ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 13, nº 3 (1977), p. 231-238
<http://www.numdam.org/item?id=AIHPB_1977__13_3_231_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XIII, n° 3, 1977, p. 231-238. *Calcul des Probabilités et Statistique.*

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by

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1. INTRODUCTION

There exist a compact metric space K and two singular Radon probability measures on K which agree on all balls ([*5*], Th. II, [*4*]). Therefore, since K is isometric to a compact subset of the Banach space C(K), we can find two singular Radon probability measures μ and ν on C(K) satisfying the condition

(C_0) *for every* $a \in C(K)$ *there exists a* $\delta > 0$ *such that*
$$\mu(B(a\,;r)) = \nu(B(a\,;r)), \qquad 0 < r < \delta .$$

Here $B(a\,;r)$ denotes the closed ball of centre a and radius r. (Compare [*6*], p. 326, and [*9*].)

The main result of this note shows that two Gaussian Radon measures on C(K) (or any Banach space) coincide whenever the condition (C_0) holds (Theorem 3.1). Moreover, we prove that two Gaussian Radon measures on a Banach space are equal, if they agree on all balls of radius one (Theorem 3.2). The same theorem also gives a positive result for dual Banach spaces, equipped with the weak* topology.

Finally, I am grateful to J. Neveu, H. Sato and F. Topsøe for a very stimulating exchange of ideas about the group of problems considered in this note.

2. THE REPRODUCING KERNEL HILBERT SPACE OF A GAUSSIAN RADON MEASURE

In this section it will always be assumed that E is a fixed locally convex Hausdorff vector space over $\mathbb{R}$. The class of all (centred) Gaussian Radon measures on E is denoted by $\mathcal{G}(E)$ ($\mathcal{G}_0(E)$). In the following, all non-trivial statements will either be proved or, otherwise, they can be found in e. g. [3].

Let $\mu \in \mathcal{G}(E)$ be fixed and denote by b the barycentre of μ. Set $\mu_0(.) = \mu(. + b)$ and $E'_2(\mu) =$ the closure of E' in $L_2(\mu_0)$, respectively. Then for every $\eta \in E'_2(\mu)$, the measure $\eta\mu_0$ has a barycentre $\Lambda(\eta) \in E$. The map $\Lambda : E'_2(\mu) \to E$ is injective. Its range is denoted by $H(\mu)$. For brevity we write $\Lambda^{-1}h = \tilde{h}$, $h \in H(\mu)$. Obviously, the scalar product

$$\langle h, k \rangle_\mu = \int \tilde{h}\tilde{k} d\mu_0, \qquad h, k \in H(\mu),$$

makes $H(\mu)$ into a Hilbert space, the so-called reproducing kernel Hilbert space of μ. The closed unit ball $O(\mu)$ of $H(\mu)$ is a compact subset of E. Moreover,

$$\max_{O(\mu)} \xi^2 = \int \xi^2 d\mu_0, \qquad \xi \in E'.$$

Observing that

$$\int \exp(i\xi) d\mu_0 = \exp\left(-\frac{1}{2} \int \xi^2 d\mu_0 \right), \qquad \xi \in E'.$$

we have the following useful

THEOREM 2.1. — *Let $\mu, \nu \in \mathcal{G}_0(E)$. Then $\mu = \nu$ if $O(\mu) = O(\nu)$.*

Our strategy from now on will be to determine $O(\mu)$ from measures of sufficiently many « balls ». A weak result in this direction follows from e. g. [3], Th. 10.1. Theorems 2.2 and 2.3 below yield stronger conclusions.

Before proceeding, let us introduce

$$\| a \|_\mu^2 = \begin{cases} \langle a, a \rangle_\mu, & a \in H(\mu), \\ +\infty, & a \in E \backslash H(\mu). \end{cases}$$

Moreover, in the following, measurable always means Borel measurable.

THEOREM 2.2. — *Let $\mu \in \mathcal{G}_0(E)$ and suppose V is a bounded, symmetric, convex, and measurable subset of E such that $\mu(rV) > 0$, $r > 0$. Then*

$$(2.1) \qquad \lim_{r \to 0^+} \frac{\mu(a + rV)}{\mu(rV)} = \exp\left(-\frac{1}{2} \| a \|_\mu^2 \right), \qquad a \in E.$$

In many special cases, the behaviour of $\mu(rV)$ for small $r > 0$ is known. For example, J. Hoffmann-Jørgensen [7] and L. A. Shepp [8] give some very precise estimates when E is a Hilbert space and V the unit ball of E.

To prove Theorem 2.2, we need two lemmas.

LEMMA 2.1. — [3], Cor. 2.1 (Cameron-Martin's formula). *For any* $\mu \in \mathcal{G}_0(E)$

$$\mu(. - h) = \left[\exp\left(\tilde{h} - \frac{1}{2}\| h \|_\mu^2\right)\right]\mu(.), \qquad h \in H(\mu).$$

LEMMA 2.2. — [2], Cor. 2.1, Th. 6.1. *For any* $\mu \in \mathcal{G}_0(E)$

$$\mu_*(\lambda A + (1 - \lambda)B) \geq \mu^\lambda(A)\mu^{1-\lambda}(B), \qquad 0 < \lambda < 1,$$

for all measurable subsets A *and* B *of* E.

In particular,

$$\mu(a + A) \leq \mu(A), \qquad a \in E,$$

whenever A *is symmetric, convex, and measurable subset of* E.

PROOF OF THEOREM 2.2. — Let us first assume that $a \in H(\mu)$. By the Cameron-Martin formula, we have

$$\mu(a + rV) = \exp\left(-\frac{1}{2}\| a \|_\mu^2\right)\int_{rV} \exp(-\tilde{a})d\mu. \tag{2.2}$$

Moreover, the Jensen inequality yields

$$\int_{rV} \exp(-\tilde{a})d\mu \geq \mu(rV)\exp\left(-(\mu(rV))^{-1}\int_{rV}\tilde{a}d\mu\right).$$

Since

$$\int_{rV}\tilde{a}d\mu = 0,$$

it follows that

$$\varliminf_{r\to 0^+}\frac{\mu(a + rV)}{\mu(rV)} \geq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right).$$

We now prove the estimate

$$\varlimsup_{r\to 0^+}\frac{\mu(a + rV)}{\mu(rV)} \leq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right). \tag{2.3}$$

To this end let $\xi \in E'$ be fixed and set $h = \Lambda\xi$. Then (2.2) gives

$$\mu(a + rV) \leq \left[\exp\left(-\frac{1}{2}\| a \|_\mu^2 - \inf_{rV}\xi\right)\right]\int_{rV}\exp(\xi - \tilde{a})d\mu.$$

Moreover, the Cameron-Martin formula yields

$$\int_{rV}\exp(\xi - \tilde{a})d\mu = \mu(a - h + rV)\exp\left(\frac{1}{2}\| h - a \|_\mu^2\right).$$

By applying Lemma 2.2, we have

$$\frac{\mu(a + rV)}{\mu(rV)} \leq \exp\left(-\frac{1}{2} \| a \|_\mu^2 - \inf_{rV} \xi + \frac{1}{2} \| h - a \|_\mu^2 \right),$$

and hence

$$\overline{\lim_{r \to 0^+}} \frac{\mu(a + rV)}{\mu(rV)} \leq \exp\left(-\frac{1}{2} \| a \|_\mu^2 + \frac{1}{2} \| h - a \|_\mu^2 \right).$$

By choosing $\xi \in E'$ close to $\tilde{a}$ in $E_2'(\mu)$, the estimate (2.3) follows at once. This proves (2.1) when $a \in H(\mu)$.

Let now $a \in \operatorname{supp}(\mu) \backslash H(\mu)$. Then, for every $n \in \mathbb{N}$, there exists a $\xi_n \in E'$ such that

$$\xi_n^2(a) > (n + 1) \int \xi_n^2 d\mu$$

and

$$\int \xi_n^2 d\mu = 1,$$

respectively. Set $a_n = \xi_n(a) \Lambda \xi_n$ and note that

$$a_n + rV \supseteq \frac{1}{2}(a + rV) + \frac{1}{2}(2a_n - a + rV).$$

By applying Lemma 2.2, we have

$$\mu^2(a_n + rV) \geq \mu(a + rV)\mu(2a_n - a + rV). \tag{2.4}$$

Furthermore, observing that μ is symmetric, the Cameron-Martin formula yields

$$\mu(2a_n - a + rV) = [\exp(-2 \| a_n \|_\mu^2)] \int_{a+rV} \exp(2\tilde{a}_n) d\mu.$$

Since $\| a_n \|_\mu^2 = \xi_n^2(a)$ and $\tilde{a}_n = \xi_n(a)\xi_n$, respectively, we get

$$\mu(2a_n - a + rV) \geq [\exp(2 \inf_{a+rV} \xi_n(a)(\xi_n - \xi_n(a)))] \mu(a + rV).$$

Using (2.4), it follows that

$$\overline{\lim_{r \to 0^+}} \frac{\mu(a_n + rV)}{\mu(rV)} \geq \overline{\lim_{r \to 0^+}} \frac{\mu(a + rV)}{\mu(rV)}.$$

Here, by the first part of the proof, the left-hand side equals $\exp(-\xi_n^2(a)/2)$. Clearly, this expression converges to zero as n tends to plus infinity. This proves (2.1) when $a \in \operatorname{supp}(\mu) \backslash H(\mu)$. Finally, the case $a \in E \backslash \operatorname{supp}(\mu)$ is trivial. This completes the proof of Theorem 2.2.

We also have

THEOREM 2.3. — *Let* $\mu \in \mathcal{G}(E)$ *and suppose* V *is a bounded measurable subset of* E *with positive* μ*-measure. Then*

$$\lim_{t\to+\infty} (\mu(ta + V))^{1/t^2} = \exp\left(-\frac{1}{2}\| a \|_\mu^2\right), \qquad a \in E. \tag{2.5}$$

Proof. — Without loss of generality it can be assumed that $\mu \in \mathcal{G}_0(E)$. Suppose first that $a \in H(\mu)$. As in the proof of Theorem 2.2, we have

$$\mu(ta + V) = \left[\exp\left(-\frac{t^2}{2}\| a \|_\mu^2\right)\right] \int_V \exp(-t\tilde{a})d\mu, \tag{2.6}$$

and

$$\int_V \exp(-t\tilde{a})d\mu \geq \mu(V)\exp\left(-t(\mu(V))^{-1}\int_V \tilde{a}d\mu\right),$$

respectively. Hence

$$\varliminf_{t\to+\infty} (\mu(ta + V))^{1/t^2} \geq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right).$$

We now prove the estimate

$$\varlimsup_{t\to+\infty} (\mu(ta + V))^{1/t^2} \leq \exp\left(-\frac{1}{2}\| a \|_\mu^2\right). \tag{2.7}$$

To this end let $\xi \in E'$ be arbitrary and set $h = \Lambda\xi$. Then, assuming $t > 0$, it follows that

$$\int_V \exp(-t\tilde{a})d\mu \leq [\exp(-t\inf_V \xi)] \int_V \exp(t(\xi - \tilde{a}))d\mu.$$

Using the trivial estimate

$$\int_V \exp(t(\xi - \tilde{a}))d\mu \leq \exp\left(\frac{t^2}{2}\| h - a \|_\mu^2\right),$$

the relation (2.6) yields

$$\varlimsup_{t\to+\infty} (\mu(ta + V))^{1/t^2} \leq \exp\left(-\frac{1}{2}\| a \|_\mu^2 + \| h - a \|_\mu^2\right).$$

By choosing ξ to close to $\tilde{a}$ in $E_2'(\mu)$, we get (2.7). This proves (2.5) when $a \in H(\mu)$.

Let now $a \in E \backslash H(\mu)$. Then, for every $n \in \mathbb{N}$, there exists a $\xi_n \in E'$ so that

$$\xi_n^2(a) \geq (n + 1)\int \xi_n^2 d\mu,$$

and $\xi_n(a) = 1$, respectively. Since

$$ta + V \subseteq \{ \xi_n \geq t + \inf_V \xi_n \}, \qquad t > 0,$$

it follows that

$$\overline{\lim_{t \to +\infty}} (\mu(ta + V))^{1/t^2} \leq \exp(-(n+1)/2).$$

By letting n tend to plus infinity, we get (2.5) for $a \in E \backslash H(\mu)$. This concludes the proof of Theorem 2.3.

3. APPLICATIONS

The results proved in Section 2 apply to any locally convex Hausdorff vector space. In order to be concrete, however, we here restrict ourselves to Banach spaces and dual Banach spaces equipped with the weak* topology respectively.

THEOREM 3.1. — *Let* E *be a Banach space and suppose* $\mu \in \mathcal{G}_0(E)$ *and* $\nu \in \mathcal{G}(E)$. *Moreover, assume there exists a function* $\delta : B(0; 1) \to]0, +\infty[$ *such that*

$$\mu(B(a; r)) = \nu(B(a; r)), \qquad 0 < r < \delta(a), \qquad \| a \| \leq 1.$$

Then $\mu = \nu$.

Proof. — Let c denote the barycentre of ν and note that

$$\nu_0(B(-c, r)) = \mu(B(0; r)) > 0, \qquad 0 < r < \delta(0),$$

by Lemma 2.2. Hence $-c \in \text{supp}\, \nu_0 = \overline{H(\nu)}$ [3], Cor. 8.2. By choosing $k \in B(c; 1) \cap H(\nu)$, we get

$$\mu(B(c-k; r)) = \nu_0(B(-k; r)), \qquad 0 < r < \delta(c-k).$$

Since $\nu_0(B(0; r)) \geq \nu_0(B(-c; r)) > 0$, $r > 0$, the relation

$$1 \geq \frac{\mu(B(c-k; r))}{\mu(B(0; r))} = \frac{\nu_0(B(-k; r))}{\nu_0(B(0; r))} \cdot \frac{\nu_0(B(0; r))}{\nu_0(B(-c; r))}$$

must be true for all $0 < r < \min(\delta(c-k), \delta(0))$. By letting r tend to zero from the right and using Theorem 2.2, we get $-c \in H(\nu)$. Moreover,

$$\frac{\mu(B(a; r))}{\mu(B(0; r))} = \frac{\nu_0(B(a-c; r))}{\nu_0(B(0; r))} \cdot \frac{\nu_0(B(0; r))}{\nu_0(B(-c; r))}$$

for every $0 < r < \min(\delta(a), \delta(0))$ and $\| a \| \leq 1$. Another application of Theorem 2.2 therefore yields that $H(\mu) = H(\nu)$ and

$$\| a \|_\mu^2 = \| a \|_\nu^2 - 2 \langle a, c \rangle_\nu, \qquad a \in H(\mu), \qquad \| a \| \leq 1.$$

Now choosing $a = tc$ and letting t tend to zero, we have $c = 0$. Moreover, $\| \cdot \|_\mu = \| \cdot \|_\nu$. Theorem 2.1 therefore implies that $\mu = \nu$. This proves Theorem 3.1.

THEOREM 3.2. — *Let* E *either be a Banach space or a dual Banach space equipped with the weak* topology. Moreover, let* $\mu \in \mathcal{G}_0(\mathrm{E}) \setminus \{$ Dirac measure at 0 $\}$ *and* $\nu \in \mathcal{G}(\mathrm{E})$ *be such that*

$$(3.1) \qquad \mu(\mathrm{B}(0\,;1)) > 0$$

and

$$\mu(\mathrm{B}(a\,;1)) = \nu(\mathrm{B}(a\,;1)), \qquad \|a\| > \mathrm{K},$$

where $\mathrm{K} > 0$ *is a fixed constant. Then* $\mu = \nu$.

The condition (3.1) is, of course, automatically fulfilled, if E is a Banach space. Note also that the closed unit ball $\mathrm{B}(0\,;1)$ is weak* measurable when E is a dual Banach space.

Proof. — Theorems 2.3 and 2.1 tell us that $\mu = \nu_0$. Let c denote the barycentre of ν. It only remains to be proved that $c = 0$. Suppose to the contrary that $c \neq 0$. Let first $a \in \mathrm{E} \setminus \{0\}$ be arbitrary and choose $p = p_a \in \mathbb{N}_+$ such that $p\|a\| \geq \|c\| + 1$. Then

$$\|npa + mc\| > \mathrm{K}, \, m = 0, \ldots, n, \qquad n > \mathrm{K}.$$

For every $n \in \mathbb{N}$, with $n > \mathrm{K}$, we therefore get the following chain of equalities

$$(3.2) \quad \mu(\mathrm{B}(npa\,;1)) = \nu_0(\mathrm{B}(npa\,;1)) = \nu(\mathrm{B}(npa + c\,;1)) = \mu(\mathrm{B}(npa + c\,;1)) = \ldots = \mu(\mathrm{B}(n(pa + c)\,;1)).$$

By assuming that $a \in \mathrm{H}(\mu) \setminus \{0\}$ and applying Theorem 2.3, we deduce that $c \in \mathrm{H}(\mu)$. In the next step, we set $a = c$ and $p = p_c$ in (3.2) and get, again using Theorem 2.3

$$\|p_c c\|_\mu = \|(p_c + 1)c\|_\mu.$$

Hence $c = 0$, which is a contradiction. This, finally, shows that $\mu = \nu$ and concludes the proof of Theorem 3.2.

REMARK 3.1. — Theorem 3.1 is true for a dual Banach space E, equipped with the weak* topology, if we assume that $\mu(\mathrm{B}(0\,;r)) > 0$, $r > 0$. However, under these conditions both μ and ν extend to Gaussian Radon measures on the Banach space E [*1*], Th. VI, 2 ; 1. The result is thus already contained in Theorem 3.1.

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(Manuscrit reçu le 4 mai 1977)