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Section B :
Calcul des Probabilités et Statistique.

Catching small sets under Flows

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ABSTRACT. — Two results are established concerning the ability of a set to cover any element of a class of sets under the action of an ergodic flow. The classes considered are the class of countable sets and the class of sets of measure zero.

RÉSUMÉ. — Deux résultats sont démontrés concernant la possibilité pour un ensemble de recouvrir, sous l'action d'un flot ergodique, tous les éléments d'une classe d'ensembles. Les classes considérées sont la classe des ensembles dénombrables et la classe des ensembles de mesure zéro.

1. INTRODUCTION

One of the earliest results in ergodic theory is Poincaré's recurrence theorem which states that if a measure preserving transformation on a finite space has no non-trivial invariant sets then every set of positive measure hits almost every point infinitely often under the action of the transformation. The first result of this paper establishes a complement to Poincaré's theorem by proving that if a measure preserving flow on a complete probability space has no non-trivial invariant sets then there exists a set of *measure zero* which completely contains each *countable* set infinitely often under the action of the flow.

To isolate the difference between this result and Poincaré's theorem we note that the representation theorem of Ambrose [*1*] easily implies there is a set of measure zero which will catch any element of Ω, but a more elaborate procedure is needed to catch every countable subset. Also we note that under the action of a set of discrete transformations $\{ T^j : j \in \mathbb{Z} \}$ one can easily show that no set other than Ω is capable of catching every countable set.

Our second theorem shows that no set of less than full measure is able to completely catch each set of *measure zero*, hence the countable set in the result mentioned above cannot be substantially enlarged. An essential step in the proof of this second result is a lemma which states that given any positive real numbers $\{ \alpha_i \}_{i=1}^{\infty}$ for which $\sum_{i=1}^{\infty} \alpha_i < 1$ there is a closed set $F \subset [0, 1]$ of measure zero such that $F \not\subset \bigcup_{i=1}^{\infty} I_i$ for any collection of intervals $\{ I_i \}$ with lengths $\{ \alpha_i \}$.

We will call a collection of measure preserving transformations $\{ T_t : t \in \mathbb{R} \}$ a *measurable flow* on the complete probability space $(\Omega, \mathfrak{F}, \mu)$ provided (1) T_t is a bimeasurable measure preserving transformation of Ω onto Ω for each $t \in \mathbb{R}$, (2) $T_{s+t} = T_s \circ T_t$ and (3) $M^* = \{ (\omega, t) : \omega \in T_t M \}$ is measurable in the product space $\Omega \times \mathbb{R}$ for each $M \in \mathfrak{F}$. A flow $\{ T_t : t \in \mathbb{R} \}$ is *ergodic* provided that $\mu(A) > 0$ implies that $\bigcup_{t \in \mathbb{R}} T_t A$ is of full measure. With these preliminaries, we now have our main results.

2. CATCHING THEOREMS

THEOREM 1. — Suppose T_t is a measurable ergodic flow on a complete probability space $(\Omega, \mathfrak{F}, \mu)$. There is a set A of measure zero such that for any countable C there is a $t = t(C)$ such that $C \subset T_t A$. In fact, for any such C, the set $\{ t : C \subset T_t A \}$ is dense in $\mathbb{R}$.

Proof. — To prove this result we will use Rudolph's representation theorem [*6*] as sharpened by Krengel [*5*]. Let $\{ T_t \}$ be an ergodic measurable flow on the complete probability $(\Omega, \mathfrak{F}, \mu)$. Further, let p, q be two positive real numbers with p/q irrational. Then there exists a finite measure space $(B, \mathfrak{F}^*, \nu)$, an ergodic, measure preserving, invertible map S from B to B and a set D in $\mathfrak{F}^*$ with the following property. Let

$$\Omega' = (D \times [0, p)) \cup (D^c \times [0, q)),$$

$\mathfrak{F}'$ be the restriction to Ω' of the completion of the product of $\mathfrak{F}^*$ with the Lebesgue measurable sets, and let μ' be the restriction to $\mathfrak{F}^*$ of the completed product of ν with Lebesgue measure. Let $\{ T'_t \}$ be the measurable flow on $(\Omega', \mathfrak{F}', \mu')$ satisfying

$$T'_t(x, r) = \begin{cases} (x, r + t) & x \in D \quad \text{and} \quad t < p - r \\ (Sx, r + t - p) & x \in D \quad \text{and} \quad t \geq p - r \\ (x, r + t) & x \in D^c \quad \text{and} \quad t < q - r \\ (Sx, r + t - q) & x \in D^c \quad \text{and} \quad t \geq q - r \end{cases}$$

for all $(x, r) \in \Omega'$ and $0 \leq t \leq \min(p, q)$. There is a set $N \in \mathfrak{F}$ with $\mu(N) = 0$, $T_t(N) = N$ for all $t \in \mathbb{R}$, and a bimeasurable measure-preserving bijection $\Phi : \Omega - N \to \Omega'$ such that $T_t(w) = \Phi^{-1}(T'_t(\Phi(\omega)))$ for all $t \in \mathbb{R}$ and $\omega \in \Omega - N$.

In words, $\{ T'_t \}$ is the flow on $(\Omega', \mathfrak{F}', \mu')$ which sends each point (x, r) in $D \times [0, p)$ (respectively, $D^c \times [0, q)$) upward at unit velocity until the second coordinate reaches p (respectively q), at which instant the point « jumps » to $(Sx, 0)$ and continues moving upward; Φ is a measure-preserving isomorphism from $(\Omega - N, \mathfrak{F}|_{\Omega - N}, \mu)$ to $(\Omega', \mathfrak{F}', \mu')$ which carries T_t to T'_t.

Let $(\Omega', \mathfrak{F}', \mu')$, $\{ T'_t \}$, N and Φ be as above. For each positive integer n let $\mathcal{O}_n$ be a dense open subset of $(0, 1)$ whose measure is less than 2^{-n}, and let

$$Q_n = \{ (x, r) : x \in D \quad \text{and} \quad rp^{-1} \in \mathcal{O}_n \} \cup \{ (x, r) : x \in D^c \quad \text{and} \quad rq^{-1} \in \mathcal{O}_n \}.$$

For each positive integer n, $\mu'(Q_n) < \mu'(\Omega')/2^n$, which implies

$$\mu'\left(\bigcap_{n=1}^{\infty} Q_n\right) = 0.$$

Furthermore, for each $(x, r) \in \Omega'$ and positive integer n one has that

$$\{ t : (x, r) \in T'_t(Q_n) \}$$

is a dense open subset of $\mathbb{R}$. The Baire Caterogy Theorem now implies that for each countable subset C of Ω',

$$\Lambda(C) = \bigcap_{n=1}^{\infty} \bigcap_{(x,r) \in C} \{ t : (x, r) \in T'_t(Q_n) \}$$

is dense in $\mathbb{R}$. We then see that $t \in \Lambda(C)$ if and only if $C \subset T'_t\left(\bigcap_{n=1}^{\infty} Q_n\right)$.
Thus $A = N \cup \Phi^{-1}\left(\bigcap_{n=1}^{\infty} Q_n\right)$ catches each countable subset of Ω at a dense set of times, and $\mu(A) = 0$. ‖

Before proving our second theorem we establish the lemma mentioned in the introduction.

LEMMA. — Let $\{ \alpha_i \}_{i=1}^{\infty}$ be a countable set of positive reals for which $\sum_{i=1}^{\infty} \alpha_i = \alpha < 1$. There is then a closed set F of measure zero, $F \subset [0, 1]$, such that $F \not\subset \bigcup_{i=1}^{\infty} I_i$ for any collection of open intervals $\{ I_i \}$ satisfying $m(I_i) = \alpha_i$.

Proof. — We will first construct closed subsets $F_k \subset [0, 1]$ such that $F_{k+1} \subset F_k$, $m(F_k) \to 0$, and such that $F_k \not\subset \bigcup_{i=1}^{\infty} I_i$ for any $\{ I_i \}$ with $m(I_i) = \alpha_i$.

We can assume the α_i are monotone decreasing and use the fact that $\sum_{i=1}^{\infty} \alpha_i < 1$ to choose and increasing sequence of positive integers $r_1, r_2, \ldots$ satisfying

$$(2.1) \qquad \left(\sum_{i=1}^{r_1} \alpha_i\right) + \left(2 \sum_{i=1+r_1}^{r_2} \alpha_i\right) + \ldots + \left(2^{l-1} \sum_{i=1+r_{l-1}}^{r_l} \alpha_i\right) + \ldots < (1 + \varepsilon)^{-1}$$

for some $\varepsilon > 0$. Sequentially choose positive integers $\{ n_i \}_{i=1}^{\infty}$ to satisfy

$$(2.2) \qquad n_1 > (2\varepsilon\alpha_{r_1})^{-1}$$

and

$$(2.3) \qquad \prod_{i=1}^{l} n_i > (2^l \alpha_{r_l+1})^{-1}$$

for all $l \geq 1$.

Let $F_1 = \bigcup_{j=0}^{n_1-1} [j/n_1, j/n_1 + 1/2n_1]$. Next F_2 is defined by partitioning each interval of F_1 into $2n_2$ equal intervals and letting F_2 be the set formed by taking every other one of these smaller intervals (including endpoints). Similarly F_l is obtained by partitioning each interval of F_{l-1} into $2n_l$ equal intervals and taking every other one (including endpoints) as an interval of F_l. We have $F_{l+1} \subset F_l$ and $m(F_l) = 2^{-l}$. We need to show that for all l, $F_l \not\subset \bigcup_{i=1}^{\infty} I_i$ when $m(I_i) = \alpha_i$.

Fix l and define a measure of the efficiency of covering F_l with an interval I of length x by

$$\text{eff}\,(x) = \sup \{ m(I \cap F_l)/xm(F_l) : I \text{ is an interval of length } x \}.$$

Inequality (2.2) was imposed precisely to guarantee that

$$\text{(2.4)} \qquad \text{eff}\,(x) < 1 + \varepsilon \quad \text{if} \quad x \geq \alpha_{r_1}.$$

Similarly, (2.3) insures that eff $(x) < 2$ if $x \geq \alpha_{r_2}$, eff $(x) < 2^2$ if $x \geq \alpha_{r_3}$, and generally

$$\text{(2.5)} \qquad \text{eff}\,(x) < 2^{k-1} \quad \text{if} \quad x \geq \alpha_{r_k} \ , \quad 2 \leq k \leq l+1.$$

Suppose now $\{ I_i \}$ is any collection of open intervals with $\alpha_i = m(I_i)$. Then (2.1), (2.4) and (2.5) imply

$$\begin{aligned} m\left(\left(\bigcup_{i=1}^{\infty} I_i\right) \cap F_l\right) &\leq \sum_{i=1}^{r_1} m(I_i \cap F_l) + \sum_{i=1+r_1}^{r_2} m(I_i \cap F_l) + \dots \\ &\leq \left\{ \sum_{i=1}^{r_1} \alpha_i \,\text{eff}\,(\alpha_i) + \sum_{i=1+r_1}^{r_2} \alpha_i \,\text{eff}\,(\alpha_i) + \dots \right\} m(F_l) \\ &\leq \left\{ \sum_{i=1}^{r_1} (1+\varepsilon)\alpha_i + \sum_{i=1+r_1}^{r_2} 2.\alpha_i + \sum_{i=1+r_2}^{r_3} 2^2.\alpha_i + \dots \right\} m(F_l) \\ &< m(F_l). \end{aligned}$$

This last inequality shows $F_l \not\subset \bigcup_{i=1}^{\infty} I_i$, as claimed.

To complete the proof we let $F = \bigcap_{i=1}^{\infty} F_i$. Since $m(F_k) = 2^{-k}$ we see $m(F) = 0$. Now for any collection of open intervals $\{ I_i \}$ with $m(I_i) = \alpha_i$ we have $G_k = F_k \cap \left(\bigcup_{i=1}^{\infty} I_i\right)^c \neq \emptyset$. Since the G_k are nested and compact we also have $F \cap \left(\bigcup_{i=1}^{\infty} I_i\right)^c \neq \emptyset$ and the proof of the lemma is complete. ∥

In the next result we use Rudolph's representation theorem to create circumstances where the preceding lemma can be effectively applied.

THEOREM 2. — Given any A of less than full measure there is an E of measure zero such that E is not contained in $T_t A$ for any $t \in \mathbb{R}$.

Proof. — As in Theorem 1 we consider the flow T'_t built under the two-step (p, q) function, where D and D^c denote the parts of the base B which lie under the heights p and q respectively. Let A' be the image of $A - N$ under the isomorphism Φ.

The product measurability of A' together with the fact that A does not have full measure imply there is an $0 \leq \alpha < 1$ such that $\mu'(A') = \alpha\mu'(\Omega')$ and one of the following must hold.

(1) There is an $\tilde{x} \in D$ such that $m\{ t : (\tilde{x}, t) \varepsilon A', \ 0 \leq t < p \} \leq \alpha p$,

or

(2) There is an $\tilde{x} \in D^c$ such that $m\{ t : (\tilde{x}, t) \varepsilon A', \ 0 \leq t < q \} \leq \alpha q$.

We can suppose without loss of generality that (1) occurs. Next we take a collection of open intervals $\{ I_i \}_{i=1}^{\infty}$ such that

$$\{ t : (\tilde{x}, t) \varepsilon A', \ 0 \leq t < p \} \subset \bigcup_{i=1}^{\infty} I_i$$

and for which $\sum_{i=1}^{\infty} m(I_i) = \alpha' < p$. Setting $\alpha_i = m(I_i)$ for $i = 1, 2, \ldots$ and $\alpha_0 = \alpha_{-1} = (p - \alpha')/3$ we apply the preceding lemma to obtain a closed $F \subset \{ t : 0 \leq t < p \}$ of measure zero which cannot be covered by any collection of intervals with lengths $\{ \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots \}$. We claim that the set $\{ (\tilde{x}, t) : t \varepsilon F \}$ is not contained in any element of the class

$$\mathscr{F} = \{ T'_t A' : | t | \leq (p - \alpha')/3 \}.$$

To prove the claim we note by the definition of a flow under a function that the image of $A' \cap \{ (x, r) : 0 \leq r < p \}$ under T_t is just a translate of $A' \cap \{ (x, r) : 0 \leq r < p \}$ up the fiber (x, r) by an amount t except for the bottom $\{ (x, r) : 0 \leq r < t \}$ and the image of the top

$$A' \cap \{ (x, r) : p - t \leq r < p \}.$$

The fact that a system of intervals of lentgh $\{ \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots \}$ cannot cover F then implies that $\{ (\tilde{x}, t) : t \varepsilon \mathrm{F} \}$ is not caught by any element of $\{ \mathrm{T}'_t \mathrm{A}' : | t | \leq (p - \alpha')/3 \}$.

To complete the proof we let $\tilde{\mathrm{F}} = \bigcup_{k=-\infty}^{\infty} \mathrm{T}'_{ks} \{ (\tilde{x}, t) : t \varepsilon \mathrm{F} \}$ where $s = (p - \alpha')/3$. Since $\{ (\tilde{x}, t) : t \varepsilon \mathrm{F} \} \not\subset \mathrm{T}'_t \mathrm{A}'$ for $| t | \leq s$ we have

$$\mathrm{T}'_{ks} \{ (\tilde{x}, t) : t \varepsilon \tilde{\mathrm{F}} \} \not\subset \mathrm{T}'_{t+ks} \mathrm{A}' \quad \text{for} \quad | t | \leq s \quad \text{and all } k.$$

This says $\tilde{\mathrm{F}} \not\subset \mathrm{T}'_t \mathrm{A}'$ for all $t \in \mathbb{R}$. For the last step we let $\mathrm{E} = \Phi^{-1} \tilde{\mathrm{F}}$. Since Φ preserves measure $\mu(\mathrm{E}) = 0$ and since $(\mathrm{T}_t \mathrm{A}) \cap \mathrm{E} = (\mathrm{T}_t(\mathrm{A} - \mathrm{N})) \cap \mathrm{E}$ for all $t \in \mathbb{R}$ the theorem is established.

3. FURTHER REMARKS

Rudolph's representation theorem [*5*], [*6*] played a key-role in the conceptualization and proof of the results given here, but one should note that at the expense of greater complexity one need only appeal to the representation theorems of Ambrose [*1*] or Ambrose and Kakutani [*2*]. The assumption of ergodicity made in our results can be weakened to aperiodicity since the representation theorem remains valid. This is mentioned in Rudolph [*6*] and the extension is described in detail in Krengel [*5*]. For simplicity of exposition we have omitted the full discussion of this more technical hypothesis.

Since we have dealt here only with flows, we should note that there are some related results for transformations. It is proved in Steele [*7*] that for an ergodic T on a Lebesgue probability space there is for any $\varepsilon > 0$ an A with $\mu(\mathrm{A}) < \varepsilon$ which satisfies the condition:

For any *finite* F there is a $j = j(\mathrm{F})$ such that $\mathrm{F} \subset \mathrm{T}^j \mathrm{A}$.

One can easily show that this result is best possible in the senses that neither can A be taken to be of measure 0 nor can F be allowed to be countable.

This result provided one of the motivations of the present paper, and it has also been extended in a quite different direction in Ellis [*3*].

Finally, we note that a much earlier contribution to covering with sets of measure zero was made by Erdos and Kakutani [*4*], but their work concerns Euclidean similarities rather than measure preserving transformations.

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