach space that has the Radon-Nikodym property, and let E be a subspace of F. If $(X_n)_{n \in \mathbb{N}}$ is an adapted, L^1-bounded, sequence of random variables in E, then

$$\int \limsup_{n,m} \| X_n - X_m \| \, dP \leq 2 \limsup_\tau \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP .$$

COROLLARY 6. — Let E be a Banach space isomorphic to a subspace of a separable conjugate space. Then there is a constant C such that for all adapted L^1-bounded sequences (X_n) of random variables in E,

$$(5) \quad \int \limsup_{n,m} \| X_n - X_m \| \, dP \leq C \limsup_\tau \sup_{\sigma \geq \tau} \int \| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| \, dP .$$

The example of McCartney and O'Brien (1979) of a separable Banach space that has the Radon-Nikodym property but is not isomorphic to a subspace of a separable dual space shows that inequality (5) fails in certain spaces with RNP. In the language of [*10*] and [*11*], an (ε, δ)-neighborly tree structure is an L^∞-bounded sequence (X_n), adapted to the dyadic σ-algebras, with $\| X_{n+1} - X_n \| \geqslant \varepsilon$ and $\| E[X_\sigma \mid \mathscr{F}_\tau] - X_\tau \| < \delta$ for σ, $\tau \in T$, $\sigma \geq \tau$. The example of McCartney and O'Brien (1979) admits a $(1,\delta)$-neighborly tree structure for all $\delta > 0$, so no constant C exists for which (5) holds.

Note also that formula (4) implies

$$\limsup_{m,n} \| X_n - X_m \| \leq 2 \limsup_{m\in\mathbb{N}} \sup_{n\geq m} \| E[X_n \mid \mathscr{F}_m] - X_m \| \qquad \text{a. s.}$$

An adapted sequence $(X_n)_{n\in\mathbb{N}}$ for which the right-hand side of this inequality is 0 is called a *martingale in the limit* (see Mucci (1973) and (1976)). As far as I know, the question of norm a. e. convergence of an L^1-bounded martingale in the limit with values in a space with the Radon-Nikodym property is still open. Partial results can be found in Millet-Sucheston (1979). The above observation yields another partial result.

COROLLARY 7. — Let E be a Banach space isomorphic to a subspace of a dual space with RNP. If (X_n) is an L^1-bounded martingale in the limit and a uniform semiamart, then X_n converges (strongly) a. s.

EXAMPLE 8. — We shall briefly sketch an example showing that (2) and (3) are not equivalent in the Banach space l^2. Let $(\Omega, \mathscr{F}, P)$ be a non-atomic probability space. It is not hard to construct a sequence of events $A_1, A_2, A_3, \ldots$ in Ω such that *i*) $p_n = P(A_n)$ decreases to 0, *ii*) almost every $\omega \in \Omega$ belongs to infinitely many of the sets A_n, *iii*) A_{n+1} is a subset of an atom of the σ-algebra $\mathscr{F}_n$ generated by $A_1, \ldots, A_n$. Let $e_1, e_2, \ldots$ denote an orthonormal sequence in l^2. Define $X_n = e_n 1_{A_n}$. Verification of the following assertions is left to the interested reader:

a) $\int \| X_n \| dP = p_n \to 0$;

b) $\| E[X_\tau \mid \mathscr{F}_m] - X_m \| \leq p_m$ a. e. if $\tau > m$;

c) $\| E[X_\tau \mid \mathscr{F}_m] \| \leq p_N$ a. e. if $\tau > N$;

d) For each m, the net $E[X_\tau \mid \mathscr{F}_m] \to 0$ a. e.;

e) $\limsup_{n\geq m} \| E[X_n \mid \mathscr{F}_m] - X_m \| > 0$ a. s., so (X_n) is not a martingale in the limit;

f) X_n diverges a. s., so (X_n) is not a uniform amart;

g) $\int \| E[X_\tau \mid \mathscr{F}_m] - X_m \| dP \to 0$ as $\tau \geq m \to \infty$, so (X_n) satisfies (2).

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