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Extension of the Birkhoff and von Neumann ergodic theorems to semigroup actions (*)

by

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In 1967, A. A. Tempelman announced generalizations of the Birkhoff and von Neumann ergodic theorems [*6*]. This paper supplies proofs of results similar to Tempelman's. The main arguments are drawn from Calderon's paper [*1*]. The author has also had the benefit of reading Mrs. J. Chatard's work on the same problem [*2*].

PRELIMINARIES

Let $(\mathrm{M}, \mathscr{M}, \mu)$ and $(\mathrm{G}, \mathscr{I}, \gamma)$ be complete measure spaces, where μ is σ-finite. Assume that G is a semigroup with product indicated by juxtaposition and that there is a map $(x, m) \mapsto x(m)$ from $\mathrm{G} \times \mathrm{M}$ to M, measurable with respect to $\mathrm{G} \times \mathscr{M}$ and such that $x(y(m)) = xy(m)$ for all $x, y \in \mathrm{G}$ and $m \in \mathrm{M}$. Assume that $\mu(x^{-1}\mathrm{F}) \leq \mu(\mathrm{F})$ for all $x \in \mathrm{G}$ and $\mathrm{F} \in \mathscr{M}$, where $x^{-1}\mathrm{F} = \{ m : x(m) \in \mathrm{F} \}$. Finally, assume that for all $x \in \mathrm{G}$ and $\mathrm{E} \in \mathscr{I}$, $x\mathrm{E}$ and $\mathrm{E}x$ are measurable, $\gamma(x\mathrm{E}) = \gamma(\mathrm{E}) = \gamma(\mathrm{E}x)$, and that $x^{-1}\mathrm{E}$ and $\mathrm{E}x^{-1}$ are measurable, where $x^{-1}\mathrm{E} = \{ y \in \mathrm{G} : xy \in \mathrm{E} \}$ and $\mathrm{E}x^{-1} = \{ y \in \mathrm{G} : yx \in \mathrm{E} \}$.

If $x \in \mathrm{G}$ and E and D are in $\mathscr{I}$, then

$$\gamma(\mathrm{E} \cap x^{-1}\mathrm{D}) = \gamma(x(\mathrm{E} \cap x^{-1}\mathrm{D})) = \gamma((x\mathrm{E}) \cap \mathrm{D}),$$

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so that

$$\int_{\mathrm{E}} \chi_{\mathrm{D}}(xy)d\gamma(y) = \int_{x\mathrm{E}} \chi_{\mathrm{D}}(y)d\gamma(y),$$

where χ_{D} is the characteristic function of D. Therefore, if f is any integrable function on G,

$$\int_{\mathrm{E}} f(xy)d\gamma(y) = \int_{x\mathrm{E}} f(y)d\gamma(y),$$

and similarly,

$$\int_{\mathrm{E}} f(yx)d\gamma(y) = \int_{\mathrm{E}x} f(y)d\gamma(y).$$

If f is any function on M and $x \in \mathrm{G}$, define f_x by $f_x(m) = f(xm)$. If f is any nonnegative integrable function on M,

$$\int f_x d\mu \leq \int f d\mu.$$

Let A_n be a sequence of measurable subsets of G such that $0 < \gamma(\mathrm{A}_n) < \infty$ for all n. We shall use the following conditions on the A_n.

I. $n < m$ implies $\mathrm{A}_n \subset \mathrm{A}_m$;

II. $\lim_n \dfrac{\gamma(\mathrm{A}_n \triangle x\mathrm{A}_n)}{\gamma(\mathrm{A}_n)} = \lim_n \dfrac{\gamma(\mathrm{A}_n \triangle \mathrm{A}_n x)}{\gamma(\mathrm{A}_n)} = 0$, for all $x \in \mathrm{G}$, where $\triangle$ denotes symmetric difference;

III. for each k and n, $\mathrm{A}_k\mathrm{A}_n$ is measurable and $\lim_n \dfrac{\gamma(\mathrm{A}_k\mathrm{A}_n \triangle \mathrm{A}_n)}{\gamma(\mathrm{A}_n)} = 0$; and

IV. there exists $\mathrm{K} > 1$ such that $\gamma(\mathrm{A}_n^{-1}\mathrm{A}_n) \leq \mathrm{K}(\mathrm{A}_n)$ for all n, where

$$\mathrm{A}_n^{-1}\mathrm{A}_n = \{ x \in \mathrm{G}\colon yx \in \mathrm{A}_n \text{ for some } y \in \mathrm{A}_n \}.$$

Let B be a real or complex Banach space with norm $\|\,.\,\|$ and dual B*. If $\lambda \in \mathrm{B}^*$ and $b \in \mathrm{B}$, $\lambda(b)$ will be denoted by $\lambda \cdot b$. If (N, ν) is a measure space and if $1 \leq p < \infty$, $\mathrm{L}_p^{\mathrm{B}}(\mathrm{N}, \nu)$ will denote the set of equivalence classes of functions $f : \mathrm{M} \to \mathrm{B}$ such that f is the limit in measure of simple functions and

$$\| f \|_p^{\mathrm{B}} = \left(\int \| f \|^p d\nu \right)^{1/p} < \infty.$$

$\mathrm{L}_p^{\mathrm{B}}$ is a Banach space, and if (N, ν) is σ-finite, its dual is $\mathrm{L}_q^{\mathrm{B}^*}$, where $q = \infty$ if $p = 1$, and $\dfrac{1}{q} + \dfrac{1}{p} = 1$ otherwise.

If $g \in L_q^{B^*}$ and $f \in L_p^B$,

$$(1) \quad \int_{A_n}\int_M |g(m)\cdot f_x(m)|\,d\mu(m)d\gamma(x) \leq \int_{A_n}\int_M \|g(m)\|\ \|f_x(m)\|\,d\mu d\gamma \leq \|g\|_q^{B^*}\|f\|_p^B\gamma(A_n).$$

If we choose g so that $\|g(m)\| > 0$ a. e., then (1) and Fubini's theorem imply that for almost every m,

$$\int_{A_n} \|f_x(m)\|\,d\gamma(x)$$

exists. It follows that for almost every m, $f_{(.)}(m) \in L_1^B(A_n, \gamma|_{A_n})$ and hence that

$$(2) \quad g(m)\cdot\int_{A_n} f_x(m)d\gamma(x) = \int_{A_n} g(m)\cdot f_x(m)d\gamma(x) \quad [3, \text{III}.6.10,\ \text{III}.2.22].$$

Applying Fubini's theorem again and using (2), we obtain

$$\int_{A_n}\int_M g(m)\cdot f_x(m)d\mu d\gamma = \int_M g(m)\cdot\left(\int_{A_n} f_x(m)d\gamma\right)d\mu.$$

Hence, the map $x \mapsto f_x$ from A_n to L_p^B is integrable in the sense of Pettis [5], and the integral is equal to $\int_{A_n} f_x(m)d\gamma(x)$ almost everywhere. Define

$$\pi_n : L_p^B \to L_p^B \quad \text{by} \quad \pi_n(f) \equiv \frac{1}{\gamma(A_n)}\int_{A_n} f_x d\gamma(x).$$

Clearly, π_n is a continuous linear operator of norm less than or equal to one.

THE ERGODIC THEOREMS

THEOREM 1 (von Neumann's Mean Ergodic Theorem). — If the A_n satisfy II and if $1 < p < \infty$ or if $p = 1$ and $\mu(M) < \infty$, then there is $\pi(f) \in L_p^B$ such that $\lim_n \|\pi_n(f) - \pi(f)\|_p^B = 0$ and such that $\pi(f_x) = \pi(f) = \pi(f)_x$ for all $x \in G$. If $p = 1$, $\int \pi(f)d\mu = \int f d\mu$. π is the projection of L_p^B onto I_p^B along M_p^B, where I_p^B is the subspace of invariant functions and M_p^B is the closed subspace generated by $\{ f_x - f : f \in L_p^B,\ x \in G \}$.

THEOREM 2 (Wiener-Calderon Dominated Convergence Theorem). — Suppose that the A_n satisfy I, III, and IV. If f is a nonnegative integrable function and if for $\alpha > 0$,

$$E_\alpha = \{ m: \sup_{n=1,\ldots,\infty} \pi_n(f)(m) \geq \alpha \},$$

then

$$\mu(E_\alpha) \leq \frac{K}{\alpha} \int f d\mu.$$

THEOREM 3 (Birkhoff's Individual Ergodic Theorem). — If the A_n satisfy I-IV and if $f \in L_p^B$, where $1 \leq p < \infty$, then $\pi_n(f)$ converges almost everywhere. If $1 < p < \infty$ or if $p = 1$ and $\mu(M) < \infty$, then $\pi_n(f)$ converges almost everywhere to the $\pi(f)$ of Theorem 1.

PROOF OF THEOREM 1

LEMMA 1. — If $f \in L_p^B$ and $1 < p < \infty$, then $C(f) = \langle \overline{\{ f_x : x \in G \}} \rangle$ is weakly compact.

Proof. — If $p > 1$, let $\frac{1}{q} + \frac{1}{p} = 1$. If $p = 1$, let $q = \infty$. $C(f)$ is weakly compact if for each sequence $x_m \in G$ and each sequence $\lambda_n \in L_q^{B^*}$ such that $\| \lambda_n \|_q^{B^*} \leq 1$ for all n, $\lim_m \lim_n \lambda_n \cdot f_{x_m} = \lim_n \lim_m \lambda_n \cdot f_{x_m}$ whenever each limit exists [4, p. 159].

Let $\varepsilon > 0$ and choose a simple function $g \in L_p^B$ such that $\| f - g \|_p^B < \varepsilon$. $| \lambda_n \cdot g_{x_m} | \leq \| g \|_p^B$ for all n and m, so that we may, by a diagonal process, choose a subsequence $g_{x_{m_k}}$ such that for each n, $\lim_k \lambda_n \cdot g_{x_{m_k}} \to a_n$. We may assume that $a_n \to a$. Similarly, we may choose a subsequence $\lambda_{n_l} \cdot g_{x_{m_k}}$ such that $\lim_l \lambda_{n_l} \cdot g_{x_{m_k}} = c_k$ for each k. Again, we may assume that $c_k \to c$. Since $\| \lambda_n \cdot f_{x_m} - \lambda_n \cdot g_{x_m} \|_p^B \leq \| \lambda_n \|_q^{B^*} \| f_{x_m} - g_{x_m} \|_p^B < \varepsilon$ for all m, n, it suffices to show that $a = c$.

Since the λ_n are uniformly bounded, the sequence λ_{n_l} has a weak star limit point, λ_0. $g = \sum_{i=1}^{s} h_i b_i$, where $h_i \in L_p$ and $b_i \in B$. Since $\| (h_i)_x \|_p \leq \| h_i \|_p$ for all $x \in G$, $\{ (h_i)_x : x \in G \}$ is weakly relatively compact for each i. Hence, x_{m_k} has a subnet $x_{m_{k(\alpha)}}$ such that $(h_i)_{x_{m_{k(\alpha)}}}$ converges weakly to some h_{i0} for each i. Then, $a = \lambda_0 \cdot \Sigma b_i h_{i0} = c$. Q. E. D.

PROOF OF THEOREM 1. — Suppose that $p > 1$. $\pi_n(f) \in C(f)$, $\forall n$. Since $C(f)$ is weakly compact, $\pi_n(f)$ has a weak cluster point $\pi(f)$. Given $\varepsilon > 0$, there are $v \in L_p^B$ with $\| v \|_p^B < \varepsilon$ and α_i, f_{x_i}, $i = 1, \ldots, m$, with

$$0 \leq \alpha_i \leq 1, \sum_{i=1}^{m} \alpha_i = 1,$$

such that

$$\pi(f) = \sum_{n=1}^{\infty} \alpha_i f_{x_i} + v.$$

Hence,

$$\pi(f) - f = \sum_{i=1}^{m} \alpha_i (f_{x_i} - f) + v.$$

For every $x \in G$,

$$\| \pi_n(f)_x - \pi_n(f) \|_p^B \leq \frac{\gamma(A_n \triangle xA_n)}{\gamma(A_n)} \| f \|_p^B \to 0,$$

so that $\pi(f)$ is invariant and $\pi_n(\pi(f)) = \pi(f)$ for all n. Hence,

$$\pi(f) - \pi_n(f) = \sum_{i=1}^{m} \alpha_i \pi_n(f_{x_i} - f) + \pi_n(v).$$

Since $\| \pi_n(v) \|_p^B < \varepsilon$ and since for all $x \in G$,

$$\lim_n \| \pi_n(f_x) - \pi_n(f) \|_p^B \leqq \lim_n \frac{\gamma(A_n \triangle A_n x)}{\gamma(A_n)} \| f \|_p^B = 0,$$

it follows that $\lim_n \| \pi_n(f) - \pi(f) \|_p^B = 0$,

The case $p = 1$ follows from the case $p = 2$, since the π_n are uniformly bounded on L_1^B and since, if $\mu(M) < \infty$, L_2^B is B_1^B-dense in L_2^B and the $\| \cdot \|_2^B$-topology is stronger than the L_1^B-topology. Q. E. D.

PROOF OF THEOREM 2

The key step of the proof is the Wiener-Calderon covering argument made in proving Lemma 1 below.

Let « \ » denote set theoretic difference.

LEMMA 1. — Let h be a real-valued γ-integrable function defined on A_kA_n. Suppose that

(1) $h(x) \geq 0$ for all $x \in A_kA_n \backslash A_n$;

(2) for all $x \in A_n$, either $h(x) \geq 0$ or $h(x) \geq -\frac{\alpha}{K}$ and

$$\frac{1}{\gamma(A_n)} \int_{A_i} h(yx)d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

for some $i = 1, \ldots, K$, where $a > 0$.

Then, $\int_{A_kA_n} hd\gamma \geq 0$.

Proof. — Let $\mathscr{M}_k$ be a maximal collection of disjoint subsets of the form A_kx, such that

$$x \in A_n \quad \text{and} \quad \frac{1}{\gamma(A_k)} \int_{A_k} h(yx)d\gamma(y) \geq \alpha - \frac{\alpha}{K}.$$

Given $\mathscr{M}_{i+1}$, where $k > i \geq 1$, let $\mathscr{M}_i$ be a maximal collection of sets of the form A_ix where $x \in A_n$ and such that

$$\frac{1}{\gamma(A_i)} \int_{A_i} h(yx)d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

and the A_ix are mutually disjoint and are disjoint from every set in $\mathscr{M}_j$ for $i + 1 \leq j \leq k$. Let $\mathscr{M} = \bigcup_{i=1}^{k} \mathscr{M}_i$.

Let $\mathscr{N} = \{ A_i^{-1}A_ix : A_ix \in \mathscr{M} \}$. Let $N = A_kA_n \backslash \cup \mathscr{N}$. Suppose $x \in N$ and $k(x) < 0$. Then $x \in A_n$ and for some $i = 1, \ldots, k$,

$$\frac{1}{\gamma(A_i)} \int_{A_ix} hd\gamma \geq \alpha - \frac{\alpha}{K}, \qquad A_ix \notin \mathscr{M}.$$

Therefore, there exists $A_jx' \in \mathscr{M}$ with $j \geq i$ such that $A_ix \cap A_jx' \neq \phi$. But then $x \in A_i^{-1}A_jx' \subset A_j^{-1}A_jx' \subset \mathscr{N}$. This contradicts $x \in N$. Hence, $h \geq 0$ on N and

$$\int_{A_kA_n} h \geq \int_N h + \int_{\cup\mathscr{M}} h - \frac{\alpha}{K}\gamma(\cup\mathscr{N}\backslash\cup\mathscr{M}) \geq \left(\alpha - \frac{\alpha}{K}\right)\sum_{\mathscr{M}}\gamma(A_ix)$$

$$-\frac{\alpha}{K}\gamma(\cup\mathscr{N}\backslash\cup\mathscr{M}) = \alpha\Sigma\gamma(A_ix) - \frac{\alpha}{K}\gamma(\cup\mathscr{N}) \geq \frac{\alpha}{K}\Sigma\gamma(A_i^{-1}A_nx) - \frac{\alpha}{K}\gamma(\cup\mathscr{N}) = 0.$$

Q. E. D.

LEMMA 2. — Suppose that $f \in L_1(M)$ and that for each $m \in M$, either $f(m) \geq 0$ or $f(m) \geq -\frac{\alpha}{K}$ and for some $i = 1, \ldots, k$,

$$\frac{1}{\gamma(A_i)} \int_{A_i} f(ym) d\gamma(y) \geq \alpha - \frac{\alpha}{K}.$$

Then, $\int f d\mu \geq 0$.

Proof. — Let n be a positive integer and for each $m \in M$, let $h_m \colon A_k A_n \to R$ be defined by

$$h_m(y) = \begin{cases} |f(ym)|, & \text{if } \ y \in A_k A_n \backslash A_n; \\ f(ym), & \text{if } \ y \in A_n. \end{cases}$$

Let $M' = \{ m \in M : |f(ym)| \text{ is } \gamma\text{-integrable on } A_k A_n \}$. By Fubini's Theorem, $\mu(M \backslash M') = 0$. For each $m \in M'$, h_m satisfies the assumptions of Lemma 1. Hence,

$$0 \leq \int_{A_k A_n} h_m d\gamma = \int_{A_n} f(ym) d\gamma(y) + \int_{A_k A_n \backslash A_n} |f(ym)| \, d\gamma(y).$$

Applying Fubini's Theorem, we have

$$0 \leq \gamma(A_n) \int_M f d\mu + \gamma(A_k A_n \backslash A_n) \int_M |f| \, d\mu,$$

or

$$0 \leq \int_M f d\mu + \frac{\gamma(A_k A_n \bigtriangleup A_n)}{\gamma(A_n)} \| f \|.$$

Let $n \to \infty$ and apply III. Q. E. D.

PROOF OF THEOREM 2. — It is sufficient to prove that

$$\mu(F \cap E_\alpha^k) \leqslant \frac{K}{\alpha} \int f d\mu,$$

where $\mu(F) < \infty$ and

$$E_\alpha^k = \{ m \in M : \max_{i=1,\ldots,k} \pi_i(f)(m) \geq \alpha \}.$$

Let

$$h = f - \frac{\alpha}{K} \chi_{F \cap E_\alpha^k}.$$

Since h satisfies the assumptions of Lemma 3, $\int h d\mu \geq 0$. Q. E. D.

PROOF OF THEOREM 3

Suppose at first that if $p = 1$, $\mu(\mathrm{M}) < \infty$.

Let $\pi(f)$ be the limit defined by the Mean Convergence Theorem. Since $\pi_n(f - \pi(f)) = \pi_n(f) - \pi(f)$, one may suppose that $\pi(f) = 0$. We show that $\pi_n(f) \to 0$ a. e.

Let $\varepsilon > 0$. Choose f^b bounded and such that $\| f^b - f \|_p^{\mathrm{B}} < \frac{\varepsilon}{3}$. Choose k such that $\| \pi_k(f^b) - \pi(f^b) \|_p^{\mathrm{B}} < \frac{\varepsilon}{3}$. Then, $f = \mathrm{H} + \mathrm{G}$, where

$$\mathrm{H} = (f - f^b) + (\pi_k(f^b) - \pi(f^b)) + \pi(f^b) \quad \text{and} \quad \mathrm{G} = f^b - \pi_k(f^b).$$

Clearly, $\| \mathrm{H} \|_p^{\mathrm{B}} < \varepsilon$.

$\pi_n(\mathrm{G})$ converges to zero almost everywhere since, for almost every m,

$$\begin{aligned} \| \pi_n(\mathrm{G})m \| &= \frac{1}{\gamma(\mathrm{A}_n)} \left\| \int_{\mathrm{A}_n} \frac{1}{\gamma(\mathrm{A}_k)} \int_{\mathrm{A}_k} f^b_{xy}(m) d\gamma(x) - \int_{\mathrm{A}_n} f^b_y(m) d\gamma(y) \right\| \\ &\leq \frac{1}{\gamma(\mathrm{A}_k)} \int_{\mathrm{A}_k} \frac{1}{\gamma(\mathrm{A}_n)} \left\| \int_{\mathrm{A}_n} (f^b_{xy}(m) - f^b_y(m)) d\gamma(y) \right\| d\gamma(x) \\ &\leq \frac{1}{\gamma(\mathrm{A}_k)} \int_{\mathrm{A}_k} \frac{1}{\gamma(\mathrm{A}_n)} \int_{\mathrm{A}_n \Delta x \mathrm{A}_n} \| f^b_y(m) \| \, d\gamma(y) d\gamma(x) \\ &\leq \frac{\gamma(\mathrm{A}_n \bigtriangleup \mathrm{A}_k \mathrm{A}_n)}{\gamma(\mathrm{A}_n)} \sup \| f^b \| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, if $\delta > 0$,

$$\mu \{ m \colon \overline{\lim_n} \, \| \pi_n(f)(m) \| > 3\delta \} \leq \mu \{ m \colon \lim_n \pi_n(\| \mathrm{H}(m) \|) > 2\delta \}.$$

If $p = 1$, by Theorem 2 we obtain

$$\mu \{ m \colon \overline{\lim_n} \, \pi_n(\| \mathrm{H}(m) \|) > 2\delta \} \leq \frac{\mathrm{K}}{2\delta} \| \mathrm{H} \|_1^{\mathrm{B}} < \frac{\mathrm{K}\varepsilon}{2\delta}.$$

Since ε is arbitrarily small, we obtain $\lim_n \pi_n(f) = 0$ a. e.

If $p > 1$, let

$$\mathrm{H}^\delta(m) = \begin{cases} \| \mathrm{H}(m) \| & \text{if} \quad \| \mathrm{H}(m) \| \geq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

$\| \mathrm{H} \| \leqslant \mathrm{H}^\delta + \delta$, so that $\mu \{ m \colon \overline{\lim} \, \pi_n(\| \mathrm{H}(m) \|) \geq 2\delta \} \leq \mu \{ m \colon \overline{\lim} \, \pi_n(\mathrm{H}^\delta(m)) \geq \delta \}$.

Furthermore,

$$H^{\delta} \in L_1 \quad \text{and} \quad \| H^{\delta} \|_1 \leq \frac{(\| H \|_1^{B})^p}{\delta^{p-1}} < \frac{\varepsilon^p}{\delta^{p-1}}$$

so that by Theorem 2,

$$\mu\{ m : \overline{\lim_n}\ \pi_n(H^{\delta}(m)) \geq \delta \} \leq K\left(\frac{\varepsilon}{\delta}\right)^p \quad \text{and} \quad \lim_n \pi_n(f) = 0$$

almost everywhere.

We now remove the assumption that $\mu(M) < \infty$ as in the case $p = 1$ and prove that $\pi_n(f)$ converges almost everywhere. We call a set $E \in \mathscr{M}$ invariant if $xE \subset E$, $\forall x \in G$. It is possible to find a sequence of invariant sets, I_k, of finite measure and such that if I is invariant and measurable and if $\mu(I \cap \bigcup_k I_k) = 0$, then either $\mu(I) = 0$ or $\mu(I) \equiv \infty$. By what we have already proved, $\pi_n(f)$ converges on each I_k and hence on $\bar{I} = \bigcup_k I_k$.

Let $\varepsilon > 0$ and let f^b be a bounded function such that $\| f - f^b \|_1^B < \varepsilon$.

$$\{ m \in M \backslash \bar{I} : \overline{\lim_n}\ \| \pi_n(f)(m) \| > 2\delta \} \subset \{ m \in M : \overline{\lim_n}\ \pi_n(\| f - f^b)(m) \|) > \delta \}$$
$$\cup \{ m \in M \backslash \bar{I} : \overline{\lim_n}\ \pi_n(\| f^b(m) \|) > \delta \}.$$

The measure of the first set on the right is bounded by $\frac{K\varepsilon}{\delta}$. It is easy to show that the second set is invariant. Since it is bounded by $\frac{K}{\delta} \| f^b \|_1^B$ it must have measure zero. Q. E. D.

BIBLIOGRAPHY

[*1*] A. P. Calderon, A General Ergodic Theorem. *Annals of Mathematics*, t. **58**, 1953, p. 182-191.

[*2*] J. Chatard, « Application des propriétés de moyenne d'un groupe localement compact à la théorie ergodique », Université de Paris (mineographed).

[*3*] N. Dunford and J. Schwartz, *Linear Operator*, *Part I*. Interscience, 1966.

[*4*] J. L. Kelly, I. Namioka and coauthors, *Linear Topological Spaces*. Van Nostrand, 1963.

[*5*] B. J. Pettis, On Integration on Vector Spaces. *Trans. Amer. Math. Soc.*, t. **44**, 1938, p. 277-304.

[*6*] A. A. Tempelman, Ergodic Theorems for General Dynamic Systems. *Soviet Math. Dokl.*, t. **8**, 1967, p. 1213-1216.

[*7*] Norbert Wiener, The Ergodic Theorem. *Duke Math. Journal*, t. **5**, 1934, p. 1-18.

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