ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 11, nº 2 (1975), p. 109-125

<http://www.numdam.org/item?id=AIHPB_1975__11_2_109_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XI, n° 2, 1975, p. 109-125. *Calcul des Probabilités et Statistique.*

Ergodic properties of marked point processes in $\mathbf{R}^r$

by

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SUMMARY. — We consider a point process in r-dimensional Euclidean space (R^r); with each point we associate an ancillary random variable Y_i. Given a region T of R^r, let $\mathrm{N_T}$ be the number of points in T, $\mathrm{S_T}$ the sum of the Y_i corresponding to these points. Under various hypothesis on the point process and the Y_i, the a. s. convergence of $\mathrm{S_T/N_T}$ (or $\mathrm{S_T/E(N_T)}$) as $\mathrm{T} \to \mathrm{R}^r$ is investigated. Particular emphasis is given to the cases where the point process is stationary or completely random and the Y_i are equidistributed.

RÉSUMÉ. — On considère un processus ponctuel dans l'espace euclidien R^r de dimension r; à chaque point, on associe une v. a. auxiliaire Y_i. Étant donné une région T de R^r, soit $\mathrm{N_T}$ le nombre de points dans T, $\mathrm{S_T}$ la somme des Y_i correspondants à ces points. Sous des hypothèses diverses sur le processus ponctuel et sur les Y_i, on étudie la convergence p. s. de $\mathrm{S_T/N_T}$ (ou $\mathrm{S_T/E(N_T)}$) lorsque $\mathrm{T} \to \mathrm{R}^r$. En particulier, on étudie les cas où le processus est, soit stationnaire, soit « complètement aléatoire » et les Y_i sont équidistribués.

0. INTRODUCTION AND NOTATION

Consider a point process in Euclidean r-space, *i. e.* for every ω belonging to a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ we have a *realization* which is a collection of points in R^r.

For any Borel set A in R^r denote by $N(A, \omega)$ the number of points in A for the realization ω. We assume that:

HYPOTHESIS 1. — For all bounded A, $E(N(A, \omega)) < \infty$.

The reader is referred to, *e. g.* [3] for details of the setup.

Given ω, we can enumerate the points of the realization as $\{ X_i(\omega) \}_{i=1}^{\infty}$. Let $\{ Y_k(\omega) \}_{k=1}^{\infty}$ be a sequene of random variables defined on $(\Omega, \mathscr{F}, P)$; suppose that with each $X_i(\omega)$ we associate $Y_i(\omega)$. We then have what is often called a *marked point process*, the Y_i being the marks (cf. [3], p. 315).

Denote by T a rectilinear region of R^r. The regions of interest to us will generally be either of the form $\bigcap_{i=1}^{r} \{ \underset{\sim}{s} : 0 \leqslant s_i \leqslant t_i \}$ where $\underset{\sim}{s} = (s_1, s_2, \ldots, s_r)$ is a point in R^r, or else $\bigcap_{i=1}^{r} \{ \underset{\sim}{s} : - t_i \leqslant s_i \leqslant t_i \}$, where $t_i > 0$ for $i = 1, 2, \ldots, r$. Given such a T, let

$$S_T(\omega) = \sum_{\{ i : X_i(\omega) \in T \}} Y_i(\omega) \tag{0.1}$$

We may regard S_T as the sum of a random number of random variables. If, for example, when $r = 2$ we think of the $X_i(\omega)$ as representing sites where trees grow in a forest, and $Y_i(\omega)$ as the number of board feet in the tree at site $X_i(\omega)$, then S_T represents the potential yield of the region T. One can then inquire about the asymptotic behavior of S_T as T increases without bound; for example, under what conditions will S_T/N_T or $S_T/E(N_T)$ converge with probability one, as the coordinates of the corners of T tend independently to ∞?

We shall consider three separate (but overlapping) cases corresponding to different hypothesis on the point processes and on the ancillary random variables Y_k; these are treated in paragraphs 1-3.

CASE 1. — The point process is stationary, *i. e.*,

$$P \{ N(A_i, \omega) = k_i, i = 1, 2, \ldots m \} = P \{ N(A_i + \underset{\sim}{y}, \omega) = k_i, i = 1, 2, \ldots m \}$$

for any $\underset{\sim}{y} \in R^r$, where $A + \underset{\sim}{y}$ is the translate of A by $\underset{\sim}{y}$.

CASE 2. — The point process is *completely random* (but not necessarily stationary) in the sense that if $A_1, \ldots, A_m$ are disjoint bounded Borel sets, then $\{ N(A_1, \ ), \ldots, N(A_m, \ ) \}$ are independent.

CASE 3. — No assumptions are made concerning the stationarity or independence properties of the point process.

We emphasize that Hypothesis 1 will always be in force.

The author would like to thank Professor J. H. B. Kemperman for a stimulating conversation which inspired this work, and Professor Ron Pyke for some helpful remarks.

1. THE STATIONARY CASE

Suppose the point process is stationary over R^r. For each $k = 1, 2, \ldots$, let there be given a sequence $\{ Y_i^k \}_{i=1}^{\infty}$ of exchangeable random variables, independent for different k, and independent of the point process. Given a realization ω, we enumerate the points $X_i(\omega)$ in some order. Thinking of our forest analogy in § 0, we would like to allow the size of the marks to depend on the density of the points in the realization. This consideration motivates our model described below.

DEFINITION. — Let $d > 0$ be fixed. Call a point $X_i(\omega)$ in the realization a « type k point » if there are k-1 other points of the realization contained within a sphere of radius d centered at $X_i(\omega)$.

We now « mark » the point process in the following way: if $X_i(\omega)$ is of type k, we mark it with $Y_i^k(\omega)$. This gives a marked point process which we can represent this way (see [3]): with each ω we associate a random set function $S(., \omega)$ defined on the bounded Borel sets of R^r, where $S(A, \omega)$ is the sum of the marks corresponding to the points of the realization ω which lie in A. If $\mathcal{S}$ is the set of such random set functions, let Σ, the σ-algebra of measurable sets, be generated by all subsets of $\mathcal{S}$ of the form $\{ S : S(A, \omega) \in B \}$ where A is bounded Borel in R^r and B is a real Borel set; we may then regard the marked point process as being defined on $(\mathcal{S}, \Sigma, P)$.

If $t \in R^1$, let $\underset{\sim}{t}_i \in R^r$ be defined by $\underset{\sim}{t}_i = (0, 0, ..0, t, 0, ..0)$ (where the t is in the ith place). For $S(., \omega) \in \mathcal{S}$, define point transformations T_t^i $(i = 1, 2, \ldots, r)$ as follows: $T_t^i(S(A, \omega)) = S(A + \underset{\sim}{t}_i, \omega)$. For each i, the set $\{ T_t^i \}_{t \in R^1}$ is a group of measurable transformations; it is evident that under the hypothesis of stationarity on the point process and the assumptions made on the sequences $\{ Y_i^k \}$ that the transformations T_t^i are *measure-preserving*.

Let C denote the unit cube centered at the origin, with edges parallel to the coordinate axes, and left $f_0(\omega) = S(C, \omega)$. The following theorem is essentially due to Zygmund [*13*]; see Calderon [*1*] for a more general formulation.

THEOREM 1.1. — With notation as above, let f be a random variable such that $E(|f|(\log^+|f|)^{r-1}) < \infty$. Then the limit

$$(1.1) \quad \lim_{t_1 \ldots t_r \to \infty} 2^{-r}(t_1 t_2 \ldots t_r)^{-1} \int_{-t_1}^{t_1} \ldots \int_{-t_r}^{t_r} f(T^1_{s_1} \ldots T^r_{s_r}\omega) ds_1 \ldots ds_r$$

exists a. s. (and is finite). The limit is an invariant random variable with respect to each group T^i_t.

With the aid of Theorem 1.1 we can prove the main result of this section. Recall that $N(C, \omega)$ is the number of the points of the realization which lie in C.

THEOREM 1.2. — With notation as above, let $T = \bigcap_{i=1}^{r} \{ \underset{\sim}{s} : -t_i \leqslant s_i \leqslant t_i \}$. If $E(N(C)(\log^+ N(C))^{r-1}) < \infty$ and $\sup_j E(|Y^j_1|(\log^+|Y^j_1|)^{r-1}) < \infty$, then S_T/N_T converges a. s. as each $t_i \to \infty$ (independently).

PROOF. — We apply Theorem 1.1 to $f_0(\omega)$ $(= S(C, \omega))$. Assume that the moment conditions of Theorem 1.2 are satisfied; then

$$(1.2) \quad E(|f_0|(\log^+|f_0|)^{r-1}) = \sum_{n=1}^{\infty} E(|f_0|(\log^+|f_0|)^{r-1}|N(C) = n)P(N(C) = n).$$

We will get an upper bound on $E(|f_0|(\log^+|f_0|)^{r-1} \mid N(C) = n)$. For each $k = 1, 2, \ldots$, let A_k be the set of all k-tuples $(n_1, \ldots n_k)$ such that $n_1 + \ldots + n_k = n$ (n is now fixed for the moment). Let B_k be the set of all k-tuples of positive integers $(i_1, i_2, \ldots i_k)$ with $i_1 < i_2 < \ldots < i_k$. Define $g_0 = |f_0|(\log^+|f_0|)^{r-1}$, and

$$(1.3) \quad H(i_1, \ldots i_k, n_1, \ldots n_k) = \bigcap_{j=1}^{k} \{ n_j \text{ points of type } i_j \text{ in C} \}$$

Then

$$(1.4) \quad E(g_0 \mid N(C) = n) = \sum_{k=1}^{n} \sum_{A_k, B_k} E(g_0 \mid H(i_1, \ldots i_k, n_1, \ldots n_k))P(H(i_1, \ldots i_k, n_1, \ldots n_k)).$$

Now by the independence of the $\{ Y^j_i \}$ and the point process,

$$(1.5) \quad E(g_0 \mid H(i_1, \ldots i_k, n_1, \ldots n_k)) = E(|T_n|(\log^+|T_n|)^{r-1})$$

where T_n is the sum of n_1 random variables distributed as $Y_1^{i_1}$, n_2 random variables distributed as $Y_2^{i_2}$, etc. Now

$$(1.6)\quad E(|T_n|(\log^+|T_n|)^{r-1}) \leqslant n(\log n)^{r-1} + nE\{(|T_n|/n)[\log^+|T_n/n| + \log n]^{r-1}\}.$$

The second term in the right-hand side of (1.6) is bounded by

$$2^{r-1}\{nE(|T_n/n|(\log^+|T_n/n|)^{r-1} + (\log n)^{r-1}E(|T_n|)\}.$$

Using the convexity of the function $x \to |x|(\log^+|x|)^{r-1}$, we get that

$$(1.7)\quad E(|T_n|(\log^+|T_n|)^{r-1}) \leqslant n(\log n)^{r-1} + 2^{r-1}n\sup_j E(|Y_1^j|(\log^+|Y_1^j|)^{r-1}) + 2^{r-1}n(\log n)^{r-1}\sup_j E(|Y_1^j|)$$

From (1.4), (1.5), and (1.7) we have that

$$E(g_0 \mid N(C) = n) \leqslant n(\log n)^{r-1}[1 + 2^{r-1}\sup_j E(|Y_1^j|)] + 2^{r-1}n\sup_j E(|Y_1^j|)(\log^+|Y_1^j|)^{r-1})$$

and from (1.2) it follows that $E(|f_0|(\log^+|f_0|)^{r-1} < \infty$.

Applying Theorem 1.1 to f_0, we have, if λ is the intensity of the point process (*i. e.*, $E(N(C)) = \lambda$),

$$(1.8)\quad \lim_{t_1..,t_r\to\infty} \lambda/E(N_T)\int_{-t_1}^{t_1}\dots\int_{-t_r}^{t_r} f_0(T_{s_1}^1 \dots T_{s_r}^r\omega)ds_1 \dots ds_r \text{ exists a. s.}$$

The integral in (1.8)—call it I_T—is not quite equal to S_T, but we will show that $(I_T - S_T)/E(N_T) \to 0$ a. s. Suppose we set up the marked point process as before, except that instead of using the marks Y_i^j, we use $|Y_i^j|$, with all else remaining the same. We use tilda to denote the accompanying quantities ($\tilde{I}_T$, $\tilde{S}_T$, etc.). Let T be defined as in the statement of the theorem, where each t_i is assumed to be large. If $T_1 = \bigcap_{i=1}^{r}\{\underset{\sim}{s} : -(t_i - 1) \leqslant s_i \leqslant t_i - 1\}$ and $T_2 = \bigcap_{i=1}^{r}\{\underset{\sim}{s} : -(t_i + 1) \leqslant s_i \leqslant t_i + 1\}$, it is easy to see that $|I_T - S_T| \leqslant |\tilde{I}_{T_2} - \tilde{I}_{T_1}|$. But $\tilde{I}_{T_2}/E(N_{T_2})$ and $\tilde{I}_{T_1}/E(N_{T_1})$ converge a. s. to the same limit as each $t_i \to \infty$ by Theorem 1.1, and $E(N_{T_2})/E(N_{T_1}) \to 1$. It follows that $|\tilde{I}_{T_2} - \tilde{I}_{T_1}|/E(N_{T_1}) \to 0$ a. s., hence that

$$|I_T - S_T|/E(N_T) \to 0$$

a. s. as each $t_i \to \infty$. To complete Theorem 1.2 we note that, since the point process itself is stationary, another application of Theorem 1.1 as above gives that $N_T/E(N_T)$ is a. s. convergent as each $t_i \to \infty$. Thus we finally have that S_T/N_T is a. s. convergent (to an invariant random variable) as each $t_i \to \infty$. ■

REMARK. — It is not essential that the sets T defined above be products of intervals symmetric about the origin; the sides of the « rectangle » defining T may grow at different rates. One way to see this is by using the one-sided version of Theorem 1.1 (in 1.1 take the integrals from $-t_i$ to 0 or from 0 to t_j) in each orthant separately; the invariants sets will be the same in each case so the limit of $S_{T_i}/E(N_{T_i})$ for $i = 1, 2, \ldots 2^r$ will be a. s. the same, where S_{T_i} is the sum over that part of T lying in the ith orthant. Since $N_T = N_{T_1} + N_{T_2} + \ldots + N_{T_{2^r}}$, the limit of $S_T/E(N_T)$ will still exist a. s.

EXAMPLE 1.1. — If the point process is a stationary Poisson process with parameter λ (*i. e.*, N(C) has a Poisson distribution with parameter λ), the first condition of Theorem 1.2 clearly holds. Hence the conclusion of Theorem 1.2 will hold whenever the condition on the $\{ Y_i^j \}$ is satisfied.

For some purposes a simplified version of our marked point process where we have only one i. i. d. sequence of marks $\{ Y_i \}$ may be more appropriate. If the point process is stationary Poisson then the invariant field will be trivial (since the Poisson process is completely random); then we clearly have $S_T/N_T \to E(Y_i)$ a. s. It follows from [*10*] that for this convergence the condition $E(| Y_1 | (\log^+ | Y_1 |)^{r-1} < \infty$ cannot be weakened.

2. THE COMPLETELY RANDOM CASE

Now suppose we have a completely random point process defined on the positive orthant (later we will indicate how to extend the results to the case where the point process is defined on all of R^r). For each lattice point with positive integer coordinates, denoted $\underset{\sim}{n}$, let $C_{\underset{\sim}{n}}$ denote the semi-open unit cube with « top » vertex at $\underset{\sim}{n}$ and edges parallel to the coordinate axes. For each $\underset{\sim}{n}$ let there be given a sequence $\left\{ Y_i^{\underset{\sim}{n}} \right\}_{i=1}^{\infty}$ of i. i. d. random variables; let the sequences corresponding to different $\underset{\sim}{n}$ be mutually independent, and let each sequence $\left\{ Y_i^{\underset{\sim}{n}} \right\}$ be independent of $N(C_{\underset{\sim}{n}})$. Let $\mathbf{X}_i^{\underset{\sim}{n}}(\omega)$ be an enumeration of the points of the realization ω which fall in C_n; with $X_i^{\underset{\sim}{n}}(\omega)$ we associate the « mark » $Y_i^{\underset{\sim}{n}}(\omega)$. An important special case is that in which we have a single i. i. d. sequence Y_i with $Y_i^{\underset{\sim}{n}} \overset{L}{=} Y_i$ for each $\underset{\sim}{n}$ (Instead of taking the integer grid on R^r_+ we could take any rectangular grid; for simplicity we limit discussion to the integer case).

Let $\mathrm{Z}_{\underset{\sim}{k}} = \sum_{\mathrm{X}_i \in \mathrm{C}_{\underset{\sim}{k}}} \mathrm{Y}_i^{\underset{\sim}{k}}$ and let $\mathrm{S}_{\underset{\sim}{k}} = \sum_{\underset{\sim}{i} \leq \underset{\sim}{k}} \mathrm{Z}_{\underset{\sim}{i}}$. The random variables $\mathrm{Z}_{\underset{\sim}{k}}$ are now mutually independent. Denote by $b_{\underset{\sim}{n}}$ the quantity $\mathrm{E}(\mathrm{N}(0, \underset{\sim}{n}])$, where $(\underset{\sim}{a}, \underset{\sim}{b}] = \{ \underset{\sim}{s} : \underset{\sim}{a} < \underset{\sim}{s} \leqslant \underset{\sim}{b} \}$. Note that $\Delta b_{\underset{\sim}{n}} \geqslant 0$, where $\Delta b_{\underset{\sim}{n}}$ denotes the differencing of b over the 2^r vertices neighboring $\underset{\sim}{n}$ which are $\leqslant \underset{\sim}{n}$.

2.1. A law of large numbers

We will need a slight generalization of a result announced without proof in [*10*] (p. 169):

THEOREM 2.1.1. — Let $\{ b_{\underset{\sim}{k}} \}_{\underset{\sim}{k} \in \mathrm{N}^r}$ be such that $b_{\underset{\sim}{k}} \to \infty$ as $\underset{\sim}{k} \to \infty$ and $\Delta b_{\underset{\sim}{k}} \geq 0$ for all $\underset{\sim}{k} \in \mathrm{N}^r$. Let $\{ \mathrm{Z}_{\underset{\sim}{k}} \}_{\underset{\sim}{k} \in \mathrm{N}^r}$ be independent random variables with zero mean. Let ϕ be a positive, even, continuous function on R^1 such that as $| x |$ increases, $\dfrac{\phi(x)}{x}$ increases and $\phi(x)/x^2$ decreases.

If $\displaystyle\sum_{\underset{\sim}{k}} \frac{\mathrm{E}[\phi(\mathrm{Z}_{\underset{\sim}{k}})]}{\phi(b_{\underset{\sim}{k}})} < \infty$, then $\displaystyle\sum_{\underset{\sim}{k}} \frac{\mathrm{Z}_{\underset{\sim}{k}}}{b_{\underset{\sim}{k}}}$ converges a. s., and $\dfrac{\mathrm{S}_{\underset{\sim}{k}}}{b_{\underset{\sim}{k}}} \xrightarrow{\underset{\sim}{k} \uparrow \infty} 0$ a. s.

PROOF. — The key to the proof is an inequality analogous to the Kolmogorov inequality for $r = 1$. This was first established by Wichura in [*12*], and the constant improved later in [*11*].

LEMMA 2.1.1. — Let $\{ \mathrm{X}_{\underset{\sim}{j}} \}_{\underset{\sim}{j} \leqslant \underset{\sim}{n}}$ be independent with zero means and finite variances, $\mathrm{S}_{\underset{\sim}{n}} = \sum_{\underset{\sim}{j} \leq \underset{\sim}{n}} \mathrm{X}_{\underset{\sim}{j}}$. Then for any $\lambda > 0$,

$$\mathrm{P}\{ \max_{\underset{\sim}{j} \leq \underset{\sim}{n}} | \mathrm{S}_{\underset{\sim}{j}} | > \lambda \} \leq \frac{4^{r-1}}{\lambda^2} \sum_{\underset{\sim}{j} \leq \underset{\sim}{n}} \mathrm{E}(\mathrm{X}_{\underset{\sim}{j}})^2. \tag{2.1.1}$$

Once the lemma is established, the proof of Theorem 2.1 is very similar to the one-dimensional theorem (see, *e. g.*, [*2*], p. 124). We truncate the $\mathrm{Z}_{\underset{\sim}{n}}$ at $b_{\underset{\sim}{n}}$ to form the truncated array $\mathrm{Y}_{\underset{\sim}{n}}$; an application of Lemma 2.1 then gives

$$\sum_{\underset{\sim}{k}} \frac{\mathrm{Y}_{\underset{\sim}{k}} - \mathrm{E}(\mathrm{Y}_{\underset{\sim}{k}})}{b_{\underset{\sim}{k}}} < \infty \text{ a. s.} \tag{2.1.2}$$

The convergence of $\sum_{\underset{\sim}{k}} \frac{E(\phi(X_{\underset{\sim}{k}}))}{\phi(b_{\underset{\sim}{k}})}$ then allows us to conclude that $\sum_{\underset{\sim}{k}} \frac{E\,|\,Y_{\underset{\sim}{k}}\,|}{b_{\underset{\sim}{k}}} < \infty$, hence that

$$\sum_{\underset{\sim}{k}} \frac{Y_{\underset{\sim}{k}}}{b_{\underset{\sim}{k}}} < \infty \text{ a. s.} \tag{2.1.3}$$

But $\{ Y_k \}$ and $\{ Z_k \}$ are equivalent arrays so that $\sum_{\underset{\sim}{k}} \frac{Z_{\underset{\sim}{k}}}{b_{\underset{\sim}{k}}}$ converges a. s. A simple generalization of Kronecker's lemma (see, *e. g.* [*2*] p. 123) to N^r then yields the result $\frac{S_{\underset{\sim}{k}}}{b_{\underset{\sim}{k}}} \xrightarrow{\underset{\sim}{k} \to \infty} 0$ a. s. (it is here that the hypothesis $\Delta b_{\underset{\sim}{k}} \geq 0$ is essential). ∎

2.2. An ergodic theorem for completely random point processes

Returning now to our point process, we center the Y_i^n by defining $\overline{Y}_i^{\underset{\sim}{n}} = Y_i^{\underset{\sim}{n}} - E\left(Y_i^{\underset{\sim}{n}}\right)$, and let

$$\overline{Z}_{\underset{\sim}{n}} = \sum_{\{ i \,:\, X_i(\omega) \in C_{\underset{\sim}{n}} \}} \overline{Y}_i^{\underset{\sim}{n}}(\omega) = Z_{\underset{\sim}{n}} - E\left(Y_i^{\underset{\sim}{n}}\right) N(C_{\underset{\sim}{n}}).$$

Then $E(\overline{Z}_{\underset{\sim}{n}}) = 0$ and we can apply Theorem 2.1 to the $\overline{Z}_{\underset{\sim}{n}}$.

THEOREM 2.2.1. — If for some α, $1 \leq \alpha \leq 2$, $\sum_{\underset{\sim}{n}} \frac{E\left(\left| \overline{Y}^{\underset{\sim}{n}} \right|^{\alpha}\right) E\{ N(C_{\underset{\sim}{n}}) \}}{\{ E[N(0, \underset{\sim}{n}] \}^{\alpha}} < \infty$, then

$$\frac{S_{\underset{\sim}{n}} - \sum_{\underset{\sim}{k} \leqslant \underset{\sim}{n}} E\left(Y^{\underset{\sim}{k}}\right) N(C_{\underset{\sim}{k}})}{E\{ N(0, \underset{\sim}{n}] \}} \xrightarrow{\underset{\sim}{n} \to \infty} 0 \text{ a. s.}$$

Proof. — $E(|\, \overline{Z}_{\underset{\sim}{n}} \,|^{\alpha}) = E\left(\sum_{j=1}^{\infty} 1_{[N(C_{\underset{\sim}{n}}) = j]} \,|\, S_j \,|^{\alpha} \right)$ where S_j is the sum of j i. i. d. random variables with the distribution $\overline{Y}^{\underset{\sim}{n}}$. If $1 \leqslant \alpha \leqslant 2$,

$$E \,|\, S_j \,|^{\alpha} \leq 2 \sum_{i=1}^{j} E\left(\left| \overline{Y}_i^{\underset{\sim}{n}} \right|\right)^{\alpha}$$

(see, *e. g.* [4]); since the $Y_i^{\underset{\sim}{n}}$ are independent of $N(C_{\underset{\sim}{n}})$ and $E\left(\left|Y^{\underset{\sim}{n}}\right|\right)^{\alpha} < \infty$, we have

$$E(|\overline{Z}_{\underset{\sim}{n}}|^{\alpha}) \leqslant 2E(|\overline{Y}^{\underset{\sim}{n}}|^{\alpha})E\left(\sum_{j=1}^{\infty} j1_{[N(C_{\underset{\sim}{n}})=j]}\right) = E(|\overline{Y}^{\underset{\sim}{n}}|^{\alpha})E[N(C_{\underset{\sim}{n}})]$$

Now

$$E(|\overline{Y}^{\underset{\sim}{n}}|^{\alpha}) \leqslant 2^{\alpha-1}\{E(|Y^{\underset{\sim}{n}}|^{\alpha}) + |E(Y^{\underset{\sim}{n}})|^{\alpha}\} \leq 2^{\alpha}E(|Y^{\underset{\sim}{n}}|^{\alpha})$$

(the first inequality is the « C_r inequality » ([7], p. 157) and the second follows by Jensen's inequality); so upon applying Theorem 2.1.1 with $b_{\underset{\sim}{n}} = E(N(0, \underset{\sim}{n}])$, the convergence of the series in Theorem 2.2.1 implies that

$$\frac{\sum_{\underset{\sim}{k} \leq \underset{\sim}{n}} \overline{Z}_{\underset{\sim}{k}}}{E(N(0, \underset{\sim}{n}])} \xrightarrow{\underset{\sim}{n} \uparrow \infty} 0 \text{ a. s., proving the theorem. } \blacksquare$$

The case of most interest, and *to which we restrict ourselves in the remainder of the paper*, is that in which the $Y_i^{\underset{\sim}{n}}$ have the same distribution Y for each $\underset{\sim}{n}$, where $E(|Y|^{\alpha}) < \infty$ (it is obvious from Theorem 2.2.1 that the weaker conditions $\sup_{\underset{\sim}{n}} E(|Y^{\underset{\sim}{n}}|^{\alpha}) < \infty$, $E(Y^{\underset{\sim}{n}}) \xrightarrow{|\underset{\sim}{n}| \uparrow \infty} C$ would permit the same conclusions (where $|\underset{\sim}{n}| = \prod_{i=1}^{r} n_i$).

Denote by $\mu(A)$ the measure $E[N(A)]$. Then the series in the theorem is just (to a constant multiple).

$$\sum_{\underset{\sim}{n}} \frac{\mu(C_{\underset{\sim}{n}})}{\{\mu(0, \underset{\sim}{n}]\}^{\alpha}}. \tag{2.2.1}$$

COROLLARY 2.2.1. — If the $\{Y^{\underset{\sim}{n}}\}$ are identically distributed with $E(|Y^{\underset{\sim}{n}}|^{\alpha}) < \infty$ for some $1 \leqslant \alpha \leqslant 2$, and

(*i*) (2.2.1) converges.

(*ii*) $\dfrac{N(0, \underset{\sim}{n}]}{E\{N(0, \underset{\sim}{n}]\}} \xrightarrow{\underset{\sim}{n} \uparrow \infty} 1$ a. s.

Then

$$\frac{S_{\underset{\sim}{n}}}{N(0, \underset{\sim}{n}]} \xrightarrow{\underset{\sim}{n} \to \infty} E(Y) \text{ a. s.}$$

Proof. — Immediate from Theorem 2.2.1 and the remarks above.

Let ν be the atomic measure on N^r which sweeps all of the μ-mass in $C_{\underset{\sim}{n}}$ to the point $\underset{\sim}{n}$, and let $U(\underset{\sim}{x}) = \nu(0, \underset{\sim}{x}]$. Then the series (2.2.1) is simply.

$$(2.2.2) \qquad \int_{R^r_+} \frac{\nu(d\underset{\sim}{x})}{U^\alpha(\underset{\sim}{x})}$$

where R^r_+ is the positive orthant of R^r. At first glance it may seem that this integral should always converge when $\alpha > 1$. This is true when $r = 1$, but not in general.

LEMMA 2.2.1. — When $\alpha > 1$, (2.2.2) converges under either of the following conditions:

(*i*) μ is a product measure,

(*ii*) $\mu\{\underset{\sim}{x} : U(\underset{\sim}{x}) < C\} = o(C^{1+\delta})$ for any $\delta > 0$.

Proof. — It follows by Fubini's theorem (see, *e. g.* [5], p. 421) that

$$(2.2.3) \qquad \int_{R^r_+} \frac{\nu(d\underset{\sim}{x})}{U^\alpha(\underset{\sim}{x})} = \int_0^\infty \alpha t^{\alpha-1} \nu \left\{ \underset{\sim}{x} : U(\underset{\sim}{x}) < \frac{1}{t} \right\} dt.$$

When $\alpha > 1$, condition (*ii*) clearly implies the convergence of the right-hand side of (2.2.3). If μ is a product measure, we apply (2.2.3) separately to each factor, noting that $\nu\left\{ x : U(x) < \frac{1}{t} \right\} \leqslant \frac{1}{t}$ in one dimension.

Note that, in the only case of interest, *i. e.*, $\mu(R^r_+) = \infty$, (2.2.1) will not converge when $\alpha = 1$.

Remark. — It is easy to check that, for (2.2.2) to converge, it is enough that μ be « asymptotically » a product measure in the sense that there exist $\nu^1, \ldots \nu^r$ such that (if $\underset{\sim}{n} = (n_1, \ldots n_r)$:

$$(2.2.4) \quad C_1 \leqslant \frac{\nu(\{\underset{\sim}{n}\})}{\nu^1(\{n_1\}) \ldots \nu^r(\{n_r\})} \leq C_2 \qquad \text{when} \qquad |\underset{\sim}{n}| > N_0.$$

It would be interesting to characterize those measures for which (2.2.1) is convergent.

2.3. The Poisson process

The most common completely random point processes on R^r are the (not necessarily stationary) Poisson processes; if $A_1, \ldots, A_k$ are bounded disjoint Borel sets, then $N(A_1), \ldots, N(A_k)$ are independent, with Poisson distributions with parameters $\mu(A_1), \ldots, \mu(A_k)$ respectively, where μ is a non-atomic Radon measure on the positive orthant. Since we are only interested in the case when the point process has a. s. an infinite number of points we will assume

$$\sum_{\underset{\sim}{n}} (1 - e^{-\mu(\underset{\sim}{C_n})}) = \infty. \tag{2.3.1}$$

which implies by Borel-Cantelli that $P(N(\underset{\sim}{C_n}) \geq 1 \text{ i. o.}) = 1$.

PROPOSITION 2.3.1. — Suppose the point process is Poisson and that μ satisfies (2.3.1) and one of the conditions of Lemma 2.2.1. Then if $E(|Y|^\alpha) < \infty$ for some $\alpha > 1$,

$$\frac{\underset{\sim}{S_n}}{N(0, \underset{\sim}{n}]} \xrightarrow{\underset{\sim}{n} \uparrow \infty} E(Y) \text{ a. s.}$$

Proof. — Under the stated conditions (2.2.2) converges by Lemma 2.2.1. Now $N(0, \underset{\sim}{n}] = \sum_{\underset{\sim}{k} \leq \underset{\sim}{n}} N(\underset{\sim}{C_k})$ and the $N(\underset{\sim}{C_k})$ are independent; further, $\text{Var}(N(\underset{\sim}{C_k})) = \mu(\underset{\sim}{C_k})$ since $N(\underset{\sim}{C_k})$ is Poisson $(\mu(\underset{\sim}{C_k}))$. Thus

$$\sum_{\underset{\sim}{k}} \frac{\sigma^2(N(\underset{\sim}{C_k}))}{(\mu(0, \underset{\sim}{k}])^2} = \sum_{\underset{\sim}{k}} \frac{\mu(\underset{\sim}{C_k})}{(\mu(0, \underset{\sim}{k}])^2} < \infty,$$

so

$$\frac{N(0, \underset{\sim}{n}] - E[N(0, \underset{\sim}{n}])}{\mu(0, \underset{\sim}{n}]} \xrightarrow{\underset{\sim}{n} \to \infty} 0 \text{ a. s. by Theorem 2.1.1}$$

and thus

$$\frac{N(0, \underset{\sim}{n}]}{E\{N(0, \underset{\sim}{n}]\}} \xrightarrow{\underset{\sim}{n} \to \infty} 1 \text{ a. s. by Corollary 2.2.1,}$$

$$\frac{\underset{\sim}{S_n}}{N(0, \underset{\sim}{n}]} \xrightarrow{\underset{\sim}{n} \to \infty} E(Y) \text{ a. s. } \blacksquare$$

Under further restrictions on μ, the hypothesis $E(|Y|^\alpha) < \infty$ for some $\alpha > 1$ can be weakened. Suppose we are working in r dimensions and that

$$E(|Y| (\log^+ |Y|)^{r+\delta} < \infty \text{ for some } \delta > 0. \tag{2.3.2}$$

Making essentially the same calculation as in § 1, we have that

$$E\{|\underset{\sim}{Z_n}| (\log^+ |\underset{\sim}{Z_n}|)^{r+\delta}\} \leqslant C\{[E(N(\underset{\sim}{C_n})(\log^+ N(\underset{\sim}{C_u}))^{r+\delta}](1 + E|Y|) + E(N(\underset{\sim}{C_n}))E(|Y| (\log^+ |Y|)^{r+\delta})\}.$$

LEMMA 2.3.1. — There exist constants C_1 and C_2 such that, for all $\underset{\sim}{n}$,

$$E\{N(C_{\underset{\sim}{n}})(\log^+ N(C_{\underset{\sim}{n}}))^{r+\delta}\} \leqslant C_1\mu(C_{\underset{\sim}{n}}) + C_2\mu(C_{\underset{\sim}{n}})(\log^+ \mu(C_{\underset{\sim}{n}}))^{r+\delta}.$$

The proof of this is elementary and unenlightening and thus will be omitted. Now let $\overline{Z}_{\underset{\sim}{n}} = Z_{\underset{\sim}{n}} - E(Z_{\underset{\sim}{n}})$.

LEMMA 2.3.2. — There exist constants K_1 and K_2 such that for all $\underset{\sim}{n}$,

$$E\{|\overline{Z}_{\underset{\sim}{n}}|(\log^+|\overline{Z}_{\underset{\sim}{n}}|)^{r+\delta}\} \leq K_1\mu(C_{\underset{\sim}{n}})E\{|Y|(\log^+|Y|)^{r+\delta}\} + K_2E(|Y|)E\{N(C_{\underset{\sim}{n}})(\log^+ N(C_{\underset{\sim}{n}}))^{r+\delta}\}.$$

Proof. — This follows from the remarks preceding Lemma 2.3.1 and an easy calculation.

PROPOSITION 2.3.2. — Suppose the point process is Poisson with a product measure satisfying (2.3.1). Then if (2.3.2) holds and

$$(2.3.3) \qquad \sum_{\underset{\sim}{n}} \frac{\mu(C_{\underset{\sim}{n}})[\log^+ \mu(C_{\underset{\sim}{n}})]^{r+\delta}}{\mu(0, \underset{\sim}{n}](\log^+ \mu(0, \underset{\sim}{n}])^{r+\delta}} < \infty,$$

it follows that $\dfrac{S_{\underset{\sim}{n}}}{N(0, \underset{\sim}{n})} \xrightarrow{\underset{\sim}{n}\to\infty} E(Y)$ a. s.

Proof. — We apply Theorem 2.1.1 to the random variables $\overline{Z}_{\underset{\sim}{n}}$, with $\varphi(x) = |x|(\log^+|x|)^{r+\delta}$. Using (2.3.2) and Lemmas (2.3.1) and (2.3.2), we have to show that the series

$$\sum_{\underset{\sim}{n}} \frac{\mu(C_{\underset{\sim}{n}})}{\mu(0, \underset{\sim}{n}](\log^+ \mu(0, \underset{\sim}{n}])^{r+\delta}} \quad \text{and} \quad \sum_{\underset{\sim}{n}} \frac{\mu(C_{\underset{\sim}{u}})(\log^+ \mu(C_{\underset{\sim}{n}}))^{r+\delta}}{\mu(0, \underset{\sim}{n}](\log^+ \mu(0, \underset{\sim}{n}])^{r+\delta}}$$

are convergent. But if μ is a product measure, the first series can easily be shown to converge by Lemma 2.2.1 and the second converges by hypothesis. Thus

$$\frac{\sum_{k\leq n}[Z_{\underset{\sim}{k}} - E(Z_{\underset{\sim}{k}})]}{E[N(0, \underset{\sim}{n}]} \xrightarrow{\underset{\sim}{n}\to\infty} 0 \text{ a. s. ; but}$$

$$\sum_{\underset{\sim}{k}\leq\underset{\sim}{n}} E(Z_{\underset{\sim}{k}}) = E(Y)E\{N(0, \underset{\sim}{n}]\} \quad \text{and} \quad \frac{N(0, \underset{\sim}{n}]}{E\{N(0, \underset{\sim}{n}]\}} \xrightarrow{\underset{\sim}{n}\to\infty} 1 \text{ a. s.}$$

as in Prop. 2.3.1, yielding the result.

COROLLARY 2.3.1. — Suppose the point process is Poisson, μ is a product measure with a bounded density, and (2.3.2) holds. Then

$$\frac{S_{\underset{\sim}{n}}}{N(0, \underset{\sim}{n}]} \xrightarrow[]{\underset{\sim}{n}\to\infty} E(Y) \text{ a. s.}$$

Proof. — Obvious from Prop. 2.3.1.

The Poisson process considered here is not far from being the most general case. If we impose the condition that the process have no multiple points, then the most general completely random point process with no fixed atoms and no deterministic component is the Poisson process [6]. When multiple points are permitted, the most general completely random point process with no fixed atoms and no deterministic component is the *compound Poisson process* [8]. For this (see [3]) the assumptions are the same as above except that the Poisson distributions there are replaced by compound Poisson distributions of the following type:

$\mu_1, \mu_2, \ldots$ are non-atomic Radon measures on the Borel sets of the positive orthant, with $\sum_k \mu_k(A) < \infty$ for all compact sets A. N(A) is the sum of a Poisson number (with parameter $\sum_k \mu_k(A)$) of non-negative i. i. d. random variables Z_i^A, independent of the Poisson number, with $P(Z^A = k) = \dfrac{\mu_k(A)}{\sum_i \mu_i(A)}$ $k = 1, 2, \ldots$. We then have

$$E(N(A)) \equiv \mu(A) = E(Z^A)\sum_k \mu_k()A = \sum_k k\mu_k(A).$$

Thus this point process is itself a marked point process, with the marks having nonnegative integer values and the underlying process Poisson.

PROPOSITION 2.3.3. — Let the point process be compound Poisson as above. Suppose that

(*i*) $E(|Y|^\alpha) < \infty$ for some $\alpha > 1$.

(*ii*) $\sum_{\underset{\sim}{n}} \{ \mu(C_{\underset{\sim}{n}})/(\mu(0, \underset{\sim}{n}])^\alpha \} < \infty$.

(*iii*) $\sum_{\underset{\sim}{n}}\sum_k k^2\mu_k(C_{\underset{\sim}{n}})/(\mu(0, \underset{\sim}{n}])^2 < \infty$.

Then $S_{\underset{\sim}{n}}/N(0, \underset{\sim}{n}] \to E(Y)$ a s. as $\underset{\sim}{n} \to \infty$.

Proof. — By Corollary 2.2.1 it suffices to show that

$$\sum_{\underset{\sim}{n}} \{ \mu(\mathrm{C}_{\underset{\sim}{n}})/(\mu(0, \underset{\sim}{n}])^\alpha \} < \infty$$

—which is true by (*ii*)—and that $\mathrm{N}(0, \underset{\sim}{n}]/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{n}]) \to 1$ a. s. as $n \to \infty$. Now $\mathrm{N}(0, \underset{\sim}{n}] = \sum_{\underset{\sim}{k} \leq \underset{\sim}{n}} \mathrm{N}(\mathrm{C}_{\underset{\sim}{k}})$ and $\mathrm{Var}(\mathrm{N}(\mathrm{C}_{\underset{\sim}{k}})) = \sum_j j^2 \mu_j(\mathrm{C}_{\underset{\sim}{k}})$ by an easy calculation. Thus the convergence of (*iii*) implies that $\sum_k \{ \sigma^2(\mathrm{N}(\mathrm{C}_k))/\mu^2(0, \underset{\sim}{k}] \}$ converges, hence by Theorem 2.1.1 that $\mathrm{N}(0, \underset{\sim}{n}]/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{n}]) \to 1$ a. s. as $\underset{\sim}{n} \to \infty$.

2.4. Generalizations

Two questions remain concerning the ergodic theorems of Sections 2.2 and 2.3. First, can the convergence $\mathrm{S}_{\underset{\sim}{n}}/\mathrm{N}(0, \underset{\sim}{n}] \to \mathrm{E}(\mathrm{Y})$ a. s. be extended to show that $\mathrm{S}_{\underset{\sim}{t}}/\mathrm{N}(0, \underset{\sim}{t}] \to \mathrm{E}(\mathrm{Y})$ a. s. as $\underset{\sim}{t} \to \infty$? Second, can we extend the results for point processes defined on the positive orthant to the whole plane?

The following result shows that the answer to the first question is affirmative if $\mathrm{E}(|\mathrm{Y}|^\alpha) < \infty$ for some $\alpha > 1$ and if $\mathrm{E}(\mathrm{N}(0, \underset{\sim}{t}])$ is reasonably well-behaved as a function of $\underset{\sim}{t}$.

Proposition 2.4.1. — Suppose that

$$(2.4.1) \quad \lim_{\underset{\sim}{n} \to \infty} \mathrm{E}(\mathrm{N}(0, \underset{\sim}{n})/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{n} + \underset{\sim}{1}) = 1, \quad (\text{where } \underset{\sim}{1} = (1, 1, \dots 1)).$$

Then the conditions (*i*) and (*ii*) of Corollary 2.2.1 are sufficient to give $\mathrm{S}_{\underset{\sim}{t}}/\mathrm{N}(0, \underset{\sim}{t}] \to \mathrm{E}(\mathrm{Y})$ a. s. as $\underset{\sim}{t} \to \infty$.

Proof. — Let $[t] = ([t_1], \dots, [t_r])$. We then have

$$\mathrm{S}_{\underset{\sim}{t}}/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{t}]) = \{ \mathrm{S}_{[\underset{\sim}{t}]} + (\mathrm{S}_{\underset{\sim}{t}} - \mathrm{S}_{[\underset{\sim}{t}]}) \}/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{t}])$$

so it will suffice to show, by (2.4.1), that $\{ \mathrm{S}_t - \mathrm{S}_{[t]} \}/\mathrm{E}(\mathrm{N}(0, t]) \to 0$ a. s. as $\underset{\sim}{t} \to \infty$. If $\underset{\sim}{n} = (n_1, \dots, n_r)$, let $\delta \underset{\sim}{n} = (0, \underset{\sim}{n} + \underset{\sim}{1})] - (0, \underset{\sim}{n}]$, and let $\Delta_{\underset{\sim}{k}} = \sum_{\mathrm{X}_i \in \mathrm{C}_{\underset{\sim}{k}}} |\mathrm{Y}_i|$.

Then $|\mathrm{S}_{\underset{\sim}{t}} - \mathrm{S}_{[\underset{\sim}{t}]}| \leq \sum_{\underset{\sim}{k} \in \delta[\underset{\sim}{t}]} \Delta_{\underset{\sim}{k}}$. By Corollary 2.2.1, $\sum_{\underset{\sim}{k} \leq \underset{\sim}{n} + \underset{\sim}{1}} \Delta_{\underset{\sim}{k}}/\mathrm{E}(\mathrm{N}(0, \underset{\sim}{n} + 1])$

and $\sum_{\underset{\sim}{k} \leqslant \underset{\sim}{n}} \Delta_{\underset{\sim}{k}}/E(N(0, \underset{\sim}{n}])$ are a. s. convergent to the same limit; by (2.4.1), $\sum_{\underset{\sim}{k} \in \delta[\underset{\sim}{t}]} \Delta_{\underset{\sim}{k}}/E(N(0, [\underset{\sim}{t}])$ converges a. s. to zero, and so therefore does

$$\{ S_{\underset{\sim}{t}} - S_{[\underset{\sim}{t}]} \}/E(N(0, [\underset{\sim}{t}]]). \quad \blacksquare$$

Next we consider the case when the point process is defined on all of R^r. As in the remark of § 1, we can treat each orthant separately, writing $S_T = S_{T_1} + S_{T_2} + \ldots + S_{T_{2^r}}$. If the conditions on the point process and the Y_j are sufficient to ensure that $S_{T_i}/N_{T_i} \to E(Y)$ a. s. for each i, then it is easy to see that S_T/N_T will converge a. s. to E(Y). To be more precise, let Z^r denote the set of r-tuples with integer coordinates; for $\underset{\sim}{n} \in Z^r$, if $\underset{\sim}{n}$ lies in the ith orthant, let θ^i be the rotation which maps the ith orthant onto the positive orthant, and tle $b_{\underset{\sim}{n}} = \mu(0, \theta^i \underset{\sim}{n}]$. We then have the following result:

THEOREM 2.4.1. — Let T be an r-dimensional rectangle with one vertex in each orthant. Suppose that for some α, with $1 \leqslant \alpha \leqslant 2$,

(*i*) $E(| Y |^\alpha) < \infty$.

(*ii*) $\sum_{\underset{\sim}{n} \in Z^r} \mu(C_{\underset{\sim}{n}})/b_{\underset{\sim}{n}}^\alpha < \infty$.

(*iii*) $b_{\underset{\sim}{n}}/b_{\underset{\sim}{n}+\underset{\sim}{1}} \to 1$ as $\underset{\sim}{n} \to \infty$.

Then $S_T/E(N_T) \to E(Y)$ a. s. as the coordinates of each corner of T tend independently to ∞. If we restrict T to rectangles whose vertices have integral coordinates, condition (*iii*) is dispensable.

COROLLARY 2.4.1. — Suppose the point process is Poisson and that, in each orthant, μ satisfies (2.3.1) and one of the conditions of Lemma 2.2.1. Then if $E(| Y |^\alpha) < \infty$ for some $\alpha > 1$,

$$S_T/N_T \to E(Y) \text{ a. s.}$$

Finally we note that in [9] we have proved a slightly sharper version of Theorem 2.1.1 which shows that for $1 \leqslant \alpha \leqslant 2$, $E(\sup_{\underset{\sim}{k}} |S_{\underset{\sim}{k}}|^\alpha/b_{\underset{\sim}{k}}^\alpha) < \infty$ if $\sum_{\underset{\sim}{k}} E(| Z_{\underset{\sim}{k}} |^\alpha/b_{\underset{\sim}{k}}^\alpha) < \infty$. Thus under the hypothesis of Theorem 2.2.1, we can conclude that $E(\sup_{\underset{\sim}{k}} | S_{\underset{\sim}{k}} |/(\mu(0, \underset{\sim}{k}])^\alpha < \infty$. An argument like that of Prop. 2.4.1 can then be made to extend this to say that

$$E(\sup_{\underset{\sim}{t}} | S_{\underset{\sim}{t}} |/(\mu(0, \underset{\sim}{t}])^\alpha < \infty.$$

3. THE GENERAL CASE

In this section we will assume neither stationarity nor complete randomness of the point process, but we will restrict ourselves to point processes on the positive orthant. It will be convenient here to assume that the point process is defined on $(\Omega, \mathscr{F}, P)$ and an i. i. d. sequence $\{ Y_i \}$ on $(\tilde{\Omega}, \mathscr{F}, \tilde{P})$. Our results in this case are very incomplete; insofar as they exist they are based on the techniques of [*11*]. Given a realization of the point process, we regard the resulting configuration of points as a « local lattice », in the terminology of [*11*]; if α is a point in this configuration, we take $| \alpha |$ to be the number of points in the realization which are $\leqslant \alpha$ (in the natural ordering of R^r). We define the numbers

$$d^\omega(n) = \text{card}\,\{ \alpha : | \alpha | = n \}, \qquad M^\omega(x) = \sum_{k \leq} d^\omega(k) \qquad \text{for} \qquad x > 0.$$

We have shown in [*11*] that if

(*i*) $M^\omega(x)$ varies dominatedly at infinity with index < 2, and
(*ii*) $E\{ M^\omega(| Y) \} < \infty$,

then a strong law of large numbers holds, *i. e.*, $\dfrac{S_{\underset{\sim}{t}}^\omega(\tilde{\omega})}{N(0, \underset{\sim}{t}]} \xrightarrow[\sim]{t \to \infty} E(Y)$ a. s. $(\tilde{P})$.

Of course this requires that $M^\omega(x)$ be finite, which is not true for many point processes (one could get around this by placing fixed atoms of the process at unit intervals on each axis). The details of the convergence are given in [*11*]; we content ourselves here with one example of a class of processes amenable to this approach.

EXAMPLE 3.1. — Suppose that, for almost all ω, we can partition the positive orthant by a rectilinear grid in such a way that there is at least one point and not more than K points in each cell. For example, suppose we take $r = 2$ and the integer grid; modify the point process temporarily by placing fixed atoms at unit intervals on each axis, and denote the modified quantities with primes. It is easy to verify that for large N,

$$K(N \log N) + 0(N) \geqslant M'^\omega(N) \geqslant (N/(K + 1)) \log (N/(K + 1)) + 0(N).$$

M'^{ω} is then of dominated variation with index 1 and if $E(|Y| \log^+ |Y|) < \infty$, it follows that $S_t'^{\omega}(\tilde{\omega})/N'(0, t] \to E(Y)$ a. s. $(\tilde{P})$ as $t \to \infty$; it is easy to deduce from this, using the ordinary strong law of large numbers and the fact that $N'(0, t]/N(0, t] \to 1$ a. s. (P), that $S_t(\tilde{\omega})/N(0, t] \to E(Y)$ a. s. $(\tilde{P})$. Thus on the product space $(\Omega \times \tilde{\Omega}, \mathscr{F} \times \tilde{\mathscr{F}}, P \times \tilde{P})$, $S_t/N(O, t] \to E(Y)$ a. s. as $t \to \infty$. The corresponding result holds in r dimensions if

$$E(|Y| (\log^+ |Y|)^{r-1} < \infty.$$

A characterization of those point processes which can be treated in this way appears to be difficult.

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(Manuscrit reçu le 7 février 1975).