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Complete solution of Hadamard's problem for the scalar wave equation on Petrov type III space-times

by

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ABSTRACT. – We prove that there are no Petrov type III space-times on which the conformally invariant scalar wave equation or the non-self-adjoint scalar wave equation satisfies Huygens' principle. © Elsevier, Paris

RÉSUMÉ. – Nous prouvons qu'il n'existe aucun espace-temps de type III de Petrov sur lequel l'équation invariante conforme des ondes scalaires ou l'équation des ondes scalaires non-auto-adjoint satisfait au principe de Huygens. © Elsevier, Paris

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1. INTRODUCTION

This paper is devoted to the solution of Hadamard's problem on Petrov type III space-times, for the conformally invariant scalar wave equation

$$\Box u + \tfrac{1}{6} R u = 0, \tag{1}$$

and the non-self-adjoint scalar wave equation

$$\Box u + A^a \partial_a u + C u = 0. \tag{2}$$

In the above equations $\Box$ denotes the Laplace–Beltrami operator corresponding to the metric g_{ab} of the background space-time V_4, u the unknown function, R the Ricci scalar, A^a the components of a given contravariant vector field and C a given scalar function. The background manifold, metric tensor, vector field and scalar function are assumed to be C^∞. All considerations of this paper are entirely local.

The homogeneous equations (1) and (2) satisfy *Huygens' principle* in the sense of Hadamard [15] if $u(x)$ depends only on the Cauchy data in an arbitrarily small neighborhood of the intersection between the backward characteristic conoid $C^-(x)$ with the vertex at x and the initial surface S, for arbitrary Cauchy data on S, arbitrary S, and for all points x in the future of S. *Hadamard's problem* for (1) and (2) is that of determining all space-times for which Huygens' principle is valid. We recall that two equations (2) are said to be *equivalent* if and only if one may be transformed into the other by any combination of the following *trivial transformations*:

(a) a general coordinate transformation;

(b) multiplication of the equation by the function $\exp(-2\phi(x))$, which induces a *conformal transformation* of the metric

$$\tilde{g}_{ab} = e^{2\phi} g_{ab};$$

(c) substitution of λu for the unknown function u, where λ is a non-vanishing function on V_4.

We note that the Huygens' character of (2) is preserved by any trivial transformation. In the case of (1) the trivial transformations reduce to conformal transformations with $\lambda = e^{\phi}$.

Carminati and McLenaghan [4] have outlined a program for the solution of Hadamard's problem for the scalar wave equation, Weyls' neutrino equation and Maxwell's equations based on the conformally invariant Petrov classification of the Weyl conformal curvature tensor. This involves the consideration of five disjoint cases which exhausts all the possibilities for non-conformally flat space-times. Hadamard's problem for (1) and (2) has been completely solved for Petrov type N space-times by Carminati and McLenaghan [3,5] and McLenaghan and Walton [20]. Their results may be summarized as follows:

Any non-self-adjoint equation (2) on any Petrov type N background space-time satisfies Huygens' principle if and only if it is equivalent to the wave equation $\Box u = 0$ on an exact plane wave space-time with metric

$$ds^2 = 2dv\{du + [D(v)z^2 + \overline{D}(v)\bar{z}^2 + e(v)z\bar{z}]\, dv\} - 2dz\, d\bar{z}. \tag{3}$$

For Petrov type D space-times the following result was obtained by Carminati and McLenaghan [6], McLenaghan and Williams [21] and Wünsch [27]:

There exist no Petrov type D space-times on which the conformally invariant scalar wave equation (1) satisfies Huygens' principle.

In the present paper we complete this program for the conformally invariant scalar wave equation (1) on Petrov type III space-times by proving the following theorem:

THEOREM 1. – *There exists no Petrov type* III *space-times on which the conformally invariant scalar wave equation* (1) *satisfies Huygens' principle.*

The results on type N and type D space-times described above and Theorem 1 lend weight to the conjecture which states that every space-time on which the conformally invariant scalar wave equation satisfies Huygens' principle is conformally related to the plane wave space-time (3) or is conformally flat [3,5].

Hadamard's problem for the general non-self-adjoint equation (2) may now be solved with the help of Theorem 1 and the results of Anderson, McLenaghan and Sasse [1] where the following theorem is proved:

THEOREM 2. – *Any non-self-adjoint scalar wave equation* (2) *which satisfies Huygens' principle on any Petrov type* III *background space-time is equivalent to the conformally invariant scalar wave equation* (1).

Combining these two theorems we obtain

THEOREM 3. – *There exist no Petrov type* III *space-times on which the non-self-adjoint scalar wave equation satisfies Huygens' principle.*

The corresponding problem for the Weyl neutrino equation and Maxwell's equations is solved in [19].

The starting point of our proof of Theorem 1 is the paper by Carminati and McLenaghan [7], where the following results are obtained for Petrov type III space-times:

THEOREM 4. – *The validity of Huygens' principle for the conformally invariant scalar wave equation* (1), *on any Petrov type* III *space-time implies that the space-time is conformally related to one in which every repeated principal spinor field o_A of the Weyl spinor is recurrent, that is*

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}}, \tag{4}$$

where $I_{B\dot{B}}$ is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^C o^E \bar{o}^{\dot{E}} = 0, \tag{5}$$

$$R = 0, \qquad \Phi_{AB\dot{A}\dot{B}} o^A o^B = 0, \tag{6}$$

where ι^A is any spinor field satisfying $o_A \iota^A = 1$.

THEOREM 5. – *If any one of the following three conditions*

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^D \iota^E \bar{o}^{\dot{E}} = 0, \tag{7}$$

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B o^D o^E \bar{o}^{\dot{E}} = 0, \tag{8}$$

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^D o^E \bar{o}^{\dot{E}} = 0, \tag{9}$$

is satisfied, then there exist no Petrov type III *space-times on which the conformally invariant scalar wave equation* (1) *satisfies Huygens' principle.*

It is important to note that these earlier results solve Hadamard's problem under what have proved to be fairly strong assumptions (namely, that one of (7), (8) or (9) is satisfied). The purpose of the present paper is to make the analysis completely general by removing these assumptions. We follow the conventions of [7], and use the results established there to obtain (most of) the basic equations needed for the proof of Theorem 1.

In Section 2 we give the necessary conditions for the validity of Huygens' principle that will be used in this paper, and give a brief summary of their implications. From these necessary conditions, we derive the further side relations needed for our analysis in terms of the

Newman–Penrose scalars. The key to our proof is the six-index necessary condition obtained by Rinke and Wünsch [23] which was not used in [7]. In Section 3 we examine these side relations in the case $\Phi_{11}=0$ and show that they lead to a contradiction. The proof of Theorem 1 is completed in Section 4, where the case $\Phi_{11}\neq 0$ is treated.

It is worth mentioning that the tools of computer algebra are used throughout this paper. Initially we employ the Maple [22] package NPspinor [10,11] to extract dyad components of spinor versions of the necessary conditions, and then to manipulate the resulting expressions in Newman–Penrose form. In the case $\Phi_{11}=0$ we use the Gröbner basis package of the Maple system to explicitly determine solutions of systems of algebraic equations. Finally, for the case $\Phi_{11}\neq 0$ we use the GB package of Faugère [12] to examine the solvability of a somewhat larger system of algebraic equations.

2. FORMALISM AND BASIC EQUATIONS

The necessary conditions for the validity of Huygens' principle for (1) which we employ are given by

$$\text{(III)}\qquad S_{abk;}{}^{k}-\tfrac{1}{2}C^{k}{}_{ab}{}^{l}L_{kl}=0, \tag{10}$$

$$\begin{aligned}\text{(V)}\qquad TS\big(&3C^{k}{}_{ab}{}^{lm}{}_{;}C_{kcdl;m}+8C^{k}{}_{ab}{}^{l}{}_{;c}S_{kld}+40S_{ab}{}^{k}S_{cdk}\\&-8C^{k}{}_{ab}{}^{l}S_{klc;d}-24C^{k}{}_{ab}{}^{l}S_{cdk;l}+4C^{k}{}_{ab}{}^{l}C_{l}{}^{m}{}_{ck}L_{dm}\\&+12C^{k}{}_{ab}{}^{l}C^{m}{}_{cdl}L_{km}\big)=0,\end{aligned} \tag{11}$$

$$\begin{aligned}\text{(VII)}\qquad TS\big(&Q^{(1)}_{abcdef}-10Q^{(2)}_{abcdef}+4Q^{(3)}_{abcdef}+5Q^{(4)}_{abcdef}\\&+Q^{(5)}_{abcdef}\big)=0,\end{aligned} \tag{12}$$

where

$$\begin{aligned}Q^{(1)}_{abcdef}={}&3C^{k}{}_{ab}{}^{lm}{}_{;}{}_{c}C_{kdel;mf}+C^{k}{}_{ab}{}^{l}{}_{;cd}\big(10S_{kle;f}+6S_{efk;l}\big)\\&+64S_{abk;c}S_{de}{}^{k}{}_{;f}-C^{k}{}_{ab}{}^{l}\big(3C^{m}{}_{cdk;ef}L_{lm}+5C_{kcdl;me}L^{m}{}_{f}\\&+7C^{m}{}_{cdk;le}L_{mf}+13S_{klc;d}L_{ef}+12S_{cdk;l}L_{ef}\\&+71S_{cdk;e}L_{lf}\big),\end{aligned} \tag{13}$$

$$\begin{aligned}Q^{(2)}_{abcdef}={}&C^{k}{}_{ab}{}^{l}{}_{;c}\big(S_{kld;ef}+3S_{dek;lf}+2S_{abk;cd}S_{ef}{}^{k}-5S_{abk}S_{cd}L_{ef}\big)\\&-\tfrac{1}{2}C^{k}{}_{ab}{}^{l}{}_{;c}\big(2C^{m}{}_{kld;e}L_{mf}+3C^{m}{}_{dek;l}L_{mf}+S_{kld}L_{ef}\end{aligned}$$

$$+3C_{kde}{}^{m}{}_{;f}L_{lm}+15S_{dek}S_{lf}\big)-C^{k}{}_{ab}{}^{l}\big(C_{kcd}{}^{m}{}_{;e}L_{(lm;f)}$$
$$+S_{cdk}L_{(le;f)}-\tfrac{1}{12}R_{;c}C_{kdel;f}\big), \tag{14}$$

$$Q^{(3)}_{abcdef}=-C^{k}{}_{ab}{}^{l}\big(2C_{k}{}^{mn}{}_{c}C_{lnmd;ef}-10C^{mn}{}_{cd}C_{kefl;mn}+20C_{lcd}{}^{m}S_{kme;f}\big)$$
$$-5C_{k}{}^{mn}{}_{a}C_{lmnb}C^{k}{}_{cd}{}^{l}{}_{;ef}+C^{k}{}_{ab}{}^{l}(7C_{k}{}^{mn}{}_{c}C_{lmnd}L_{ef}$$
$$-10C_{kefl}C^{m}{}_{cd}{}^{n}L_{mn}), \tag{15}$$

$$Q^{(4)}_{abcdef}=-C^{k}{}_{ab}{}^{l}\big(2C_{k}{}^{mn}{}_{c;d}C_{lmnd;ef}+54C_{lcd}{}^{m}{}_{;e}S_{kmf}+74C_{lcd}{}^{m}{}_{;k}S_{efm}$$
$$-\tfrac{76}{3}C_{ckl}{}^{m}{}_{;d}S_{efm}-\tfrac{404}{3}S_{cdk}S_{efl}\big)+6C_{k}{}^{mn}{}_{a}C^{k}{}_{bc;d}C_{lefm;n}, \tag{16}$$

$$Q^{(5)}_{abcdef}=-C^{k}{}_{ab}{}^{l}C_{lcd}{}^{m}L_{km}L_{ef}+\tfrac{1}{6}C^{k}{}_{ab}{}^{l}C_{kcdl}\big(87L^{m}{}_{e}L_{mf}+19RL_{ef}\big), \tag{17}$$

where

$$C_{abcd}:=R_{abcd}-2g_{[a[d}L_{b]c]}, \tag{18}$$
$$S_{abc}:=L_{a[b;c]}, \tag{19}$$
$$L_{ab}:=-R_{ab}+\frac{R}{6}g_{ab}. \tag{20}$$

Here $A_a:=g_{ab}A^b$, R_{abcd} denotes the Riemann tensor, C_{abcd} the Weyl tensor, $R_{ab}:=g^{cd}R_{cabd}$, the Ricci tensor, and $R:=g^{ab}R_{ab}$ the Ricci scalar associated to the metric g_{ab}. The conditions (III), (V) and (VII) are necessarily conformally invariant. Spinor versions of conditions (III) and (V), and the conventions used for conversion from the original tensor form, are given in [7].

Mathisson [17], Hadamard [16], and Asgeirson [2] obtained condition (III) for (2) in the case g^{ij} constant. Condition (III) was obtained in the general case for (2) by Günther [14]. Condition V was obtained by McLenagham [18] in the case $R_{ab}=0$, and by Wünsch [26] for the general case. Condition (VII) was obtained by Rinke and Wünsch [23].

Petrov type III space-times are characterized by the existence of a spinor field o^A satisfying

$$\Psi_{ABCD}o^C o^D=0, \qquad \Psi_{ABCD}o^D\neq 0. \tag{21}$$

Such a spinor field is called a *repeated principal spinor* of the Weyl spinor and is determined by the latter up to an arbitrary variable complex factor.

Let ι^A be any spinor field satisfying

$$o_A \iota^A = 1. \tag{22}$$

The ordered set o_A, ι_A, called a dyad, defines a basis for the 1-spinor fields on V_4.

It was shown in [7] that the necessary conditions (III) and (V) imply that there exists a dyad $\{o_A, \iota_A\}$ and a conformal transformation ϕ such that

$$\kappa = \sigma = \rho = \tau = \varepsilon = 0, \tag{23}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \qquad \Psi_3 = -1, \tag{24}$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0, \tag{25}$$

$$\mathrm{D}\alpha = \mathrm{D}\beta = \mathrm{D}\pi = 0, \tag{26}$$

$$\delta\Phi_{11} = \mathrm{D}\Phi_{11} = 0. \tag{27}$$

We notice that the expressions (24) determine the tetrad uniquely. On the other hand, conditions (23) are invariant under any conformal transformation satisfying

$$\mathrm{D}\phi = 0, \qquad \delta\phi = 0, \tag{28}$$

which implies that we still have some conformal freedom. Under a conformal transformation we have [25]:

$$\widetilde{\Phi}_{11} = e^{-2\phi}\Phi_{11}. \tag{29}$$

Thus, we can choose ϕ such that

$$\Phi_{11} = c, \tag{30}$$

where c is a constant. The conditions (28) are satisfied in view of (27).

Let us now derive some side relations that follow from the previously obtained Eqs. (23)–(30) and the necessary conditions (10)–(12); these will be required in the analysis of the following sections. We may assume that $\alpha\beta\pi \neq 0$, since the case in which this is not true was already considered in [7]. By contracting condition (III) with $\iota^A o^B \bar{\iota}^{\dot{A}\dot{B}}$ we get

$$\delta\beta = -\beta(\overline{\alpha} + \beta). \tag{31}$$

From the Bianchi identities, using the above conditions, we obtain

$$D\Phi_{12} = 2\overline{\pi}\,\Phi_{11}, \tag{32}$$
$$D\Phi_{22} = -2(\beta + \overline{\beta}) + 2\Phi_{21}\,\overline{\pi} + 2\Phi_{12}\pi, \tag{33}$$
$$\delta\Phi_{12} = 2\overline{\alpha} + 4\overline{\pi} + 2\overline{\lambda}\,\Phi_{11} - 2\overline{\alpha}\,\Phi_{12}, \tag{34}$$
$$\overline{\delta}\,\Phi_{12} = -2\beta + 2\overline{\mu}\,\Phi_{11} - 2\overline{\beta}\,\Phi_{12}. \tag{35}$$

From the Ricci identities we get the following relevant Pfaffians:

$$D\gamma = \overline{\pi}\,\alpha + \beta\pi + \Phi_{11}, \tag{36}$$
$$\mathrm{D}\lambda = (1/2)\delta\alpha - (11/2)\,\overline{\beta}\,\alpha + \pi^2 - 2\pi\alpha - 11\pi\,\overline{\beta} - (3/2)\alpha^2, \tag{37}$$
$$\delta\,\overline{\pi} = D\overline{\lambda} - \overline{\pi}^{\,2} - \overline{\pi}\overline{\alpha} + \overline{\pi}\,\beta, \tag{38}$$
$$D\overline{\nu} = \Delta\overline{\pi} + \overline{\pi}\overline{\mu} + \overline{\lambda}\pi + \overline{\pi}\overline{\gamma} - \overline{\pi}\,\gamma - 1 + \Phi_{12}, \tag{39}$$
$$\delta\alpha = \overline{\delta}\,\beta + \alpha\,\overline{\alpha} + \beta\,\overline{\beta} - 2\beta\alpha + \Phi_{11}, \tag{40}$$
$$\delta\pi = D\mu - \overline{\pi}\,\pi + \pi\,\overline{\alpha} - \beta\pi. \tag{41}$$

We can obtain useful integrability conditions for the above Pfaffians, by using Newman–Penrose (NP) commutation relations. By substituting them in the commutator expression $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$, we get

$$\delta\,\overline{\beta} = -2\Phi_{11} - \overline{\beta}\,\overline{\alpha} - 4\overline{\beta}\,\overline{\pi} - 2D\,\overline{\mu} - \beta\,\overline{\beta} + 2\overline{\pi}\,\pi. \tag{42}$$

By contracting condition (V) with $\iota^{ABCD}\bar{\iota}^{\dot{A}\dot{B}}\bar{o}^{\dot{C}\dot{D}}$, we find

$$20\,\overline{\beta}\,\pi + 12\,\overline{\beta}\,\alpha + 6\pi\alpha + 3\alpha^2 + \overline{\delta}\,\alpha + 2\overline{\delta}\,\pi + \overline{\delta\beta} + \overline{\beta}^{\,2} = 0. \tag{43}$$

By substituting (30) into this equation we get

$$\delta(2\overline{\pi} + \overline{\alpha}) = -20\overline{\pi}\,\beta - 11\beta\,\overline{\alpha} - 6\overline{\pi}\overline{\alpha} - 3\overline{\alpha}^{\,2}. \tag{44}$$

From (40), (41) and (30) we then obtain:

$$\delta(2\pi + \alpha) = 2\pi\,\overline{\alpha} + \alpha\,\overline{\alpha} - 6\beta\pi - 3\beta\alpha - \Phi_{11}. \tag{45}$$

By contracting condition (V) with $\iota^{ABC}o^{D}\bar{\iota}^{\dot{A}\dot{B}\dot{C}}\bar{o}^{\dot{D}}$, we find

$$\begin{aligned}&-6\delta\pi - 15\alpha\,\overline{\pi} - 10\alpha\,\overline{\alpha} - 68\pi\,\overline{\pi} - 15\pi\,\overline{\alpha} - 3\delta\alpha - 126\,\overline{\beta}\,\beta\\ &+ 5\mathrm{D}\,\overline{\gamma} + 10\mathrm{D}\,\overline{\mu} - 24\,\overline{\beta}\,\overline{\alpha} - 3\,\overline{\delta}\,\overline{\alpha} - 6\,\overline{\delta}\,\overline{\pi} - 15\delta\,\overline{\beta} - 3\,\overline{\beta}\,\overline{\pi}\\ &+ 5\mathrm{D}\gamma + 10\mathrm{D}\mu - 15\,\overline{\delta}\,\beta - 24\beta\alpha - 3\beta\pi - 4\Phi_{11} = 0.\end{aligned} \tag{46}$$

Using (36), (41) and the complex conjugate of (42), we get

$$9\Phi_{11}+10\overline{\beta}\,\overline{\pi}+5[\mathrm{D}\mu+\mathrm{D}\overline{\mu}]-2\overline{\pi}\,\alpha-12\beta\,\overline{\beta}-2\alpha\,\overline{\alpha}$$
$$-16\overline{\pi}\,\pi+10\beta\pi-2\pi\,\overline{\alpha}=0. \tag{47}$$

On the other hand, the NP commutator $[\overline{\delta},\delta](\alpha+2\pi)=(\alpha-\overline{\beta})\delta(\alpha+2\pi)+(-\overline{\alpha}+\beta)\overline{\delta}(\alpha+2\pi)$, yields the following expression

$$2\pi\beta\,\overline{\beta}+22\pi\,\overline{\beta}\,\overline{\alpha}+43\pi\,\overline{\beta}\,\overline{\pi}-22\overline{\pi}\,\pi^2+\overline{\beta}\mathrm{D}\mu+22\pi\mathrm{D}\overline{\mu}$$
$$+12\alpha\,\overline{\beta}\,\overline{\alpha}+6\overline{\beta}\,\Phi_{11}-12\alpha\,\overline{\pi}\,\pi+11\alpha\Phi_{11}+18\pi\Phi_{11}$$
$$+24\overline{\pi}\,\overline{\beta}\alpha+12\alpha\mathrm{D}\overline{\mu}=0. \tag{48}$$

Eliminating $\mathrm{D}\overline{\mu}$ between (47) and (48), and solving for $\mathrm{D}\mu$, we get

$$\mathrm{D}\mu=-\tfrac{1}{5}\big(108\pi\Phi_{11}-44\pi^2\overline{\alpha}-24\overline{\pi}\,\alpha^2-68\pi\alpha\,\overline{\alpha}$$
$$+144\alpha\beta\,\overline{\beta}+53\alpha\Phi_{11}-274\pi\beta\,\overline{\beta}+120\pi\beta\alpha-24\alpha^2\overline{\alpha}$$
$$-242\overline{\pi}\,\pi^2+220\beta\pi^2-176\alpha\,\overline{\pi}\,p-60\alpha\,\overline{\beta}\,\overline{\alpha}-30\overline{\beta}\,\Phi_{11}$$
$$+5\pi\,\overline{\pi}\,\overline{\beta}-110\overline{\alpha}\,\pi\,\overline{\beta}\big)/\big(-\overline{\beta}+12\alpha+22\pi\big), \tag{49}$$

where we have assumed that the denominator of the expression above, given by

$$d_1:=-\overline{\beta}+12\alpha+22\pi, \tag{50}$$

is non-zero. The case $d_1=0$ will be considered later.

Substituting expression (49) for $\mathrm{D}\mu$ into (47) we obtain

$$\mathrm{D}\overline{\mu}=-\tfrac{1}{5}\big(90\pi\Phi_{11}+12\beta\,\overline{\beta}^2-10\overline{\beta}^2\overline{\pi}+55\alpha\Phi_{11}+122\alpha\,\overline{\beta}\,\overline{\pi}$$
$$-110\overline{\pi}\,\pi^2-60\alpha\,\overline{\pi}\,\pi+62\alpha\,\overline{\beta}\,\overline{\alpha}+21\overline{\beta}\,\Phi_{11}+231\pi\,\overline{\beta}\,\overline{\pi}$$
$$+112\overline{\alpha}\,\pi\,\overline{\beta}\big)/\big(-\overline{\beta}+12\alpha+22\pi\big). \tag{51}$$

One side relation can now be obtained by subtracting the complex conjugate of (49) from (51). We obtain:

$$S_1:=\tfrac{1}{5}\big(720\overline{\alpha}^2\overline{\beta}\alpha+2904\pi^2\overline{\pi}^2-12\beta^2\overline{\beta}^2+288\overline{\alpha}^2\alpha^2+528\pi^2\overline{\alpha}^2$$
$$+528\overline{\pi}^2\alpha^2+2420\overline{\alpha}\,\pi\,\overline{\beta}\,\overline{\pi}+3056\overline{\alpha}\,\pi\beta\,\overline{\beta}+2888\overline{\alpha}\,\alpha\,\overline{\pi}\,\pi$$
$$+1320\overline{\alpha}\,\pi\beta\alpha+1320\overline{\pi}\,\beta\alpha^2+1606\overline{\alpha}\,\beta\,\overline{\beta}\,\alpha+5802\overline{\pi}\,\pi\beta\,\overline{\beta}$$
$$+1320\overline{\pi}\,\overline{\alpha}\,\overline{\beta}\,\alpha+2420\overline{\pi}\,\beta\alpha\pi+1320\pi\,\overline{\beta}\,\overline{\alpha}^2+3056\overline{\pi}\,\beta\alpha\,\overline{\beta}$$
$$+2552\pi\,\overline{\pi}^2\alpha+305\beta\Phi_{11}\alpha+570\beta\Phi_{11}\pi-51\beta\Phi_{11}\overline{\beta}$$
$$+24\overline{\alpha}\,\Phi_{11}\alpha-86\overline{\alpha}\,\Phi_{11}\pi+305\overline{\alpha}\,\Phi_{11}\overline{\beta}+816\pi\alpha\,\overline{\alpha}^2$$
$$+2552\overline{\alpha}\,\overline{\pi}\,\pi^2+816\overline{\alpha}\,\overline{\pi}\,\alpha^2-86\overline{\pi}\,\alpha\Phi_{11}-396\overline{\pi}\,\pi\Phi_{11}$$
$$+720\beta\,\overline{\alpha}\,\alpha^2+570\overline{\pi}\,\overline{\beta}\,\Phi_{11}\big)$$
$$/\big((-12\alpha-22\pi+\overline{\beta})(12\overline{\alpha}+22\overline{\pi}-\beta)\big)=0. \tag{52}$$

We notice that (49) and (51) have the same denominator. Thus, if we keep these expressions for $D\mu$ and $D\overline{\mu}$, the Pfaffians $\delta\overline{\beta}$, $\delta\alpha$, $\delta\pi$, given by (42), (38) and (41), respectively, and their complex conjugates, also have the same denominator. This procedure is crucial to keep the expressions to be obtained from the integrability conditions within a reasonable size. Except for $\overline{\delta}\alpha$, all Pfaffians involving δ, $\overline{\delta}$, applied to α, β, π are explicitly determined.

The following expression for $\overline{\delta}\alpha$ can be obtained from the NP commutator $[\overline{\delta},\delta]\overline{\beta} = (\overline{\mu}-\mu)D\overline{\beta} + (\alpha-\overline{\beta})\delta\overline{\beta} + (\beta-\overline{\alpha})\overline{\delta}\overline{\beta}$:

$$\begin{aligned}\overline{\delta}\alpha = \big(&2\pi D\overline{\mu} - 2\overline{\delta}(D\overline{\mu}) - 3\overline{\pi}\alpha^2 - 8\overline{\beta}D\overline{\mu} - 11\alpha\overline{\beta}\overline{\pi} - 2\overline{\pi}\pi^2 \\ &- 4\alpha\overline{\pi}\pi - 4\overline{\beta}\Phi_{11} - 14\overline{\pi}\,\overline{\beta}\,\overline{\pi}\big)/\overline{\pi}. \qquad (53)\end{aligned}$$

By substituting (49) and (51) into this equation we get:

$$\begin{aligned}\overline{\delta}\alpha := -\tfrac{1}{5}\big(&19519\Phi_{11}\overline{\beta}^2\alpha + 8570\Phi_{11}\overline{\beta}\alpha^2 + 1950\Phi_{11}\pi\alpha^2 \\ &+ 35850\Phi_{11}\pi\overline{\beta}^2 + 2900\Phi_{11}\pi^2\overline{\beta} - 3180\pi^2\overline{\beta}^2\overline{\pi} - 210\alpha^3\overline{\beta}\overline{\pi} \\ &+ 150\alpha^3\overline{\beta}\overline{\alpha} - 1307\alpha^2\overline{\beta}^2\overline{\pi} - 180\alpha^2\beta\overline{\beta}^2 + 628\alpha^2\overline{\beta}^2\overline{\alpha} \\ &- 1280\pi^2\overline{\beta}^2\overline{\alpha} + 1950\overline{\alpha}\,\overline{\beta}^3\alpha + 4668\alpha\overline{\beta}^3\beta + 3520\overline{\beta}^3\overline{\alpha}\pi \\ &+ 8160\beta\pi\overline{\beta}^3 + 330\overline{\beta}^3\overline{\pi}\pi + 975\Phi_{11}\alpha^3 - 860\overline{\beta}^3\Phi_{11} \\ &+ 17950\Phi_{11}\overline{\beta}\alpha\pi - 420\alpha^2\overline{\beta}\overline{\pi}\pi + 300\alpha^2\overline{\beta}\pi\overline{\alpha} - 4116\alpha\overline{\beta}^2\overline{\pi}\pi \\ &+ 588\alpha\pi\overline{\alpha}\overline{\beta}^2 + 175\overline{\beta}^3\alpha\overline{\pi}\big) \\ /\big(&\overline{\beta}^2\overline{\pi} - 14\alpha\overline{\beta}\overline{\pi} + 10\overline{\beta}\alpha\overline{\alpha} + 20\overline{\beta}\pi\overline{\alpha} + 65\alpha\Phi_{11} \\ &- 12\beta\overline{\beta}^2 + 130\pi\Phi_{11} - 31\overline{\beta}\Phi_{11} - 28\overline{\beta}\overline{\pi}\pi - 6\overline{\beta}^2\overline{\alpha}\big), \qquad (54)\end{aligned}$$

where, for now, we assume that the denominator in the expression above:

$$\begin{aligned}d_2 := \ &\overline{\beta}^2\overline{\pi} - 14\alpha\overline{\beta}\overline{\pi} + 10\overline{\beta}\alpha\overline{\alpha} + 20\overline{\beta}\pi\overline{\alpha} + 65\alpha\Phi_{11} - 12\beta\overline{\beta}^2 \\ &+ 130\pi\Phi_{11} - 31\overline{\beta}\Phi_{11} - 28\overline{\beta}\overline{\pi}\pi - 6\overline{\beta}^2\overline{\alpha}, \qquad (55)\end{aligned}$$

is non-zero.

Contracting condition (VII) with $\iota^{ABCDE}o^F\overline{o}^{\dot{B}\dot{C}\dot{D}}\overline{\iota}^{\dot{A}\dot{E}\dot{F}}$ yields

$$\begin{aligned}\mathrm{VII}_{13} := \ &648\overline{\beta}^2\alpha\overline{\pi} - 165\pi\overline{\pi}\,\overline{\delta}\beta - 40\pi D\Phi_{21} + 1557\delta\overline{\beta}\alpha\pi \\ &+ 270\alpha\overline{\beta}\delta\pi + 144\delta\alpha\alpha\overline{\beta} + 378\alpha\overline{\beta}D\overline{\gamma} + 2058\alpha^2\overline{\pi}\pi \\ &+ 570D\mu\,\overline{\delta}\beta + 630D\mu\overline{\beta}^2 + 21D\lambda\overline{\delta}\overline{\alpha} - 201D\overline{\gamma}\,\overline{\delta}\pi \\ &- 36\pi^2\overline{\delta}\overline{\alpha} - 954\overline{\delta}\beta\overline{\beta}^2 - 864\overline{\delta}\beta\,\overline{\delta}\beta - 864\beta\overline{\beta}^3 \\ &- 132\overline{\delta}\alpha\overline{\delta}\overline{\alpha} + 1116\alpha^3\overline{\alpha} - 366\alpha^2\overline{\delta}\overline{\alpha} - 900\alpha\overline{\beta}\delta\overline{\beta} \\ &- 20\alpha D\Phi_{21} + 1454\Phi_{11}\pi\overline{\beta} + 771\overline{\delta}\overline{\pi}\pi\overline{\beta} + 423\overline{\delta}\alpha\overline{\alpha}\pi\end{aligned}$$

$$
\begin{aligned}
&+939\overline{\alpha}\alpha\overline{\delta}\pi-1038\mathrm{D}\overline{\mu}\alpha\pi-90\lambda\mathrm{D}(\mathrm{D}\overline{\mu})+45\pi\overline{\delta}(\delta\overline{\beta})\\
&-15\pi\overline{\delta}(\mathrm{D}\overline{\gamma})-30\pi\overline{\delta}(\mathrm{D}\overline{\mu})+84\alpha\overline{\beta}\overline{\delta}\overline{\pi}-774\overline{\delta}\alpha\overline{\beta}\overline{\pi}\\
&-570\delta\overline{\beta}\,\overline{\alpha}\pi+180\overline{\delta}\alpha\beta\overline{\beta}+135\mathrm{D}(\delta\overline{\beta})\lambda-45\lambda\mathrm{D}(\mathrm{D}\overline{\gamma})\\
&-186\overline{\delta}\,\overline{\beta}\alpha\overline{\pi}+570\alpha\overline{\beta}\mathrm{D}\overline{\mu}+30\mathrm{D}\lambda\alpha\overline{\pi}-39\mathrm{D}\lambda\overline{\alpha}\pi\\
&-78\mathrm{D}\lambda\overline{\pi}\pi-744\overline{\alpha}\alpha\delta\overline{\beta}-924\overline{\alpha}\alpha^{2}\overline{\beta}-387\pi^{2}\overline{\alpha}\,\overline{\beta}\\
&-21\mathrm{D}\lambda\overline{\alpha}\alpha+351\mathrm{D}\overline{\gamma}\pi\overline{\beta}-1773\overline{\beta}\,\overline{\pi}\,\overline{\delta}\pi+216\overline{\alpha}\pi\overline{\delta}\pi\\
&+870\mathrm{D}\mu\alpha\overline{\beta}-198\overline{\delta}\,\overline{\alpha}\alpha\overline{\beta}+726\overline{\delta}\alpha\overline{\alpha}\alpha+378\beta\alpha\delta\overline{\beta}\\
&+828\alpha^{2}\overline{\beta}\beta-1674\alpha\overline{\beta}\,\overline{\delta}\beta-594\beta\overline{\beta}\delta\overline{\beta}+636\pi^{2}\overline{\alpha}\alpha\\
&+2289\alpha^{2}\pi\overline{\alpha}+36\mathrm{D}\lambda\overline{\alpha}\,\overline{\beta}-660\alpha\pi\overline{\delta}\,\overline{\alpha}+1773\overline{\beta}^{2}\overline{\pi}\pi\\
&-555\mathrm{D}\overline{\gamma}\alpha\pi+2187\beta\pi\overline{\beta}^{2}+2619\beta\pi\delta\overline{\beta}-888\overline{\alpha}\,\overline{\beta}^{2}\alpha\\
&-126\overline{\delta}\alpha\overline{\alpha}\,\overline{\beta}+3681\beta\pi\alpha\overline{\beta}-2055\alpha\pi\overline{\alpha}\,\overline{\beta}-6951\overline{\beta}\,\overline{\pi}\alpha\pi\\
&-1476\alpha\overline{\beta}^{2}\beta+150\overline{\delta}\alpha\overline{\pi}\pi-63\mathrm{D}\lambda\overline{\beta}\,\overline{\pi}-30\overline{\beta}^{2}\overline{\alpha}\pi\\
&+934\Phi_{11}\alpha\pi-465\pi^{2}\overline{\beta}\,\overline{\pi}-1608\alpha\pi\overline{\delta}\,\overline{\pi}+63\mathrm{D}\lambda\beta\overline{\beta}\\
&+20\overline{\beta}\mathrm{D}\Phi_{21}+567\pi^{2}\beta\overline{\beta}-426\overline{\delta}\,\overline{\pi}\,\overline{\delta}\pi+18\mathrm{D}\lambda\mathrm{D}\overline{\mu}\\
&-39\pi^{2}\mathrm{D}\overline{\gamma}+39\pi^{3}\overline{\alpha}-78\pi^{2}\mathrm{D}\overline{\mu}+78\pi^{3}\overline{\pi}\\
&+9\mathrm{D}\lambda\mathrm{D}\overline{\gamma}+324\delta\alpha\delta\overline{\beta}+324\delta\alpha\overline{\beta}^{2}-189\overline{\delta}\,\overline{\alpha}\,\overline{\delta}\pi\\
&+276\delta\overline{\beta}\mathrm{D}\gamma+306\overline{\beta}^{2}\mathrm{D}\gamma+432\alpha^{3}\overline{\pi}+1080\alpha^{2}\delta\overline{\beta}\\
&-144\overline{\delta}\alpha\overline{\delta}\,\overline{\pi}+360\overline{\delta}\alpha\delta\overline{\beta}-372\alpha^{2}\overline{\delta}\,\overline{\pi}+630\delta\overline{\beta}\delta\pi\\
&+630\overline{\beta}^{2}\delta\pi+528\mathrm{D}\overline{\mu}\pi\overline{\beta}+408\overline{\pi}\pi\overline{\delta}\pi+246\alpha\overline{\pi}\,\overline{\delta}\pi\\
&+639\overline{\delta}\,\overline{\alpha}\pi\overline{\beta}+156\alpha\overline{\beta}\mathrm{D}\gamma-2592\alpha^{2}\overline{\beta}\,\overline{\pi}+204\overline{\delta}\alpha\alpha\overline{\pi}\\
&+117\beta\overline{\beta}\,\overline{\delta}\pi-657\delta\overline{\beta}\pi\overline{\beta}+1128\pi^{2}\alpha\overline{\pi}-108\overline{\delta}\alpha\mathrm{D}\overline{\gamma}\\
&-240\overline{\delta}\alpha\mathrm{D}\overline{\mu}-720\alpha^{2}\mathrm{D}\overline{\mu}-324\alpha^{2}\mathrm{D}\overline{\gamma}+567\delta\overline{\beta}\,\overline{\delta}\pi\\
&-30\mathrm{D}\lambda\overline{\delta}\,\overline{\pi}-27\mathrm{D}\lambda\delta\overline{\beta}+117\pi^{2}\delta\overline{\beta}-378\mathrm{D}\overline{\mu}\,\overline{\delta}\pi\\
&-102\Phi_{11}\mathrm{D}\lambda+182\Phi_{11}\pi^{2}+366\Phi_{11}\overline{\delta}\pi-600\Phi_{11}\delta\overline{\beta}\\
&-600\Phi_{11}\overline{\beta}^{2}+1008\Phi_{11}\alpha^{2}+336\Phi_{11}\overline{\delta}\alpha+1704\Phi_{11}\alpha\overline{\beta}\\
&+30\pi\mathrm{D}(\overline{\delta}\,\overline{\gamma})-90\pi\delta(\delta\overline{\beta})+60\pi\mathrm{D}(\overline{\delta}\,\overline{\mu})-90\mathrm{D}(\delta\overline{\beta})\mu\\
&-90\pi\mathrm{D}(\Delta\overline{\beta})-660\overline{\beta}\pi\mathrm{D}\gamma-270\alpha\overline{\delta}(\delta\overline{\beta})-30\alpha\overline{\delta}(\overline{\delta}\,\overline{\alpha})\\
&-60\alpha\overline{\delta}(\overline{\delta}\,\overline{\pi})+30\overline{\beta}\,\overline{\delta}(\mathrm{D}\gamma)+60\overline{\beta}\,\overline{\delta}(\mathrm{D}\mu)+90\alpha\overline{\delta}(\mathrm{D}\overline{\gamma})\\
&+180\alpha\overline{\delta}(\mathrm{D}\overline{\mu})-180\overline{\beta}\,\overline{\delta}(\delta\pi)-90\overline{\beta}\,\overline{\delta}(\delta\alpha)+45\pi\overline{\delta}(\overline{\delta}\,\overline{\alpha})\\
&+90\pi\overline{\delta}(\overline{\delta}\,\overline{\pi})+30\overline{\beta}\mathrm{D}(\overline{\delta}\gamma)+60\overline{\beta}\mathrm{D}(\overline{\delta}\mu)+15\lambda\mathrm{D}(\overline{\delta}\,\overline{\alpha})
\end{aligned}
$$

$$
\begin{aligned}
&+30\lambda\mathrm{D}(\overline{\delta}\,\overline{\pi})+60\overline{\alpha}\,\overline{\delta}(\overline{\delta}\alpha)-90\overline{\beta}\,\overline{\delta}(\overline{\delta}\beta)-90\overline{\beta}\mathrm{D}(\Delta\alpha)\\
&-180\overline{\beta}\mathrm{D}(\Delta\pi)+180\beta\overline{\delta}(\delta\overline{\beta})+120\overline{\alpha}\,\overline{\delta}(\overline{\delta}\pi)+60\gamma\mathrm{D}(\overline{\delta}\alpha)\\
&+120\gamma\mathrm{D}(\overline{\delta}\pi)-180\overline{\alpha}\,\overline{\beta}\,\overline{\delta}\pi-60\overline{\beta}\mathrm{D}\mu\pi-315\overline{\beta}\,\overline{\delta}\beta\pi\\
&+180\overline{\beta}\delta\pi\pi=0. \qquad (56)
\end{aligned}
$$

The second-order terms $\mathrm{D}(\overline{\delta}\,\overline{\gamma})$, $\mathrm{D}(\overline{\delta}\gamma)$, $\mathrm{D}(\overline{\delta}\,\overline{\mu})$, $\mathrm{D}(\Delta\overline{\beta})$, $\mathrm{D}(\Delta\alpha)$, $\mathrm{D}(\Delta\pi)$, can be expressed in terms of known Pfaffians and $\overline{\delta}\alpha$, by using the NP commutation relations involving each pair of operators. After the substitutions we obtain

$$
\begin{aligned}
\mathrm{VII}_{13} := \tfrac{2}{5}\big(&760\overline{\beta}^2\overline{\pi}\pi\overline{\delta}\alpha+9600\overline{\beta}\Phi_{11}\overline{\delta}\alpha\pi+25040\overline{\beta}\,\overline{\delta}\alpha\overline{\alpha}\pi^2\\
&-23440\overline{\beta}\,\overline{\pi}\pi^2\overline{\delta}\alpha+2400\beta\pi\overline{\delta}\alpha\overline{\beta}^2-240\overline{\beta}^2\overline{\delta}\alpha\overline{\alpha}\pi\\
&+2880\alpha\overline{\beta}^2\beta\overline{\delta}\alpha+8600\alpha\Phi_{11}\overline{\delta}\alpha\pi+360\alpha\overline{\beta}^2\overline{\pi}\,\overline{\delta}\alpha\\
&+600\alpha\overline{\delta}\alpha\overline{\alpha}\,\overline{\beta}^2+24880\alpha\overline{\beta}\,\overline{\delta}\alpha\overline{\alpha}\pi-22160\alpha\overline{\beta}\,\overline{\pi}\pi\overline{\delta}\alpha\\
&+8940\alpha\Phi_{11}\overline{\delta}\alpha\overline{\beta}-5280\alpha^2\overline{\beta}\,\overline{\pi}\,\overline{\delta}\alpha+6240\overline{\beta}\,\overline{\alpha}\alpha^2\overline{\delta}\alpha\\
&-9885300\alpha\Phi_{11}\pi^2\overline{\beta}+113040\alpha\beta\overline{\beta}^4-1200\Phi_{11}\overline{\delta}\alpha\alpha^2\\
&+48400\alpha\pi^3\Phi_{11}+40200\Phi_{11}\pi\alpha^3+22400\Phi_{11}\overline{\delta}\alpha\pi^2\\
&-720\overline{\beta}^3\beta\overline{\delta}\alpha-300\overline{\delta}\alpha\overline{\alpha}\,\overline{\beta}^3+142635\alpha\overline{\beta}^3\Phi_{11}\\
&-910512\alpha^2\overline{\beta}^3\beta-351912\overline{\beta}^2\alpha^3\overline{\pi}-5085800\overline{\beta}\pi^3\Phi_{11}\\
&-1467420\Phi_{11}\overline{\beta}\alpha^3-1760\Phi_{11}\overline{\delta}\alpha\overline{\beta}^2-3600\Phi_{11}\alpha^4\\
&+35640\overline{\beta}^3\alpha^2\overline{\pi}-360060\overline{\alpha}\,\overline{\beta}^3\alpha^2+120000\Phi_{11}\pi^2\alpha^2\\
&-633396\Phi_{11}\overline{\beta}^2\alpha^2+8640\alpha^3\beta\overline{\beta}^2-15840\alpha^4\overline{\beta}\,\overline{\pi}\\
&-155352\alpha^3\overline{\beta}^2\overline{\alpha}+18720\alpha^4\overline{\beta}\,\overline{\alpha}+176880\pi^2\overline{\beta}^3\overline{\pi}\\
&+205920\beta\pi\overline{\beta}^4+85800\overline{\beta}^4\overline{\alpha}\pi+47100\overline{\alpha}\,\overline{\beta}^4\alpha\\
&+227170\Phi_{11}\pi\overline{\beta}^3-932620\Phi_{11}\pi^2\overline{\beta}^2-2702400\pi^2\beta\overline{\beta}^3\\
&-1041760\pi^3\overline{\beta}^2\overline{\alpha}-2977440\pi^3\overline{\beta}^2\overline{\pi}-1064800\pi^2\overline{\beta}^3\overline{\alpha}\\
&-835840\alpha^2\pi\overline{\alpha}\,\overline{\beta}^2-6584360\Phi_{11}\overline{\beta}\alpha^2\pi+74640\alpha^3\overline{\beta}\pi\overline{\alpha}\\
&-66480\alpha^3\overline{\beta}\,\overline{\pi}\pi+75120\pi^2\overline{\beta}\alpha^2\overline{\alpha}-70320\pi^2\alpha^2\overline{\beta}\,\overline{\pi}\\
&-2193544\alpha^2\overline{\beta}^2\overline{\pi}\pi+7200\alpha^2\beta\pi\overline{\beta}^2-4461264\alpha\pi^2\overline{\beta}^2\overline{\pi}\\
&-3128928\alpha\beta\pi\overline{\beta}^3-1237920\alpha\overline{\beta}^3\overline{\alpha}\pi+161720\alpha\overline{\beta}^3\overline{\pi}\pi\\
&-1662504\alpha\Phi_{11}\pi\overline{\beta}^2-1572528\alpha\pi^2\overline{\beta}^2\overline{\alpha}-25\overline{\beta}^4\Phi_{11}\big)\\
&/(-\overline{\beta}+12\alpha+22\pi)^2=0. \qquad (57)
\end{aligned}
$$

Solving this equation for $\overline{\delta}\alpha$ we get

$$
\begin{aligned}
\overline{\delta}\alpha := -\big(&227170\pi\,\overline{\beta}^{3}\Phi_{11} + 40200\pi\alpha^{3}\Phi_{11} + 176880\pi^{2}\,\overline{\beta}^{3}\,\overline{\pi} \\
&+ 18720\,\overline{\beta}\,\alpha^{4}\overline{\alpha} - 1467420\,\overline{\beta}\,\alpha^{3}\Phi_{11} - 5085800\,\overline{\beta}\,\pi^{3}\Phi_{11} \\
&+ 85800\pi\,\overline{\beta}^{4}\overline{\alpha} - 1064800\pi^{2}\,\overline{\beta}^{3}\,\overline{\alpha} + 8640\,\overline{\beta}^{2}\alpha^{3}\beta \\
&- 932620\pi^{2}\,\overline{\beta}^{2}\Phi_{11} - 15840\,\overline{\beta}\,\alpha^{4}\,\overline{\pi} + 205920\pi\,\overline{\beta}^{4}\beta \\
&- 2977440\pi^{3}\,\overline{\beta}^{2}\,\overline{\pi} - 1041760\pi^{3}\,\overline{\alpha}\,\overline{\beta}^{2} - 2702400\pi^{2}\beta\,\overline{\beta}^{3} \\
&- 360060\alpha^{2}\,\overline{\beta}^{3}\,\overline{\alpha} - 351912\alpha^{3}\,\overline{\beta}^{2}\,\overline{\pi} - 155352\alpha^{3}\,\overline{\beta}^{2}\,\overline{\alpha} \\
&- 633396\alpha^{2}\,\overline{\beta}^{2}\Phi_{11} - 910512\alpha^{2}\beta\,\overline{\beta}^{3} + 35640\alpha^{2}\,\overline{\beta}^{3}\,\overline{\pi} \\
&+ 47100\,\overline{\beta}^{4}\alpha\,\overline{\alpha} + 142635\,\overline{\beta}^{3}\alpha\Phi_{11} + 113040\,\overline{\beta}^{4}\alpha\beta \\
&- 3600\alpha^{4}\Phi_{11} - 1572528\pi^{2}\,\overline{\beta}^{2}\alpha\,\overline{\alpha} - 1662504\pi\,\overline{\beta}^{2}\alpha\Phi_{11} \\
&+ 161720\pi\,\overline{\beta}^{3}\alpha\,\overline{\pi} - 835840\pi\,\overline{\beta}^{2}\alpha^{2}\overline{\alpha} - 1237920\pi\,\overline{\beta}^{3}\alpha\,\overline{\alpha} \\
&- 4461264\pi^{2}\alpha\,\overline{\beta}^{2}\,\overline{\pi} - 2193544\alpha^{2}\pi\,\overline{\beta}^{2}\,\overline{\pi} + 7200\,\overline{\beta}^{2}\alpha^{2}\pi\beta \\
&- 3128928\,\overline{\beta}^{3}\alpha\pi\beta - 9885300\,\overline{\beta}\,\pi^{2}\alpha\Phi_{11} + 75120\,\overline{\beta}\,\pi^{2}\alpha^{2}\overline{\alpha} \\
&- 70320\,\overline{\beta}\,\pi^{2}\,\overline{\pi}\,\alpha^{2} - 6584360\,\overline{\beta}\,\alpha^{2}\pi\Phi_{11} + 74640\,\overline{\beta}\,\alpha^{3}\pi\,\overline{\alpha} \\
&- 66480\,\overline{\beta}\,\alpha^{3}\,\overline{\pi}\,\pi + 48400\pi^{3}\alpha\Phi_{11} \\
&+ 120000\alpha^{2}\pi^{2}\Phi_{11} - 25\,\overline{\beta}^{4}\Phi_{11}\big) \\
&/\big(-1760\,\overline{\beta}^{2}\Phi_{11} - 300\,\overline{\beta}^{3}\,\overline{\alpha} - 720\beta\,\overline{\beta}^{3} + 360\,\overline{\beta}^{2}\alpha\,\overline{\pi} \\
&- 240\,\overline{\beta}^{2}\,\overline{\alpha}\,\pi + 2400\beta\pi\,\overline{\beta}^{2} + 600\,\overline{\alpha}\,\overline{\beta}^{2}\alpha + 2880\alpha\,\overline{\beta}^{2}\beta \\
&+ 760\,\overline{\beta}^{2}\,\overline{\pi}\,\pi - 5280\alpha^{2}\,\overline{\beta}\,\overline{\pi} - 22160\,\overline{\beta}\,\overline{\pi}\,\alpha\pi \\
&+ 25040\pi^{2}\,\overline{\alpha}\,\overline{\beta} + 8940\,\overline{\beta}\,\alpha\Phi_{11} + 9600\,\overline{\beta}\,\pi\Phi_{11} \\
&+ 6240\,\overline{\alpha}\,\alpha^{2}\,\overline{\beta} - 23440\pi^{2}\,\overline{\beta}\,\overline{\pi} + 24880\alpha\pi\,\overline{\alpha}\,\overline{\beta} \\
&+ 22400\pi^{2}\Phi_{11} - 1200\alpha^{2}\Phi_{11} + 8600\pi\alpha\Phi_{11}\big),
\end{aligned}
\tag{58}
$$

where the denominator of (58),

$$
\begin{aligned}
d_{3} := &-1760\,\overline{\beta}^{2}\Phi_{11} - 300\,\overline{\beta}^{3}\,\overline{\alpha} - 720\beta\,\overline{\beta}^{3} + 2880\alpha\,\overline{\beta}^{2}\beta \\
&+ 360\,\overline{\beta}^{2}\alpha\,\overline{\pi} - 240\,\overline{\beta}^{2}\,\overline{\alpha}\,\pi + 2400\beta\pi\,\overline{\beta}^{2} + 600\,\overline{\alpha}\,\overline{\beta}^{2}\alpha \\
&+ 760\,\overline{\beta}^{2}\,\overline{\pi}\,\pi - 5280\alpha^{2}\,\overline{\beta}\,\overline{\pi} - 22160\,\overline{\beta}\,\overline{\pi}\,\alpha\pi \\
&+ 25040\pi^{2}\,\overline{\alpha}\,\overline{\beta} + 8940\,\overline{\beta}\,\alpha\Phi_{11} + 9600\,\overline{\beta}\,\pi\Phi_{11} \\
&+ 6240\,\overline{\alpha}\,\alpha^{2}\,\overline{\beta} - 23440\pi^{2}\,\overline{\beta}\,\overline{\pi} + 24880\alpha\pi\,\overline{\alpha}\,\overline{\beta} \\
&+ 22400\pi^{2}\Phi_{11} - 1200\alpha^{2}\Phi_{11} + 8600\pi\alpha\Phi_{11},
\end{aligned}
\tag{59}
$$

is assumed to be non-zero for now.

Subtracting (54) from (58) and taking the numerator,

$$
\begin{aligned}
N_1 := {} & 684288\,\overline{\beta}^5\beta^2\alpha + 286000\pi^3\alpha\Phi_{11}^2 \\
& - 7290900\pi^2\,\overline{\beta}^2\Phi_{11}^2 - 7913100\alpha^3\Phi_{11}^2\,\overline{\beta} + 165600\,\overline{\beta}^5\,\overline{\alpha}^2\alpha \\
& - 655680\pi^3\,\overline{\beta}^3\,\overline{\alpha}^2 + 295488\alpha^3\,\overline{\beta}^3\,\overline{\pi}^2 - 205632\alpha^3\,\overline{\beta}^3\,\overline{\alpha}^2 \\
& - 582090\pi\,\overline{\beta}^3\Phi_{11}^2 - 1157435\,\overline{\beta}^2\Phi_{11}^2\alpha^2 - 1517760\pi^2\,\overline{\beta}^4\,\overline{\alpha}^2 \\
& - 447840\,\overline{\beta}^4\,\overline{\alpha}^2\alpha^2 + 299000\pi^2\alpha^2\Phi_{11}^2 + 123540\,\overline{\beta}^5\Phi_{11}\beta \\
& + 303600\pi\,\overline{\beta}^5\,\overline{\alpha}^2 + 51450\,\overline{\beta}^5\,\overline{\alpha}\,\Phi_{11} - 126720\pi^2\,\overline{\beta}^4\,\overline{\pi}^2 \\
& - 30643000\pi^3\,\overline{\beta}\,\Phi_{11}^2 - 23040\,\overline{\beta}^4\,\overline{\pi}^2\alpha^2 + 78000\pi\alpha^3\Phi_{11}^2 \\
& + 3111840\pi^3\,\overline{\beta}^3\,\overline{\pi}^2 - 361718\,\overline{\beta}^3\Phi_{11}^2\alpha + 25\,\overline{\beta}^5\,\overline{\pi}\,\Phi_{11} \\
& + 1296000\pi\,\overline{\beta}^5\beta^2 - 14398260\pi\,\overline{\beta}^2\Phi_{11}\,\overline{\alpha}\,\alpha^2 - 57600\,\overline{\beta}^5\,\overline{\alpha}\,\alpha\,\overline{\pi} \\
& + 781920\,\overline{\beta}^4\,\overline{\alpha}\,\alpha^2\,\overline{\pi} + 89856\alpha^3\,\overline{\beta}^3\,\overline{\alpha}\,\overline{\pi} + 301945\,\overline{\beta}^4\Phi_{11}^2 \\
& - 138240\,\overline{\beta}^5\beta\alpha\,\overline{\pi} + 1886976\,\overline{\beta}^4\beta\alpha^2\,\overline{\pi} - 1119744\,\overline{\beta}^4\beta\,\overline{\alpha}\,\alpha^2 \\
& + 682560\,\overline{\beta}^5\,\overline{\alpha}\,\alpha\beta + 972820\,\overline{\beta}^4\,\overline{\alpha}\,\alpha\Phi_{11} - 3755902\,\overline{\beta}^3\,\overline{\alpha}\,\alpha^2\Phi_{11} \\
& - 266205\,\overline{\beta}^4\Phi_{11}\alpha\,\overline{\pi} + 1749108\,\overline{\beta}^4\Phi_{11}\alpha\beta + 3627158\,\overline{\beta}^3\Phi_{11}\alpha^2\,\overline{\pi} \\
& - 3766320\,\overline{\beta}^3\beta\alpha^2\Phi_{11} + 599880\alpha^3\Phi_{11}\,\overline{\beta}^2\,\overline{\pi} \\
& - 3021720\alpha^3\Phi_{11}\,\overline{\alpha}\,\overline{\beta}^2 + 6871680\pi^2\,\overline{\beta}^4\,\overline{\pi}\,\beta \\
& - 14919600\pi^2\,\overline{\beta}^3\Phi_{11}\beta - 13552520\pi^2\,\overline{\beta}^3\Phi_{11}\,\overline{\alpha} \\
& - 1389792\pi^2\,\overline{\beta}^3\,\overline{\alpha}^2\alpha + 4357632\pi^2\,\overline{\beta}^3\,\overline{\pi}^2\alpha + 2915280\pi^2\,\overline{\beta}^4\,\overline{\alpha}\,\overline{\pi} \\
& + 13578100\pi^2\,\overline{\beta}^3\Phi_{11}\,\overline{\pi} - 3718080\pi^2\,\overline{\beta}^4\,\overline{\alpha}\,\beta \\
& - 60662200\pi^2\,\overline{\beta}\,\Phi_{11}^2\alpha - 713952\pi^2\,\overline{\beta}^3\,\overline{\pi}\,\alpha\,\overline{\alpha} \\
& - 8470120\pi^2\,\overline{\beta}^2\Phi_{11}\,\overline{\pi}\,\alpha - 22343840\pi^2\,\overline{\beta}^2\Phi_{11}\alpha\,\overline{\alpha} \\
& + 46000\pi^2\alpha^2\Phi_{11}\,\overline{\alpha}\,\overline{\beta} - 64400\pi^2\alpha^2\Phi_{11}\,\overline{\beta}\,\overline{\pi} \\
& - 26400\pi^2\alpha\Phi_{11}\beta\,\overline{\beta}^2 - 105600\pi\,\overline{\beta}^5\,\overline{\alpha}\,\overline{\pi} + 1806460\pi\,\overline{\beta}^4\Phi_{11}\,\overline{\alpha} \\
& - 38468250\pi\,\overline{\beta}\,\Phi_{11}^2\alpha^2 - 111360\pi\,\overline{\beta}^4\,\overline{\pi}^2\alpha + 3379920\pi\,\overline{\beta}^4\Phi_{11}\beta \\
& - 474200\pi\,\overline{\beta}^4\Phi_{11}\,\overline{\pi} - 253440\pi\,\overline{\beta}^5\,\overline{\pi}\,\beta + 1268640\pi\,\overline{\beta}^5\,\overline{\alpha}\,\beta \\
& - 1647600\pi\,\overline{\beta}^4\,\overline{\alpha}^2\alpha - 6010420\pi\,\overline{\beta}^2\Phi_{11}^2\alpha - 939744\pi\,\overline{\beta}^3\,\overline{\alpha}^2\alpha^2 \\
& + 1993248\pi\,\overline{\beta}^3\,\overline{\pi}^2\alpha^2 + 3021840\pi\,\overline{\beta}^4\,\overline{\alpha}\,\overline{\pi}\,\alpha + 51264\pi\,\overline{\beta}^3\,\overline{\pi}\,\overline{\alpha}\,\alpha^2 \\
& + 14037554\pi\,\overline{\beta}^3\Phi_{11}\alpha\,\overline{\pi} - 14255872\pi\,\overline{\beta}^3\Phi_{11}\,\overline{\alpha}\,\alpha \\
& - 15000840\pi\,\overline{\beta}^3\Phi_{11}\alpha\beta - 592500\pi\,\overline{\beta}^2\Phi_{11}\alpha^2\,\overline{\pi} \\
& - 4074624\pi\,\overline{\beta}^4\,\overline{\alpha}\,\alpha\beta + 7198848\pi\,\overline{\beta}^4\,\overline{\pi}\,\alpha\beta
\end{aligned}
$$

$$
\begin{aligned}
&-14400\pi\alpha^2\Phi_{11}\beta\overline{\beta}^2-16800\pi\alpha^3\Phi_{11}\overline{\beta}\,\overline{\pi}+12000\pi\alpha^3\Phi_{11}\overline{\alpha}\,\overline{\beta}\\
&-9855600\pi^3\overline{\beta}^2\Phi_{11}\overline{\pi}-11178800\pi^3\overline{\beta}^2\Phi_{11}\overline{\alpha}\\
&-929760\pi^3\overline{\beta}^3\overline{\pi}\,\overline{\alpha}+44000\pi^3\alpha\Phi_{11}\overline{\alpha}\,\overline{\beta}\\
&-61600\pi^3\alpha\Phi_{11}\overline{\beta}\,\overline{\pi}=0.
\end{aligned}
\tag{60}
$$

3. THE CASE $\Phi_{11}=0$

Carminati and McLenaghan [7] used the conditions (III) and (V) given in Section 2 to prove that Huygens' principle is not satisfied if any of the spin coefficients α, β or π vanish. We now extend the proof for the case in which $\alpha\beta\pi\neq 0$ and $\Phi_{11}=0$; i.e., we shall prove the following theorem:

THEOREM 6. – *Let V_4 be any space-time which admits a spinor dyad with the properties*

$$o_{A;B\dot{B}}=o_A I_{B\dot{B}}, \tag{61}$$

where $I_{B\dot{B}}$ is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C o^D o^E\bar{o}^{\dot{E}}=0, \tag{62}$$

$$R=0,\qquad \Phi_{AB\dot{A}\dot{B}}o^A o^B=0. \tag{63}$$

Then the validity of Huygens' principle for the conformally invariant equation (1) *implies that*

$$\Phi_{AB\dot{A}\dot{B}}o^A o^B\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}}\neq 0. \tag{64}$$

Proof. – When $\Phi_{11}=0$ the quantity N_1, given by (60), factors in the following form:

$$N_1:=-12\overline{\beta}\,p_1p_2, \tag{65}$$

where

$$p_1:=12\beta\overline{\beta}+2\pi\overline{\alpha}+2\alpha\overline{\alpha}+5\overline{\beta}\,\overline{\alpha}+6\overline{\pi}\pi+2\overline{\pi}\alpha, \tag{66}$$

$$
\begin{aligned}
p_2:=\;&1188\overline{\beta}\beta\alpha+240\alpha\overline{\beta}\,\overline{\pi}+440\overline{\beta}\,\overline{\pi}\pi-1265\overline{\beta}\pi\overline{\alpha}\\
&-2250\overline{\beta}\beta\pi+6830\pi^2\overline{\alpha}+7647\alpha\pi\overline{\alpha}+2142\alpha^2\overline{\alpha}\\
&-690\overline{\beta}\alpha\overline{\alpha}-10805\overline{\pi}\pi^2-11529\alpha\overline{\pi}\pi-3078\overline{\pi}\alpha^2.
\end{aligned}
\tag{67}
$$

Let us consider first the case in which $p_2=0$. Applying $\overline{\delta}$ to (67) and solving for $\overline{\delta}\alpha$, we obtain:

$$\begin{aligned}\overline{\delta}\alpha := -\big(&690120\overline{\beta}\beta\alpha^3 - 177100\overline{\beta}^3\overline{\alpha}\pi + 2475000\overline{\beta}\alpha\pi^2\beta\\
&-97175\overline{\alpha}\,\overline{\beta}^3\alpha - 186390\alpha\overline{\beta}^3\beta + 1716210\overline{\alpha}\alpha^2\overline{\beta}^2\\
&-1131915\alpha^2\overline{\beta}^2\overline{\pi} - 4470219\overline{\beta}\alpha^3\overline{\alpha} - 3875990\pi^2\overline{\beta}^2\overline{\pi}\\
&-341280\beta\pi\overline{\beta}^3 + 5791820\pi^2\overline{\alpha}\,\overline{\beta}^2 + 9784170\alpha^3\overline{\pi}\pi\\
&+2573586\alpha^2\beta\overline{\beta}^2 + 8639361\alpha^3\overline{\beta}\,\overline{\pi} + 8903160\pi^2\beta\overline{\beta}^2\\
&-28683640\pi^3\overline{\beta}\,\overline{\alpha} - 11970480\pi^2\alpha^2\overline{\alpha} + 18687060\pi^2\overline{\pi}\alpha^2\\
&-6346350\alpha^3\pi\overline{\alpha} + 9567000\beta\pi\alpha\overline{\beta}^2 + 2615220\overline{\beta}\beta\pi\alpha^2\\
&+33800\overline{\beta}^3\alpha\overline{\pi} + 61600\overline{\beta}^3\overline{\pi}\pi + 90203190\pi^2\alpha\overline{\beta}\,\overline{\pi}\\
&-1119420\alpha^4\overline{\alpha} - 4188240\overline{\beta}^2\overline{\pi}\alpha\pi + 6302070\alpha\overline{\beta}^2\pi\overline{\alpha}\\
&-24866544\overline{\beta}\alpha^2\pi\overline{\alpha} - 46210320\overline{\beta}\alpha\pi^2\overline{\alpha} + 48333948\alpha^2\overline{\beta}\,\overline{\pi}\pi\\
&+56154880\pi^3\overline{\beta}\,\overline{\pi} + 1705860\alpha^4\overline{\pi} - 7513000\pi^3\alpha\overline{\alpha}\\
&+11885500\overline{\pi}\pi^3\alpha\big)\\
&/\big((-12\alpha - 22\pi + \overline{\beta})(115\overline{\beta}\,\overline{\alpha} + 126\beta\overline{\beta} - 40\overline{\beta}\,\overline{\pi} + 783\overline{\pi}\alpha\\
&-1634\pi\overline{\alpha} + 1448\pi\overline{\pi} - 921\alpha\overline{\alpha})\big),\end{aligned}\tag{68}$$

where the denominator of the expression above, given by

$$\begin{aligned}d_4 := (-12\alpha - 22\pi + \overline{\beta})(&115\overline{\beta}\,\overline{\alpha} + 126\beta\overline{\beta} - 40\overline{\beta}\,\overline{\pi} + 783\overline{\pi}\alpha\\
&-1634\pi\overline{\alpha} + 1448\pi\overline{\pi} - 921\alpha\overline{\alpha}),\end{aligned}\tag{69}$$

is assumed to be non-zero, for now.

Here N_1, and all equations obtained by comparing different expressions for $\overline{\delta}\,\overline{\alpha}$, are polynomials in three complex variables α, β and π. One complex variable can be eliminated by introducing the following new variables:

$$x_1 := \frac{\alpha}{\pi}, \qquad x_2 := \frac{\beta}{\pi}.\tag{70}$$

In what follows we first prove that the necessary conditions imply that both x_1 and x_2 are constants. Then, later, we shall prove that this leads to a contradiction.

In the new variables defined by (70), the expression (67) assumes the form

$$\begin{aligned}p_2 = -2250\overline{x_2}\,x_2 - 1188\overline{x_2}\,x_2x_1 - 3078x_1^2 - 11529x_1 - 10805\\
+7647x_1\overline{x_1} + 6830 + 2142x_1^2\overline{x_1}\overline{x_1} - 1265\overline{x_2}\,\overline{x_1} + 440\overline{x_2}\\
-690\overline{x_2}\,x_1\overline{x_1} + 240x_1\overline{x_2} + 2142x_1^2\overline{x_1} = 0.\end{aligned}\tag{71}$$

Subtracting (68) from (58) (with $\Phi_{11} = 0$), and taking the numerator, gives

$$
\begin{aligned}
N_2 := {} & -75130000\,\overline{x_1}^2 x_1 - 568034312\,\overline{x_2}\,x_1\,\overline{x_1}^2 - 162662000 x_1^2\,\overline{x_1}^2 \\
& - 263829680\,\overline{x_2}\,\overline{x_1}^2 + 69828000\,\overline{x_1}\,\overline{x_2}\,x_2 x_1 + 91299060\,\overline{x_2}^2 \\
& - 105963000\,\overline{x_2}\,x_2 x_1 + 328900\,\overline{x_2}^4\,\overline{x_1}^2 - 3248115\,\overline{x_2}^3 \\
& - 97977600 x_1^4 - 12927600 x_1^5 - 278399100 x_1^3 \\
& + 37400\,\overline{x_2}^4 - 16070400 x_1^4\,\overline{x_2}\,x_2 - 277285752 x_1^2\,\overline{x_2}^2 x_2 \\
& + 6646212 x_1^2\,\overline{x_2}^3 x_2 + 78647544 x_1^4\,\overline{x_2}\,\overline{x_1} - 48437136 x_1^3\,\overline{x_2}^2 x_2 \\
& + 15461670 x_1\,\overline{x_1}\,\overline{x_2}^3 - 39517398 x_1^3\,\overline{x_2}^2\,\overline{x_1} \\
& + 12081744\,\overline{x_1}\,x_1^3\,\overline{x_2}^2 x_2 + 593329572 x_1^3\,\overline{x_2}\,\overline{x_1} \\
& + 4160700 x_1^2\,\overline{x_2}^3\,\overline{x_1} - 221403987 x_1^2\,\overline{x_2}^2\,\overline{x_1} \\
& - 42924720\,\overline{x_1}\,\overline{x_2}^3 x_2 - 459117172\,\overline{x_1}^2\,\overline{x_2}\,x_1^2 \\
& + 1679968716\,\overline{x_1}\,\overline{x_2}\,x_1^2 + 112691520\,\overline{x_1}\,\overline{x_2}^2 x_2 \\
& + 163297776\,\overline{x_1}\,\overline{x_2}^2 x_2 x_1 - 413617894\,\overline{x_1}\,\overline{x_2}^2 x_1 \\
& + 125141836\,\overline{x_1}^2\,\overline{x_2}^2 x_1 + 14366310\,\overline{x_1}\,\overline{x_2}^3 - 169509600 x_1^2\,\overline{x_2}\,x_2 \\
& - 15940800 x_1^2\,\overline{x_2}^2 x_2^2 + 114001200 x_1^2\,\overline{x_2}\,x_2\,\overline{x_1} \\
& + 62078400 x_1^3\,\overline{x_2}\,x_2\,\overline{x_1} + 11275200 x_1^4\,\overline{x_2}\,x_2\,\overline{x_1} \\
& - 45650088 x_1\,\overline{x_1}\,\overline{x_2}^3 x_2 - 213660\,\overline{x_2}^4 x_1 x_2 \\
& + 77697000\,\overline{x_1}\,x_1^2\,\overline{x_2}^2 x_2 - 12134448\,\overline{x_1}\,x_1^2\,\overline{x_2}^3 x_2 \\
& - 165106878\,\overline{x_1}^2 x_1^3\,\overline{x_2} + 66914052\,\overline{x_1}^2 x_1^2\,\overline{x_2}^2 \\
& + 383646000 x_1^3\,\overline{x_1} + 136587600 x_1^4\,\overline{x_1} - 1405233644 x_1^2\,\overline{x_2} \\
& - 495836310 x_1^3\,\overline{x_2} + 78021681 x_1^2\,\overline{x_2}^2 - 257647540\,\overline{x_1}\,\overline{x_2}^2 \\
& - 44763192 x_2^2\,\overline{x_2}^3 x_1 - 14850000 x_2^2 x_1\,\overline{x_2}^2 - 121050\,\overline{x_2}^4 x_1\,\overline{x_1} \\
& - 11901168 x_1^2\,\overline{x_2}^3 x_2^2 - 4276800 x_1^3\,\overline{x_2}^2 x_2^2 - 22290588\,\overline{x_1}^2 x_1^4\,\overline{x_2} \\
& + 11929896\,\overline{x_1}^2 x_1^3\,\overline{x_2}^2 - 3427140\,\overline{x_1}^2 x_1^2\,\overline{x_2}^3 - 65630268 x_1^4\,\overline{x_2} \\
& + 13886406 x_1^3\,\overline{x_2}^2 - 939210 x_1^2\,\overline{x_2}^3 - 11744220\,\overline{x_1}^2\,\overline{x_2}^3 \\
& + 78035680\,\overline{x_1}^2\,\overline{x_2}^2 + 20400\,\overline{x_2}^4 x_1 + 1533600 x_2^2\,\overline{x_2}^4 \\
& + 146164809\,\overline{x_2}^2 x_1 + 750960 x_1\,\overline{x_2}^4 x_2\,\overline{x_1} + 179400\,\overline{x_1}^2\,\overline{x_2}^4 x_1 \\
& - 90396000 x_1^3\,\overline{x_2}\,x_2 - 12687610 x_1\,\overline{x_1}^2\,\overline{x_2}^3 + 25076106\,\overline{x_2}^3 x_1 x_2 \\
& - 528822552\,\overline{x_2}^2 x_1 x_2 - 3492935\,\overline{x_2}^3 x_1 + 23644920\,\overline{x_2}^3 x_2
\end{aligned}
$$

$$
\begin{aligned}
&-397830\,\overline{x_2}^{\,4}x_2-336008520\,\overline{x_2}^{\,2}x_2-6426000\,\overline{x_1}^{\,2}x_1^5\\
&-131992500x_1^3\,\overline{x_1}^{\,2}+18230400x_1^5\,\overline{x_1}+824256x_1\,\overline{x_1}^{\,4}x_2^2\\
&+1395480\,\overline{x_2}^{\,4}\,\overline{x_1}\,x_2-221925\,\overline{x_2}^{\,4}\,\overline{x_1}-47574000x_1^4\,\overline{x_1}^{\,2}\\
&-166397000x_1-42109200x_2^2\,\overline{x_2}^{\,3}-836819360\,\overline{x_2}\\
&-1770562898x_1\,\overline{x_2}+224037000x_1\,\overline{x_1}+478803300x_1^2\,\overline{x_1}\\
&-351507100x_1^2+999889240\,\overline{x_2}\,\overline{x_1}\\
&+2115717928\,\overline{x_2}\,x_1\,\overline{x_1}=0. \qquad (72)
\end{aligned}
$$

We now wish to determine the solutions of the system of algebraic equations $\{p_2=0,\ N_2=0\}$. This may be accomplished in principle using the Gröbner basis method of Buchberger [13] as follows. First, we treat the quantities $x_1,\ x_2,\ \overline{x_1},\ \overline{x_2}$ as independent variables, and view the quantities $p_2,\ N_2$ as polynomials in these indeterminates over the field of rational numbers. (In the subsequent analysis we may use the fact that some variables are complex conjugates of each other, but this will not be necessary for our immediate purpose.) Then, by computing a Gröbner basis for the set $\{p_2, N_2\}$ (actually, the ideal $\langle p_2, N_2\rangle$) with respect to a purely lexicographic ordering of terms (see [13]) we obtain a new set of polynomials with the same solutions but in which the variables have been successively eliminated as far as possible. In order to speed the computations, we use a special variant of the algorithm [9] which combines the nonlinear elimination with factorization of intermediate results. (This algorithm is available in the Maple system as the function `gsolve`.) For the polynomials $\{p_2, N_2\}$, the algorithm produces the following components, which collectively contain all solutions:

$$G_1 := [-8+23\,\overline{x_1},\ 11\,\overline{x_2}+8,\ 66x_1+125], \qquad (73)$$

$$G_2 := [9108x_2\,\overline{x_2}+247,\ -8+23\,\overline{x_1},\ 66x_1+125], \qquad (74)$$

$$G_3 := [828x_2\,\overline{x_2}+75\,\overline{x_2}+77,\ -8+23\,\overline{x_1},\ 66x_1+125], \qquad (75)$$

$$G_4 := [271x_1+138x_2\,\overline{x_2}+517,\ -8+23\,\overline{x_1}], \qquad (76)$$

$$G_5 := [36x_2\,\overline{x_2}+7-5\,\overline{x_1},\ 6x_1+11], \qquad (77)$$

$$
\begin{aligned}
G_6 := \big[&671514624x_2^2\,\overline{x_2}^{\,2}+488374272x_2^2\,\overline{x_2}-220446720\,\overline{x_1}\,x_2\\
&+35785728\,\overline{x_2}^{\,2}x_2+88473600x_2\,\overline{x_2}-27979776x_1x_2
\end{aligned}
$$

$$
\begin{aligned}
&+69101568x_2 - 167878656\overline{x_2}^2\overline{x_1}^2 + 181020672\overline{x_1}^2\overline{x_2}\\
&-26599040\overline{x_1}^2 + 73852416\overline{x_2}^2\overline{x_1} - 132857600\overline{x_1}\,\overline{x_2}\\
&+23168456\overline{x_1} - 26978094x_1 - 49722705 - 2204136x_1^2\\
&+48043776x_1\overline{x_2} + 111324800\overline{x_2} - 7645440\overline{x_2}^2,\\
&847872\overline{x_1}\,\overline{x_2}x_2 - 294912x_2\overline{x_2} + 423936\overline{x_1}^2\overline{x_2}\\
&-382720\overline{x_1}^2 - 218112\overline{x_1}\,\overline{x_2} + 323928\overline{x_1}\\
&-54450x_1 - 169493 + 24576\overline{x_2},\\
&139392\overline{x_2}^3x_2 + 202752\overline{x_2}^2x_2 + 73728x_2\overline{x_2}\\
&+69696\overline{x_2}^3\overline{x_1} + 38456\overline{x_2}^2\overline{x_1} - 54656\overline{x_1}\,\overline{x_2}\\
&-33280\overline{x_1} - 4224x_1 + 10432 - 11616x_1\overline{x_2}\\
&+22544\overline{x_2} + 2827\overline{x_2}^2 - 11616\overline{x_2}^3 - 7986\overline{x_2}^2x_1,\\
&304128x_1x_2\overline{x_2} + 576000x_2\overline{x_2} + 66240\overline{x_1}\,\overline{x_2}\\
&-59800\overline{x_1} - 191598x_1 - 279575 - 17424x_1^2\\
&+198000x_1\overline{x_2} + 351960\overline{x_2},\\
&192x_1\overline{x_1} - 282x_1 - 505 + 280\overline{x_1}\,]. \qquad (78)
\end{aligned}
$$

Using the fact that the pair (x_1, x_2) and $(\overline{x_1}, \overline{x_2})$ are complex conjugates of each other, we conclude that the sets G_1 to G_5 provide solutions which are either impossible or in which x_1 and x_2 are constant. In the case of G_6, this is not immediately obvious. Its smallest term is:

$$192x_1\overline{x_1} - 282x_1 - 505 + 280\overline{x_1} = 0. \qquad (79)$$

Subtracting (79) from its complex conjugate we obtain the conclusion that x_1 is real, which implies that it must be constant. It follows that x_2 must be constant as well.

Let us consider now the case

$$p_1 = 12x_2\overline{x_2} + 6 + 2x_1 + 2\overline{x_1} + 2x_1\overline{x_1} + 5\overline{x_1x_2} = 0. \qquad (80)$$

We then use the side relation S_1 given by (52), whose numerator takes the form:

$$
\begin{aligned}
P_3 := &-6x_2^2\overline{x_2}^2 + 1210x_2x_1 + 1276x_1 + 1276\overline{x_1}\\
&+360x_2x_1^2\overline{x_1} + 2901x_2\overline{x_2} + 1528x_1x_2\overline{x_2} + 660x_1\overline{x_2}\,\overline{x_1}\\
&+660x_2x_1^2 + 408x_1\overline{x_1}^2 + 660\overline{x_1}^2\overline{x_2} + 408\overline{x_1}x_1^2
\end{aligned}
$$

$$+264\overline{x_1}^{\,2}+144x_1^2\overline{x_1}^{\,2}+1452+264x_1^2$$
$$+1210\overline{x_1}\,\overline{x_2}+1444\overline{x_1}\,x_1+1528x_2\overline{x_1}\,\overline{x_2}$$
$$+660x_2\overline{x_1}\,x_1+803x_2x_1\overline{x_2}\,\overline{x_1}+360x_1\overline{x_2}\,\overline{x_1}^{\,2}=0. \qquad (81)$$

Applying our nonlinear elimination algorithm as before to p_1, p_3 we obtain the following equivalent system of equations:

$$s_1 := 6x_2\overline{x_2}+31x_1\overline{x_1}+56x_1+\overline{x_1}+3=0, \qquad (82)$$
$$s_2 := 72x_2x_1\overline{x_1}+132x_1x_2+31x_1\overline{x_1}^{\,2}+56\overline{x_1}\,x_1$$
$$+\overline{x_1}^{\,2}+3\overline{x_1}=0, \qquad (83)$$
$$s_3 := \overline{x_1x_2}-22x_1-12\overline{x_1}\,x_1=0. \qquad (84)$$

Subtracting (82) from its complex conjugate yields $\overline{x_1}=x_1$. Subtracting (84) from its complex conjugate now gives $x_2=\overline{x_2}=12x_1+22$. Substituting these relations back in (82) and (83) results in a system with no solution.

It thus follows that in either of the cases which arise from Eq. (65), x_1 and x_2 must necessarily be constant. However, it may be shown (though we postpone the details until the following section) that this too leads to a contradiction.

We must finally consider the case in which the denominator of $D\mu$, given by (49), is zero. Here we shall suppose that Φ_{11} is not necessarily zero, so that the side relations derived in Section 2 will remain valid in the following section as well. According to (50) and (70),

$$d_1 := 22+12x_1-\overline{x_2}=0. \qquad (85)$$

From (49) we obtain

$$E_1 := -53x_1\phi_{11}+242-220x_2+24\overline{x_1}\,x_1^2+30\overline{x_2}\,\phi_{11}+274x_2\overline{x_2}$$
$$-5\overline{x_2}+110\overline{x_2}\,\overline{x_1}+24x_1^2+44\overline{x_1}+144x_1x_2\overline{x_2}-108\phi_{11}$$
$$-120x_1x_2+68x_1\overline{x_1}+176x_1+60\overline{x_1}\,\overline{x_2}\,x_1=0, \qquad (86)$$

where ϕ_{11} is defined as follows:

$$\phi_{11} := \frac{\Phi_{11}}{\pi\,\overline{\pi}}. \qquad (87)$$

Applying $\overline{\delta}$ to f_1, using (30), (38) and (36), and solving for $\overline{\delta}\alpha$, we get

$$\overline{\delta}\alpha = 120\overline{\beta}\,\alpha+66\pi\alpha+220\pi\,\overline{\beta}+33\alpha^2-\overline{\beta}^2. \qquad (88)$$

By applying δ to d_1, now using (40)–(42) and (47) and solving for $\mathrm{D}\mu$, we get

$$\mathrm{D}\mu = (4\overline{\pi}\alpha + 29\beta\overline{\beta} + 64\alpha\overline{\alpha} - 370\beta\pi + 32\overline{\pi}\pi - 190\beta\alpha + 114\pi\overline{\alpha} + 5\overline{\beta}\overline{\alpha} - 68\Phi_{11})/20. \tag{89}$$

Subtracting $D\overline{\mu}$, given by (47), from the complex conjugate of (89), gives

$$E_2 := 20\phi_{11} - 2x_2\overline{x_2} - 22x_1 - 24x_1\overline{x_1} + 66x_2 + 35x_1x_2 - 22\overline{x_1} + 35\overline{x_2}\,\overline{x_1} + 66\overline{x_2} = 0. \tag{90}$$

Applying $\overline{\delta}$ to (90) gives

$$\begin{aligned}E_3 := {} & 1168x_1\phi_{11} + 3980x_2x_1^2 - 264 + 14520x_2 - 3584\overline{x_1}\,x_1^2 \\ & - 255\overline{x_2}\,\phi_{11} - 2838x_2\overline{x_2} - 9482x_1\overline{x_1} - 24200\overline{x_2} \\ & - 12870\overline{x_2}\,\overline{x_1} - 12870x_1\overline{x_2} - 3784x_1^2 - 1210\overline{x_2}^{\,2} \\ & - 4928\overline{x_1} - 1514x_1x_2\overline{x_2} + 20x_2\overline{x_2}^{\,2} - 625\overline{x_2}^{\,2}\,\overline{x_1} \\ & + 2046\phi_{11} + 15180x_1x_2 - 7480x_1 - 6835\overline{x_1}\,\overline{x_2}\,x_1 = 0. \end{aligned} \tag{91}$$

Applying nonlinear elimination to d_1, $\overline{f_1}$, E_1, $\overline{E_1}$, E_2 and E_3, we find that this system has no solution.

The cases where each of the denominators d_2, d_3 and d_4, that appeared in the preceeding equations are zero lead to contradictions, according to [24]. The demonstration of this fact follows the steps described above and will not be presented here for brevity.

Thus, for Huygens' principle to be satisfied on Petrov type III space-times we must have $\Phi_{11} \neq 0$, and Theorem 6, which states this result in a conformally invariant way, is proved.

4. THE CASE $\Phi_{11} \neq 0$

We shall now examine the sole case which remains after the analysis of the previous section, namely that in which $\alpha\beta\pi \neq 0$ and $\Phi_{11} \neq 0$. This, in view of Theorem 6, will complete the proof of Theorem 1. Our approach is related to that of the previous section, in that we reduce the problem to an issue of solvability of a purely algebraic system of equations.

We first observe that, in addition to the algebraic equations given by (52) and (60), an extra independent equation may be obtained by applying the NP operator δ to (52). All of the Pfaffians which result are known explicitly, and may be replaced using the expressions found in Section 2 to obtain a (very large) expression in the complex variables α, β, and π

and the real quantity Φ_{11}. Upon transforming variables according to (70) and (87), we obtain a complex quantity in the new variables x_1, x_2, $\overline{x_1}$, $\overline{x_2}$, and ϕ_{11}. (This polynomial contains 408 terms of maximum total degree 9.) Together with the equations which similarly follow from (52) and (60), we have in effect a system of five equations in five real variables. Let us denote the set of polynomials which arise in this system (i.e., when the equations are written with a right hand side of 0) by F.

It must be mentioned that the approach of the previous section, namely computing the solutions by explicit elimination, is impossible in the present case due to the intrinsic computational complexity of nonlinear elimination and the high degree of our polynomials. It is possible and will suffice, however, to *bound* the number of solutions using the following result due to Buchberger [13]:

THEOREM 7. – *Let G be a Gröbner basis for $\langle F\rangle$ (the polynomial ideal generated by F) with respect to a given ordering of terms, and let H denote the set of leading terms of the elements of G with respect to the chosen term ordering. Then the system of equations corresponding to F has finitely many solutions if and only if for every indeterminate x in F there is a natural number m such that $x^m \in H$.*

The key to using this result is that we may use an ordering of terms based on *total degree* (i.e., a non-elimination ordering) for which the computational complexity of Buchberger's algorithm for Gröbner bases is much lower. Unfortunately, even in this setting a Gröbner basis for F cannot easily be computed due to the extreme size of intermediate results produced by the algorithm.

It would be highly desirable to apply modular homomorphisms in the manner used in algorithms for factorization (i.e., so-called Chinese remainder, or Hensel algorithms [13]) in the present situation. This is not currently possible due to a number of unresolved problems with the approach. Nonetheless, it provides a useful probabilistic experimental approach: treat the elements of F as polynomials over a prime field Z_p (rather than the rationals), where p is of modest size, and compute the Gröbner basis of F *modulo* p over Z_p. For a *single* prime, it is possible that the result so obtained may have no useful relationship with the Gröbner basis of F over the rationals. However, if the basis polynomials computed using a large number of different primes all exhibit identical monomial structures, it is extremely likely that they each represent a distinct homomorphic image of the true Gröbner basis of F. The question of accurately computing the probability of success for a specific series of primes remains an open problem. However, the individual prime field computations are comparatively easy since (unlike the rational case) no single coefficient may be larger than the chosen prime. This provides an experimental "sampling" method which gives clues on how best to compute the true result, and what that result will likely be.

We must also consider that if we were able to compute a Gröbner basis for F over the rationals, we would derive information on all solutions of the corresponding system *including* those which were examined in the previous section (i.e., for which $\Phi_{11} = 0$). It is possible to exclude those solutions entirely by adding an additional constraint and variable,

$$\phi_{11} z - 1 = 0 \tag{92}$$

to our equations to produce the augmented system $\widetilde{F}$. Still, only an actual computation reveals whether this improves or worsens the tractability of the problem. In our case, a large number (a few thousand) prime field "sample" computations (done using the GB package of Faugère [12], which is far more efficient than the general-purpose Maple system) all suggested that the addition of Eq. (92) made the Gröbner basis calculation much more efficient. More importantly, once the solutions examined in the previous section were in effect discarded, only a *finite number* remained when Theorem 7 is taken into account. With this in mind, it was possible (and worthwhile) to compute the *true* Gröbner basis of $\widetilde{F}$ over the rationals in the indeterminates x_1, x_2, $\overline{x_1}$, $\overline{x_2}$, r_{11}, z using a total degree ordering of terms. Since this basis contains polynomials with leading terms

$$x_1^6, x_2^5, \overline{x_1}^5, \overline{x_2}^5, \phi_{11}^4, z^5, \tag{93}$$

we may conclude that there are only finitely many solutions for which ϕ_{11}, and hence Φ_{11} as well, is nonzero. (For this last computation the latest and most efficient version of Faugère's GB package, known as FGB, was required.) It follows that x_1, x_2, $\overline{x_1}$, $\overline{x_2}$, ϕ_{11} must be constants; it remains only to show that this yields a contradiction.

Since ϕ_{11} must be constant (including the case in which $\Phi_{11} = 0$) it follows from (70), (87) that the quantities $\pi\overline{\pi}$, $\beta\overline{\beta}$ and $\overline{\alpha}/\beta$ are all constant as well. From the equation $\delta(\beta\overline{\beta}) = 0$ we obtain, in the variables x_1, x_2, ϕ_{11} given by (70), (87), the side relation

$$\begin{aligned}&(7\overline{x_1} + 25x_2 + 2)\phi_{11} + 374x_2\overline{x_2} + 199\overline{x_1}\,x_2\overline{x_2} - 5x_2^2\overline{x_2}\\&+ 60x_1\overline{x_1}\,x_2 + 60\overline{x_1}^2\overline{x_2} + 110x_1x_2 + 110\overline{x_1x_2} + 68x_1\overline{x_1}\\&+ 24x_1\overline{x_1}^2 + 24\overline{x_1}^2 + 44x_1 + 116\overline{x_1} + 132 = 0.\end{aligned} \tag{94}$$

Next, from $\delta(x_2) = 0$ we obtain the Pfaffian

$$\delta\overline{\pi} = -\overline{\pi}(\overline{\alpha} + \beta). \tag{95}$$

Using this, along with the previously determined Pfaffians, we then obtain from $\delta(\pi\overline{\pi}) = 0$ another side relation; on subtracting this result from (94) (and ignoring the possibility that $d_1 = 0$, which has already been considered) we obtain

$$x_2(x_1 + \overline{x_2} + 4) + \phi_{11} = 0. \tag{96}$$

Finally, from $\overline{\delta}(\overline{\alpha}/\beta) = 0$ we obtain

$$x_1(x_1 + 5\overline{x_2} + 2) + 9\overline{x_2} = 0. \tag{97}$$

The collection of polynomials given by (94), (96), (97), (52) and their complex conjugates has a Gröbner basis (computed easily using Maple) containing only the polynomial 1. This is equivalent to showing that there exists a combination of these polynomials which equals 1, and hence that they cannot vanish simultaneously (see [13]); i.e., the associated system of equations has *no* solutions. This completes the proof.

5. CONCLUSION

In completing the proof of Theorem 1, we have fully solved Hadamard's problem for the scalar wave equation in the case of Petrov type III space-times. Essential to our proof were use of the six-index necessary condition obtained by Rinke and Wünsch [23], and separate analyses (and different ideal-theoretic tools) for the cases $\Phi_{11} = 0$ and $\Phi_{11} \neq 0$. To complete the proof of the conjecture stated in the Introduction it remains to consider the space-times of Petrov types I and II. A partial result for type II has been obtained by Carminati, Czapor, McLenaghan and Williams [8]. However, it is not yet clear whether the complicated equations which arise from conditions (III), (V), and (VII) can be solved by the method used in the present paper.

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