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## Extension of the Birkhoff and von Neumann ergodic theorems to semigroup actions (\*)

by

Truman BEWLEY

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In 1967, A. A. Tempelman announced generalizations of the Birkhoff and von Neumann ergodic theorems [6]. This paper supplies proofs of results similar to Tempelman's. The main arguments are drawn from Calderon's paper [1]. The author has also had the benefit of reading Mrs. J. Chatard's work on the same problem [2].

### PRELIMINARIES

Let  $(M, \mathcal{M}, \mu)$  and  $(G, \mathcal{J}, \gamma)$  be complete measure spaces, where  $\mu$  is  $\sigma$ -finite. Assume that  $G$  is a semigroup with product indicated by juxtaposition and that there is a map  $(x, m) \mapsto x(m)$  from  $G \times M$  to  $M$ , measurable with respect to  $G \times \mathcal{M}$  and such that  $x(y(m)) = xy(m)$  for all  $x, y \in G$  and  $m \in M$ . Assume that  $\mu(x^{-1}F) \leq \mu(F)$  for all  $x \in G$  and  $F \in \mathcal{M}$ , where  $x^{-1}F = \{m : x(m) \in F\}$ . Finally, assume that for all  $x \in G$  and  $E \in \mathcal{J}$ ,  $xE$  and  $Ex$  are measurable,  $\gamma(xE) = \gamma(E) = \gamma(Ex)$ , and that  $x^{-1}E$  and  $Ex^{-1}$  are measurable, where  $x^{-1}E = \{y \in G : xy \in E\}$  and  $Ex^{-1} = \{y \in G : yx \in E\}$ .

If  $x \in G$  and  $E$  and  $D$  are in  $\mathcal{J}$ , then

$$\gamma(E \cap x^{-1}D) = \gamma(x(E \cap x^{-1}D)) = \gamma((xE) \cap D),$$

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so that

$$\int_E \chi_D(xy) d\gamma(y) = \int_{xE} \chi_D(y) d\gamma(y),$$

where  $\chi_D$  is the characteristic function of  $D$ . Therefore, if  $f$  is any integrable function on  $G$ ,

$$\int_E f(xy) d\gamma(y) = \int_{xE} f(y) d\gamma(y),$$

and similarly,

$$\int_E f(yx) d\gamma(y) = \int_{Ex} f(y) d\gamma(y).$$

If  $f$  is any function on  $M$  and  $x \in G$ , define  $f_x$  by  $f_x(m) = f(xm)$ . If  $f$  is any nonnegative integrable function on  $M$ ,

$$\int f_x d\mu \leq \int f d\mu.$$

Let  $A_n$  be a sequence of measurable subsets of  $G$  such that  $0 < \gamma(A_n) < \infty$  for all  $n$ . We shall use the following conditions on the  $A_n$ .

- I.  $n < m$  implies  $A_n \subset A_m$ ;
- II.  $\lim_n \frac{\gamma(A_n \Delta xA_n)}{\gamma(A_n)} = \lim_n \frac{\gamma(A_n \Delta A_n x)}{\gamma(A_n)} = 0$ , for all  $x \in G$ , where  $\Delta$  denotes symmetric difference;
- III. for each  $k$  and  $n$ ,  $A_k A_n$  is measurable and  $\lim_n \frac{\gamma(A_k A_n \Delta A_n)}{\gamma(A_n)} = 0$ ; and
- IV. there exists  $K > 1$  such that  $\gamma(A_n^{-1} A_n) \leq K \gamma(A_n)$  for all  $n$ , where

$$A_n^{-1} A_n = \{ x \in G: yx \in A_n \text{ for some } y \in A_n \}.$$

Let  $B$  be a real or complex Banach space with norm  $\| \cdot \|$  and dual  $B^*$ . If  $\lambda \in B^*$  and  $b \in B$ ,  $\lambda(b)$  will be denoted by  $\lambda \cdot b$ . If  $(N, \nu)$  is a measure space and if  $1 \leq p < \infty$ ,  $L_p^B(N, \nu)$  will denote the set of equivalence classes of functions  $f: M \rightarrow B$  such that  $f$  is the limit in measure of simple functions and

$$\| f \|_p^B = \left( \int \| f \|^p d\nu \right)^{1/p} < \infty.$$

$L_p^B$  is a Banach space, and if  $(N, \nu)$  is  $\sigma$ -finite, its dual is  $L_q^{B^*}$ , where  $q = \infty$  if  $p = 1$ , and  $\frac{1}{q} + \frac{1}{p} = 1$  otherwise.

If  $g \in L_q^{B^*}$  and  $f \in L_p^B$ ,

$$(1) \int_{A_n} \int_M |g(m) \cdot f_x(m)| d\mu(m) d\gamma(x) \leq \int_{A_n} \int_M \|g(m)\| \|f_x(m)\| d\mu d\gamma \leq \|g\|_q^{B^*} \|f\|_p^B \gamma(A_n).$$

If we choose  $g$  so that  $\|g(m)\| > 0$  a. e., then (1) and Fubini's theorem imply that for almost every  $m$ ,

$$\int_{A_n} \|f_x(m)\| d\gamma(x)$$

exists. It follows that for almost every  $m$ ,  $f_{(\cdot)}(m) \in L_1^B(A_n, \gamma|_{A_n})$  and hence that

$$(2) g(m) \cdot \int_{A_n} f_x(m) d\gamma(x) = \int_{A_n} g(m) \cdot f_x(m) d\gamma(x) \quad [3, \text{III.6.10, III.2.22}].$$

Applying Fubini's theorem again and using (2), we obtain

$$\int_{A_n} \int_M g(m) \cdot f_x(m) d\mu d\gamma = \int_M g(m) \cdot \left( \int_{A_n} f_x(m) d\gamma \right) d\mu.$$

Hence, the map  $x \mapsto f_x$  from  $A_n$  to  $L_p^B$  is integrable in the sense of Pettis [5], and the integral is equal to  $\int_{A_n} f_x(m) d\gamma(x)$  almost everywhere. Define

$$\pi_n : L_p^B \rightarrow L_p^B \quad \text{by} \quad \pi_n(f) \equiv \frac{1}{\gamma(A_n)} \int_{A_n} f_x d\gamma(x).$$

Clearly,  $\pi_n$  is a continuous linear operator of norm less than or equal to one.

### THE ERGODIC THEOREMS

**THEOREM 1** (von Neumann's Mean Ergodic Theorem). — If the  $A_n$  satisfy II and if  $1 < p < \infty$  or if  $p = 1$  and  $\mu(M) < \infty$ , then there is  $\pi(f) \in L_p^B$  such that  $\lim_n \|\pi_n(f) - \pi(f)\|_p^B = 0$  and such that  $\pi(f_x) = \pi(f) = \pi(f)_x$

for all  $x \in G$ . If  $p = 1$ ,  $\int \pi(f) d\mu = \int f d\mu$ .  $\pi$  is the projection of  $L_p^B$  onto  $I_p^B$  along  $M_p^B$ , where  $I_p^B$  is the subspace of invariant functions and  $M_p^B$  is the closed subspace generated by  $\{f_x - f : f \in L_p^B, x \in G\}$ .

**THEOREM 2 (Wiener-Calderon Dominated Convergence Theorem).** — Suppose that the  $A_n$  satisfy I, III, and IV. If  $f$  is a nonnegative integrable function and if for  $\alpha > 0$ ,

$$E_\alpha = \left\{ m: \sup_{n=1, \dots, \infty} \pi_n(f)(m) \geq \alpha \right\},$$

then

$$\mu(E_\alpha) \leq \frac{K}{\alpha} \int f d\mu.$$

**THEOREM 3 (Birkhoff's Individual Ergodic Theorem).** — If the  $A_n$  satisfy I-IV and if  $f \in L_p^B$ , where  $1 \leq p < \infty$ , then  $\pi_n(f)$  converges almost everywhere. If  $1 < p < \infty$  or if  $p = 1$  and  $\mu(M) < \infty$ , then  $\pi_n(f)$  converges almost everywhere to the  $\pi(f)$  of Theorem 1.

### PROOF OF THEOREM 1

**LEMMA 1.** — If  $f \in L_p^B$  and  $1 < p < \infty$ , then  $C(f) = \langle \overline{\{f_x: x \in G\}} \rangle$  is weakly compact.

*Proof.* — If  $p > 1$ , let  $\frac{1}{q} + \frac{1}{p} = 1$ . If  $p = 1$ , let  $q = \infty$ .  $C(f)$  is weakly compact if for each sequence  $x_m \in G$  and each sequence  $\lambda_n \in L_q^{B*}$  such that  $\|\lambda_n\|_q^{B*} \leq 1$  for all  $n$ ,  $\lim_m \lim_n \lambda_n \cdot f_{x_m} = \lim_n \lim_m \lambda_n \cdot f_{x_m}$  whenever each limit exists [4, p. 159].

Let  $\varepsilon > 0$  and choose a simple function  $g \in L_p^B$  such that  $\|f - g\|_p^B < \varepsilon$ .  $|\lambda_n \cdot g_{x_m}| \leq \|g\|_p^B$  for all  $n$  and  $m$ , so that we may, by a diagonal process, choose a subsequence  $g_{x_{m_k}}$  such that for each  $n$ ,  $\lim_k \lambda_n \cdot g_{x_{m_k}} \rightarrow a_n$ . We may assume that  $a_n \rightarrow a$ . Similarly, we may choose a subsequence  $\lambda_{n_l} \cdot g_{x_{m_k}}$  such that  $\lim_l \lambda_{n_l} \cdot g_{x_{m_k}} = c_k$  for each  $k$ . Again, we may assume that  $c_k \rightarrow c$ . Since  $\|\lambda_n \cdot f_{x_m} - \lambda_n \cdot g_{x_m}\|_p^B \leq \|\lambda_n\|_q^{B*} \|f_{x_m} - g_{x_m}\|_p^B < \varepsilon$  for all  $m, n$ , it suffices to show that  $a = c$ .

Since the  $\lambda_n$  are uniformly bounded, the sequence  $\lambda_{n_l}$  has a weak star limit point,  $\lambda_0$ .  $g = \sum_{i=1}^s h_i b_i$ , where  $h_i \in L_p$  and  $b_i \in B$ . Since  $\|(h_i)_x\|_p \leq \|h_i\|_p$  for all  $x \in G$ ,  $\{(h_i)_x: x \in G\}$  is weakly relatively compact for each  $i$ . Hence,  $x_{m_k}$  has a subnet  $x_{m_k(\alpha)}$  such that  $(h_i)_{x_{m_k(\alpha)}}$  converges weakly to some  $h_{i0}$  for each  $i$ . Then,  $a = \lambda_0 \cdot \sum b_i h_{i0} = c$ . Q. E. D.

PROOF OF THEOREM 1. — Suppose that  $p > 1$ .  $\pi_n(f) \in C(f), \forall n$ . Since  $C(f)$  is weakly compact,  $\pi_n(f)$  has a weak cluster point  $\pi(f)$ . Given  $\varepsilon > 0$ , there are  $v \in L_p^B$  with  $\|v\|_p^B < \varepsilon$  and  $\alpha_i, f_{x_i}, i = 1, \dots, m$ , with

$$0 \leq \alpha_i \leq 1, \sum_{i=1}^m \alpha_i = 1,$$

such that

$$\pi(f) = \sum_{n=1}^{\infty} \alpha_i f_{x_i} + v.$$

Hence,

$$\pi(f) - f = \sum_{i=1}^m \alpha_i (f_{x_i} - f) + v.$$

For every  $x \in G$ ,

$$\|\pi_n(f)_x - \pi_n(f)\|_p^B \leq \frac{\gamma(A_n \Delta x A_n)}{\gamma(A_n)} \|f\|_p^B \rightarrow 0,$$

so that  $\pi(f)$  is invariant and  $\pi_n(\pi(f)) = \pi(f)$  for all  $n$ . Hence,

$$\pi(f) - \pi_n(f) = \sum_{i=1}^m \alpha_i \pi_n(f_{x_i} - f) + \pi_n(v).$$

Since  $\|\pi_n(v)\|_p^B < \varepsilon$  and since for all  $x \in G$ ,

$$\lim_n \|\pi_n(f_x) - \pi_n(f)\|_p^B \leq \lim_n \frac{\gamma(A_n \Delta A_n x)}{\gamma(A_n)} \|f\|_p^B = 0,$$

it follows that  $\lim_n \|\pi_n(f) - \pi(f)\|_p^B = 0$ ,

The case  $p = 1$  follows from the case  $p = 2$ , since the  $\pi_n$  are uniformly bounded on  $L_1^B$  and since, if  $\mu(M) < \infty$ ,  $L_2^B$  is  $B_1^B$ -dense in  $L_2^B$  and the  $\|\cdot\|_2^B$ -topology is stronger than the  $L_1^B$ -topology. Q. E. D.

### PROOF OF THEOREM 2

The key step of the proof is the Wiener-Calderon covering argument made in proving Lemma 1 below.

Let « \ » denote set theoretic difference.

LEMMA 1. — Let  $h$  be a real-valued  $\gamma$ -integrable function defined on  $A_k A_n$ . Suppose that

$$(1) \quad h(x) \geq 0 \quad \text{for all } x \in A_k A_n \setminus A_n;$$

$$(2) \quad \text{for all } x \in A_n, \text{ either } h(x) \geq 0 \text{ or } h(x) \geq -\frac{\alpha}{K} \text{ and}$$

$$\frac{1}{\gamma(A_n)} \int_{A_i} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

for some  $i = 1, \dots, K$ , where  $\alpha > 0$ .

$$\text{Then, } \int_{A_k A_n} h d\gamma \geq 0.$$

*Proof.* — Let  $\mathcal{M}_k$  be a maximal collection of disjoint subsets of the form  $A_k x$ , such that

$$x \in A_n \quad \text{and} \quad \frac{1}{\gamma(A_k)} \int_{A_k} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}.$$

Given  $\mathcal{M}_{i+1}$ , where  $k > i \geq 1$ , let  $\mathcal{M}_i$  be a maximal collection of sets of the form  $A_i x$  where  $x \in A_n$  and such that

$$\frac{1}{\gamma(A_i)} \int_{A_i} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

and the  $A_i x$  are mutually disjoint and are disjoint from every set in  $\mathcal{M}_j$

for  $i + 1 \leq j \leq k$ . Let  $\mathcal{M} = \bigcup_{i=1}^k \mathcal{M}_i$ .

Let  $\mathcal{N} = \{A_i^{-1} A_i x : A_i x \in \mathcal{M}\}$ . Let  $N = A_k A_n \setminus \cup \mathcal{N}$ . Suppose  $x \in N$  and  $k(x) < 0$ . Then  $x \in A_n$  and for some  $i = 1, \dots, k$ ,

$$\frac{1}{\gamma(A_i)} \int_{A_i x} h d\gamma \geq \alpha - \frac{\alpha}{K}, \quad A_i x \notin \mathcal{M}.$$

Therefore, there exists  $A_j x' \in \mathcal{M}$  with  $j \geq i$  such that  $A_i x \cap A_j x' \neq \emptyset$ . But then  $x \in A_i^{-1} A_j x' \subset A_j^{-1} A_j x' \subset \mathcal{N}$ . This contradicts  $x \in N$ . Hence,  $h \geq 0$  on  $N$  and

$$\begin{aligned} \int_{A_k A_n} h &\geq \int_N h + \int_{\cup \mathcal{M}} h - \frac{\alpha}{K} \gamma(\cup \mathcal{N} \cup \mathcal{M}) \geq \left(\alpha - \frac{\alpha}{K}\right) \sum_{\mathcal{M}} \gamma(A_i x) \\ &- \frac{\alpha}{K} \gamma(\cup \mathcal{N} \cup \mathcal{M}) = \alpha \sum \gamma(A_i x) - \frac{\alpha}{K} \gamma(\cup \mathcal{N}) \geq \frac{\alpha}{K} \sum \gamma(A_i^{-1} A_n x) - \frac{\alpha}{K} \gamma(\cup \mathcal{N}) = 0. \end{aligned}$$

Q. E. D.

LEMMA 2. — Suppose that  $f \in L_1(M)$  and that for each  $m \in M$ , either  $f(m) \geq 0$  or  $f(m) \geq -\frac{\alpha}{K}$  and for some  $i = 1, \dots, k$ ,

$$\frac{1}{\gamma(A_i)} \int_{A_i} f(y) d\gamma(y) \geq \alpha - \frac{\alpha}{K}.$$

Then,  $\int f d\mu \geq 0$ .

*Proof.* — Let  $n$  be a positive integer and for each  $m \in M$ , let  $h_m: A_k A_n \rightarrow \mathbb{R}$  be defined by

$$h_m(y) = \begin{cases} |f(y)|, & \text{if } y \in A_k A_n \setminus A_n; \\ f(y), & \text{if } y \in A_n. \end{cases}$$

Let  $M' = \{m \in M: |f(y)| \text{ is } \gamma\text{-integrable on } A_k A_n\}$ . By Fubini's Theorem,  $\mu(M \setminus M') = 0$ . For each  $m \in M'$ ,  $h_m$  satisfies the assumptions of Lemma 1. Hence,

$$0 \leq \int_{A_k A_n} h_m d\gamma = \int_{A_n} f(y) d\gamma(y) + \int_{A_k A_n \setminus A_n} |f(y)| d\gamma(y).$$

Applying Fubini's Theorem, we have

$$0 \leq \gamma(A_n) \int_M f d\mu + \gamma(A_k A_n \setminus A_n) \int_M |f| d\mu,$$

or

$$0 \leq \int_M f d\mu + \frac{\gamma(A_k A_n \setminus A_n)}{\gamma(A_n)} \|f\|.$$

Let  $n \rightarrow \infty$  and apply III. Q. E. D.

PROOF OF THEOREM 2. — It is sufficient to prove that

$$\mu(F \cap E_\alpha^k) \leq \frac{K}{\alpha} \int f d\mu,$$

where  $\mu(F) < \infty$  and

$$E_\alpha^k = \{m \in M: \max_{i=1, \dots, k} \pi_i(f)(m) \geq \alpha\}.$$

Let

$$h = f - \frac{\alpha}{K} \chi_{F \cap E_\alpha^k}.$$

Since  $h$  satisfies the assumptions of Lemma 3,  $\int h d\mu \geq 0$ . Q. E. D.



### PROOF OF THEOREM 3

Suppose at first that if  $p = 1$ ,  $\mu(M) < \infty$ .

Let  $\pi(f)$  be the limit defined by the Mean Convergence Theorem. Since  $\pi_n(f - \pi(f)) = \pi_n(f) - \pi(f)$ , one may suppose that  $\pi(f) = 0$ . We show that  $\pi_n(f) \rightarrow 0$  a. e.

Let  $\varepsilon > 0$ . Choose  $f^b$  bounded and such that  $\|f^b - f\|_p^B < \frac{\varepsilon}{3}$ . Choose  $k$  such that  $\|\pi_k(f^b) - \pi(f^b)\|_p^B < \frac{\varepsilon}{3}$ . Then,  $f = H + G$ , where

$$H = (f - f^b) + (\pi_k(f^b) - \pi(f^b)) + \pi(f^b) \quad \text{and} \quad G = f^b - \pi_k(f^b).$$

Clearly,  $\|H\|_p^B < \varepsilon$ .

$\pi_n(G)$  converges to zero almost everywhere since, for almost every  $m$ ,

$$\begin{aligned} \|\pi_n(G)m\| &= \frac{1}{\gamma(A_n)} \left\| \int_{A_n} \frac{1}{\gamma(A_k)} \int_{A_k} f_{xy}^b(m) d\gamma(x) - \int_{A_n} f_y^b(m) d\gamma(y) \right\| \\ &\leq \frac{1}{\gamma(A_k)} \int_{A_k} \frac{1}{\gamma(A_n)} \int_{A_n} (f_{xy}^b(m) - f_y^b(m)) d\gamma(y) \left\| d\gamma(x) \right. \\ &\leq \frac{1}{\gamma(A_k)} \int_{A_k} \frac{1}{\gamma(A_n)} \int_{A_n \Delta x A_n} \|f_y^b(m)\| d\gamma(y) d\gamma(x) \\ &\leq \frac{\gamma(A_n \Delta A_k A_n)}{\gamma(A_n)} \sup \|f^b\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, if  $\delta > 0$ ,

$$\mu \{ m: \overline{\lim}_n \|\pi_n(f)(m)\| > 3\delta \} \leq \mu \{ m: \lim_n \pi_n(\|H(m)\|) > 2\delta \}.$$

If  $p = 1$ , by Theorem 2 we obtain

$$\mu \{ m: \overline{\lim}_n \pi_n(\|H(m)\|) > 2\delta \} \leq \frac{K}{2\delta} \|H\|_1^B < \frac{K\varepsilon}{2\delta}.$$

Since  $\varepsilon$  is arbitrarily small, we obtain  $\lim_n \pi_n(f) = 0$  a. e.

If  $p > 1$ , let

$$H^\delta(m) = \begin{cases} \|H(m)\| & \text{if } \|H(m)\| \geq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

$\|H\| \leq H^\delta + \delta$ , so that  $\mu \{ m: \overline{\lim}_n \pi_n(\|H(m)\|) \geq 2\delta \} \leq \mu \{ m: \overline{\lim}_n \pi_n(H^\delta(m)) \geq \delta \}$ .

Furthermore,

$$H^\delta \in L_1 \quad \text{and} \quad \|H^\delta\|_1 \leq \frac{(\|H\|_1^B)^p}{\delta^{p-1}} < \frac{\varepsilon^p}{\delta^{p-1}}$$

so that by Theorem 2,

$$\mu \left\{ m: \overline{\lim}_n \pi_n(H^\delta(m)) \geq \delta \right\} \leq K \left( \frac{\varepsilon}{\delta} \right)^p \quad \text{and} \quad \lim_n \pi_n(f) = 0$$

almost everywhere.

We now remove the assumption that  $\mu(M) < \infty$  as in the case  $p = 1$  and prove that  $\pi_n(f)$  converges almost everywhere. We call a set  $E \in \mathcal{M}$  invariant if  $x \in E, \forall x \in G$ . It is possible to find a sequence of invariant sets,  $I_k$ , of finite measure and such that if  $I$  is invariant and measurable and if  $\mu(I \cap \bigcup_k I_k) = 0$ , then either  $\mu(I) = 0$  or  $\mu(I) \equiv \infty$ . By what we have already proved,  $\pi_n(f)$  converges on each  $I_k$  and hence on  $\bar{I} = \bigcup_k I_k$ .

Let  $\varepsilon > 0$  and let  $f^b$  be a bounded function such that  $\|f - f^b\|_1^B < \varepsilon$ .

$$\begin{aligned} \{ m \in M \setminus \bar{I}: \overline{\lim}_n \|\pi_n(f)(m)\| > 2\delta \} &\subset \{ m \in M: \overline{\lim}_n \pi_n(\|f - f^b\|) > \delta \} \\ &\cup \{ m \in M \setminus \bar{I}: \overline{\lim}_n \pi_n(\|f^b(m)\|) > \delta \}. \end{aligned}$$

The measure of the first set on the right is bounded by  $\frac{K\varepsilon}{\delta}$ . It is easy

to show that the second set is invariant. Since it is bounded by  $\frac{K}{\delta} \|f^b\|_1^B$  it must have measure zero. Q. E. D.

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