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Renewal theorem and Markov chains

by

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RÉSUMÉ. — Une nouvelle démonstration des théorèmes de renouvellement de Orey-Feller-Blackwell est donnée; elle utilise les propriétés des fonctions harmoniques d'un processus markovien *ad hoc*.

We will explore a remark by Feller [1], a remark unfortunately not initially remarked by the author, concerning a Markov process associated with the renewal process. From considerations of the asymptotic properties of this Markov process we obtain the Orey-Feller-Blackwell renewal theorems in a unified « simple » way.

Consider a sequence of iid random variables $T_1, T_2, \dots$ with distribution functions F concentrated on $(0, \infty)$, that is $F(0) = 0$. Following Feller [2] we may have a non-negative variable S_0 with a proper distribution G and we put

$$S_n = S_0 + T_1 + \dots + T_n$$

The renewal process S_n is called pure if $S_0 = 0$ and delayed otherwise (for brevity a delayed process with starting distribution G will be called a G -process as opposed to a pure process). Also for any $t > 0$ there is a unique subscript N_t [3] such that $S_{N_t} < t \leq S_{N_t+1}$ (note this definition for the stopping time N_t+1 differs slightly from Feller's). We define the excess (waiting) time at t as $S_{N_t+1} - t$ and we define

$Y^t(x)$ = the probability the excess time at t for the pure process $\leq x$,

$Y_G^t(x)$ = the probability the excess time at t for the G -process $\leq x$,

$u(t)$ = expected number of renewals for the pure process by t ,

and

$V_G(t)$ = expected number of renewals for the G-process by t .

We now consider $R_+ = [0, \infty]$ as the state space of a Markov process where the transition probabilities are defined as follows. For $A \in \mathcal{B}(R_+)$, the Borel sets on R_+ , $t > 0$ and $x \in R_+$

$$\begin{aligned} P_t(x, A) &= \Pr(\text{the excess time at } t \text{ for a } \delta_x\text{-process} \in A) & \text{if } x < t \\ P_t(x, A) &= 1 & \text{if } x \in t + A \\ P_t(x, A) &= 0 & \text{if } x \notin t + A \\ P_0(x, A) &= 1 & \text{if } x \in A \\ P_0(x, A) &= 0 & \text{if } x \notin A. \end{aligned}$$

(The renewal δ_x -process gives S_0 distribution δ_x where δ_x means the distribution where all the probability is concentrated at x). In more picturesque language we say $P_t(x, A)$ is the probability of starting at x and with variable steps of length T finally jumping right over t and into the set $t + A$. Of course if $x \geq t$ we are already passed t and we only ask if we are already in $t + A$. Also we note that $P_t(x, A) = \Pr(\text{the excess time for the pure process at time } t - A)$ since these probabilities are invariant under translation.

It is quickly seen that for all $x \in R_+$, $t \geq 0$, $A \rightarrow P_t(x, A)$ is a probability on $\mathcal{B}(R_+)$. Also for all $A \in \mathcal{B}(R_+)$, $x \rightarrow P_t(x, A)$ is measurable. Thus we need only establish the Chapman-Kolmogorov relation to show that $(P_t)_{t \geq 0}$ is a semi-group of transition probabilities.

LEMMA 1.

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_1}(x, ds)P_{t_2}(s, A).$$

Proof. — For $x < t_1$.

$P_{t_1+t_2}(x, A) = \Pr(\text{the excess time for the } \delta_x\text{-process at } t_1 + t_2 \in A)$. Conditioning on the excess time at t_1 we have

$$\begin{aligned} P_{t_1+t_2}(x, A) &= \int_0^\infty \Pr(\text{excess time for } \delta_x\text{-process at } t_1 + t_2 \in A \mid \text{excess time} \\ &\quad \text{for } \delta_x\text{-process at } t_1 = s) dY^{t_1-x}(s) \\ &= \int_0^{t_2} \Pr(\text{excess time for } \delta_s\text{-process at } t_2 \in A) dY^{t_1-x}(s) \\ &\quad + \int_{t_2}^\infty \chi_{\{s \in t_2 + A\}} dY^{t_1-x}(s) \text{ by definition of } P_{t_2}(s, A). \end{aligned}$$

Finally we notice that if $I_y = [0, y]$ then $P_t(x, I_y) = Y^{t-x}(y)$. $x < t$. Thus for $x < t$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds).$$

For $t_1 \leq x < t_1 + t_2$.

$$\begin{aligned} P_{t_1+t_2}(x, A) &= \Pr(\text{the excess time for } \delta_{x-t_1}\text{-process at } t_2 \in A) \\ &= P_{t_2}(x - t_1, A) \\ &= \int_0^\infty \delta_{x-t_1}(ds)P_{t_2}(s, A). \end{aligned}$$

Now for $x \geq t_1$, $P_{t_1}(x, x - t_1) = 1$ by definition and we see that for $t_1 \leq x < t_1 + t_2$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds).$$

Lastly for $t_1 + t_2 \leq x$.

$$\begin{aligned} P_{t_1+t_2}(x, A) &= 1 && \text{if } x - t_1 - t_2 \in A \\ &= 0 && \text{if } x - t_1 - t_2 \notin A. \end{aligned}$$

Now $P_{t_1}(x, ds) = \delta_{x-t_1}(ds)$, which gives

$$\int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds) = P_{t_2}(x - t_1, A).$$

Since $x - t_1 \geq t_2$, we have

$$\begin{aligned} P_{t_2}(x - t_1, A) &= 1 && \text{if } (x - t_1) - t_2 \in A \\ &= 0 && \text{if } (x - t_1) - t_2 \notin A. \end{aligned}$$

This is precisely $P_{t_1+t_2}(x, A)$. Thus for $x \geq t_1 + t_2$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds),$$

and we have the result for all x .

Given the initial distribution on the state space (or delay distribution if you like) we may now construct a Markov process $(X_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P_G)$, where G is the delay distribution, such that

$$P_G(X_{t_n} \in A / X_{t_0} = x_0, \dots, X_{t_k} = x_k, t_0 < \dots, t_k < t_n) = P_{t_n - t_k}(x_{t_k}, A).$$

We note in passing that

$$P_G(X_t \in [0, y]) = Y_G^t(y).$$

We are interested in the limits of these distributions as t goes to infinity.

We approach this problem by looking at the properties of the harmonic

functions on the related space-time process. We pick an arbitrary sequence $I = \{t_n\}_{n=0}^\infty$ $0 = t_0 < t_1 < t_2 < \dots < t_n \dots t_n \rightarrow \infty$. The random variables X_{t_n} are by construction all measurable with respect to the measure space $(\Omega, \mathcal{F}, P_G)$ and we define $\mathcal{F}^n$ to be the σ -field generated by $\{X_{t_m}, m \geq n\}$,

and let $\mathcal{F}^\infty = \bigcap_{n=0}^\infty \mathcal{F}^n$. A random variable Y is (following [4]) a tail random variable if there exists a sequence (f_n) of $\mathcal{F}$ -measurable functions on Ω such that $Y = f_n(X_{t_n}, X_{t_{n+1}}, \dots)$. If f can be chosen independently of n so that

$$Y = f(X_{t_n}, X_{t_{n+1}}, \dots)$$

then Y is called invariant. If $Y = \chi_A$ where Y is a tail random variable (respectively an invariant random variable) then A is called a tail event (respectively an invariant event). The class of all tail events (invariant events) form a σ -field called the tail σ -field (the invariant σ -field). We will call the pair $(X_{t_n}, t_n)_{n=0}^\infty$, the space-time chain associated with (X_{t_n}) and we see that we can construct a Markov chain on $\mathcal{R}_+ \times I$ with transition probabilities

$$\tilde{P}((x, t_n), \{A, t_{n+1}\}) = P_{t_{n+1}-t_n}(x, A), \quad x \in \mathcal{R}_+, t_n, t_{n+1} \in I, A \in \mathcal{B}(\mathcal{R}_+).$$

A real valued measurable function g on $\mathcal{R} \times I$ is called space-time harmonic if

$$g(x, t_n) = \int P_{t_{n+1}-t_n}(x, ds) g(s, t_{n+1}) \quad x \in \mathcal{R}_+, t_n, t_{n+1} \in I.$$

We should note that the restriction of our space-time process to the times t is not obligatory and is only done to avoid certain measurability questions and to apply more easily the theorems in [4].

We now examine the space-time harmonic functions which are bounded.

LEMMA 2. — If h is a bounded space-time harmonic function then

$$h(x, t_n) = h(x - (t_{n+1} - t_n), t_{n+1}), \quad x \geq t_{n+1} - t_n.$$

Proof. — A space-time harmonic function satisfies

$$\begin{aligned} h(x, t_n) &= \int_0^\infty h(s, t_{n+1}) \tilde{P}((x, t_n), ds \times t_{n+1}) \\ &= \int_0^\infty h(s, t_{n+1}) P_{t_{n+1}-t_n}(x, ds). \end{aligned}$$

Now for $t_{n+1} - t_n \leq x$, $P_{t_{n+1}-t_n}(x, ds) = \delta_{x-(t_{n+1}-t_n)}(ds)$. Therefore

$$h(x, t_n) = h(x - (t_{n+1} - t_n), t_{n+1}) \quad \text{for} \quad t_{n+1} - t_n \leq x.$$

We may extend this equality right along the diagonal.

We need now only consider $h(x, 0)$ to establish results on all space-time.

LEMMA 3. — (a) If F is periodic with period p $h(x, 0)$ is constant on the periods of F . That is $h(x, 0) = h(x + p, 0) \forall x \geq 0$.

(b) If F is aperiodic $h(x, 0)$ is almost surely constant with respect to Lebesgue measure.

(c) If F is aperiodic and there exists some convolution F^{*n} of F which is not singular with respect to Lebesgue measure then $h(x, 0)$ is constant.

Proof.

$$\begin{aligned} h(0, t_n) &= \int_0^\infty \tilde{P} \{ (0, t_n), ds \times t_{n+1} \} h(s, t_{n+1}) \\ &= \int_0^\infty P_{t_{n+1}-t_n}(0, ds) h(s, t_{n+1}). \end{aligned}$$

From the equality along the diagonals we have

$$h(t_n, 0) = \int_0^\infty P_{t_{n+1}-t_n}(0, ds) h(s + t_{n+1}, 0).$$

Letting $q(x) = h(x, 0)$ we have

$$q(t_n) = \int_0^\infty P_{t_{n+1}-t_n}(0, ds) q(s + t_{n+1})$$

where $q(t_n)$ is measurable and bounded. Since the t_n were chosen arbitrarily we have in general that

$$q(x) = \int_0^\infty P_t(0, ds) q(x + t + s) \quad \forall t, x \geq 0. \quad (1)$$

(a) Now if F has period p we may set $t = p$ to get $q(x) = \int_0^\infty q(x+s) dF(s)$ and by Choquet-Deny [5] we have $q(x)$ is constant on the periods of F .

(b) If F is aperiodic we remark that by regularization of (Eq. 1) we have functions $q^\varepsilon(x)$ which are bounded, uniformly continuous solutions of (Eq. 1) and which tend almost surely to $q(x)$ (w. r. t. Lebesgue measure) as $\varepsilon \rightarrow 0$. Also letting $t \rightarrow 0$ in $q^\varepsilon(x) = \int_0^\infty P_t(0, ds) q^\varepsilon(x + t + s)$ we have $q^\varepsilon(x) = \int_0^\infty q^\varepsilon(x + s) dF(s)$ and again by [5] $q^\varepsilon(x)$ is constant. Thus $q(x)$ is constant almost surely.

(c) We note that if F^{*n} is not singular w. r. t. Lebesgue measure we can pick a $\bar{t}$ sufficiently big so that $F_e^{*n}(\bar{t}) > 0$ (F_e^{*n} is the absolutely conti-

nuous part of F^{**}). Hence $P_i(0, ds)$ is not singular and we may employ the method in Note 1 to prove that $q(x)$ is constant.

We remark that the aperiodic distribution F giving mass $\frac{1}{2}$ to 1 and $\sqrt{2}$ and the function

$$\begin{aligned} h(x, 0) &= 1 & \text{if } x &= p + q\sqrt{2} \text{ } p, q \text{ positive integers} \\ &= 0 & \text{if } x &\neq p + q\sqrt{2} \text{ for any } p, q. \end{aligned}$$

generate a counter example to an extension of Lemma 3 (b).

It is now useful to distinguish between the cases when F is arithmetic (without loss of generality having period 1) and when F is non-arithmetic. Henceforth if F is arithmetic the t_n and x 's will be restricted to the positive integers. Moreover for the arithmetic case we will define the measure m to be the counting measure on the positive integers while in the non-arithmetic case m will be Lebesgue measure. With this in mind we note that the distribution,

$$E(r) = \frac{1}{\mu} \int_0^r (1 - F(y))m(dy)$$

$$\mu = \text{mean of } F$$

called the equilibrium distribution, gives a stationary delayed renewal process [6]. Hence $Y_E^t(r) = E(r) \forall t \geq 0$, and see X^t is stationary w. r. t. P_E .

The utility of the space-time chain is seen in the following.

PROPOSITION 1. — The following conditions are equivalent.

(a) For all probability measures μ and ν on $\mathcal{B}(\mathcal{R}_+)$

$$\lim_{n \rightarrow \infty} \|\mu P_{t_n} - \nu P_{t_n}\| \rightarrow 0. \quad \text{Where } \mu P(A) = \int P(x, A)\mu(dx) \quad \text{for } A \in \mathcal{B},$$

and $\|\nu\|$ is the total variation of ν .

(b) The only bounded space-time harmonic functions are constants.

Proof. — The proof is an adaptation of the proof for Proposition 4.3 in [4].

THEOREM 1. — If (a) F is arithmetic and α is any probability measure on the positive integers or if (b) F^{**} is not singular w. r. t. Lebesgue measure for some n and α is a probability measure on $[0, \infty)$ then $\lim_{t \rightarrow \infty} \|\alpha P_t - e\| = 0$ (in the arithmetic case t is an integer), where e is the equilibrium measure having distribution $E(r)$.

Proof. — In the arithmetic case we restrict attention to the positive integers. Hence by Lemmas 3 (a) and 3 (c) respectively we see the bounded space-time harmonic functions are constant. Hence by Proposition 1, we have for probability measures α and e

$$\lim_{t \rightarrow \infty} \|\alpha P_t - e P_t\| = 0.$$

However

$$\begin{aligned} e P_t(A) &= \int_0^\infty P_t(x, A) dE(x) \\ &= e(A) \end{aligned}$$

and we have the result. Q. E. D.

COROLLARY 1 (Feller). — If F is arithmetic with period 1 and $F(0) = 0$ then

$$\lim_{n \rightarrow \infty} \Pr \{ \text{renewal at } n \} = 1/\mu (\mu \text{ is mean of } F)$$

Proof.

$$\Pr \{ \text{renewal at } n \} = P_n(0, 0).$$

Thus

$$\lim_{n \rightarrow \infty} \{ \text{renewal at } n \} = \frac{1}{\mu} e \{ 0 \} = 1/\mu, \text{ from Theorem 1.}$$

If F is aperiodic but not absolutely continuous w. r. t. Lebesgue measure we must be a little more subtle.

PROPOSITION 2. — If F is aperiodic and if G is a probability measure which is absolutely continuous with respect to Lebesgue measure ($G \ll m$) then the tail field of $\{X_{t_n}\}$ is trivial w. r. t. P_G .

Proof. — Let A be a tail event of $\{X_{t_n}\}$. Then by Proposition 4.1 of [4] A is an invariant event of $\{X_{t_n}, t_n\}$. Consider the function defined on space-time by

$$h_A(Z) = \tilde{E}_Z(A) \quad (Z \text{ of the form } Z = (x, t_n)).$$

By Proposition 4.2 of [4] $h_A(Z)$ is bounded and space-time harmonic. Also

$$h_A(X_n, t_n) = \tilde{E}_{[X_n, t_n]}[A] = \tilde{E}_\eta[\theta^n A \mid \mathcal{F}_n] = \tilde{E}_\eta[A \mid \mathcal{F}_n]$$

(where again η is the initial distribution given by

$$\eta \{ A x t_0 \} = G(A).$$

Thus $h_A(X_n, t_n)$ converges a. s. w. r. t. $\tilde{P}_\eta$ to A . Moreover by Lemma 3 (b)

we know $h_A(Z) = C$ (constant) a. s. w. r. t. Lebesgue measure.

$$\text{Let } B = \{ (x, t) \mid h(x, t) \neq C \}.$$

Since $G \ll m$ we have $\tilde{P}_\eta \{ (X_n, t_n) \in B \} = 0$. Thus a. s.- $\tilde{P}_\eta$ $h_A(X_n, t_n)$ converges to C and hence A is trivial w. r. t. $\tilde{P}_\eta$. Thus A is trivial w. r. t. P_G .

THEOREM 2. — If F is aperiodic and if α is a probability measure which is absolutely continuous w. r. t. Lebesgue measure then

$$\lim_{t_n \rightarrow \infty} \| \alpha P_{t_n} - e \| = 0.$$

Proof. — By Proposition 2, the tail field of $\{ X_{t_n} \}$ is trivial w. r. t. P_e (since the equilibrium distribution $e \ll m$). Hence as in Theorem 4.1 of [4].

$$\lim_{t_n \rightarrow \infty} \sup_{A \in \mathcal{F}^n} | P_e(A \cap B) - P_e(A)P_e(B) | = 0 \quad \text{for every } B \in \mathcal{F}. \quad (2)$$

$$\text{Let } A = \{ X_{t_n} \in F \}; \quad B = \{ X_{t_0} \in G \}.$$

Thus we have

$$\lim_{t_n \rightarrow \infty} \sup_F \left| \int_G P_{t_n}(x, F) e(dx) - e(F)e(G) \right| = 0.$$

Now $\alpha \ll m$. Also $m \ll e$ on $\{ x \mid F(x) < 1 \}$ by the construction of e . Moreover by truncation there exists a probability measure $\tilde{\alpha} \ll m$ concentrated on $[0, T]$ such that $\| \tilde{\alpha} - \alpha \| < \varepsilon$ for any ε . Hence

$$\| \tilde{\alpha} P_{t_n} - \alpha P_{t_n} \| < \varepsilon.$$

Also by the construction of our chain we see $\tilde{\alpha} P_T$ is concentrated on $\{ x \mid F(x) < 1 \}$, and hence for t_n sufficiently large $\tilde{\alpha} P_{t_n} \ll e$. We may therefore consider the case $\alpha \ll e$. Let $\mathcal{F}(x) = \frac{d\alpha}{de}(x)$ and hence $\int \mathcal{F} de = 1$. Let $\mathcal{F}_k(x)$ be step functions such that $\mathcal{F}_k \uparrow \mathcal{F}$. Thus by Eq. 2 we have

$$\lim_{t_n \rightarrow \infty} \sup_F \left| \int P_{t_n}(x, F) \mathcal{F}_k(x) e(dx) - \int \mathcal{F}_k(x) e(dx) \cdot e(F) \right| = 0.$$

However we remark that

$$\begin{aligned} & \left| \int P_{t_n}(x, F) \mathcal{F}_k(x) e(dx) - \int P_{t_n}(x, F) \mathcal{F}(x) e(dx) \right| \\ & \leq \int P_{t_n}(x, F) | \mathcal{F}_k(x) - \mathcal{F}(x) | e(dx) \leq \int | \mathcal{F}_k(x) - \mathcal{F}(x) | e(dx). \end{aligned}$$

This uniform bound implies

$$\lim_{t_n \rightarrow \infty} \sup_F \left| \int P_{t_n}(x, F) \mathcal{F}(x) e(dx) - \int \mathcal{F}(x) e(dx) \cdot e(F) \right| = 0.$$

and $\lim_{t_n \rightarrow \infty} \sup_F \left| \int P_{t_n}(x, F) \alpha(dx) - e(F) \right| = 0$

and $\lim_{t_n \rightarrow \infty} \sup_F |\alpha P_{t_n}(F) - e(F)| = 0$

and hence

$$\lim_{n \rightarrow \infty} \|\alpha P_{t_n} - e\| = 0.$$

COROLLARY 2. — For F aperiodic $Y^t(x)$ converges weakly to $E(x)$.

Proof. — For any interval $[x, y]$ we pick $0 < \varepsilon < x$ and by considering the possible trajectories of our process, as well as the translation invariance of our transition probabilities we see

$$P_{t_n}(s, [x + \varepsilon, y]) \leq P_{t_n}(0, [x, y]) \leq P_{t_n}(s, [x, y + \varepsilon]) + P_{t_n}(s, [0, \varepsilon])$$

for $0 \leq s \leq \varepsilon$. Consider the uniform probability measure u_ε on $[0, \varepsilon]$. Integrating our inequality we have

$$u_\varepsilon P_{t_n}[x + \varepsilon, y] \leq P_{t_n}(0, [x, y]) \leq u_\varepsilon P_{t_n}[x, y + \varepsilon] + u_\varepsilon P_{t_n}[0, \varepsilon].$$

Hence by Theorem 2. We have for all $\varepsilon < x$

$$e[x + \varepsilon, y] \leq \liminf P_{t_n}(0, [x, y]) \leq \overline{\lim} P_{t_n}(0, [x, y]) \leq e[x, y + \varepsilon] + e[0, \varepsilon].$$

Thus by the continuity of $E(x)$ we have $\lim_{t_n \rightarrow \infty} P_{t_n}(0, [x, y]) = e[x, y]$ which implies

$$\lim_{t_n \rightarrow \infty} Y^{t_n}(x) = E(x).$$

COROLLARY 3 (Blackwell). — If F is aperiodic then $\lim_{t \rightarrow \infty} u(t+h) - u(t) = h/\mu$.

Proof. — $u(t+h) - u(t)$ is the expected number of renewals in $[t, t+h]$. If we condition on the excess time at t we have

$$u(t+h) - u(t) = \int_0^{h-} \{1 + u(h-s)\} dY^t(s)$$

(if we step over t and land at $t+s < t+h$ then we restart the process at $t+s$ and we take on the average $u(h-s)$ more steps before $t+h$). Next we remark that $1 + u(h-s)$ is a decreasing function of s and hence

has a countable number of discontinuities on $[0, h]$. Moreover by Corollary 2, Y^t converges weakly to E and $E \ll m$. Hence the set of discontinuities of $1 + u(h - s)$ has measure 0 w. r. t. E and we have

$$\lim_{t \rightarrow \infty} u(t + h) - u(t) = \int_0^h (1 + u(h - s)) dE(s)$$

(see Theorem 5.2 (iii) in [7] for example).

However since E is the equilibrium measure

$$\int_0^h (1 + u(h - s)) dE(s) = V_E(h) = h/\mu \quad (\text{see } [6]).$$

Remarks. — If F has infinite mean we have no equilibrium probability measure. We still have however an invariant measure e with distribution

$$E(x) = \int_0^x (1 - F(s)) m(ds).$$

We have for every $\gamma > 0$ and every $x \in \mathbb{R}_+$

$$\frac{P_{t_n}(x, F)}{e(F) + \gamma} \rightarrow 0 \quad \text{uniformly in } F \in \mathcal{B}(\mathbb{R}_+) \quad \text{as } t_n \rightarrow \infty.$$

(See Theorem 7.3 in [4]). We have all the obvious extensions of Corollaries 1, 2 and 3.

If our distribution F is not concentrated on the half line but has $\mu > 0$ we can still consider the excess time at $t \in \mathbb{R}_+$ (since the walk drifts to the right) and we can prove our theorems based on the strict ladder distribution [8].

NOTE 1. — If h is measurable and bounded and satisfies

$$h(x) = \int_0^\infty h(x + y) dF(y) \quad (3)$$

and if F^{*n} is not singular (w. r. t. Lebesgue measure), then $h(x)$ is constant.

Proof. — By Choquet and Deny's lemma $h(x)$ is almost surely constant (say C). Subtracting C from both sides of (Eq. 3) we have

$$g(x) = \int_0^\infty g(x + y) dF(y) \quad (4)$$

where $g(x) = h(x) - C$ is almost surely 0.

By convolution of (Eq. 4) we have

$$g(x) = \int_0^\infty g(x + y) dF^{*n}(y).$$

Now $F^{**} = G = pG_e + qG_d$ where G_e generates a measure absolutely continuous with respect to Lebesgue measure and G_d generates a measure singular with respect to Lebesgue measure [7] and $p > 0$, $q > 0$, $p + q = 1$. Now

$$\begin{aligned} g(x) &= p \int_0^\infty g(x+y) dG_e(y) + q \int_0^\infty g(x+y) dG_d(y) \\ &= q \int_0^\infty g(x+y) dG_d(y). \end{aligned}$$

since $g(x) = 0$ a. e.

Again using

$$g(x+y) = \int_0^\infty g(x+y+y_1) dG(y_1)$$

we have

$$\begin{aligned} g(x) &= q^2 \int_0^\infty g(x+y) dG_d^{*2}(y) \\ &= q^n \int_0^\infty g(x+y) dG_d^{*n}(y) \downarrow 0. \end{aligned}$$

Thus $g(x) = 0$ everywhere and hence $h(x) = C$ everywhere.

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