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On solutions of the KZ and qKZ equations at level zero

by

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ABSTRACT. – We consider the Knizhnik–Zamolodchikov differential (KZ) and the quantum Knizhnik–Zamolodchikov (qKZ) difference equations at level 0 associated with rational solutions of the Yang–Baxter equation. The relations between different formulae for solutions are determined. © Elsevier, Paris

RÉSUMÉ. – Nous considérons l'équation différentielle de Knizhnik–Zamolodchikov (KZ) et l'équation aux differences de Knizhnik–Zamolodchikov (qKZ) au niveau 0 associées aux solutions rationnelles de l'équation de Yang–Baxter. Les relations entre les formules différentes pour les solutions sont determinées. © Elsevier, Paris

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1. INTRODUCTION

The quantum Knizhnik–Zamolodchikov (qKZ) difference equation has recently attracted much attention. The qKZ equations appear naturally in the representation theory of quantum affine algebras as equations for matrix elements of products of vertex operators [4]. Later the qKZ equations were derived as equations for traces of products of vertex operators [7], their specializations being equations for correlation functions in solvable lattice models. An important special case of the qKZ equations was introduced earlier by Smirnov [17] as equations for form factors in two-dimensional massive integrable models of quantum field theory.

In this paper we consider the rational qKZ equation associated with the Lie algebra sl_2. We will address the trigonometric qKZ equation in a separate paper. We also restrict ourselves to the case of the qKZ equation at level zero, which is precisely the case of Smirnov's equations for form factors. Besides this important application, this case is peculiar itself, which will become clear in the paper. Another important special case is given by the qKZ equation at level -4, which corresponds to equations for correlation functions. We are going to discuss this case elsewhere. It is not much known about solutions of the qKZ equation in the other cases than sl_2. Only a few results are available for the sl_N case [17,11,13,22].

There are several approaches to integral formulae for solutions of the qKZ equation at level zero. The first one is given in [17]. Solutions of the qKZ equation are obtained there in a rather straightforward way. They are expressed via certain polynomials and they are enumerated by periodic functions, which are arbitrary polynomials of exponentials of bounded degree. Smirnov's construction can be seen as a deformation of hyperelliptic integrals, the periodic functions being "deformations" of cycles on the corresponding hyperelliptic curve [18]. From the view point of constructing solutions, the main disadvantage of this approach is that it fails to work in the case of the qKZ equation at nonzero level. As we understand now, the reason is that Smirnov's formulae are intimately related to specific features of the qKZ equation at level zero.

Another way to produce integral formulae for solutions of the qKZ equation is given in [7,12,10]. One has to calculate a trace of a product of vertex operators over an infinite-dimensional representation of the quantum affine algebra $U_q(\widehat{sl_2})$ or the centrally extended Yangian double $\widehat{\mathcal{D}Y}_\hbar(sl_2)$ [12,10], using the bosonization technique, thus getting integral solutions of the trigonometric and rational qKZ equation, respectively.

This approach is not restricted only to the qKZ equation at level zero, but still cannot be applied to producing solutions of the qKZ equation in general. The traces of products of vertex operators satisfy the qKZ equation by construction and there is no need to verify independently that the final integral formula gives a solution. On the other hand the resulting parametrization of solutions by the products of vertex operators is very hard to control effectively until now.

A general approach to integral representations for solutions of the qKZ equation is developed in [23,24] combining ideas of [21,25,17]. It allows to describe effectively the total space of solutions of the qKZ equation at generic position, as well as to compute its transition matrices between asymptotic solutions, which are substitutes for monodromy matrices of a differential equation in the case of a difference equation. Unfortunately, one cannot apply immediately these general results in the case of the qKZ equation at level zero, because the genericity assumptions imposed in [23] are not fulfilled in this case. But it is possible to repeat the consideration following the same lines as in [23] making necessary modifications and obtain a general formula for solutions of the qKZ equation at level zero. As in Smirnov's formula, solutions are parametrized by periodic functions which are polynomials in exponentials of bounded degree. We beleive that Theorem 6.3 describes all solutions of the qKZ equation at level zero, though we have no proof of this conjecture yet. For the recent development concerning general solutions of the qKZ equations with values in finite-dimensional representantions and at rational levels see [14].

The general aim of this paper is to compare three above described types of integral formulae. We will show that Smirnov's formula can be obtained from a general formula (6.2) by a certain specialization of a periodic function due to a certain trick available only at level zero. This trick was first observed for the KZ differential equation. It turns out that the integrand in a general integral formula for solutions of the KZ equation [3,21] becomes an exact form in the case in question, but it is possible to produce nonzero solutions by integrating over suitable unclosed contours. And a similar effect happens to occur in the difference case.

An open challenging problem is to describe traces of products of vertex operators in terms of solutions of the qKZ equations given by the formula (6.2). Namely, for any product of vertex operators one must find the corresponding periodic function. This will give a description of form factors of local operators in an alternative way to Smirnov's axiomatic approach [18]. In this paper we make this identification in the simplest examples of the energy-momentum tensor and currents of the $SU(2)$-invariant Thirring model.

The plan of the paper is as follows. In Section 2 we recall the definition of the KZ and qKZ equations at level zero. We describe the necessary spaces of rational functions in Section 3. Solutions of the KZ equation at level zero are considered in Section 4. In Section 5 we describe the space of "deformed cycles" = periodic functions, and the pairing between the spaces of rational and periodic functions given by the hypergeometric integral. We obtain solutions of the qKZ equation at level zero in Section 6. The traces of products of vertex operators are considered in Section 7.

There are four appendices in the paper which contain technical details and some proofs.

2. THE KZ AND QKZ EQUATIONS AT LEVEL ZERO

Let $\hbar$ be a nonzero complex number. Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$, and $R(z) \in \mathrm{End}\left(V^{\otimes 2}\right)$ be the following R-matrix:

$$R(z) = \frac{z + \hbar P}{z + \hbar}, \tag{2.1}$$

where $P \in \mathrm{End}(V^{\otimes 2})$ is the permutation operator. We have

$$R(z) = 1 + \hbar r(z) + \mathrm{O}(\hbar^2), \quad \hbar \to 0,$$

where $r(z)$ is the corresponding classical r-matrix:

$$r(z) = \frac{-1 + P}{z}.$$

Fix a nonzero complex number p called *step*. We consider the qKZ equation for a $V^{\otimes n}$-valued function $\psi(z_1, \dots, z_n)$:

$$\begin{aligned} \psi(z_1, \dots, z_j + p, \dots, z_n) &= K_j(z_1, \dots, z_n)\psi(z_1, \dots, z_n), \\ K_j(z_1, \dots, z_n) &= R_{j,j-1}(z_j - z_{j-1} + p) \cdots R_{j,1}(z_j - z_1 + p) \\ &\quad \times R_{j,n}(z_j - z_n) \cdots R_{j,j+1}(z_j - z_{j+1}), \end{aligned} \tag{2.2}$$

$j = 1, \dots, n$. The number $-2 + p/\hbar$ is called *level* of the qKZ equation.

Let $p = \hbar\kappa$ and consider the limit $\hbar \to 0$. Then the qKZ difference equation turns into the KZ differential equation:

$$\kappa \frac{\partial \psi}{\partial z_j} = \sum_{\substack{k=1 \\ k\neq j}}^{n} r_{jk}(z_j - z_k)\psi, \quad j = 1, \dots, n. \tag{2.3}$$

The number $-2+\kappa$ is called *level* of the KZ equation.

Let

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\Sigma^j = \sum_{i=1}^{n} \sigma_i^j, \quad j = \pm, 3.$$

The operators $\Sigma^\pm$, $\Sigma^3 \in \mathrm{End}(V^{\otimes n})$ define an sl_2 action on $V^{\otimes n}$. For any k such that $0 \leqslant k \leqslant n$, denote by $(V^{\otimes n})_k$ the weight subspace

$$(V^{\otimes n})_k = \{v \in V^{\otimes n} \mid \Sigma^3 v = (n-2k)v\}.$$

Notice that the operators $K_j(z_1, \dots, z_n)$, $j = 1, \dots, n$, see (2.2), commute with the sl_2 action on $V^{\otimes n}$.

In this paper we consider only the case of the KZ and qKZ equations at level 0, i.e., all over the paper we assume that

$$\kappa = 2 \quad \text{and} \quad p = 2\hbar$$

unless otherwise stated. Furthermore, fixing an integer ℓ such that $0 \leqslant 2\ell \leqslant n$, we discuss solutions of (2.2) and (2.3) taking values in $(V^{\otimes n})_\ell$ and obeying an extra condition

$$\Sigma^+ \psi(z_1, \dots, z_n) = 0, \tag{2.4}$$

that is, the solutions taking values in singular vectors with respect to the sl_2 action.

In this paper we assume that $\mathrm{Im}\,\hbar < 0$. Notice that we do not use this assumption until the definition of the hypergeometic integral, see (5.7).

3. THE SPACES OF RATIONAL FUNCTIONS

Let M be a subset of $\{1,\dots,n\}$ such that $\#M=\ell$. We write $M=\{m_1,\dots,m_\ell\}$ assuming that $m_1<\dots<m_\ell$. The subset M defines a point $\hat{z}_M\in\mathbb{C}^\ell$:

$$\hat{z}_M=(z_{m_1},\dots,z_{m_\ell}). \tag{3.1}$$

For any subsets $M,N\subset\{1,\dots,n\}$ we say that $M\leqslant N$ if $\#M=\#N$ and $m_i\leqslant n_i$ for any $i=1,\dots,\#M$. For any function $f(t_1,\dots,t_\ell)$ we set

$$\operatorname{Asym} f(t_1,\dots,t_\ell)=\sum_{\sigma\in\mathbf{S}_\ell}\operatorname{sgn}(\sigma)f(t_{\sigma_1},\dots,t_{\sigma_\ell}),$$

and for any point $u=(u_1,\dots,u_\ell)\in\mathbb{C}^\ell$ we define

$$\operatorname{Res} f(u)=\operatorname{res}(\dots\operatorname{res} f(t_1,\dots,t_\ell)|_{t_\ell=u_\ell}\dots)|_{t_1=u_1}.$$

From now on until the end of the section we assume that $z_1,\dots,z_n$ are pairwise distinct complex numbers. Denote by $\mathcal{F}$ the space of rational functions in t with at most simple poles at points $z_1,\dots,z_n$. Let

$$\mathcal{F}^{(k)}=\{f\in\mathcal{F}\mid f(t)=\mathrm{O}(t^{k-2}),\ t\to\infty\},\quad k\in Z_{\geqslant 0}.$$

We consider $\mathcal{F}^{\otimes\ell}$ as a space of rational functions in ℓ variables so that $\bigwedge^\ell\mathcal{F}\subset\mathcal{F}^{\otimes\ell}$ is the subspace of antisymmetric functions. Let

$$\mathcal{F}[\ell]=\{f\in\bigwedge^\ell\mathcal{F}^{(\ell)}\mid \operatorname*{res}_{t=z_m} f(t,t+\hbar,t_3,\dots,t_\ell)=0,\ m=1,\dots,n\}$$

and

$$\mathcal{F}_{cl}[\ell]=\Bigg\{f\in\bigwedge^\ell\mathcal{F}^{(\ell)}\ \Bigg|\operatorname*{res}_{t_1=z_m}\Bigg(\operatorname*{res}_{t_2=z_m}\Bigg(f(t_1,t_2,\dots,t_\ell)\times\prod_{1\leqslant a<b\leqslant\ell}\frac{1}{t_a-t_b}\Bigg)\Bigg)=0,\ m=1,\dots,n\Bigg\}.$$

For any $M=\{m_1,\dots,m_\ell\}\subset\{1,\dots,n\}$ and $m\in M$, let $\mu_M^{(m)}$, g_M, w_M and $\widetilde{w}_M$ be the following functions:

$$\mu_M^{(m)}(t)=\frac{1}{t-z_m}\prod_{\substack{l\in M\\ l\neq m}}\frac{t-z_l-\hbar}{z_m-z_l-\hbar},$$

$$\widetilde{w}_M(t_1,\dots,t_\ell) = \operatorname{Asym}\left(\prod_{a=1}^{\ell} \mu_M^{(m_a)}(t_a)\right) = \det\left[\mu_M^{(m_a)}(t_b)\right]_{a,b=1}^{\ell},$$

$$g_M(t_1,\dots,t_\ell) = \prod_{a=1}^{\ell}\left(\frac{1}{t_a - z_{m_a}} \prod_{1\leqslant l<m_a} \frac{t_a - z_l - \hbar}{t_a - z_l}\right) \prod_{1\leqslant a<b\leqslant \ell} (t_a - t_b - \hbar),$$

$$w_M = \operatorname{Asym} g_M. \tag{3.2}$$

The functions $\mu_M^{(m)cl}$, g_M^{cl}, w_M^{cl} and $\widetilde{w}_M^{cl}$ are defined by the same formulae at $\hbar = 0$.

LEMMA 3.1. – *Let $M \subset \{1,\dots,n\}$, $\#M = \ell$. Then $w_M, \widetilde{w}_M \in \mathcal{F}[\ell]$ and $w_M^{cl}, \widetilde{w}_M^{cl} \in \mathcal{F}_{cl}[\ell]$.*

LEMMA 3.2. – *Let $M, N \subset \{1,\dots,n\}$, $\#M = \#N$. Then* $\operatorname{Res} w_M(\hat{z}_N) = 0$ *unless $N \leqslant M$,*

$$\operatorname{Res} \widetilde{w}_M(\hat{z}_N) = \operatorname{Res} w_M^{cl}(\hat{z}_N) = \operatorname{Res} \widetilde{w}_M^{cl}(\hat{z}_N) = 0, \quad M \neq N,$$

$$\operatorname{Res} w_M(\hat{z}_M) = \operatorname{Res} g_M(\hat{z}_M), \qquad \operatorname{Res} w_M^{cl}(\hat{z}_M) = \operatorname{Res} g_M^{cl}(\hat{z}_M),$$

$$\operatorname{Res} \widetilde{w}_M(\hat{z}_M) = \operatorname{Res} \widetilde{w}_M^{cl}(\hat{z}_M) = 1.$$

PROPOSITION 3.3. – *Let $|z_j - z_k| \neq 0, \hbar$ for any $1 \leqslant j < k \leqslant n$. Then the functions w_M, $M \subset \{1,\dots,n\}$, $\#M = \ell$, form a basis in the space $\mathcal{F}[\ell]$, and the same do the functions $\tilde{w}_M$, $M \subset \{1,\dots,n\}$, $\#M = \ell$.*

PROPOSITION 3.4. – *Let $z_1,\dots,z_n$ be pairwise distinct. Then the functions w_M^{cl}, $M \subset \{1,\dots,n\}$, $\#M = \ell$, form a basis in the space $\mathcal{F}_{cl}[\ell]$, and the same do the functions $\widetilde{w}_M^{cl}$, $M \subset \{1,\dots,n\}$, $\#M = \ell$.*

The propositions are proved in Appendix A.

From Lemma 3.2 and Propositions 3.3, 3.4 it is clear that

$$w_M = \sum_{\substack{N\subset\{1,\dots,n\}\\ \#N=\#M}} \widetilde{w}_N \operatorname{Res} w_M(\hat{z}_N). \tag{3.3}$$

$$w_M^{cl} = \widetilde{w}_M^{cl} \operatorname{Res} w_M^{cl}(\hat{z}_M). \tag{3.4}$$

LEMMA 3.5. – *For any $M \subset \{1,\dots,n\}$, $\#M = \ell - 1$, the following relations hold*:

$$\hbar \sum_{k\notin M} w_{M\cup\{k\}}(t_1,\dots,t_\ell) = \mathrm{Asym}\Bigg(\Bigg(\prod_{a=2}^{\ell}(t_1 - t_a - \hbar) - \prod_{a=2}^{\ell}(t_1 - t_a + \hbar)\prod_{m=1}^{n}\frac{t_1 - z_m - \hbar}{t_1 - z_m}\Bigg) g_M(t_2,\dots,t_\ell)\Bigg), \tag{3.5}$$

$$\sum_{k\notin M} w^{cl}_{M\cup\{k\}}(t_1,\dots,t_\ell) = \mathrm{Asym}\Bigg(\Bigg(\sum_{m=1}^{n}\frac{1}{t_1 - z_m} - \sum_{a=2}^{\ell}\frac{2}{t_1 - t_a}\Bigg)\prod_{a=2}^{\ell}(t_1 - t_a) g^{cl}_M(t_2,\dots,t_\ell)\Bigg). \tag{3.6}$$

The lemma is proved in Appendix B.

Let $M_{\rm ext} = \{1,\dots,\ell\}$. Say that $M_{\rm ext}$ is the *extremal* subset. Due to Lemma 3.2 the right hand side of 3.3 for the extremal subset contains only one summand and

$$w_{M_{\rm ext}}(t_1,\dots,t_\ell) = \prod_{1\leqslant a<b\leqslant \ell}\frac{(z_a - z_b - \hbar)(z_a - z_b + \hbar)}{z_a - z_b}\widetilde{w}_{M_{\rm ext}}(t_1,\dots,t_\ell). \tag{3.7}$$

4. SOLUTIONS OF THE KZ EQUATION AT LEVEL ZERO

Let

$$\phi_{cl} = \prod_{m=1}^{n}(t - z_m)^{-1/2}$$

be the *phase function*. Consider the hyperelliptic curve $\mathcal{E}$:

$$y^2 = \prod_{m=1}^{n}(t - z_m). \tag{4.1}$$

Say that a contour γ on the curve $\mathcal{E}$ is *admissiblle* if $\int_\gamma \phi_{cl} f_1\, dt$ is convergent and

$$\int_\gamma \frac{\partial}{\partial t}(\phi_{cl} f_2)\, dt = 0$$

for any $f_1 \in \mathcal{F}^{(\ell)}$ and $f_2 \in \mathcal{F}^{(\ell+1)}$.

Denote by $\xi : \mathcal{E} \to \mathbb{C}P^1$ the canonical projection: $\xi : (t, y) \mapsto t$.

LEMMA 4.1. – *Let γ be a smooth simple contour on $\mathcal{E}$ avoiding the branching points $z_1, \ldots, z_n$. Assume that either γ is a cycle or γ admits a parametrisation $\rho : \mathbb{R} \to \mathcal{E}$, such that $|\xi(\rho(u))| \to \infty$ and $\arg \xi(\rho(u))$ has finite limits as $u \to \pm\infty$. Then γ is admissible.*

Proof. – If γ is a cycle, then the claim is clear. Since $2\ell \leqslant n$ we have that $\phi_{cl}(t) f(t) = \mathrm{O}(t^{\ell-2-n/2}) = \mathrm{O}(t^{-2})$ for $f \in \mathcal{F}^{(\ell)}$ and $\phi_{cl}(t) f(t) = \mathrm{O}(t^{-1})$ for $f \in \mathcal{F}^{(\ell+1)}$ as $t \to \infty$, which proves the claim in the second case. □

We denote by $\widehat{\mathcal{C}}$ the set of all admissible contours and by $\mathcal{C} \subset \widehat{\mathcal{C}}$ the set of cycles on $\mathcal{E}$. With any subset $M \subset \{1, \ldots, n\}$ we associate a vector $v_M \in (V^{\otimes n})_\ell$ by the rule:

$$v_M = v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n}, \tag{4.2}$$

where $\varepsilon_i = +$ for $i \notin M$ and $\varepsilon_i = -$ for $i \in M$. For any $\gamma_1, \ldots, \gamma_\ell \in \widehat{\mathcal{C}}$ we define a function $\psi_{\gamma_1,\ldots,\gamma_\ell}(z_1, \ldots, z_n)$ as follows:

$$\psi_{\gamma_1,\ldots,\gamma_\ell}(z_1, \ldots, z_n) = \sum_{\substack{M \subset \{1,\ldots,n\} \\ \#M = \ell}} v_M \int\limits_{\gamma_1 \times \cdots \times \gamma_\ell} w_M^{cl}(t_1, \ldots, t_\ell) \times \prod_{a=1}^{\ell} \phi_{cl}(t_a)\, dt_1 \wedge \cdots \wedge dt_\ell. \tag{4.3}$$

THEOREM 4.2 [3,21]. – *For any $\gamma_1, \ldots, \gamma_\ell \in \widehat{\mathcal{C}}$ the function $\psi_{\gamma_1,\ldots,\gamma_\ell}(z_1, \ldots, z_n)$ is a solution of the KZ equation* (2.3) *taking values in $(V^{\otimes n})_\ell$ and satisfying the condition* (2.4).

Remark. – Formula (4.3) gives a solution of the KZ equation for arbitrary n, ℓ. For $2\ell = n$, there exists another integral representation for solutions of the KZ equation, see the formula (4.5) and Theorem 4.6, which is proved in [15] for a general sl_N case in a straightforward way. This integral representation is a quasiclassical limit of Smirnov's formulae for solutions of the qKZ equation. We will show that the formula (4.5) is a specialization of the formula (4.3) for a certain choice of the integration contours, cf. (4.6).

Assume until the end of this section that $2\ell = n$. This means that we are looking for a singlet solution $\psi(z_1, \ldots, z_n)$ of the KZ equation (2.3):

$$\Sigma^j \psi(z_1, \ldots, z_n) = 0, \quad j = \pm, 3.$$

For any $M \subset \{1, \dots, n\}$, $\# M = \ell$, define functions ν_M and $\tilde{\nu}_M$ by the formulae:

$$\nu_M^{cl}(t) = \prod_{j \notin M} (t - z_j), \qquad \tilde{\nu}_M^{cl}(t) = \nu_M^{cl}(t) \Big(\sum_{j \in M} \frac{1}{t - z_j} - \sum_{j \notin M} \frac{1}{t - z_j} \Big).$$

It is clear that

$$\phi_{cl} \tilde{\nu}_M^{cl} = -2 \frac{d}{dt} \big(\phi_{cl} \nu_M^{cl} \big). \tag{4.4}$$

Since n is even, the curve $\mathcal{E}$ has no branching point at infinity. We distinguish two infinity points $\infty^{\pm} \in \mathcal{E}$ as follows:

$$y = \pm t^{\ell} (1 + \mathrm{o}(1)), \quad (t, y) \to \infty^{\pm}.$$

Fix an admissible contour γ^{∞} going from ∞^{+} to ∞^{-}, cf. Lemma 4.1.

LEMMA 4.3. – *Let $2\ell = n$. Then*

$$\int_{\gamma^{\infty}} \phi_{cl} \tilde{\nu}_M^{cl} \, dt = 4.$$

Proof. – The statement immediately follows from the formula (4.4) and the definition of γ^{∞}. □

LEMMA 4.4. – *Let $2\ell = n$. Then, for any $M \subset \{1, \dots, n\}$, $\# M = \ell$, we have*

$$\tilde{\nu}_M^{cl}(t) = \sum_{m \in M} \mu_M^{(m)cl}(t) \operatorname{res} \tilde{\nu}_M^{cl}(z_m).$$

Proof. – The claim follows from Lemma 6.6 since, using explicit formulae, it is easy to see that

$$\hbar^{-1} \tilde{\nu}_M \to \tilde{\nu}_M^{cl}, \quad \hbar^{-1} \operatorname{res} \tilde{\nu}_M(z_m) \to \operatorname{res} \tilde{\nu}_M^{cl}(z_m), \quad \mu_M^{(m)} \to \mu_M^{(m)cl},$$

as $\hbar \to 0$. □

COROLLARY 4.5. – *Let $2\ell = n$. Then, for any $M \subset \{1, \dots, n\}$, $\# M = \ell$, we have*

$$\tilde{w}_M^{cl}(t_1, \dots, t_\ell) = \frac{1}{\operatorname{res} \tilde{\nu}_M^{cl}(z_{m_\ell})} \operatorname{Asym} \big(\mu_M^{(m_1)cl}(t_1) \cdots \mu_M^{(m_{\ell-1})cl}(t_{\ell-1}) \tilde{\nu}_M^{cl}(t_\ell) \big).$$

For any $\gamma_1, \ldots, \gamma_{\ell-1} \in \mathcal{C}$ we define a function $\psi^S_{\gamma_1,\ldots,\gamma_{\ell-1}}(z_1, \ldots, z_n)$:

$$\psi^S_{\gamma_1,\ldots,\gamma_{\ell-1}}(z_1, \ldots, z_n) = \sum_{\substack{M\subset\{1,\ldots,n\}\\ \#M=\ell}} \prod_{1\leqslant a<b\leqslant \ell} (z_{m_a} - z_{m_b}) \prod_{k\notin M} (z_{m_\ell} - z_k)^{-1}$$
$$\times \det\left[\int_{\gamma_b} \phi^{cl} \mu_M^{(m_a)cl}\, dt\right]_{a,b=1}^{\ell-1} v_M. \tag{4.5}$$

THEOREM 4.6 ([15]). – *For any $\gamma_1, \ldots, \gamma_{\ell-1} \in \mathcal{C}$ the function $\psi^S_{\gamma_1,\ldots,\gamma_{\ell-1}}(z_1, \ldots, z_n)$ is a solution of the KZ equation* (2.3) *taking values in $(V^{\otimes n})_\ell$ and satisfying condition* (2.4).

Proof. – Subsequently using formulae (4.3), (3.4), Corollary 4.5, Lemmas 4.3 and 3.2 we obtain

$$\begin{aligned}
&\psi_{\gamma_1,\ldots,\gamma_{\ell-1},\gamma^\infty}(z_1, \ldots, z_n)\\
&= \sum_M v_M \operatorname{Res} w_M^{cl}(\hat{z}_M) \det\left[\int_{\gamma_b} \phi^{cl} \mu_M^{(m_a)cl}\, dt\right]_{a,b=1}^{\ell}\\
&= 4\sum_M v_M \frac{\operatorname{Res} w_M^{cl}(\hat{z}_M)}{\operatorname{res} \tilde{v}_M^{cl}(z_{m_\ell})} \det\left[\int_{\gamma_b} \phi^{cl} \mu_M^{(m_a)cl}\, dt\right]_{a,b=1}^{\ell-1}\\
&= 4\psi^S_{\gamma_1,\ldots,\gamma_{\ell-1}}(z_1, \ldots, z_n).
\end{aligned} \tag{4.6}$$

Here $\gamma_\ell = \gamma^\infty$. Since $\gamma_1, \ldots, \gamma_{\ell-1}, \gamma^\infty \in \widehat{\mathcal{C}}$, the theorem follows from Theorem 4.2. □

Remark. – Formula (4.4) and Corollary 4.5 show that for $n = 2\ell$, that is in the zero weight case, the integrand in the right hand side of the formula (4.3) for solutions of the KZ equation is an exact form. So that, for any $\gamma_1, \ldots, \gamma_\ell \in \mathcal{C}$ we have

$$\psi^S_{\gamma_1,\ldots,\gamma_\ell}(z_1, \ldots, z_n) = 0.$$

Therefore, to produce nonzero solutions of the qKZ equation we inevitably have to consider not only cycles on the curve $\mathcal{E}$, but also certain unclosed contours, which we call admissible.

5. THE SPACE OF "DEFORMED CYCLES" AND THE HYPERGEOMETRIC INTEGRAL

Let $\widehat{\mathcal{F}}_q$ be the space of functions $F(t; z_1, \ldots, z_n)$ such that

$$F(t; z_1, \ldots, z_n) \prod_{j=1}^{n} \left(1 - e^{2\pi i (t - z_j)/p}\right) = P\left(e^{2\pi i t/p}\right) \tag{5.1}$$

is a polynomial in $e^{2\pi i t/p}$ of degree at most n and

$$F(t; z_1, \ldots, z_j + p, \ldots, z_n) = F(t; z_1, \ldots, z_n), \quad j = 1, \ldots, n.$$

Notice that the definition (5.1) implies that any $F(t) \in \widehat{\mathcal{F}}_q$ is a periodic function of t: $F(t + p) = F(t)$.

For any function $F \in \widehat{\mathcal{F}}_q$ we set

$$F(\pm\infty; z_1, \ldots, z_n) = \lim_{t \to \pm\infty} F(t; z_1, \ldots, z_n). \tag{5.2}$$

Let $\mathcal{F}_q \subset \widehat{\mathcal{F}}_q$ be the subspace of functions $F(t; z_1, \ldots, z_n)$ such that $F(\pm\infty; z_1, \ldots, z_n) = 0$, that is the polynomial P in (5.1) obeys extra conditions

$$P(0) = 0 \quad \text{and} \quad \deg P < n.$$

Let $\mathcal{Q} \subset \widehat{\mathcal{F}}_q$ be the following subspace: $\mathcal{Q} = \mathbb{C}1 \oplus \mathbb{C}\Theta$, where

$$\Theta(t) = \prod_{j=1}^{n} \frac{1 + e^{2\pi i (t - z_j)/p}}{1 - e^{2\pi i (t - z_j)/p}}. \tag{5.3}$$

Let $\phi(t)$ be the *phase function*:

$$\phi(t) = \frac{\prod_{j=1}^{n} \Gamma\left(\frac{t - z_j - \hbar}{p}\right)}{\Gamma\left(\frac{t - z_j}{p}\right)}. \tag{5.4}$$

LEMMA 5.1. – *For any $\varepsilon > 0$ the phase function $\phi(t)$ has the following asymptotics*

$$\phi(t) = (t/p)^{-n\hbar/p}\left(1 + o(1)\right)$$

as $t \to \infty$, $\varepsilon < |\arg(t/p)| < \pi - \varepsilon$.

The statement follows from the Stirling formula.

Denote by D the operator defined by

$$Df(t) = f(t) - f(t+p)\prod_{j=1}^{n}\frac{t-z_j-\hbar}{t-z_j}, \tag{5.5}$$

which means

$$\phi(t)Df(t) = \phi(t)f(t) - \phi(t+p)f(t+p).$$

We call the functions of the form Df the *total differences*. For any $s \in \mathbb{Z}$ let $\bigcup_s^{\pm}$ be the following sets:

$$\bigcup_s^{+} = \bigcup_{j=1}^{n}(z_j + \hbar + p\mathbb{Z}_{\leqslant s}), \qquad \bigcup_s^{-} = \bigcup_{j=1}^{n}(z_j - p\mathbb{Z}_{\leqslant s}). \tag{5.6}$$

Let $I(w, W)$ be the *hypergeometric integral*:

$$I(w, W) = \int_C \phi w W\, dt, \tag{5.7}$$

where C is a simple curve separating the sets $\bigcup_1^{+}$ and $\bigcup_1^{-}$, and going from $-\infty$ to $+\infty$. More precisely, the contour C admits a parametrization $\rho : \mathbb{R} \to \mathbb{C}$ such that $\rho(u) \to \pm\infty$ and $\operatorname{Im}\rho(u)$ has finite limits as $u \to \pm\infty$. Recall that we assume $\operatorname{Im} p < 0$ and $p = 2\hbar$.

Remark. – A construction of the hypergeometric integral for the qKZ equation at general level is given in [23] for the rational case and in [24] for the trigonometric case. In this paper we adapt the general construction from [23] to the case of the qKZ equation at level zero.

LEMMA 5.2. – *Let $w \in \mathcal{F}$, $W \in \mathcal{F}_q$ or $w \in \mathcal{F}^{(\ell)}$, $W \in \widehat{\mathcal{F}}_q$. Then the integral $I(w, W)$ is absolutely convergent and does not depend on a particular choice of the contour C.*

Proof. – The integrand $\phi(t)w(t)W(t)$ of the integral $I(w, W)$ behaves like $O(t^{-2})$ as $t \to \pm\infty$, which proves the convergence of the integral.

The poles of the integrand $\phi w W$ belong to the set $\bigcup_0^{+} \cup \bigcup_0^{-}$. Therefore, a homotopy class of the contour C in the complement of the singularities of the integrand does not depend on a particular choice of the contour and the same does the value of the integral $I(w, W)$. $\square$

LEMMA 5.3. – *Assume that $w \in \mathcal{F}^{(\ell)}$, $W \in \mathcal{Q}$. Then we have $I(w, W) = 0$.*

Proof. – Let $W = 1$. Then the integrand of the integral $I(w, 1)$ equals $\phi(t)w(t)$ and has no poles at points of the set $\bigcup_1^-$. Moreover, the integrand uniformly behaves like $O(t^{-2})$ as $t \to \infty$ in the semiplane $\operatorname{Im} t \leqslant 0$. Therefore, for any large negative A, the contour C can be replaced by the line $\{t \in \mathbb{C} \mid \operatorname{Im} t = A\}$ without changing the integral $I(w, 1)$. Tending A to $-\infty$ we obtain that $I(w, 1) = 0$.

If $W = \Theta$, then the integrand $\phi(t)w(t)\Theta(t)$ has no poles at points of the set $\bigcup_1^+$ and uniformly behaves like $O(t^{-2})$ as $t \to \infty$ in the semiplane $\operatorname{Im} t \geqslant 0$. Therefore, for any large positive A, the contour C can be replaced by the line $\{t \in \mathbb{C} \mid \operatorname{Im} t = A\}$ without changing the integral $I(w, \Theta)$, and tending A to $+\infty$ we obtain that $I(w, \Theta) = 0$. □

LEMMA 5.4. – *Assume that $w \in D\mathcal{F}$, $W \in \mathcal{F}_q$ or $w \in D\mathcal{F}^{(\ell+1)}$, $W \in \widehat{\mathcal{F}}_q$. Then we have $I(w, W) = 0$.*

Proof. – Let C be the contour in the formula (5.7). For any $A > 0$ set $C^A = \{t \in C \mid |\operatorname{Re} t| \leqslant A\}$. Let C_p^A be the contour C^A shifted by p: $t \in C^A \Leftrightarrow (t + p) \in C_p^A$. Let $\Delta_\pm^A$ be segments of length $|p|$ such that the contour $C^A + \Delta_+^A - C_p^A - \Delta_-^A$ is closed and $\pm\operatorname{Re} t > 0$ for $t \in \Delta_\pm^A$.

Let $w = D\widetilde{w}$. The integral defining $I(w, W)$ is convergent. Thus we have

$$I(w, W) = \lim_{A\to\infty} \int_{C^A} \phi w W\, dt.$$

Using formulae (5.5) we obtain

$$\begin{aligned}\int_{C^A} \phi w W\, dt &= \int_{C^A - C_p^A} \phi \widetilde{w} W\, dt \\ &= \int_{C^A + \Delta_+^A - C_p^A - \Delta_-^A} \phi \widetilde{w} W\, dt + \int_{\Delta_-^A - \Delta_+^A} \phi \widetilde{w} W\, dt.\end{aligned}$$

Since there is no poles of the integrand $\phi\widetilde{w}W$ inside the contour $C^A - \Delta_+^A - C_p^A + \Delta_-^A$, the corresponding integral equals zero. Moreover,

$$\begin{aligned}&\lim_{A\to\infty} \int_{\Delta_-^A - \Delta_+^A} \phi \widetilde{w} W\, dt \\ &\quad = p \lim_{A\to\infty} \big(\phi(-A)\widetilde{w}(-A)W(-A) - \phi(A)\widetilde{w}(A)W(A)\big) = 0\end{aligned}$$

under the assumptions of the lemma. The lemma is proved. □

LEMMA 5.5. – *Let a function f be such that $(f - Df) \in \mathcal{F}^{(\ell+1)}$. Then for any $W \in \widehat{\mathcal{F}}_q$ we have that $I(Df, W) = 0$.*

The proof is similar to the proof of the second part of Lemma 5.4.

Remark. – Lemmas 5.2, 5.3 and 6.5 below remain valid under weaker assumptions that the contour C separates the sets $\bigcup_0^+$ and $\bigcup_0^-$. For Lemma 5.4 it suffices to assume that C separates the sets $\bigcup_0^+$ and $\bigcup_1^-$, and for Lemma 5.5, that C separates the sets $\bigcup_1^+$ and $\bigcup_0^-$.

6. SOLUTIONS OF THE QKZ EQUATION AT LEVEL 0

Denote by $I^{\otimes\ell}(w, W)$ the following integral:

$$I^{\otimes\ell}(w, W) = \int\limits_{C^\ell} w(t_1, \ldots, t_\ell) W(t_1, \ldots, t_\ell) \prod_{a=1}^{\ell} \phi(t_a)\, dt_a. \qquad (6.1)$$

For any $W \in \widehat{\mathcal{F}}_q^{\otimes\ell}$, define a function $\psi_W(z_1, \ldots, z_n)$ with values in $(V^{\otimes n})_\ell$ as follows:

$$\psi_W(z_1, \ldots, z_n) = \sum_{\substack{M \subset \{1,\ldots,n\} \\ \# M = \ell}} I^{\otimes\ell}(w_M, W) v_M, \qquad (6.2)$$

where v_M is defined by (4.2).

Remark. – It is clear that for any $W \in \widehat{\mathcal{F}}_q^{\otimes\ell}$ we have $\psi_W = \frac{1}{\ell!} \psi_{\mathrm{Asym}\, W}$.

PROPOSITION 6.1. – *For any $W \in \mathcal{Q} \otimes \widehat{\mathcal{F}}_q^{\otimes(\ell-1)}$ we have $\psi_W(z_1, \ldots, z_n) = 0$.*

The claim follows from Lemma 5.3.

PROPOSITION 6.2. – *For any $W \in \widehat{\mathcal{F}}_q^{\otimes\ell}$ the function $\psi_W(z_1, \ldots, z_n)$ satisfies the condition*

$$\Sigma^+ \psi_W(z_1, \ldots, z_n) = 0.$$

The claim follows from Lemmas 3.5 and 5.4.

THEOREM 6.3. – *For any $W \in \widehat{\mathcal{F}}_q^{\otimes\ell}$ the function $\psi_W(z_1, \ldots, z_n)$ is a solution of the qKZ equation* (2.2) *taking values in* $(V^{\otimes n})_\ell$.

Proof. – The statement follows from the results on formal integral representations for solutions of the qKZ equation [25] and Lemmas 5.4, 5.5. We give more details in Appendix C. □

Let

$$\tilde{v}_M = \sum_{\substack{N\subset\{1,\dots,n\}\\ \#N=\ell}} v_N \operatorname{Res} w_N(\hat{z}_M), \qquad M\subset\{1,\dots,n\},\ \#M=\ell, \tag{6.3}$$

be another basis in $(V^{\otimes n})_\ell$. Then we have

$$\psi_W(z_1,\dots,z_n) = \sum_{\substack{M\subset\{1,\dots,n\}\\ \#M=\ell}} I^{\otimes\ell}(\tilde{w}_M, W)\tilde{v}_M. \tag{6.4}$$

For $W=\prod_{a=1}^n W_a$, $W_a\in\widehat{\mathcal{F}}_q$, $a=1,\dots,\ell$, the last formula can be written in the determinant form

$$\psi_W(z_1,\dots,z_n) = \sum_{\substack{M\subset\{1,\dots,n\}\\ \#M=\ell}} \det\big[I(\mu_M^{(m_a)}, W_b)\big]_{a,b=1}^{\ell}\tilde{v}_M. \tag{6.5}$$

The solutions $\psi_W(z_1,\dots,z_n)$ can be written also via suitable polynomials rather than rational functions. For any $M\subset\{1,\dots,n\}$ set

$$P_M^+(t) = \prod_{m\in M}(t-z_m-2\hbar), \qquad P_M^-(t) = \prod_{k\notin M}(t-z_k-2\hbar). \tag{6.6}$$

Denote by $T_\hbar$ the operator defined by $T_\hbar f(t)=f(t)-f(t+\hbar)$. For any rational function $f(t)$ let $[f(t)]_+\in\mathbb{C}[t]$ be its polynomial part

$$f(t) = [f(t)]_+ + \mathrm{o}(1), \qquad t\to\infty.$$

Let $Q_M^{(1)},\dots,Q_M^{(\ell)}$ be the polynomials given by

$$\begin{aligned} Q_M^{(a)}(t) = {}& P_M^+(t+\hbar)\left[T_\hbar\left(\frac{P_M^-(t)}{P_M^+(t+\hbar)}\left[\frac{P_M^+(t+\hbar)}{(t+\hbar)^a}\right]_+\right)\right]_+ \\ & + P_M^-(t)\left[T_\hbar\left(\frac{P_M^+(t)}{t^a}\right)\right]_+. \end{aligned} \tag{6.7}$$

Remark. – The polynomials $Q_M^{(a)}$ coincide with the polynomials introduced in [17] modulo a certain change of variables. Let us write the dependence of the polynomials $Q_M^{(a)}$ on $z_1, \ldots, z_n$ explicitly:

$$Q_M^{(a)}(t) = Q_M^{(a)}(t; z_1, \ldots, z_n).$$

Let $A_a^c(t|\lambda_1, \ldots, \lambda_\ell|\mu_1, \ldots, \mu_{\ell-2c})$ be the polynomial defined by (118) in [17]. Let $c = 0$ or $c = -1$ and $n = 2\ell - 2c$. Then

$$\begin{aligned} &A_a^c(t|\hat{z}_M|\hat{z}_{\overline{M}}) = Q_M^{(\ell-c-a)}(t - \hbar/2; z_1 - 2\hbar, \ldots, z_n - 2\hbar), \\ &\quad a = 1, \ldots, \ell - c - 1, \end{aligned}$$

where $\overline{M} = \{1, \ldots, n\} \setminus M$ and $\hat{z}_{\overline{M}}$ is defined similarly to $\hat{z}_M$, cf. (3.1). Notice that there is a misprint in (118) in [17].

The polynomials $Q_M^{(a)}$ are the rational analogues of the polynomials $A_b^{(j,k)}$ introduced in [6] for the trigonometric case. The precise correspondence is as follows:

$$\begin{aligned} &A_a^{(\ell,n-\ell)}(t|\hat{z}_M|\hat{z}_{\overline{M}}) \sim Q_M^{(\ell+1-a)}(t - \hbar/2; z_1 - 2\hbar, \ldots, z_n - 2\hbar), \\ &\quad a = 1, \ldots, \ell. \end{aligned}$$

LEMMA 6.4. – *For any $M \subset \{1, \ldots, n\}$, $\#M = \ell$, and any $m \in M$ the following identity holds*:

$$\begin{aligned} D\left(\prod_{\substack{k=1 \\ k\neq m}}^{n} (t - z_k - 2\hbar)\right) &= \hbar \prod_{\substack{k=1 \\ k\neq m}}^{n} (z_m - z_k - \hbar)\mu_M^{(m)}(t) \\ &\quad + \sum_{a=1}^{\ell} Q_M^{(a)}(t)(z_m + 2\hbar)^{a-1}. \end{aligned}$$

The lemma is proved in Appendix D.

For any $M \subset \{1, \ldots, n\}$ set

$$v_M^S = \prod_{1\leqslant a<b\leqslant \ell} \frac{(z_{m_a} - z_{m_b})}{(z_{m_a} - z_{m_b} - \hbar)(z_{m_a} - z_{m_b} + \hbar)} \tilde{v}_M, \tag{6.8}$$

cf. (3.7), (6.3), which provides that $v_{M_{\mathrm{ext}}}^S = v_{M_{\mathrm{ext}}} + \cdots$, where dots stand for a linear combination of vectors v_M, $M \neq M_{\mathrm{ext}}$, $\#M = \ell$. The main property of this basis is given by Corollary C.1.

Using Lemmas 5.4, 6.4 and formulae (6.5), (6.8) we can rewrite the solution $\psi_W(z_1, \ldots, z_n)$, $W \in \mathcal{F}_q^{\otimes \ell}$, via the polynomials $Q_M^{(1)}, \ldots, Q_M^{(\ell)}$ and the basis $\{v_M^S\}_{\#M=\ell}$. Let $W_1, \ldots, W_\ell \in \mathcal{F}_q$ and $W(t_1, \ldots, t_\ell) = W_1(t_1) \cdots W_\ell(t_\ell)$. Then

$$\psi_W(z_1,\dots,z_n)=(-\hbar)^{-\ell}\sum_{\substack{M\subset\{1,\dots,n\}\\ \#M=\ell}}\prod_{\substack{k\notin M\\ m\in M}}\frac{1}{z_m-z_k-\hbar}$$

$$\times\det\bigl[I(Q_M^{(a)},W_b)\bigr]_{a,b=1}^{\ell}v_M^S. \tag{6.9}$$

Observe that if Q is a polynomial and $W\in\mathcal{F}_q$, then the integrand of the integral $I(Q,W)$ has no poles at points $z_m, z_m-p, z_m+\hbar+p$, $m=1,\dots,n$, and we have

$$I(Q,W)=\int\limits_{C'}\phi QW\,dt \tag{6.10}$$

for any contour C' going from $-\infty$ to $+\infty$ and separating the sets $\bigcup_0^+$ and $\bigcup_{-1}^-$. For instance, if $z_1,\dots,z_n$ are real, then we can take $C'=3\hbar/2+\mathbb{R}$.

Let $2\ell=n$. This means that we consider a singlet solution $\psi(z_1,\dots,z_n)$ of the qKZ equation (2.2):

$$\Sigma^j\psi(z_1,\dots,z_n)=0,\quad j=\pm,3.$$

For any $M\subset\{1,\dots,n\}$, $\#M=\ell$, let ν_M and $\tilde\nu_M$ be the following functions:

$$\nu_M(t)=\prod_{k\notin M}(t-z_k-2\hbar),$$

$$\tilde\nu_M(t)=\prod_{k\notin M}(t-z_k-2\hbar)-\prod_{k\notin M}(t-z_k-\hbar)\prod_{m\in M}\frac{(t-z_m-\hbar)}{(t-z_m)}.$$

It is clear that

$$\tilde\nu_M=D\nu_M. \tag{6.11}$$

LEMMA 6.5. – *Let $2\ell=n$. Then*

$$I(\tilde\nu_M,W)=p^{\ell+1}\bigl(W(-\infty)-W(+\infty)\bigr)$$

for any $W\in\widehat{\mathcal{F}}_q$.

Proof. – We use notations from the proof of Lemma 5.4. The integrand $\phi(t)\tilde\nu_M(t)W(t)$ behaves like $O(t^{-2})$ as $t\to\pm\infty$. Hence, the integral $I(\tilde\nu_M,W)$ is convergent and we have

$$I(\tilde\nu_M,W)=\lim_{A\to\infty}\int\limits_{C^A}\phi\tilde\nu_M W\,dt.$$

Similarly to the proof of Lemma 5.4, using the formula (6.11), we obtain

$$\begin{aligned}
I(\tilde{\nu}_M, W) &= \lim_{A\to\infty} \int\limits_{\Delta_-^A - \Delta_+^A} \phi \nu_M W \, dt \\
&= p \lim_{A\to\infty} \bigl(\phi(-A)\nu_M(-A)W(-A) - \phi(A)\nu_M(A)W(A)\bigr) \\
&= p^{\ell+1}\bigl(W(-\infty) - W(+\infty)\bigr).
\end{aligned}$$

The lemma is proved. □

LEMMA 6.6. – *Let $2\ell = n$. For any $M \subset \{1, \ldots, n\}$, $\#M = \ell$, we have*

$$\tilde{\nu}_M(t) = \sum_{m\in M} \mu_M^{(m)}(t) \operatorname{res} \tilde{\nu}_M(z_m).$$

Proof. – Both sides of the formula are rational functions in t with at most simple poles at points z_m, $m \in M$, and have the same growth $O(t^{\ell-2})$ as $t \to \infty$. Moreover, they have the same residues at points z_m, $m \in M$, and the same values at the points $z_m + \hbar$, $m \in M$, which completes the proof. □

COROLLARY 6.7. – *Let $2\ell = n$. Then for any $M \subset \{1, \ldots, n\}$, $\#M = \ell$, we have*

$$\tilde{w}_M(t_1, \ldots, t_\ell) = \frac{1}{\operatorname{res} \tilde{\nu}_M(z_{m_\ell})} \operatorname{Asym}\bigl(\mu_M^{(m_1)}(t_1) \cdots \mu_M^{(m_{\ell-1})}(t_{\ell-1}) \tilde{\nu}_M(t_\ell)\bigr).$$

The last corollary, the formula (6.4) and Lemma 6.5 imply the next theorem.

THEOREM 6.8. – *Let $2\ell = n$. Let $W_1, \ldots, W_{\ell-1} \in \mathcal{F}_q$, $W_\ell \in \widehat{\mathcal{F}}_q$ and*

$$W(t_1, \ldots, t_\ell) = W_1(t_1) \cdots W_\ell(t_\ell).$$

Then

$$\begin{aligned}
\psi_W(z_1, \ldots, z_n) = p^{\ell+1}&\bigl(W_\ell(+\infty) - W_\ell(-\infty)\bigr) \\
&\times \sum_{\substack{M\subset\{1,\ldots,n\} \\ \#M=\ell}} \frac{\prod_{a=1}^{\ell-1}(z_{m_\ell} - z_{m_a})}{\prod_{k=1}^{n}(z_{m_\ell} - z_k - \hbar)} \\
&\times \det\bigl[I\bigl(\mu_M^{(m_a)}, W_b\bigr)\bigr]_{a,b=1}^{\ell-1} \tilde{\nu}_M.
\end{aligned}$$

Now we can rewrite the solution of the qKZ equation at level 0 given by Theorem 6.8 via the polynomials $Q_M^{(1)}, \ldots, Q_M^{(\ell)}$ and recover the formulae from [17] for the singlet form-factors in the $SU(2)$-invariant Thirring model. For $2\ell = n$ the polynomials can be written in the form which appeared for the first time in the book [17]:

$$Q_M^{(a)}(t) = P_M^+(t+\hbar)\left[T_\hbar\left((t+\hbar)^{-a}P_M^-(t)\right)\right]_+ + P_M^-(t)\left[T_\hbar\left(t^{-a}P_M^+(t)\right)\right]_+, \tag{6.12}$$

cf. (6.6), (6.7). Notice that the polynomial $Q_M^{(\ell)}$ vanishes identically in this case.

Finally, using Lemmas 5.4, 6.4 and formulae (6.4), (6.8), we have the following statement.

THEOREM 6.9. – *Let $2\ell = n$. Let $W_1, \ldots, W_{\ell-1} \in \mathcal{F}_q$, $W_\ell \in \widehat{\mathcal{F}}_q$ and*

$$W(t_1, \ldots, t_\ell) = W_1(t_1)\cdots W_\ell(t_\ell).$$

Then

$$\psi_W(z_1, \ldots, z_n) = 2^\ell p\left(W_\ell(+\infty) - W_\ell(-\infty)\right) \sum_{\substack{M\subset\{1,\ldots,n\}\\ \#M=\ell}} \prod_{\substack{k\notin M\\ m\in M}} \frac{1}{z_k - z_m + \hbar}$$
$$\times \det\left[I\left(Q_M^{(a)}, W_b\right)\right]_{a,b=1}^{\ell-1} v_M^S.$$

The last formula coinsides with Smirnov's formula for the singlet solutions of the qKZ equation at level 0 given in [17].

Remark. – F. Smirnov used another basis $\{\omega_M\}$ in his construction of solutions of the qKZ equation:

$$\omega_M = \prod_{\substack{k\notin M\\ m\in M}} \frac{z_k - z_m}{z_k - z_m + \hbar} v_M^S. \tag{6.13}$$

Set $M'_{\text{ext}} = \{n - \ell + 1, \ldots, n\}$. Then $\omega_{M'_{\text{ext}}} = v_{M'_{\text{ext}}}$. The main property of this basis is given by Corollary C.1.

Remark. – In the difference case the space of periodic functions $\widehat{\mathcal{F}}_q$ plays the role of the set of admissible contours $\widehat{C}$ in the differential case and the subspace $\mathcal{F}_q$ is an analogue of the set of cycles C. By the formula (6.11) and Corollary 6.7 we observe that for $n = 2\ell$, which is the case of zero weight, the integrand in the right hand side of the formula (6.4) for solutions of the qKZ equation is a total difference. So that, for any $W \in \mathcal{F}_q^{\otimes \ell}$ we have

$$\psi_W(z_1, \ldots, z_n) = 0.$$

Therefore, to produce nonzero solutions of the qKZ equation we inevitably have to consider difference analogues of unclosed admissible contours in the differential case.

7. SOLUTIONS OF THE QKZ EQUATION FROM THE REPRESENTATION THEORY

The aim of this section is to investigate the solutions of (2.2) which is obtained from the representation theory of the centrally extended Yangian double $(\widehat{\mathcal{D}Y}_\hbar(sl_2))$ [9,5]. It is a Hopf algebra associated with the R-matrix $R^{\pm}(u) = \rho^{\pm}(u)R(u)$, where $\rho^{\pm}(u)$ are certain scalar factors and $R(u)$ is given by (2.1) [9].

The algebra $\widehat{\mathcal{D}Y}_\hbar(sl_2)$ possesses two-dimensional evaluation representation V_z and the level one infinite-dimensional representation $\mathcal{H}$. In [9,5] the intertwining operators $\Phi(y): \mathcal{H} \to \mathcal{H} \otimes V_y$ and $\Psi(z): \mathcal{H} \to V_z \otimes \mathcal{H}$ are constructed. We define the components of the intertwining operators by

$$\Phi(z)v = \sum_\nu \Phi_\nu(z)v \otimes v_\nu, \qquad \Psi(z)v = \sum_\varepsilon v_\varepsilon \otimes \Psi_\varepsilon(z)v,$$

where $v \in \mathcal{H}$ and $v_\varepsilon \in V$, $\varepsilon = \pm$.

Let us consider $(V^{\otimes n})_\ell$ and $(V^{\otimes n'})_{\ell'}$-valued functions:

$$\psi^K(z_1, \ldots, z_n; y_1, \ldots, y_{n'}) = \sum_{\substack{M \subset \{1,\ldots,n\} \\ \#M = \ell}} \Omega_M^K(z_1, \ldots, z_n; y_1, \ldots, y_{n'}) v_M,$$

$$\widetilde{\psi}_M(z_1, \ldots, z_n; y_1, \ldots, y_{n'}) = \sum_{\substack{K \subset \{1,\ldots,n'\} \\ \#K = \ell'}} \Omega_M^K(z_1, \ldots, z_n; y_1, \ldots, y_{n'}) v_K,$$

where $\Omega_M^K(z; y)$ is a certain function proportional to the ratio of traces

$$\frac{\operatorname{tr}_{\mathcal{H}}(e^{pd}\Psi_{\varepsilon_n}(z_n) \cdots \Psi_{\varepsilon_1}(z_1)\Phi_{\nu_1}(y_1) \cdots \Phi_{\nu_{n'}}(y_{n'}))}{\operatorname{tr}_{\mathcal{H}}(e^{pd})},$$

where $M = \{j \mid \varepsilon_j = -,\ j = 1, \ldots, n\} \subset \{1, \ldots, n\}$, $K = \{i \mid \nu_i = +,\ i = 1, \ldots, n'\} \subset \{1, \ldots, n'\}$ and the proportionality coefficient does not depend on M and K, cf. [10]. Notice that the sets M and K can be empty. The trace of the composition of the intertwining operators was calculated in [10,1]. The formula for $\Omega_M^K(z; y)$ is

$$\begin{aligned}
\Omega_M^K(z_1, \ldots, z_n; y_1, \ldots, y_{n'}) &= \delta_{n-2\ell, n'-2\ell'} \prod_{j=1}^{n} \mathrm{e}^{i\pi(\ell-\ell')z_j/p} \prod_{i=1}^{n'} \mathrm{e}^{i\pi(\ell'-\ell)y_i/p} \\
&\times \int_{\overline{C}} \cdots \int_{\overline{C}} dt_1 \cdots dt_\ell \int_{\widetilde{C}} \cdots \int_{\widetilde{C}} du_1 \cdots du_{\ell'} P_M(t; z) P_K(u; y) \\
&\times \prod_{s=1}^{\ell} \prod_{j=1}^{n} \left[\Gamma\left(\frac{z_j - t_s}{p}\right) \Gamma\left(\frac{t_s - z_j - \hbar}{p}\right) \right] \\
&\times \prod_{r=1}^{\ell'} \prod_{i=1}^{n'} \left[\Gamma\left(\frac{u_r - y_i}{p}\right) \Gamma\left(\frac{y_i - u_r + \hbar}{p}\right) \right] \\
&\times \prod_{1 \leqslant s' < s'' \leqslant \ell} \left[\frac{\sin(\pi(t_{s'} - t_{s''})/p)}{\Gamma((t_{s'} - t_{s''} - \hbar)/p) \Gamma(1 + (t_{s''} - t_{s'} - \hbar)/p)} \right] \\
&\times \prod_{1 \leqslant r'' < r' \leqslant \ell'} \left[\frac{\sin(\pi(u_{r'} - u_{r''})/p)}{\Gamma((u_{r'} - u_{r''} + \hbar)/p) \Gamma(1 + (u_{r''} - u_{r'} + \hbar)/p)} \right] \\
&\times \frac{\prod_{r=1}^{\ell'} \prod_{j=1}^{n} \sin(\pi(u_r - z_j)/p) \prod_{s=1}^{\ell} \prod_{i=1}^{n'} \sin(\pi(t_s - y_i - \hbar)/p)}{\prod_{s=1}^{\ell} \prod_{r=1}^{\ell'} \sin(\pi(u_r - t_s)/p) \sin(\pi(u_r - t_s + \hbar)/p)}.
\end{aligned} \tag{7.1}$$

In the last line of (7.1) we correct a misprint made in [10]. Formula (7.1) can also be obtained from the corresponding formula for the quantum affine algebra in [7] by taking the scaling limit. Particular specializations of the formula (7.1) are given in [12,16].

For $M = \{m_1 < \cdots < m_\ell\}$ the polynomial $P_M(t; z)$ is defined by the formula:

$$P_M(t; z) = \prod_{a=1}^{\ell} \left(\prod_{j > m_a} (t_a - z_j) \prod_{j < m_a} (t_a - z_j - \hbar) \right).$$

The contour $\overline{C}$ and $\widetilde{C}$ are specified as follows. The contour $\overline{C}$ for the integration over t_a, $a = 1, \ldots, \ell$, is a simple curve separating the sets of the points

$$\bigcup_{j=1}^{n}(z_j+\hbar-p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{b=1}^{\ell'}(u_b+\hbar-p\mathbb{Z}_{>0}),\quad \bigcup_{b=1}^{\ell'}(u_b-p\mathbb{Z}_{>0})$$

and

$$\bigcup_{j=1}^{n}(z_j+p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{b=1}^{\ell'}(u_b+\hbar+p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{b=1}^{\ell'}(u_b+p\mathbb{Z}_{\geqslant 0}),$$

and going from $-\infty$ to $+\infty$. More precisely the contour $\overline{C}$ admits a parametrization $\rho:\mathbb{R}\to\mathbb{C}$ such that $\rho(u)\to\pm\infty$ and $\operatorname{Im}\rho(u)$ has finite limits as $u\to\pm\infty$. Similarly the contour $\widetilde{C}$ separates the sets of the points

$$\bigcup_{i=1}^{n'}(y_i-p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{a=1}^{\ell}(t_a-\hbar-p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{a=1}^{\ell}(t_a-p\mathbb{Z}_{\geqslant 0})$$

and

$$\bigcup_{i=1}^{n'}(y_i+\hbar+p\mathbb{Z}_{\geqslant 0}),\quad \bigcup_{a=1}^{\ell}(t_a-\hbar+p\mathbb{Z}_{>0}),\quad \bigcup_{a=1}^{\ell}(t_a+p\mathbb{Z}_{>0}),$$

and going from $-\infty$ to $+\infty$.

Notice that for $\ell'=0$ the contour $\overline{C}$ coincides with the contour C used in the definition of the hypergeometric integral (5.7).

Due to the commutation relations of the intertwining operators and the cyclic property of the trace, the function $\psi^K(z_1,\dots,z_n;y_1,\dots,y_{n'})$ solves the qKZ equation (2.2) at level $-2+p/\hbar$ with respect to the variables z_j, $j=1,\dots,n$, for any set K and any values of the parameters y_i, $i=1,\dots,n'$. On the other hand the function $\widetilde{\psi}_M(z_1,\dots,z_n;y_1,\dots,y_{n'})$ solves the qKZ equation at level $-2-p/\hbar$ with respect to the variables y_i, $i=1,\dots,n'$, for any set M and any values of the parameters z_j, $j=1,\dots,n$, see [7,10].

From now on we assume that $p=2\hbar$ and we consider the solutions $\psi^K(z_1,\dots,z_n;y_1,\dots,y_{n'})$ of the qKZ equation at level zero.

It follows from (7.1) that $\psi^K(\cdot\,;y_1,\dots,y_{n'})\in(V^{\otimes n})_\ell$ with $2\ell=n-n'+2\ell'$. In general $\psi^K(z_1,\dots,z_n;y_1,\dots,y_{n'})$ does not satisfy the highest weight condition (2.4). Therefore we decompose it into the isotypic components with respect to the sl_2 action:

$$\psi^K(\cdot\,;y_1,\dots,y_{n'})=\sum_{j=0}^{n}(\Sigma^-)^j\psi^{[K,j]}(\cdot\,;y_1,\dots,y_{n'}),\tag{7.2}$$

$$\Sigma^+\psi^{[K,j]}=0,\qquad \psi^{[K,j]}\in(V^{\otimes n})_{\ell-j}.$$

Since the operators $K_j(z_1, \ldots, z_n)$, cf. (2.2), commute with the sl_2 action on $V^{\otimes n}$, each component $\psi^{[K,j]}(\cdot\,; y_1, \ldots, y_{n'})$ is a solution of the qKZ equation (2.2). Our general aim is to find functions

$$W[K, j; y_1, \ldots, y_{n'}] \in \widehat{\mathcal{F}}_q^{\otimes(\ell-j)}$$

such that

$$\psi^{[K,j]}(\cdot\,; y_1, \ldots, y_{n'}) = \psi_{W[K,j;y_1,\ldots,y_{n'}]},$$

where $\psi_{W[K,j;y_1,\ldots,y_{n'}]}$ is defined by (6.2). Notice that a function $W[K, j; y_1, \ldots, y_{n'}]$ is not uniquely defined, see Proposition 6.1 and the remark before it.

Up to now the formula of $W[K, j; y_1, \ldots, y_{n'}]$ for a general K is not yet found. Below we give simple examples.

Example 1. – Let $n' = n - 2\ell \geqslant 0$, $K = \emptyset$. Then we have

$$\psi^{\phi}(\cdot\,; y_1, \ldots, y_{n'}) = \psi_{W[\phi,0;y_1,\ldots,y_{n'}]}, \tag{7.3}$$

where

$$\begin{aligned} &W[\phi, 0; y_1, \ldots, y_{n'}](t_1, \ldots, t_\ell) \\ &= c_1 \prod_{a=1}^{\ell} \frac{\mathrm{e}^{2\pi i(2a-1)t_a/p} \prod_{k=1}^{n-2\ell} (1 + \mathrm{e}^{2\pi i(t_a - y_k)/p})}{\prod_{j=1}^{n} (1 - \mathrm{e}^{2\pi i(t_a - z_j)/p})}, \end{aligned}$$

$$c_1 = (-1)^{\ell(\ell-1)/2} 2^{\ell(n-\ell+1)} (\pi i p)^{\ell(2n-\ell+1)/2}.$$

For $n' = 0$, the empty product over k equals 1. In this case $W[\phi, 0] \in \mathcal{F}_q$ and $\psi_{W[\phi,0]} \equiv 0$ due to Lemmas 5.4 and 6.6.

Example 2. – Let $2\ell = n$, $n' = 2$, $\ell' = 1$, and $K = \{1\}$ or $\{2\}$. We have

$$\psi^{K}(\cdot\,; y_1, y_2) = \psi_{W[K,0;y_1,y_2]} + \Sigma^{-}(\psi_{W[K,1;y_1,y_2]}), \tag{7.4}$$

where

$$\begin{aligned} &W[K, 0; y_1, y_2](t_1, \ldots, t_\ell) \\ &= \sigma_K(y_2 - y_1 - \hbar) W[K, 1; y_1, y_2](t_1, \ldots, t_{\ell-1}) \widetilde{W}(t_\ell), \end{aligned} \tag{7.5}$$

$$\sigma_{\{1\}} = 1, \qquad \sigma_{\{2\}} = -1,$$

$$W[K,1;y_1,y_2](t_1,\dots,t_{\ell-1}) = c_2 \prod_{a=1}^{\ell-1} \frac{e^{2\pi i(2a-1)t_a/p} \prod_{k=1}^{2}(1+e^{2\pi i(t_a-y_k)/p})}{\prod_{j=1}^{n}(1-e^{2\pi i(t_a-z_j)/p})},$$

$$c_2 = i(-1)^{(\ell-1)(\ell-2)/2} 2^{\ell(\ell+1)} \pi^{2(1-\ell)} (\pi i p)^{(3\ell^2+3\ell+4)/2},$$

and $\widetilde{W}$ is an arbitrary function from $\widehat{\mathcal{F}}_q$ such that $2^\ell p(\widetilde{W}(+\infty) - \widetilde{W}(-\infty)) = 1$. For example we can take

$$\widetilde{W}(t) = (2^\ell p)^{-1} \prod_{j=1}^{n} \left(1 - e^{2\pi i(t-z_j)/p}\right)^{-1}.$$

Let us briefly discuss the meaning of Examples 1 and 2 in the context of integrable models of quantum field theory. If we set $\hbar = -\pi i$, $p = -2\pi i$, the model which corresponds to our consideration in this paper is the $SU(2)$ invariant Thirring model (ITM). Following the idea developed in [2] the function $\psi^K(z_1,\dots,z_n;y_1,\dots,y_{n'})$ is equal modulo a scalar factor to the n-particle form factor of the operator specified by $\Phi_{\nu_1}(y_1)\cdots\Phi_{\nu_{n'}}(y_{n'})$.

In [12] Lukyanov has introduced certain local operators. Some of them, up to normalizations, are

$$\Lambda_m(y) = \tfrac{1}{2}\left(\Phi_{\varepsilon_1}(y+\pi i)\Phi_{\varepsilon_2}(y) + \Phi_{\varepsilon_2}(y+\pi i)\Phi_{\varepsilon_1}(y)\right), \tag{7.6}$$

$$T(y) = \tfrac{1}{2}\left(\Phi_+(y+\pi i)\partial_y\Phi_-(y) - \Phi_-(y+\pi i)\partial_y\Phi_+(y)\right), \tag{7.7}$$

where $m = 0, \pm 1$, $\varepsilon_j = \pm$ and $m = (\varepsilon_1+\varepsilon_2)/2$. We will show that the form factors of (7.6) and (7.7) include the form factors obtained in [17, 20].

Remark. – We change the signs of the second terms in the definitions of $\Lambda_0(y)$ and $T(y)$ compared with [12] since we consider the S-matrix with a nonsymmetric crossing symmetry matrix [17] as opposed to [12].

We first present the formulae for form factors obtained in [17,20] in terms of the functions ψ_W. Define the funtions

$$W_\sigma \in \mathcal{F}_q^{\otimes(\ell-1)} \quad \text{and} \quad \widetilde{W}_\sigma \in \mathcal{F}_q^{\otimes(\ell-1)} \otimes \widehat{\mathcal{F}}_q, \quad \sigma = \pm,$$

by

$$W_\sigma(t_1,\dots,t_{\ell-1}) = \prod_{a=1}^{\ell-1}\left(e^{-(2a\mp 1)t_a} \prod_{j=1}^{n}\left(1 - e^{-(t_a-z_j)}\right)^{-1}\right),$$

$$\widetilde{W}_\sigma(t_1,\dots,t_\ell)=i2^{-\ell-1}\pi^{-1}W_\sigma(t_1,\dots,t_{\ell-1})\prod_{j=1}^{n}\left(1-e^{-(t_\ell-z_j)}\right)^{-1}.$$

Let f_σ^τ, $\tau=\pm,3$, $\sigma=\pm$, be the form factors of $SU(2)$ currents in the lightcone coordinates, cf. p. 38 in [20], and $f_{\mu\nu}$, $\mu,\nu=0,1$, the form factors of the energy-momentum tensor, cf. p. 106 in [17]. Then, using the formula (6.9) and Theorem 6.9, we have

$$\begin{aligned}f_\sigma^-(z_1,\dots,z_n)=\,&\mathrm{const}\prod_{i<j}\zeta(z_i-z_j)\exp\left(\frac{\ell-1-\sigma}{2}\sum_{j=1}^{n}z_j\right)\\&\times\psi_{W_\sigma}(z_1,\dots,z_n),\end{aligned}$$

$$\begin{aligned}&f_{\mu\nu}(z_1,\dots,z_n)\\&=\mathrm{const}\prod_{i<j}\zeta(z_i-z_j)\sum_{j=1}^{n}\left(e^{z_j}-(-1)^\nu e^{-z_j}\right)\exp\left(\frac{\ell-2}{2}\sum_{j=1}^{n}z_j\right)\\&\quad\times\left(\psi_{\widetilde{W}_+}(z_1,\dots,z_n)-(-1)^\mu\exp\left(\sum_{j=1}^{n}z_j\right)\psi_{\widetilde{W}_-}(z_1,\dots,z_n)\right),\end{aligned}$$

where $2\ell=n$, $\zeta(z)$ is defined in [17], p. 107, and the constants are independent of $z_1,\dots,z_n$. Formulae for the form factors f_σ^3 and f_σ^+ can be obtained by applying the operator Σ^- once or twice to f_σ^-: $f_\sigma^3=\Sigma^-f_\sigma^-$, $f_\sigma^+=(\Sigma^-)^2f_\sigma^-$. Notice that the difference equation satisfied by form factors of the $SU(2)$ ITM differs from the qKZ equation (2.2) by the sign $(-1)^{n/2}$, cf. (110) in [17]. This explains the appearance of the factors

$$\exp\left(\frac{\ell-1-\sigma}{2}\sum_{j=1}^{n}z_j\right)\quad\text{and}\quad\exp\left(\frac{\ell-2}{2}\sum_{j=1}^{n}z_j\right)$$

which, in principle, are not $2\pi i$ periodic functions of $z_1,\dots,z_n$.

Example 1 for $n=2\ell+2$, $n'=2$ shows that the n-particle form factor of the operator $\Lambda_{-1}(y)$ is proportional to $\psi_{W[\phi,0;y_1,y_2]}$. Notice that

$$W_\sigma(t_1,\dots,t_\ell)=\sigma^\ell\lim_{y\to-\sigma\infty}e^{(\sigma-1)\ell y}c_1^{-1}W[\phi,0;y+\pi i,y](t_1,\dots,t_\ell).$$

Thus f_σ^- is obtained from the form factor of $\Lambda_{-1}(y)$ by the specialization of the value of y. Similarly, f_σ^3 can be obtained from the form factor of $\Lambda_0(y)$, see Example 2. In the same way, it is possible to show from

Example 2 that $f_{\mu\nu}$, $\mu, \nu = 1, 2$, can be obtained from the form factor of $T(y)$ as suitable linear combinations of the limits $\lim_{y\to\pm\infty} T(y)$.

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APPENDIX A

Proof of Proposition 3.4. – Due to Lemmas 3.1 and 3.2, both $\{w^{cl}_M\}_{\#M=\ell}$ and $\{\widetilde{w}^{cl}_M\}_{\#M=\ell}$ are families of linearly independent functions from the space $\mathcal{F}_{cl}[\ell]$, because $\operatorname{Res} g^{cl}_M(\hat{z}_M) \neq 0$ under assumptions of the proposition. Hence,

$$\dim \mathcal{F}_{cl}[\ell] \geqslant \binom{n}{\ell},$$

and it suffices to prove the opposite inequality:

$$\dim \mathcal{F}_{cl}[\ell] \leqslant \binom{n}{\ell}.$$

Consider the space

$$\mathcal{G}_{cl} = \left\{ f(t_1, \dots, t_\ell) \mid f(t_1, \dots, t_\ell) \prod_{1\leqslant a<b\leqslant\ell} (t_a - t_b) \in \mathcal{F}_{cl}[\ell] \right\}$$

which is obviously isomorphic to $\mathcal{F}_{cl}[\ell]$. The definition of $\mathcal{F}_{cl}[\ell]$ implies that $f \in \mathcal{G}_{cl}$ iff f has the following properties:

(i) $f(t_1, \dots, t_\ell)$ is a symmetric rational function with at most simple poles at the hyperplanes $t_a = z_m$, $a = 1, \dots, \ell$, $m = 1, \dots, n$,

(ii) $f(t_1, \dots, t_\ell) \to 0$ as $t_1 \to \infty$ and $t_2, \dots, t_\ell$ are fixed,

(iii) $\operatorname{res}_{t_1=z_m}(\operatorname{res}_{t_2=z_m} f(t_1, \dots, t_\ell)) = 0$ for any $m = 1, \dots, n$.

Let $\mathfrak{X}$ be the set of sequences $(m_1, \dots, m_\ell)$ such that $m_1, \dots, m_\ell \in \{1, \dots, n\}$ are pairwise distinct. For any permutation σ of $1, \dots, \ell$ and any

$\mathfrak{m} = (m_1, \ldots, m_\ell) \in \mathfrak{X}$ set

$$\mathfrak{m}^\sigma = (m_{\sigma_1}, \ldots, m_{\sigma_\ell}).$$

LEMMA A.1. – *Let a function* $f(t_1, \ldots, t_\ell)$ *have the properties* (i)–(iii). *Then it has the form*:

$$f(t_1, \ldots, t_\ell) = \sum_{\mathfrak{m} \in \mathfrak{X}} \frac{A_\mathfrak{m}}{(t_1 - z_{m_1}) \cdots (t_\ell - z_{m_\ell})}$$

for suitable constants $\{A_\mathfrak{m}\}$ *such that* $A_\mathfrak{m} = A_{\mathfrak{m}^\sigma}$ *for any* $\mathfrak{m} \in \mathfrak{X}$ *and any permutation* σ.

Proof. – The lemma can be proved by the induction on ℓ using the partial fraction expansion of a rational function of one variable, cf. the proof of Lemma A.2. □

Since Lemma A.1 implies that $\dim \mathcal{G}_{cl} \leqslant \binom{n}{\ell}$, Proposition 3.4 is proved. □

Proof of Proposition 3.3. – Due to Lemmas 3.1 and 3.2, both $\{w_M\}_{\#M=\ell}$ and $\{\tilde{w}_M\}_{\#M=\ell}$ are families of linearly independent functions from the space $\mathcal{F}[\ell]$, because $\operatorname{Res} g_M(\hat{z}_M) \neq 0$ under the assumptions of the proposition. Hence,

$$\dim \mathcal{F}[\ell] \geqslant \binom{n}{\ell},$$

and it suffices to prove the opposite inequality:

$$\dim \mathcal{F}[\ell] \leqslant \binom{n}{\ell}.$$

Consider the space

$$\mathcal{G} = \left\{ f(t_1, \ldots, t_\ell) \mid f(t_1, \ldots, t_\ell) \prod_{1 \leqslant a < b \leqslant \ell} (t_a - t_b) \in \mathcal{F}[\ell] \right\}$$

which is obviously isomorphic to $\mathcal{F}[\ell]$. The definition of $\mathcal{F}[\ell]$ implies that $f \in \mathcal{G}$ iff f has the following properties:

(a) $f(t_1, \ldots, t_\ell)$ is a symmetric rational function with at most simple poles at the hyperplanes $t_a = z_m$, $a = 1, \ldots, \ell$, $m = 1, \ldots, n$,

(b) $f(t_1,\ldots,t_\ell)\to 0$ as $t_1\to\infty$ and $t_2,\ldots,t_\ell$ are fixed,
(c) $\operatorname{res}_{t=z_m} f(t,t+\hbar,t_3,\ldots,t_\ell)=0$ for any $m=1,\ldots,n$.

For any $\mathfrak{m}\in\mathfrak{X}$, set

$$h_{\mathfrak{m}}(t_1,\ldots,t_\ell)=\prod_{a=1}^{\ell}\frac{1}{t_a-z_{m_a}}\prod_{b=1}^{a-1}\frac{t_a-z_{m_b}-\hbar}{t_a-z_{m_b}}.$$

LEMMA A.2. – *Let a function $f(t_1,\ldots,t_\ell)$ have the properties* (a)–(c). *Then it has the form*:

$$f(t_1,\ldots,t_\ell)=\sum_{\mathfrak{m}\in\mathfrak{X}}A_{\mathfrak{m}}h_{\mathfrak{m}}(t_1,\ldots,t_\ell)\tag{A.1}$$

for suitable constants $\{A_{\mathfrak{m}}\}$ such that for any $\mathfrak{m}\in\mathfrak{X}$ and any permutation σ

$$A_{\mathfrak{m}^\sigma}=A_{\mathfrak{m}}\prod_{\substack{1\leqslant a<b\leqslant\ell\\ \sigma_a>\sigma_b}}\frac{z_{m_{\sigma_a}}-z_{m_{\sigma_b}}-\hbar}{z_{m_{\sigma_a}}-z_{m_{\sigma_b}}+\hbar}.\tag{A.2}$$

Proof. – Consider the partial fraction expansion of $f(t_1,\ldots,t_\ell)$ as a function of t_1:

$$f(t_1,\ldots,t_\ell)=\sum_{m=1}^{n}\frac{f_m(t_2,\ldots,t_\ell)}{t_1-z_m}.$$

The function $f_m(t_2,\ldots,t_\ell)$ has the properties:
(a′) $f_m(t_2,\ldots,t_\ell)$ is a symmetric rational function with at most simple poles at the hyperplanes $t_a=z_j$, $a=2,\ldots,\ell$, $j=1,\ldots,n$,
(b′) $f_m(t_2,\ldots,t_\ell)\to 0$ as $t_2\to\infty$ and $t_3,\ldots,t_\ell$ are fixed,
(c′) $f_m(z_m+\hbar,t_3,\ldots,t_\ell)=0$ and $\operatorname{res}_{t=z_l} f_m(t,t+\hbar,t_4,\ldots,t_\ell)=0$ for any $l=1,\ldots,n$.

Hence,

$$f_m(t_2,\ldots,t_\ell)=\sum_{\substack{l=1\\ l\neq m}}^{n}\frac{f_{lm}(t_3,\ldots,t_\ell)(t_2-z_m-\hbar)}{(t_2-z_l)(t_2-z_m)}.$$

The function $f_{lm}(t_3,\ldots,t_\ell)$ has the properties:
(a″) $f_{lm}(t_3,\ldots,t_\ell)$ is a symmetric rational function with at most simple poles at the hyperplanes $t_a=z_j$, $a=3,\ldots,\ell$, $j=1,\ldots,n$,
(b″) $f_{lm}(t_3,\ldots,t_\ell)\to 0$ as $t_3\to\infty$ and $t_4,\ldots,t_\ell$ are fixed,

(c$''$) $f_{lm}(z_l + \hbar, t_4, \dots, t_\ell) = 0$, $f_{lm}(z_m + \hbar, t_4, \dots, t_\ell) = 0$ and $\mathrm{res}_{t=z_k} f_{lm}(t, t+\hbar, t_5, \dots, t_\ell) = 0$ for any $k = 1, \dots, n$.

Hence,

$$f_{lm}(t_3, \dots, t_\ell) = \sum_{\substack{k=1 \\ k \neq l,m}}^{n} \frac{f_{klm}(t_4, \dots, t_\ell)(t_3 - z_l - \hbar)(t_3 - z_m - \hbar)}{(t_3 - z_k)(t_3 - z_l)(t_3 - z_m)},$$

etc. Finally we obtain the formula (A.1).

For any $\mathfrak{m} \in \mathfrak{X}$, let $\hat{z}_{\mathfrak{m}} \in \mathbb{C}^\ell$ be the point defined by $\hat{z}_{\mathfrak{m}} = (z_{m_1}, \dots, z_{m_\ell})$. It is clear that $\operatorname{Res} h_l(\hat{z}_{\mathfrak{m}}) = 0$ for $l \neq \mathfrak{m}$ and

$$\operatorname{Res} h_{\mathfrak{m}}(\hat{z}_{\mathfrak{m}}) = \prod_{1 \leqslant a < b \leqslant \ell} \frac{z_{m_a} - z_{m_b} + \hbar}{z_{m_a} - z_{m_b}}. \tag{A.3}$$

Since $f(t_1, \dots, t_\ell)$ is a symmetric function, we have $\operatorname{Res} f(\hat{z}_{\mathfrak{m}}) = \operatorname{Res} f(\hat{z}_{\mathfrak{m}^\sigma})$ for any $\mathfrak{m} \in \mathfrak{X}$ and any permutation σ. Therefore,

$$A_{\mathfrak{m}} \operatorname{Res} h_{\mathfrak{m}}(\hat{z}_{\mathfrak{m}}) = A_{\mathfrak{m}^\sigma} \operatorname{Res} h_{\mathfrak{m}^\sigma}(\hat{z}_{\mathfrak{m}^\sigma}),$$

which coincides with (A.2) because of (A.3). The lemma is proved. □

Since Lemma A.2 implies that $\dim \mathcal{G} \leqslant \binom{n}{\ell}$, Proposition 3.3 is proved. □

APPENDIX B

Proof of Lemma 3.5. – We give the proof only for the formula (3.5). The proof of the formula (3.6) is similar.

Fix a subset $M = \{m_2 < \dots < m_\ell\} \subset \{1, \dots, n\}$. Say that $a \prec k$ if $m_a < k$ and $a \succ k$ if $m_a > k$. Let $f_1, \dots, f_n$ be the following functions:

$$f_k(t_1, \dots, t_\ell) = \left(1 - \frac{t_1 - z_k - \hbar}{t_1 - z_k}\right) \prod_{1 \leqslant l < k} \frac{t_1 - z_l - \hbar}{t_1 - z_l} \times \prod_{\substack{a=2 \\ a \prec k}}^{\ell} (t_1 - t_a + \hbar) \prod_{\substack{a=2 \\ a \succ k}}^{\ell} (t_1 - t_a - \hbar), \quad k \notin M,$$

$$f_{m_b}(t_1, \dots, t_\ell) = \left(t_1 - t_b - \hbar - (t_1 - t_b + \hbar)\frac{t_1 - z_{m_b} - \hbar}{t_1 - z_{m_b}}\right)$$

$$\times \prod_{1\leqslant l<m_b} \frac{t_1 - z_l - \hbar}{t_1 - z_l} \prod_{2\leqslant a<b} (t_1 - t_a + \hbar)$$
$$\times \prod_{b<a\leqslant \ell} (t_1 - t_a - \hbar), \quad b = 2, \ldots, \ell,$$

so that

$$\prod_{a=2}^{\ell}(t_1 - t_a - \hbar) - \prod_{a=2}^{\ell}(t_1 - t_a + \hbar)\prod_{j=1}^{n}\frac{(t_1 - z_j - \hbar)}{(t_1 - z_j)} = \sum_{k=1}^{n} f_k(t_1, \ldots, t_\ell).$$

Set $m_1 = 0$ and $m_{\ell+1} = n + 1$ until the end of the proof. Let $k \notin M$. Then there is a unique a, $1 \leqslant a \leqslant \ell + 1$, such that $m_a < k < m_{a+1}$, and we get

$$f_k(t_1, \ldots, t_\ell) g_M(t_2, \ldots, t_\ell) = \hbar(-1)^{a-1} g_{M\cup\{k\}}(t_2, \ldots, t_a, t_1, t_{a+1}, \ldots, t_\ell),$$
$$\operatorname{Asym}\big(f_k(t_1, \ldots, t_\ell) g_M(t_2, \ldots, t_\ell)\big) = \hbar w_{M\cup\{k\}}(t_1, \ldots, t_\ell).$$

On the other hand, for any $a = 2, \ldots, \ell$, the product $f_{m_a}(t_1, \ldots, t_\ell) g_M(t_2, \ldots, t_\ell)$ is invariant with respect to the permutation of t_1 and t_a. Hence,

$$\operatorname{Asym}\big(f_{m_a}(t_1, \ldots, t_\ell) g_M(t_2, \ldots, t_\ell)\big) = 0,$$

which completes the proof. □

APPENDIX C

We give here the proof of Theorem 6.3 in order to make the paper selfcontained.

Proof of Theorem 6.3. – We identify a subset $M \subset \{1, \ldots, n\}$ with the sequence of signs $(\varepsilon_1, \ldots, \varepsilon_n)$ by the rule:

$$M = \{i \mid \varepsilon_i = -\},$$

cf. (4.2). Abusing notations we set $g_{\varepsilon_1,\ldots,\varepsilon_n} = g_M$ and $w_{\varepsilon_1,\ldots,\varepsilon_n} = w_M$ if the sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ corresponds to the subset M. We will indicate explicitly that the functions g_M and w_M depend on $z_1, \ldots, z_n$, that is, for $\#M = \ell$ we will write $g_M(t_1, \ldots, t_\ell; z_1, \ldots, z_n)$ and $w_M(t_1, \ldots, t_\ell; z_1, \ldots, z_n)$.

Notice that the integration contour C in the definition of the hypergeometric integral (5.7) obeys conditions which depends on $z_1, \ldots, z_n$. To indicate this we will write $C(z_1, \ldots, z_n)$.

Let us introduce the twisted shift operators $Z_1, \ldots, Z_n$ as follows

$$Z_m f(t_1, \ldots, t_\ell; z_1, \ldots, z_n) = f(t_1, \ldots, t_\ell; z_1, \ldots, z_m + p, \ldots, z_n) \times \prod_{a=1}^{\ell} \frac{t_a - z_m - p}{t_a - z_m - \hbar - p},$$

which means

$$f(t_1, \ldots, t_\ell; z_1, \ldots, z_m + p, \ldots, z_n) \prod_{a=1}^{\ell} \phi(t_a; z_1, \ldots, z_m + p, \ldots, z_n)$$
$$= Z_m f(t_1, \ldots, t_\ell; z_1, \ldots, z_n) \prod_{a=1}^{\ell} \phi(t_a; z_1, \ldots, z_n).$$

We denote by D_a the operator D acting on the variable t_a, cf. (5.5). Let $\mathcal{F}'$ be the space of functions $f(t)$ such that $(f - Df) \in \mathcal{F}^{(\ell+1)}$. Set $\widetilde{\mathcal{F}} = \left(\mathcal{F}^{(\ell+1)} + \mathcal{F}'\right)^{\otimes \ell}$. For any $a = 1, \ldots, \ell$, $f \in \widetilde{\mathcal{F}}$ and $W \in \widehat{\mathcal{F}}_q^{\otimes \ell}$ we have $I(W, D_a f) = 0$, due to Lemmas 5.4 and 5.5. We define the components of $R(z)$ by

$$R(z)_{\varepsilon,\varepsilon}^{\varepsilon,\varepsilon} = 1, \quad R(z)_{-\varepsilon,-\varepsilon}^{\varepsilon,\varepsilon} = \frac{z}{z+\hbar}, \quad R(z)_{\varepsilon,-\varepsilon}^{-\varepsilon,\varepsilon} = \frac{\hbar}{z+\hbar}, \quad \varepsilon = \pm.$$

It is easy to see that the function $\psi_W(z_1, \ldots, z_n)$ defined by (6.2) is a solution of the qKZ equation (2.2) if the following relations hold:

$$\begin{aligned} &w_{\varepsilon_1,\ldots,\varepsilon_{i+1},\varepsilon_i,\ldots,\varepsilon_n}(\cdot\,; z_1, \ldots, z_{i+1}, z_i, \ldots, z_n) \\ &= \sum_{\varepsilon_i',\varepsilon_{i+1}'=\pm} R(z_i - z_{i+1})_{\varepsilon_i,\varepsilon_{i+1}}^{\varepsilon_i',\varepsilon_{i+1}'} \\ &\quad \times w_{\varepsilon_1,\ldots,\varepsilon_i',\varepsilon_{i+1}',\ldots,\varepsilon_n}(\cdot\,; z_1, \ldots, z_i, z_{i+1}, \ldots, z_n) \end{aligned} \tag{C.1}$$

for any $i = 1, \ldots, n$, and

$$\begin{aligned} &Z_1 w_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\cdot\,; z_1, z_2, \ldots, z_n) - w_{\varepsilon_2,\ldots,\varepsilon_n,\varepsilon_1}(\cdot\,; z_2, \ldots, z_n, z_1) \\ &= \sum_{a=1}^{\ell} D_a f_{\varepsilon_1,\ldots,\varepsilon_n}^{(a)} \end{aligned} \tag{C.2}$$

for some functions $f_{\varepsilon_1,\ldots,\varepsilon_n}^{(a)} \in \widetilde{\mathcal{F}}$, $a = 1, \ldots, \ell$. Indeed, let $K_1(z_1, \ldots, z_n)$ be the operator introduced in (2.2). Then relations (C.1) and (C.2) imply that

$$Z_1 w_{\varepsilon_1,\ldots,\varepsilon_n} - \sum_{\varepsilon_1',\ldots,\varepsilon_n'=\pm} (K_1)_{\varepsilon_1,\ldots,\varepsilon_n}^{\varepsilon_1',\ldots,\varepsilon_n'} w_{\varepsilon_1',\ldots,\varepsilon_n'} = \sum_{a=1}^{\ell} D_a f_{\varepsilon_1,\ldots,\varepsilon_n}^{(a)}.$$

Notice that

$$\begin{aligned}\int\limits_{C(z_1+p,z_2,\dots,z_n)} & Z_1 w_{\varepsilon_1,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_n)\phi(t_a;z_1,\dots,z_n) \\ & \times W(t_1,\dots,t_\ell;z_1,\dots,z_n)\,dt_a \\ = \int\limits_{C(z_1,\dots,z_n)} & Z_1 w_{\varepsilon_1,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_n)\phi(t_a;z_1,\dots,z_n) \\ & \times W(t_1,\dots,t_\ell;z_1,\dots,z_n)\,dt_a\end{aligned}$$

for any $a=1,\dots,\ell$ and $W\in\widehat{\mathcal{F}}_q^{\otimes\ell}$, because the integrand has no poles at $t_a=z_1-2\hbar$ and $t_a=z_1+5\hbar$. Therefore, the claim follows from Lemmas 5.4 and 5.5. Eq. (2.2) for $j>1$ can be proved similarly.

We first prove the relation (C.1). If $(\varepsilon_i,\varepsilon_{i+1})\neq(-,-)$, then we have

$$\begin{aligned}& g_{\varepsilon_1,\dots,\varepsilon_{i+1},\varepsilon_i,\dots,\varepsilon_n}(\cdot\,;z_1,\dots,z_{i+1},z_i,\dots,z_n) \\ &= \sum_{\varepsilon_i',\varepsilon_{i+1}'=\pm} R(z_i-z_{i+1})_{\varepsilon_i,\varepsilon_{i+1}}^{\varepsilon_i',\varepsilon_{i+1}'} g_{\varepsilon_1,\dots,\varepsilon_i',\varepsilon_{i+1}',\dots,\varepsilon_n}(\cdot\,;z_1,\dots,z_i,z_{i+1},\dots,z_n),\end{aligned}$$

which implies (C.1). Assume that $(\varepsilon_i,\varepsilon_{i+1})=(-,-)$. Let a be such that $m_a=i$ and $m_{a+1}=i+1$. Then we have

$$\begin{aligned}& g_{\varepsilon_1,\dots,\varepsilon_{i+1},\varepsilon_i,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_{i+1},z_i,\dots,z_n) \\ &\quad - g_{\varepsilon_1,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_n) \\ &= \prod_{\substack{b=1\\ b\neq a,a+1}}^{\ell}\left(\frac{1}{t_b-z_{m_b}}\prod_{1\leqslant l<m_b}\frac{t_b-z_l-\hbar}{t_b-z_l}\right)\prod_{\substack{1\leqslant b<c\leqslant\ell\\ (b,c)\neq(a,a+1)}}(t_b-t_c-\hbar) \\ &\quad\times\prod_{1\leqslant l<i}\frac{(t_a-z_l-\hbar)(t_{a+1}-z_l-\hbar)}{(t_a-z_l)(t_{a+1}-z_l)} \\ &\quad\times\frac{(z_{i+1}-z_i)(t_a-t_{a+1}-\hbar)(t_a-t_{a+1}+\hbar)}{(t_a-z_i)(t_a-z_{i+1})(t_{a+1}-z_i)(t_{a+1}-z_{i+1})}\end{aligned}$$

which is symmetric with respect to the permutation of t_a and t_{a+1}. Therefore

$$\begin{aligned}& w_{\varepsilon_1,\dots,\varepsilon_{i+1},\varepsilon_i,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_{i+1},z_i,\dots,z_n) \\ &\quad = w_{\varepsilon_1,\dots,\varepsilon_n}(t_1,\dots,t_\ell;z_1,\dots,z_n)\end{aligned}$$

in this case. Relation (C.1) is proved.

To prove the relation (C.2) we observe that for $\varepsilon_1=+$ we have

$$Z_1 g_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}(\cdot\,;z_1,z_2,\dots,z_n)=g_{\varepsilon_2,\dots,\varepsilon_n,\varepsilon_1}(\cdot\,;z_2,\dots,z_n,z_1),$$

and for $\varepsilon_1 = -$ we have

$$\begin{aligned}&Z_1 g_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(t_1,\ldots,t_\ell;z_1,z_2,\ldots,z_n)\\&\quad+(-1)^\ell g_{\varepsilon_2,\ldots,\varepsilon_n,\varepsilon_1}(t_2,\ldots,t_\ell,t_1;z_2,\ldots,z_n,z_1)\\&=D_1 Z_1 g_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(t_1,\ldots,t_\ell;z_1,z_2,\ldots,z_n).\end{aligned}$$

It is clear that $Z_1 g_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\cdot\,;z_1,z_2,\ldots,z_n)\in\widetilde{\mathcal{F}}$. Since $w_M = \mathrm{Asym}\, g_M$, we obtain

$$\begin{aligned}&Z_1 w_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(t_1,\ldots,t_\ell;z_1,z_2,\ldots,z_n)\\&\quad- w_{\varepsilon_2,\ldots,\varepsilon_n,\varepsilon_1}(t_1,\ldots,t_\ell;z_2,\ldots,z_n,z_1)\\&=\sum_{\sigma\in\mathbf{S}_\ell}\mathrm{sgn}(\sigma)D_{\sigma_1} f_{\varepsilon_1,\ldots,\varepsilon_n}(t_{\sigma_1},\ldots,t_{\sigma_\ell}),\end{aligned}$$

for some functions $f_{\varepsilon_1,\ldots,\varepsilon_n}\in\widetilde{\mathcal{F}}$, which completes the proof. □

As a corollary of the proof of Theorem 6.3 given above we can describe the properties of the bases $\{v_M^S\}$ and $\{\omega_M\}$. If M corresponds to $(\varepsilon_1,\ldots,\varepsilon_n)$ we set $v^S_{\varepsilon_1,\ldots,\varepsilon_n} = v^S_M$ and $\omega_{\varepsilon_1,\ldots,\varepsilon_n} = \omega_M$.

COROLLARY C.1. – *The bases $\{v_M^S\}$ and $\{\omega_M\}$ satisfy the following equations*:

$$\begin{aligned}&u_{\varepsilon_1,\ldots,\varepsilon_{i+1},\varepsilon_i,\ldots,\varepsilon_n}(z_1,\ldots,z_{i+1},z_i,\ldots,z_n)\\&\quad= P_{i,i+1}R_{i,i+1}(z_i-z_{i+1})u_{\varepsilon_1,\ldots,\varepsilon_i,\varepsilon_{i+1},\ldots,\varepsilon_n}(z_1,\ldots,z_i,z_{i+1},\ldots,z_n)\end{aligned}\tag{C.3}$$

for any $i=1,\ldots,n$, where u stands for either v^S or ω and $P_{i,i+1}$ is the permutation operator acting on the ith and $(i+1)$th components.

Proof of Corollary C.1. – Set

$$w(t_1,\ldots,t_\ell;z_1,\ldots,z_n)=\sum_M w_M(t_1,\ldots,t_\ell;z_1,\ldots,z_n)v_M.$$

Then the equation (C.1) is equivalent to

$$\begin{aligned}&w(t_1,\ldots,t_\ell;z_1,\ldots,z_{i+1},z_i,\ldots,z_n)\\&\quad= P_{i,i+1}R_{i,i+1}(z_i-z_{i+1})w(t_1,\ldots,t_\ell;z_1,\ldots,z_n).\end{aligned}\tag{C.4}$$

Since $\tilde{v}_M(z_1,\ldots,z_n)=\mathrm{Res}\, w(\hat{z}_M;z_1,\ldots,z_n)$, cf. (6.3), we have

$$\begin{aligned}&\pm\tilde{v}_{\varepsilon_1,\ldots,\varepsilon_{i+1},\varepsilon_i,\ldots,\varepsilon_n}(z_1,\ldots,z_{i+1},z_i,\ldots,z_n)\\&\quad= P_{i,i+1}R_{i,i+1}(z_i-z_{i+1})\tilde{v}_{\varepsilon_1,\ldots,\varepsilon_i,\varepsilon_{i+1},\ldots,\varepsilon_n}(z_1,\ldots,z_i,z_{i+1},\ldots,z_n),\end{aligned}$$

with the sign $-$ for the case $(\varepsilon_i, \varepsilon_{i+1}) = (-, -)$ and $+$, otherwise. Corollary C.1 follows from this relation and formulae (6.8), (6.13). □

One can show that the bases $\{v_M^S\}$ and $\{\omega_M\}$ are uniquely determined by Eq. (C.3) and the respective normalizations $v^S_{M_{\rm ext}} = v_{M_{\rm ext}} + \cdots$, $\omega_{M'_{\rm ext}} = v_{M'_{\rm ext}}$.

APPENDIX D

Proof of Lemma 6.4. – Fix a subset $M \subset \{1, \ldots, n\}$, $\#M = \ell$. Consider a function

$$f(t,y) = \frac{P_M^+(t)P_M^-(t)}{t-y} - \frac{P_M^+(t+\hbar)P_M^-(t+\hbar)}{t-y+2\hbar} - \frac{\hbar P_M^+(y)P_M^-(t)}{(t-y)(t-y+\hbar)} - \frac{\hbar P_M^+(t+\hbar)P_M^-(y-\hbar)}{(t-y+\hbar)(t-y+2\hbar)}. \tag{D.1}$$

It is easy to see that $f(t,y)$ is a polynomial in t, y. Moreover, for any $m \in M$ we have

$$f(t, z_m + 2\hbar) = D\left(\prod_{\substack{k=1\\k\neq m}}^{n} (t - z_k - 2\hbar)\right) - \hbar \prod_{\substack{k=1\\k\neq m}}^{n} (z_m - z_k - \hbar)\mu_M^{(m)}(t). \tag{D.2}$$

Let

$$g(t,y) = \left[\frac{f(t,y)}{P_M^+(y)}\right]_{+,y}, \qquad q(t,y) = f(t,y) - P_M^+(y)g(t,y), \tag{D.3}$$

where we take the polynomial part with respect to y. Then $q(t, y)$ is a polynomial of degree less than ℓ with respect to y. We define polynomials $q^{(1)}(t), \ldots, q^{(\ell)}(t)$ by the rule:

$$q(t,y) = \sum_{a=1}^{\ell} q^{(a)}(t) y^{a-1}. \tag{D.4}$$

Since $P_M^+(z_m + 2\hbar) = 0$, using formulae (D.2) and (D.3) we obtain

$$D\left(\prod_{\substack{k=1\\k\neq m}}^{n} (t - z_k - 2\hbar)\right) = \hbar \prod_{\substack{k=1\\k\neq m}}^{n} (z_m - z_k - \hbar)\mu_M^{(m)}(t) + \sum_{a=1}^{\ell} q^{(a)}(t)(z_m + 2\hbar)^{a-1}.$$

Rewrite $f(t, y)$ in the form:

$$f(t, y) = P_M^+(t+\hbar)\big(h(t, y) - h(t+\hbar, y)\big) + P_M^-(t)\big(\tilde{h}(t, y) - \tilde{h}(t+\hbar, y)\big),$$

where

$$h(t, y) = \frac{P_M^-(t) - P_M^-(y-\hbar)}{t - y + \hbar}, \qquad \tilde{h}(t, y) = \frac{P_M^+(t) - P_M^+(y)}{t - y}.$$

Let

$$q(t, y) = P_M^+(t+\hbar)\big(r(t, y) - r(t+\hbar, y)\big) + P_M^-(t)\big(\tilde{r}(t, y) - \tilde{r}(t+\hbar, y)\big)$$

be the corresponding decomposition of $q(t, y)$, cf. (D.3), where

$$\begin{aligned} r(t, y) &= h(t, y) - P_M^+(y)\left[\frac{h(t, y)}{P_M^+(y)}\right]_{+,y}, \\ \tilde{r}(t, y) &= \tilde{h}(t, y) - P_M^+(y)\left[\frac{\tilde{h}(t, y)}{P_M^+(y)}\right]_{+,y}. \end{aligned}$$

Obviously we have

$$\tilde{r}(t, y) = \frac{P_M^+(t) - P_M^+(y)}{t - y} = \left[\frac{P_M^+(t)}{t - y}\right]_{+,t}.$$

To rewrite appropriately $r(t, y)$ we use Lemma D.1 below for replacing a polynomial part with respect to y by a polynomial part with respect to t:

$$\begin{aligned} r(t, y) &= \frac{P_M^-(t) - P_M^-(y-\hbar)}{t - y + \hbar} + P_M^+(y)\left[\frac{P_M^-(y-\hbar)}{P_M^+(y)(t - y + \hbar)}\right]_{+,y} \\ &= \left[\frac{P_M^-(t)}{t - y + \hbar}\right]_{+,t} - P_M^+(y)\left[\frac{P_M^-(t)}{P_M^+(t+\hbar)(t - y + \hbar)}\right]_{+,t} \\ &= \left[\frac{P_M^-(t)}{P_M^+(t+\hbar)} \cdot \frac{P_M^+(t+\hbar) - P_M^+(y)}{t - y + \hbar}\right]_{+,t} \\ &= \left[\frac{P_M^-(t)}{P_M^+(t+\hbar)}\left[\frac{P_M^+(t+\hbar)}{t - y + \hbar}\right]_{+,t}\right]_{+,t}. \end{aligned}$$

Finally

$$q(t,y) = P_M^+(t+h)\left[T_\hbar\left(\frac{P_M^-(t)}{P_M^+(t+h)}\left[\frac{P_M^+(t+h)}{t-y+\hbar}\right]_{+,t}\right)\right]_{+,t}$$
$$+ P_M^-(t)\left[T_\hbar\left(\frac{P_M^+(t)}{t-y}\right)\right]_{+,t}.$$

Therefore, the polynomials $q^{(1)}, \ldots, q^{(\ell)}$ defined by (D.4) coincide with the polynomilas $Q_M^{(1)}, \ldots, Q_M^{(\ell)}$ given by (6.7). Lemma 6.4 is proved. □

LEMMA D.1. – *For any rational function $f(u)$ we have*

$$\left[\frac{f(u)}{u-x}\right]_{+,u} = \left[\frac{f(x)}{x-u}\right]_{+,x}.$$

Proof. –

$$\left[\frac{f(u)}{u-x}\right]_{+,u} = \left[\frac{[f]_+(u)}{u-x}\right]_{+,u} = \frac{[f]_+(u) - [f]_+(x)}{u-x}$$
$$= \left[\frac{[f]_+(x)}{x-u}\right]_{+,x} = \left[\frac{f(x)}{x-u}\right]_{+,x}. \quad \square$$

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