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Integrals related to stationary processes and cylindrical martingales

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Integrals related to stationary processes and cylindrical martingales

by

J. PELLAUMAIL and A. WERON

SUMMARY. — The aim of this paper is to analyse a relationship between integrals related to stationay processes and square integrable martingales. We consider the case when processes take their values in Banach spaces.

RÉSUMÉ. — L'objet de ce travail est de mettre en évidence l'étroite corrélation existant entre diverses intégrales qui interviennent dans l'étude, d'une part, des processus stationnaires, d'autre part, des martingales cylindriques.

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INTRODUCTION

The aim of this paper is to analyse a relationship between integrals related to stationary processes and square integrable martingales. We consider the case when processes take their values in Banach spaces.

We establish here that integrals (stochastic and deterministic) used for the above two classes of stochastic processes are closely related. First we discuss such called quadratic integral for vector valued functions and for their tensor products. After we direct our attension at non-negative measures with values in the space $(B' \hat{\otimes}_1 B')'$, where B is a Banach space.

In sections 3.5 two types of integrals for operator valued functions are described. Finally, in the last sections 7 and 8 we show how the above general approach is used for stationary processes and for square integrable cylindrical martingales.

In the paper we use the langage of tensor products and the dilation theory, which has been recently developed for non-negative B-to-B′ valued operators. We hope that this gives a new light for problems discussed here.

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1. QUADRATIC INTEGRAL

In this first section, we recall some definitions and results on integration with respect to vector measures and on tensor product: we apply this notions to the study of quadratic integrals.

1.1. Bilinear integral.

We consider

— a set S and a σ-algebra Σ of subsets of S,
— a complex Banach space V and its topological dual V′,
— an additive V′-valued function M defined on Σ,
— a V-valued function f defined on S.

The first problem is to define the integral $\int_S f dM$. For an extensive study on this problem, we refer to [Bar].

It is convenient for our purpose to consider the total variation which is defined as follows: if A belongs to Σ

$$v_M(A) = \text{Sup} \, . \sum_{i \in I} \| M(A_i) \|_{V'}$$

where this supremum is taken over all the finite Σ-measurable partitions $(A_i)_{i \in I}$ of A.

In general situation, this total variation is not finite (see 2.7), but in the following we often will suppose that this total variation v_M is finite (or σ-finite).

If the total variation v_M is finite and σ-additive, M is strongly σ-additive and there exists a V'-valued weak Radon-Nikodym derivative $M' = \dfrac{dM}{dv_M}$ of M with respect to v_M; this R. N. derivative is V'-valued and for each element v of V, $\langle M', v \rangle$ is the classical real R. N. derivative of $\langle M, v \rangle$ with respect to (briefly w. r. t.) v_M. Conversely, if this R. N. derivative M' exists w. r. t. a positive measure μ, then the total variation v_M is finite (see [Pel]).

In the following, when we say M is a measure, it means that M is a σ-additive measure.

For the convenience we note the following elementary lemma:

1.2. LEMMA.

Let M be a function defined on an algebra $\mathscr{A}$ and with values in the dual V' of a Banach space V. We suppose that the total variation v_M of M is finite and that, for each element v of V, the real function $\langle v, M(.) \rangle$ is σ-additive. Then the total variation v_M is σ-additive.

The proof is left to the reader.

1.3. TENSOR PRODUCT.

Let B and C be two locally convex vector spaces: sometimes, we have to use a bilinear continuous mapping g defined on $B \times C$: often, it is convenient to consider this *bilinear* mapping as a *linear* mapping defined on the « tensor product » $B \otimes C$.

For definitions and properties of tensor product, we refer to [Tre]. Let us recall only some properties when B and C are Banach spaces. Let D be a Banach space, $B \hat{\otimes}_1 C$ is a Banach space such that, for each D-valued bilinear continuous mapping g defined on $B \times C$, there exists a D-valued

linear mapping f defined on $B \hat{\otimes}_1 C$ such that the following diagram is commutative

$$\begin{tikzcd} B \times C \arrow[r, "j"] \arrow[dr, "g"'] & B \hat{\otimes}_1 C \arrow[d, "f"] \\ & D \end{tikzcd}$$

where j is a canonical imbedding. The norm in $B \hat{\otimes}_1 C$ is called the projective norm or the trace-norm.

There is an isometry from $(B \hat{\otimes}_1 C)$ onto $LN(B', C)$, the Banach space of linear nuclear operators from B into C with the trace norm; this isometry is the extension of the mapping which, to $x \otimes y$, associates the operator $h \to \langle x, h \rangle . y$.

On the other hand, there is an isometry from the dual $(B \hat{\otimes}_1 C)'$ of $(B \hat{\otimes}_1 C)$ onto $CL(B, C')$, the Banach space of continuous linear operators from B into C' with the usual norm; this isometry associates to the linear form f on $B \hat{\otimes}_1 C$, the continuous linear operator $\tilde{f}$ defined by $\langle \tilde{f}(x), y \rangle = f(x, y)$.

1.4. QUADRATIC INTEGRAL.

For studying second order processes (see [MaS], [ChW$_2$], or [Wer$_3$]) or integral w. r. t. quadratic Doléans measures (see [MeP] and [Met$_1$]), it is necessary to consider integrals of the type $\int X dMY^*$ where M is a σ-additive function with values in the space of continuous antilinear operators $C\bar{L}(B', B'')$ and X and Y are functions defined on S and with values in B'.

Actually, M can be considered as a $V' = (B' \hat{\otimes}_1 B')'$-valued additive function and $f = X \hat{\otimes} Y$ can be considered as a $B' \otimes_1 B' = V$-valued function; then, the integral $\int X dMY^* = \int X \otimes Y dM$ is a special case of the general integral which was discussed in 1.1; thus, we can use all the classical results on such bilinear integrals.

In the next sections we will study more precisely the integral $\int X \otimes Y dM$, when M is a non-negative measure.

2. NON-NEGATIVE MEASURE

In this section we shall consider the space $V = B' \hat{\otimes}_1 B'$ where B is a complex Banach space. If $X: S \to B'$ and $Y: S \to B'$ are measurable mappings then

$$f = X \otimes Y : S \to B' \hat{\otimes}_1 B' = V$$

is measurable too. We are interested in measures with values in $(B' \hat{\otimes}_1 B')' = V'$ which are non-negative i. e., $\forall X \in B'$ $\forall \Delta \in \Sigma$

$$[M(\Delta)]\ (X \otimes X) = (M(\Delta)X)(X) \geqslant 0 .$$

Such measures in natural way arise in the theory of stationary processes (the measure which corresponds to the correlation function) and in the theory of square integrable cylindrical martingales (quadratic Doléans' measure), see (§ 8) below.

We need the following definition. If $B = H$ is a Hilbert space then non-negative measure $E(.)$ with values in the set of all orthogonal projections in H is called a spectral measure. Thus $\forall \Delta, \Delta_1, \Delta_2 \in \Sigma$.

i) $E^2(\Delta) = E(\Delta) = E^*(\Delta)$

ii) $E(\Delta_1)E(\Delta_2) = 0 \quad \text{if} \quad \Delta_1 \cap \Delta_2 = \phi$

2.1. LEMMA.

For each non-negative measure M with values in $(B' \hat{\otimes}_1 B')'$ there exist a Hilbert space H, an operator $R \in CL(B', H)$ and a spectral measure $E(.)$ such that

$$M(\Delta) = R^*E(\Delta)R \qquad \forall \Delta \in \Sigma$$

Moreover if H is minimal i. e., $H = \bigvee_{\Delta \in \Sigma} E(\Delta)RB'$ then H, R and $E(.)$ are unique up to unitary equivalence. This means that if $M(\Delta) = R_1^*E_1(\Delta)R_1$, where $E_1(.)$ is a spectral measure in a Hilbert Space H_1 and $R_1 : B' \to H_1$, then there exists an unitary operator $U : H_1 \to H$ such that $R = UR_1$ and $UE_1(\Delta) = E(\Delta)U$ for $\forall \Delta \in \Sigma$.

Proof. — It follows from ([Wer_2], Prop. 3) by using the fact that $(B' \hat{\otimes}_1 B')'$ is isometric to CL (B', B").

The triple (H, R, $E(.)$) is called dilation of measure $M(.)$. This dilation may be interpretated in the following diagram:

$$\begin{tikzcd} B' \arrow[r, "M(.)"] \arrow[d, "R"'] & B'' \\ H \arrow[r, "E(.)"] & H \arrow[u, "R^*"'] \end{tikzcd}$$

which shows that the measure $M(.)$ is factorized by the Hilbert Space H.

2.2. LEMMA.

The Hilbert Space H in the above diagram is separable if $M(S)B'$ is separable subset of B".

Proof. — Since $M(S)B' = R^*E(S)RB' \subset R^*M$ the necessity is clear. Conversely, let $M(S)B'$ be separable. It is sufficient to prove that the image $R(C) \subset H$ of the unit ball C in B' is separable.

Let $\{ b_k \}$ be a sequence in C such that $\{ M(S)b_k \}$ is dense in M(S)C. We shall to prove that $\{ Rb_k \}$ is dense in R(C). Let $n \in R(C)$. $\exists\, b \in C$ such that $Rb = n$. Choosing a subsequence $\{ b_{k_n} \}$ such that $M(S)b_{k_n} \to M(S)b$ in B'' we have

$$\| n - Rb_{k_n} \|^2 = \| R(b - b_{k_n}) \|^2 = M(S)[(b - b_{k_n}) \otimes | (b - b_{k_n})] \to 0$$

which completes the proof. In the last equality we use the fact that $E(S) = I_H$ and consequently $M(S) = R^*R$.

2.3. REMARK.

Let v_M be the total variation of M. Then by lemma 1.2 $v_M < \infty$ iff there exists a non negative finite measure μ on Σ such that $\| M(A)X \| \leqslant \mu(A) \| X \|$ for each $X \in B'$.

Measures with such property was called μ-bounded in [Mas].

2.4. LEMMA.

The following conditions on measure M are equivalent:

i) the total variation v_M is infinite:

ii) $\sup \left\{ \sum_{i \in I} [M(A_i)](X_i \otimes X_i) \right\} = +\infty$, where the supremum is taken over all finite collections of vectors from B' with $\| X_i \| \leqslant 1$ and all partitions of S into a finite number of disjoint sets in Σ;

iii) there exists a B'-valued measurable function X such that $\operatorname{Sup}_S \| X \| \leqslant 1$ and $\int (X \otimes X) dM = +\infty$.

Proof. — From the definition of the total variation of M immediately follows that if $v_M = \infty$ then (*ii*) holds. To prove the converse we observe that the norm of $\| M(A) \|$ in $(B' \hat{\otimes}_1 B')'$ is equal to the sup $[M(A)](X \otimes X)$, where the supremum is taken over all vectors X from B' with $\| X \| \leqslant 1$. Thus (*ii*) implies (*i*). Is is easy to see from the definition of the integral $\int f dM$ that (*ii*) is equivalent to (*iii*). Which completes the proof.

2.5. DEFINITION.

We say that M is absolutely continuous w. r. t. a non-negative measure μ on Σ if for each $x, y \in B'$, $[M(.)](x \otimes y) \ll \mu$.

2.6. LEMMA

If for measure M the image of B′ under M(S) : M(S)B′ $\subset$ B″ is separable, then there exists a non-negative finite measure μ on Σ such that M is absolutely continuous w. r. t. μ.

Proof. — By lemma 2.1 M(.) has a minimal dilation M(.) = R*E(.)R. Moreover our assumption on M implies by lemma 2.2 that the Hilbert space H in above dilation is separable.

Observe that we have $H = \bigvee_{\Delta\in\Sigma} E(\Delta)RB'$. Let μ be given by the formula:

$$\mu(\Delta) = \sum_{n=1}^{\infty} 2^{-n} \| E(\Delta)e_n \|^2 , \qquad \forall \Delta \in \Sigma$$

where (e_n) is CONS in H. Let us now use the fact that E(.) is spectral measure in CL(H, H). Thus for each n, $\| E(\Delta)e_n \|^2$ is non-negative finite scalar-valued measure on Σ and consequently μ is non-negative measure on (S, Σ) and $\mu(S) = 1$. Moreover, $\forall n$, $\| E(.)e_n \|^2 \ll \mu$. Since

$$(E(.)e_k, e_n) = (E^2(.)e_k, e_n) = (E(.)e_k, E(.)e_n) \leqslant \| E(.)e_k \|^2 \| E(.)e_n \|^2$$

we conclude that $\forall h, g \in H$

$$(E(.)h, g) \ll \mu$$

But for $x, y \in B'$, $h = E(.)Rx$ and $g = E(.)Ry$ belong to H and $[M(.)](x \otimes y) = (R^*E(.)Rx)(y) = (E^2(.)Rx, E(.)Ry) = (E(.)h, g)$.

In the last equalities, we use the fact that for spectral measure $E^2(.) = E(.)$. Hence by using the above facts we obtain that $\forall x, y \in B'$

$$[M(.)](x \otimes y) \ll \mu$$

2.7. EXAMPLE.

Now we give an example of a measure M with values in $(B' \hat{\otimes}, B')'$, where B′ = H is a separable Hilbert space, which is strongly σ-additive but its total variation v_M is not σ-finite.

Let S = [0, 1] × [0, 1], $\Sigma = \sigma$-algebra of the Borel sets of S. μ- the Lebesgue measure on S and α-the Lebesgue measure on [0, 1]. We put

$$H = L_2([0, 1], \mathscr{G}, \mu),$$

where $\mathscr{G}$ is the σ-algebra of the Borel sets on [0, 1]. For each element $A \in \Sigma$ and for each finite family $(f_i, g_i)_{i\in I}$ of elements from H × H we define

$$m(A)\left[\sum_{i\in I} (f_i \otimes g_i)\right] = \sum_{i\in I} \int\int_A f_i(x)g_i(y)\alpha(dx)\alpha(dy)$$

Now, m induces an additive mapping M from Σ into $(H \otimes_1 H)'$. If $A = B \times C$, where $B, C \subset [0, 1]$, and

$$f = 1_B \alpha(B)^{-1/2}, \qquad g = 1_C \alpha(C)^{-1/2},$$

then we have the following inequality for the norm in $(H \otimes_1 H)'$:

$$\| M(A) \| \geqslant M(A)(f \otimes g) = [\alpha(B)\alpha(C)]^{1/2},$$

which shows that the total variation of M is not σ-finite.

It remains to prove that M is strongly σ-additive. Let A be an element of Σ. It is easy to see that the norm of M(A) in $(H \otimes_1 H)'$ is equal to the supremum of $\iint_A f(x)\overline{g(y)}\alpha(dx)\alpha(dy)$, where the supremum is taken over all the pairs (f, g) of elements from H with $\| f \|_H \leqslant 1$ and $\| g \|_H \leqslant 1$. But by the Cauchy-Schwartz inequality, we have

$$\left[\iint_A f(x)\overline{g(y)}\alpha(dx)\alpha(dy)\right]^2 \leqslant \iint 1_A^2 \alpha(dx)\alpha(dy) \iint \| f.g \|^2 \alpha(dx).\alpha(dy) \leqslant \mu(A)$$

Hence

$$\| M(A) \|_{(H \times_1 H)'} \leqslant [\mu(A)]^{1/2}$$

and consequently M is a σ-additive function on Σ onto $(H \otimes_1 H)'$ with the strong topology.

2.8. REMARK.

The above example shows that there exists a measure $M : \Sigma \to (B' \otimes_1 B')'$ which is absolutely continuous w. r. t. a σ-finite measure, but its total variation v_M is infinite.

3. OPERATOR-VALUED FUNCTIONS AND QUADRATIC INTEGRAL

3.1. NOTATIONS.

In this section, the assumptions and notations are as in the previous section 2. More precisely:

Σ is a σ-algebra of subsets of S, B is a Banach space, M is a $(B' \otimes_1 B')'$-valued measure defined on Σ with total variation v_M, this total variation v_M is finite and σ-additive and $M' = \dfrac{dM}{dv_M}$.

Moreover we suppose that B′ is separable and that G is a Banach space.

3.2. ADJOINT OPERATOR (see [Yos] p. 196).

Let x be an element of L(B, G) with domain D dense in B; then, the adjoint operator x^* is an element of L(G′, B′) defined by

$$\langle x^*(g'), b \rangle = \langle x(b), g' \rangle$$

for $b \in D$;

3.3. WEAK * MEASURABILITY.

Let X be an L(B, G)-valued function defined on S; we say that X is weakly* measurable if, for every element (b, g') of $(B \times G')$, $\langle X(b), g' \rangle$ is measurable.

Remark. — The situation in this section is more general than in the previous ones where we consider the case G = C and consequently L(B, G) = B′. For studying infinite-dimensional stochastic processes in Banach spaces the necessity arises of using such operator functions (see sections 7 and 8).

3.4. LEMMA.

Let u be an element of $(B' \hat{\otimes}_1 B')'$, g' be an element of G′ and v be an element of L(B, G) with domain D dense in B. Let v^* be the adjoint of v. Let $(C_n)_{n>0}$ be a sequence of elements of B′, dense in B′. Let B'_n the vector space generated by $\{ C_k \}_{1 \leqslant k \leqslant n}$. Let π_n a be a linear idempotent contraction from B′ onto B'_n. Then we have:

$$u \{ [v^*(g)]^{\otimes 2} \} = \lim_{n \to \infty} . u \{ [\pi_n \circ v^*(g)]^{\otimes 2} \}$$

Proof. — By our assumptions, $\pi_n \in CL(B', B'_n)$, $\pi_n(C) = C$ if $C \in B'_n$ and $\| \pi'_n(C) \| \leqslant \| C \|$ if $C \in B'$. Such operator exists according to the Hahn-Banach theorem.

If we consider v, g and $\varepsilon > 0$, there exist $k > 0$ and $C \in B'_k$ such that $\| v^*(g) - C \| \leqslant \varepsilon$; this implies, for every $n \geqslant k$:

$$\begin{aligned} \| \pi_n \circ v^*(g) - v^*(g) \| &\leqslant \| \pi_n \circ v^*(g) - c \| + \| c - v^*(g) \| \\ &\leqslant \| \pi_n \circ [v^*(g) - c] \| + \varepsilon \leqslant 2\varepsilon \end{aligned}$$

Thus $\{ \pi_n \circ v^*(g) \}_{n>0}$ converges strongly to $v^*(g)$ in B′.

Now the lemma follows from the continuity of u and from the continuity of the mapping $(x, y) \rightsquigarrow (x \otimes y)$ for the « trace norm » on $(B' \hat{\otimes}_1 B')$.

3.5. PRELIMINARY PROPOSITION.

Let X be an L(B, G)-valued function defined on S and weakly* measurable. For every element s of S, we suppose that the domain of $X(s)$ is dense in B. Let X* be the adjoint of X. Then, for every element g of G′, the func-

tion $M'\{[X^*(g)]^{\otimes 2}\}$ is a (real non negative) measurable function. Thus, the integral $\int M'\{[X^*(g)]^{\otimes 2}\}.\,dv_M$ is well defined (finite or infinite).

Proof. — Let $(\pi_n)_{n>0}$ be a sequence of operators as in the above lemma. This lemma implies:

$$\lim_{n\to\infty}.M'\{[\pi_n\circ X^*(g)]^{\otimes 2}\} = M'\{[X^*(g)]^{\otimes 2}\}$$

But, for every integer n, $M'\{[\pi_n\circ X^*(g)]^{\otimes 2}\}$ is measurable according to the weak* measurability of X. Then, the same property holds for $M'\{[X^*(g)]^{\otimes 2}$.

3.6. Definitions of $\mathscr{L}_g^2$ and $\mathscr{L}_\pi^2$.

Let g be an element of G'.

We say that a L(B, G)-valued function X defined on S is an element of $\mathscr{L}_g^2$, if:

i) X is weak* measurable,

ii) for every element s of S, the domain of $X(s)$ is dense in B,

iii) $N_g(X) < +\infty$, where

$$N_g(X) = \int_S M'\{[X^*(g)]^{\otimes 2}\}\,dv_M$$

Moreover, we denote by $\mathscr{L}_\pi^2$ the vector space of all elements X from $\bigcap_{g\in G'}\mathscr{L}_g^2$ such that $N_\pi(X) < +\infty$, where

$$N_\pi(X) = \operatorname*{Sup}_{\|g\|\leqslant 1}.N_g(X)$$

4. HILBERT SPACE CASE

Here we sketch a construction of a quadratic integral which was given in [Mas]. H, K are separable Hilbert space.

Let X be an L(H, K) valued function on S; in [Mas] X is said measurable if there exists a sequence of simple measurable CL(H, K) fonctions Φ_n such that $\forall\in S$ and for each $x\in H$ we have $\lim_n \|\Phi_n(S)x - X(S)x\| = 0$.

Let M be a non-negative CL(H, H)-valued measure on Σ and M' be a weak Radon-Nikodym derivative w. r. t. the total variation v_M, which by the assumption is finite. We note that in [Mas] the author assumed that M is μ-bounded. But by (2.3) this is equivalent to our assumption.

4.1. DEFINITION ([Mas]).

The pair (X, Y) is M integrable if

i) $XM'^{1/2}$ and $YM'^{1/2}$ are CL(H, K) valued and measurable (see above),
ii) the function $(XM'^{1/2}).(YM'^{1/2})^*$ is Bochner ν_M integrable.

We denote

$$\int X \otimes Y dM = \int X dM Y^* = \int (XM'^{1/2})(YM'^{1/2})^* d\nu_M .$$

4.2. DEFINITION.

$$L_{2,M} = \{ X : \int X \otimes X dM \text{ exists and trace } \int X \otimes X dM \text{ is finite} \} .$$

The functions X, Y with $(X - Y)M'^{1/2} = 0$ a. s. $[\nu_M]$ are identified.
From the definition of $L_{2,M}$ one may see that

4.3. $\left\{ \begin{array}{l} XM'^{1/2} \text{ is a Hilbert-Schmidt operator a. s. } [\nu_M] \\ \text{and} \\ X \in L_{2,M} \text{ iff } | XM'^{1/2} |_{H-S} \in L_{2,\nu_M}, \text{ where } | \, . \, |_{H-S}\text{-denotes the Hilbert-} \\ \text{Schmidt norm, see [Mas].} \end{array} \right.$

4.4. PROPOSITION ([Mas]).

$L_{2,M}$ is a Hilbert space over the ring CL(K, K) with the norm

$$\| X \| = \operatorname{tr} \int X \otimes X dM ,$$

and moreover the simple functions are dense in $L_{2,M}$.

4.5. PROPOSITION.

If H = K, then the measure M has a spectral dilation in $L_{2,M}$.

Proof. — Let $E_1(.)$ be a spectral measure in $L_{2,M}$ defined as an operator of multiplication by the indicator 1_Δ. Let R be the inclusion of K into $L_{2,M}$. Then R^* is an orthogonal projection onto K and for each $\Delta \in \Sigma$

$$F(\Delta) = R^* E_1(\Delta) R = \operatorname{Proj}_K E_1(\Delta) . \qquad \text{cf. [Wer}_3\text{]}$$

4.6. REMARK.

It is interesting to compare the above integral with one in [Met_1], Sect. IV, which is a special case of the integral defined in (3.5); we note that by (4.3) the space $L_{2,M}$ and $\Lambda_T^{*Q_\lambda}(H, K)$ (defined in [Met_1], p. 54) are very similar.

5. STOCHASTIC INTEGRAL

5.1. Introduction.

In this section, we give a construction of the integral $\int X dW$ in a general context which includes stochastic integrals with respect to W where W can be a stationary process (see [ChW_2] and [Wer_1]) or a cylindrical martingale (see [Met] and [MeP]). A similar approach, for stationnary processes with values in Hilbert space only, can be found in [Mas], In our situation, X is an L(B, G)-valued function (or process). The assumptions on W are as follows:

5.2. Notations.

Let H be a Hilbert space and B and G two Banach spaces where B′ is separable. Let $\mathscr{A}$ be an algebra of subsets of the set S and Σ the σ-algebra generated by $\mathscr{A}$. We denote by W an additive function defined on Σ and with values in CL(B′, H) such that, for every element (u, v) of $(B' \times B')$, and for every pair (E, F) of elements of $\mathscr{A}$ such that $E \cap F = \phi$, we have $\langle W(E)(u), W(F)(v) \rangle_H = 0$.

Then, for every element (F, u, v) of $(\Sigma \times B' \times B')$ we define:

$$M(F)(u \otimes v) = \langle W(F)(u), W(F)(v) \rangle_H$$

M is an additive mapping on $\mathscr{A}$ which induces an additive mapping with values in $(B' \hat{\otimes}_1 B')'$ that we denote also by M.

We denote by v_M the total variation of M (for the norm in $(B' \hat{\otimes}_1 B')'$). We suppose that this total variation is finite and σ-additive.

In this case, M can be extended into a strongly σ-additive mapping, defined on Σ and with values in $(B' \hat{\otimes}_1 B')'$ that we also denote by M.

Then, all the assumptions given on M in section 3 are fulfilled. As before, we denote by M′ the R. N. derivative of M w. r. t. v_M.

5.3. $\mathscr{A}$-simple functions.

We denote by $\mathscr{E}$ the set of the functions such that $X = \sum_{i \in I} x_i . 1_{A(i)}$ where $(x_i)_{i \in I}$ is a finite family of elements of L(B, G) with domain dense in B and $(A(i))_{i \in I}$ is an associated family of elements of $\mathscr{A}$.

In this case, for every element g of G′, we define:

$$\left(\int X dW\right)(g) = \sum_{i \in I} W[A(i)][x_i^*(g)]$$

and $\left(\int X dW\right)$ is an element of L(G′, H) that we call the integral of X w. r. t. W.

Of course, if, for every element i of I, x_i^* belongs to CL(G′, B′), then $\left(\int X dW\right)$ is an element of CL(G′, H); but $\left(\int X dW\right)$ can be an element of CL(G′, H) for elements x_i^* which do not belong to CL(G′, B′).

Moreover, for every element (E, g) of ($\mathscr{A} \times$ G′), we define:

$$\left(\int_E X dW\right)(g) = \sum_{i \in I} W[A(i) \cap E][x_i^*(g)]$$

Then, the mapping $E \rightsquigarrow \left(\int_E X dW\right)$ is an additive mapping defined on $\mathscr{A}$ and with values in L(G′, H).

The problem is to extend the mapping $X \rightsquigarrow \int X dW$ defined on $\mathscr{E}$ to a sufficiently large class of functions.

For this extension the following elementary lemma is needed:

5.4. Lemma.

For every element X of $\mathscr{E}$ and for every element g of G′, we have:

$$\left\| \left(\int X dW\right)(g) \right\|_H^2 = \int M' \{ [X^*(g)]^{\otimes 2} \} \, dv_M$$

Proof. — X belonging to $\mathscr{E}$, we have $X = \sum_{i \in I} x_i . 1_{A(i)}$. We can suppose that the sets $(A(i))_{i \in I}$ are pairwise disjoint; in this case, we have:

$$\left\| \int (X dW)(g) \right\|_H^2 = \sum_{i \in I} \| W[A(i)][x_i^*(g)] \|_H^2$$

by using the fact that W(.) is orthogonal on disjoint sets.

Then, the definition of M implies

$$\left\|\left(\int X dW\right)(g)\right\|_H^2 = \sum_{i \in I} M[A(i)]\{[x_i^*(g)]^{\otimes 2}\}$$
$$= \int [X^*(g)]^{\otimes 2} dM$$
$$= \int M'\{[X^*(g)]^{\otimes 2}\}.dv_M$$

5.5. CONSTRUCTION OF THE INTEGRAL.

The lemma 5-4 above shows exactly that, for every element g of G', the H-valued linear mapping $X \rightsquigarrow \left(\int X dW\right)(g)$ defined on $\mathscr{E}$ is continuous for the semi-norm N_g on $\mathscr{E} \subset \mathscr{L}_g^2$ and the strong topology of H.

Thus, this mapping can be extended into an unique H-valued linear mapping defined on the closure of $\mathscr{E}$ in $\mathscr{L}_g^2$ for the semi-norm N_g. Consequently the integral $\int X dW$ is defined, in a unique way, for all the elements of the closure $\bar{\mathscr{E}}_\pi^2$ of $\mathscr{E}$ in $\mathscr{L}_\pi^2$, i. e. this integral is defined for all the L(B, G)-valued functions X which are weakly* measurable, $X(s)$ has a domain dense in B for every element s of S and such that $N_\pi(X) < +\infty$ (see 3-6 above). Moreover, if X belongs to $\bar{\mathscr{E}}_\pi^2$, $\int X dW$ is an element of CL(G′, H).

6. LOYNES SPACES

Our attention is now directed at spaces on which a vector-valued inner product can be defined.

Let $\tilde{Z}$ be a complex Hausdorff topological vector space satisfying the following conditions:

(6.1) $\tilde{Z}$ has an involution: i. e. a mapping $Z \to Z^+$ of $\tilde{Z}$ to itself with the properties

$$(Z^+)^+ = Z, \qquad (aZ_1 + bZ_2)^+ = \bar{a}Z_1^+ + \bar{b}Z_2^+ \qquad \text{for all complex } a, b;$$

(6.2) there is a closed convex cone $\tilde{P}$ in $\tilde{Z}$ such that $\tilde{P} \cap -\tilde{P} = 0$: then we define a partial ordering in $\tilde{Z}$ by writing $Z_1 \geqslant Z_2$ if $Z_1 - Z_2 \in \tilde{P}$;

(6.3) the topology in $\tilde{Z}$ is compatible with the partial ordering, in the sense that there exists a basic set $\{ N_0 \}$ of convex neighbourhoods of the origin such that if $x \in N_0$ and $x \geqslant y \geqslant 0$ then $y \in N_0$;

(6.4) if $x \in \tilde{P}$ then $x = x^+$;

(6.5) $\tilde{Z}$ is complete as a locally convex space;

(6.6) if $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$ then the sequence x_n is convergent.

We remark that it is clear that these conditions are satisfied by the complex numbers and by the space af all $q \times q$ matrices with the usual topologies.

Now, suppose that $\tilde{H}$ is a complex linear space. A vector inner product on $\tilde{H}$ is a map $x, y \to [x, y]$ from $\tilde{H} \times \tilde{H}$ into an admissible space $\tilde{Z}$ (i. e., satisfying (6.1)-(6.6)) with the following properties:

$$[x, x] \geqslant 0 \quad \text{and} \quad [x, x] = 0 \quad \text{implies } x = 0, \tag{6.7}$$

$$[x, y] = [y, x]^+, \tag{6.8}$$

$$[ax_1 + bx_2, y] = a[x_1, y] + b[x_2, y] \tag{6.9}$$

for complex a and b. When a vector inner product is defined on the space $\tilde{H}$ there is a natural way in which $\tilde{H}$ may be made into a locally convex topological vector space. Namely a basic set of neighbourhoods of the origin $\{ u_0 \}$ is defined by

$$\{ u_0 \} = \{ x : [x, x] \in N_0 \}. \tag{6.10}$$

(6.11) Definition.

The space $\tilde{H}$ which is complete in the topology defined by (6.10) with the admissible space $\tilde{Z}$ satisfying conditions (6.1)-(6.6) will be called a Loynes space. This space was introduced in [Loy].

A complex Hilbert space and the space of all $p \times q$ matrices with the usual topologies are simple examples of Loynes spaces.

(6.12) Proposition ([ChW$_1$]).

Let B be a complex Hilbert space and H a complex Hilbert space. Then the space CL(B, H) with the strong operator topology is a Loynes space if we define A^+ as an operator adjoint to A, $\tilde{P}$ as the set of all non-negative operators and

$$[A, C] = C^+A$$

with values in an admissible space $C\bar{L}(B, B)$ with the weak operator topology.

Using the remark from sect. 1, we have that $C\bar{L}(B, B)$ is isometric to $(B' \hat{\otimes}_1 B')'$. We will see below that the above Loynes space $L(B, H)$ is closely related to stationary processes if $H = L^2(\Omega)$ and to 2-integrable cylindrical martingales if $H = \mathscr{M}$-the Hilbert space of real square integrable martingales.

7. STATIONARY PROCESSES

Let $H = L^2(\Omega)$. By second order stochastic processes with values in a Banach space B, briefly B-process, we mean the trajectory X_t in the Loynes spaces $L(B', H)$. We refer the reader to [Wer_1] for the motivation and examples of such processes. Its correlation function

$$K(t, s) = [X_t, X_s] = [X_s^* X_t]$$

takes values in the admissible space $(B' \hat{\otimes}_1 B')'$.

If T is an Abelian group then a B-process is stationary if the function of two variables $K(t, s)$ depends only on $(t - s)$. In the case when T is a topological group, we will assume that $K(t)$ is weakly continuous i. e., $\forall\, b \in B'$, $K(t)\,(b \otimes b)$ is continuous. If T is a locally compact Abelian (LCA) group then we have, cf. [ChW_2]

$$X_t = \int_G \langle t, g \rangle E(dg) X_e, \tag{7.1}$$

where $E(.)$ is a spectral measure in the space $H_x = \overline{S_p}\{X_t b,\ t \in T,\ b \in B'\}$, defined on the Borel σ-algebra of the dual group G and $\langle t, g \rangle$-a value of the character $g \in G$ at point t. In the case $T = R$ we have $G = R$ and $\langle t, g \rangle = e^{itg}$, $t, g \in R$, similary for $T = \mathbb{Z}$, $G = [0, 2\pi]$ and $\langle t, g \rangle = e^{itg}$, $t \in z$, $g \in [0, 2\pi]$.

We note that the integral (7.1) is a special case of integral w. r. t. an orthogonal measure, which was studied in section 5. Indeed

$$W(\Delta) = E(\Delta) X_e$$

is an orthogonal measure with values in $CL(B', H)$. Here X is equal to $\langle t, g \rangle$.

(7.2) PROPOSITION.

B-process X_t is stationary if its correlation function has the form

$$K(t) = \int_G \langle t, g \rangle M(dg), \qquad t \in T \tag{7.3}$$

where M is non-negative $(B' \hat{\otimes}_1 B')'$-valued measure defined on the Borel σ-algebra B_G of the dual group G.

Proof. — If X_t is stationary then from the definition of the correlation function and (7.1) we have

$$K(t) = [X_t, X_e] = \int_T \langle t, g \rangle \langle \overline{e, g} \rangle X_e^* E(dg) X_e .$$

Thus if we put

$$M(\Delta) = X_e^* E(\Delta) X_e$$

we obtain (7.3). M(.) is non-negative because, according to the properties of the spectral measure E(s) (cf. Sect. 2), we have that

$$M(\Delta) = (E(\Delta)X_e)^*(E(\Delta)X_e) .$$

Conversely, Let M(.) be a non-negative $(B' \hat{\otimes}_1 B')'$-valued measure on B_G. Then, by lemme 2.1 there exist a Hilbert-space H, an operator $R \in CL(B', H)$ and a spectral measure $E_1(.)$ in H such that

$$M(\Delta) = R^* E_1(\Delta) R$$

We may take $H = L^2(\Omega)$. Let we put $U_t = \int_G \langle t, g \rangle E_1(dg)$ for each $t \in T$. Then, U_t is a unitary operator. Now we define a process X_t by the following formula

i) $X_e = R$
ii) $X_t = U_t X_e$.

Consequently, X_t has the form (7.1) and is stationary.

(7.4) REMARK.

Observe that $W(.) = E(.)X_e$ is an orthogonal random measure and that each stationary B-process has the representation as integral (stochastic) w. r. t. an orthogonal random measure

$$X_t = \int_G \langle t, g \rangle W(dg) .$$

Consequently, the measure M connected with the correlation function of a stationary B-process (by 7.2) has the form:

$$M(\Delta) = W^*(\Delta) W(\Delta) .$$

We note that the above fact is general and holds for all non-negative $(B' \hat{\otimes}_1 B')'$ valued measure on (S, Σ). It follows immediately from the dilation theorem (Lemma 2.1).

(7.5) PROPOSITION.

If $M(.)$ is a non-negative $(B' \hat{\otimes}_1 B')'$-valued measure on (S, Σ) then there exists a Hilbert space H and a orthogonaly $CL(B', H)$-valued measure $W(.)$ on (S, Σ) such that for each $\Delta \in \Sigma$

$$M(\Delta) = W^*(\Delta)W(\Delta).$$

Moreover, if H is minimal $H = \bigvee_{\Delta \in \Sigma} \{ W(\Delta)B' \}$ then H and W are determined up to unitary equivalence.

(7.6) EXAMPLE.

Let B be a Hilbert space. Then stochastic integral (as in 7.4) is defined as mapping from $L_{2,M}$ into H_x^0: closed subspace of Hilbert-Schmidt operators spanned by $\left\{ \sum_{i=1}^{N} X_{t_i} A_i,\ A_i \in CL(H),\ t_i \in T \text{ and N is integer} \right\}$ cf. [MaS].

We note only that in view of (2.1) and (4.5) this stochastic integral realized unitary equivalence between two spectral dilations of the measure M defined by the correlation function of the stationary process.

8. CYLINDRICAL MARTINGALES

Let $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t \in T})$ be a probabilized stochastic basis (see, for example [MeP$_2$]). Let $\mathscr{M}$ be the space of all the complex square integrable martingales (with respect to this stochastic basis); if $(U_t)_{t \in T}$ and $(V_t)_{t \in T}$ are two such martingales, we can define

$$\langle U, V \rangle = E[U_\infty . V_\infty] \qquad (\text{if } \infty = \operatorname{Sup}_{t \in T} t)$$

For this calar product $\langle .,. \rangle$, $\mathscr{M}$ is an Hilbert space.

Let B be a Banach space. Following Prop. (6.12), $CL(B', \mathscr{M})$ is a Loynes space. Then, an element $\tilde{W}$ of $CL(B', \mathscr{M})$ is called a square-integrable cylindrical martingales (see [Met] or [MeP$_1$]).

We define $S = \Omega \times T$ and we denote by Σ the σ-algebra of the predictable

sets (with respect to the stochastic basis given above). We denote by $\mathscr{A}$ the algebra, included in Σ, generated by the sets $(D \times]r, s])$ with $r \in T$, $s \in T$ and $D \in \mathscr{F}_r$. For such an element $F = D \times]r, s]$ and for every element u of B', we denote by $W(F)(u)$ the real martingale which is defined by:

$$[W(F)(u)](t, \omega) = 1_F(\omega) . 1_{]r,s]}(t) . [\tilde{W}(u)](t, \omega)$$

Then, W is an additive function which satisfies all the properties given in 5.2 above with $H = \mathscr{M}$.

In this case, the function M, associated to W as in 5-2 above, is the quadratic Doléans measure of the cylindrical martingale W. If X is a weakly* predictable process (with values in L(B, G)), X can be considered as a weakly* Σ-measurable function; in this case, $\int X dW$ is the stochastic integral of X with respect to W.

For other details and results, see [Met] and [MeP_1].

REFERENCES

[Bar] R. G. BARTLE, *A general bilinear vector integral. Studia Math.*, t. **15**, 1956, p. 337-352.

[ChW_1] S. A. CHOBANYAN and A. WERON, Stationary processes in pseudo-Hilbertian space (in Russian), *Bull. Acad. Polon.*, t. **21**, 1973, p. 847-854.

[ChW_2] S. A. CHOBANYAN and A. WERON, Banach space valued stationary processes and their linear prediction. *Dissertationes Math.*, t. **125**, 1975, p. 1-45.

[Dou] R. G. DOUGLAS, On factoring positive operator functions. *J. Math. Mech.*, t. **16**, 1966, p. 119-126.

[DuS] N. DUNFORD and J. T. SCHWARTZ, *Linear operators*. Part. 1. Intersciences Publishers, 1967.

[Loy] R. M. LOYNES, Linear operator in Vh-spaces, *Trans. Amer. Math. Soc.*, t. **116**, 1965, p. 167-180.

[Mas] V. MANDREKAR and H. SALEHI, The square integrability of operator-valued functions with respect to a non-negative operator valued measure and the Kolmogorov isomorphism theorem. Indiana University Mathematical Journal. vol. **20**, n° 6, December 1970.

[Mas] P. MASANI, Quasi isometric measures and their applications. *Bull. Amer. Math. Soc.*, vol. **76**, n° 3, 1970.

[Met_1] M. METIVIER, *Integration with respect to processes of linear fonctionnals*. Séminaire de Probabilité de Rennes, 1975.

[Met_2] M. METIVIER, Cylindrical martingales and stochastic integrals. Lecture on the conference. *Probability Theory on vector spaces*. Trzebieszowice, 1977 (non published).

[MeP_1] M. METIVIER and J. PELLAUMAIL, *Cylindrical stochastic integral*. Séminaire de Rennes, 1976.

[MeP_2] M. METIVIER and J. PELLAUMAIL, *A basic course on stochastic integration*. Séminaire de Rennes, 1978.

[Pel] J. PELLAUMAIL, Sur la dérivation d'une mesure vectorielle. *Bull. Soc. Math. France*, t. **98**, 1970, p. 305-318.

[Tre] F. Treves, *Topological vector spaces, distributions and kernels.* Academic Press, 1967.

[Wel] J. N. WELSH, Constructions of the Hilbert space $L_{2,M}$ for operator measures. *Proc. Symp. on vector and Operator measures.* A. Utah, 1972.

[Wer_1] A. WERON, Prediction theory in Banach spaces. *Lecture Notes in Math.*, vol. **472**, 1975, p. 207-227.

[Wer_2] A. WERON, Remarks on positive-definite operator valued functions in Banach spaces. *Bull. Acad. Polon.*, t. **24**, 1976, p. 873-876.

[Wer_3] A. WERON, *Second order stochastic processes and the dilation theory in Banach spaces* (to appear).

[Yos] K. YOSIDA, Functionnal analysis. *Second printing*, Springer Verlag, 1966.

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