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# Contributions to polynomial conformal tensors

by

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ABSTRACT. – By means of a certain conformal covariant differentiation process we construct generating systems for conformally invariant tensors in a pseudo-Riemannian manifold and use these invariants to derive necessary conditions for the validity of Huygens' principle of some conformally invariant field equations as well as for a space-time to be conformally related to an Einstein space-time. © Elsevier, Paris

*Key words:* pseudo-Riemannian manifold, conformally invariant tensor, conformal covariant derivative, moment, Huygens' principle, conformal Einstein space-time.

RÉSUMÉ. – Grâce à un certain processus de différentiation covariante conforme, nous construisons des systèmes générateurs de tenseurs invariants conformes dans une variété pseudo-Riemannienne et nous utilisons ces invariants pour en déduire les conditions nécessaires à la validité du principe d'Huyghens pour quelques équations invariantes conformes de champ et pour l'application conforme d'un espace-temps à l'espace-temps d'Einstein. © Elsevier, Paris

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## 1. INTRODUCTION

Let $(M, g)$ be a pseudo-Riemannian $C^\infty$ manifold of dimension $n(n \geq 3)$ and $g_{ab}$, $g^{ab}$, $\nabla_a$, $R_{abcd}$, $R_{ab}$, $R$, $C_{abcd}$ the local components of the covariant and contravariant metric tensor, the Levi-Civita connection, the curvature tensor, the Ricci tensor, the scalar curvature and the Weyl curvature tensor, respectively [1]. We consider polynomial tensors, *i.e.* tensors whose components are polynomials in $g^{ab}$ and the partial derivatives of $g_{ab}$. These tensors are just the elements of the tensor algebra $\mathcal{R}$ generated by the tensors

$$g_{ab},\ g^{ab},\ \nabla_{(i_1} \cdots \nabla_{i_r} R_{i_{r+1}|ab|i_{r+2})}, \qquad r = 0, 1, 2, \ldots \tag{1.1}$$

by means of the usual tensor operations [Scho; dP; GüW1].

A tensor $T[g] \in \mathcal{R}$ is said to be *conformally invariant* (or more briefly a *conformal tensor*) with weight $\omega$ if $T[g]$ has under a conformal transformation

$$\overline{g}_{ab} = e^{2\Phi} g_{ab}, \quad \Phi \in C^\infty(M) \tag{1.2}$$

the transformation law [Scho; GüW1]

$$T[\overline{g}] = e^{2\omega\Phi} T[g]. \tag{1.3}$$

*Example.* – The Bach tensor

$$B_{i_1 i_2} := \nabla_a \nabla_b C^{a}_{\ .i_1 i_2 .}{}^{b} + \frac{1}{n-2} C^{a}_{\ .i_1 i_2 .}{}^{b} R_{ab} \tag{1.4}$$

is a polynomial conformal tensor with $\omega = -1$ if $n = 4$ [Scho; GüW1].

General algebraic and differential properties of conformal tensors were investigated by Thomas [Tho], Szekeres [Sz], du Plessis [dP] and Günther/Wünsch [GüW1,2]. In [Sz; dP] particular sequences of tensors satisfying (1.3) are given which generate all tensors with this property. However, in general the elements of these sequences are not polynomial. Conformal transformations and in particular the polynomial conformal tensors have a great variety of applications, e.g. for a Lagrangian formulation with locality principle of both general relativity and conformal field theories and in the propagation theory of conformally invariant field equations [AMLW; BØ; Gü; McL; Ø; PR; Wü1-5]. It is an important problem to give a survey of all polynomial conformal tensors or, with less pretension, to give a method for constructing special classes of such tensors. Such a method was developed in [GüW1,2; Schi] using the *infinitesimal generator* and the *conformal covariant derivative* of a tensor $T \in \mathcal{R}$.

[1] All investigations in this paper are of local nature.

The paper is organized as folows. In Section 2, the basic ideas and results of [GüW1] are given with the help of which we derive further classes of generating systems for conformal tensors in Section 3-5. In Section 4 we transfer this method for the physically important case $n = 4$ to an extended tensor algebra by involving the Levi-Civita pseudo tensor. In Section 5 we consider the pseudo Riemannian space of dimension 3. In this case one has to replace the Weyl tensor by the conformal tensor $S_{abc} := \nabla_{[c}L_{b]a}$. Finally, in Section 6 we use generating systems for the conformal tensors for the derivation of further explicit moment equations for conformally invariant field equations for space-times and necessary conditions for a space-time to be conformally related to an Einstein space-time.

REMARK 1.1. – The Sections 3 and 4 are an English version of results which have already been published in [GeW1,2] in German, however, the journal "Wissenschaftliche Zeitschrift der Pädagogischen Hochschule Erfurt/Mühlhausen" has ceased publication and can neither be found at the Libraries and report organs nor be orderd. On the other hand some colleagues working on this field recommented the publication of these invariants in an international renowned journal.

## 2. SOME KNOWN RESULTS

In Section 3-5 we use the following basic ideas and results (see [GüW1,2]):

PROPOSITION 2.1. – (i) *$T \in \mathcal{R}$ has, under the conformal transformation (1.2), a transformation law of the form*

$$T[\overline{g}] = e^{2\omega\Phi}\left(T[g] + \sum_{k=1}^{m} P_k[g, \Phi]\right),$$

where the $P_k[g, \Phi]$ are tensor-valued homogeneous polynomials of degree $k$ in the derivatives of $\Phi$ up to a certain order.

(ii) $T \in \mathcal{R}$ is conformally invariant iff this is true under infinitesimal conformal transformations, i.e. iff

$$P_1[g, \Phi] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(T[e^{2\varepsilon\Phi} g] - e^{2\varepsilon\omega\Phi} T[g]) = 0.$$

REMARK 2.1. – Because of this proposition it follows from $P_1[g, \Phi] = 0$ that $P_k[g, \Phi] = 0$ $(k = 2, ..., m)$, *i.e.* for the construction of conformal tensors it is sufficient to calculate only "*to the first order in the derivatives of* $\Phi$"

*Example.* – For the *Schouten tensor* $L_{ab}$ defined by

$$L_{ab} = -R_{ab} + \frac{1}{2(n-1)} R\, g_{ab} \tag{2.1}$$

we have

$$P_1[g,\phi]_{ab} = \frac{n-2}{2} \nabla_a \nabla_b \Phi, \; P_2[g,\phi]_{ab} = \frac{n-2}{8}(g_{ab} g^{lk} - 2\delta^l_a \delta^k_b) \nabla_l \Phi \nabla_k \Phi.$$

Let $\tau$ be the subalgebra of those elements of $\mathcal{R}$ which contain only *first* order derivatives of $\Phi$ in their transformation law and let $i$ be the ideal of $\mathcal{R}$ which is generated by the tensors

$$\nabla_{(i_1} \cdots \nabla_{i_k} L_{i_{k+1} i_{k+2})}, \qquad k = 0, 1, 2, \ldots \tag{2.2}$$

Now we define two linear operators in $\tau$: The first one comes from infinitesimal conformal transformations, the second one comes from differentiation.

It follows from Proposition 2.1 that $T[g] \in \tau$ iff $P_1[g, \Phi]$ has the form

$$P_1[g, \Phi] = X^\gamma(T) \nabla_\gamma \Phi. \tag{2.3}$$

DEFINITION 2.1. – The linear operator $X^\gamma$ defined on $\tau$ by (2.3) is called the infinitesimal generator of $T$.

For $X^\gamma$ the Leibniz rule holds and one has $X^\gamma(g_{ab}) = 0$.

COROLLARY 2.1. – *$T[g] \in \mathcal{R}$ is conformally invariant iff $T[g] \in \tau$ and $X^\gamma(T) = 0$.*

*Examples.*

$$X^\gamma(C_{abcd}) = 0, \;\; X^\gamma(\nabla_u C^u{}_{abc}) = (n-3) C^\gamma{}_{abc}, \tag{2.4}$$

$$X^\gamma(\nabla_e C_{abcd}) = 2[-\delta_e{}^\gamma C_{abcd} + \delta_{[a}{}^\gamma C_{b]ecd} + \delta_{[c|}{}^\gamma C_{ab|d]e} - g_{e[a} C_{b].cd}^{\;\;\;\gamma} - g_{e[c} C_{ab|d].}{}^\gamma] \tag{2.5}$$

If $T \in \tau$, then we have in general $\nabla_a T \notin \tau$.

DEFINITION 2.2. – *For $T \in \tau$ the tensor*

$$\overset{c}{\nabla}_a T := \nabla_a T - \frac{1}{n-2} L_{a\gamma} X^\gamma(T) \tag{2.6}$$

is called the conformal covariant derivative of $T$ [2].

PROPOSITION 2.2. – *(i) The conformal covariant derivative $\overset{c}{\nabla}_a$ is linear, obeys Leibniz's rule and commutes with contractions.*

(ii) $\overset{c}{\nabla}_a : \tau \longrightarrow \tau$

(iii) *If $n \geq 4$, then $\tau$ is generated by the tensors*

$$g_{ab},\ g^{ab},\ \overset{c}{\nabla}_{(i_1} \cdots \overset{c}{\nabla}_{i_r} C^{a}_{\ .i_{r+1} i_{r+2} .)}{}^{b}, \qquad r = 0, 1, 2, \ldots \tag{2.7}$$

(iv) *$\mathcal{R}$ is generated by the tensors (2.7) and (2.2).*

(v) *For every $T \in \mathcal{R}$ there exists one and only one element $T_1 \in \tau$ with $(T - T_1) \in i$. It is $\tau \cap i = \{0\}$, and $\mathcal{R}/i$ is isomorphic to $\tau$.*

(vi) *If $T \in \tau$ has the weight $\omega$, then*

$$X^k(\overset{c}{\nabla}_a T) - \overset{c}{\nabla}_a(X^k T) = 2\omega \delta^k_a T + \triangle^k_a T, \tag{2.8}$$

*where*

$$\triangle^k_a (T_{ij\ldots}{}^{lm\ldots}) := \triangle^{kl}_{as} T_{ij\ldots}{}^{sm\ldots} + \triangle^{km}_{as} T_{ij\ldots}{}^{ls\ldots} \cdots - \triangle^{ks}_{ai} T_{sj\ldots}{}^{lm\ldots} - \triangle^{ks}_{aj} T_{is\ldots}{}^{lm\ldots} \cdots \tag{2.9}$$

*and*

$$\triangle^{ks}_{ai} := \delta^k_a \delta^s_i + \delta^k_i \delta^s_a - g_{ai} g^{ks}.$$

(vii) *The Ricci-identity for $\overset{c}{\nabla}_a$ has the form*

$$\overset{c}{\nabla}_{[a} \overset{c}{\nabla}_{b]} T = (C, T)_{\alpha\beta} - \frac{1}{n-2} \nabla_{[\alpha} L_{\beta]\gamma} X^\gamma(T), \tag{2.10}$$

*where $(C, T)_{\alpha\beta}$ is the term one obtains from the right hand side of the usual Ricci identity*

$$\nabla_{[a} \nabla_{b]} T = (R, T)_{\alpha\beta}$$

*by substitution of $R$ by $C$.*

*Examples.* – (i) $\overset{c}{\nabla}_k C_{abcd} = \nabla_k C_{abcd}$

(ii) $B_{ab} = \overset{c}{\nabla}_k \overset{c}{\nabla}_l C^{k}_{\ .ab.}{}^{l}$ (Bach tensor) (2.11)

Using (2.5) and (2.8), we get $X^k(B_{ab}) = 2(n-4) \overset{c}{\nabla}_k C^{k}_{\ .ab.}{}^{\gamma}$.

[2] This definition is different from the definition of the conformal covariant derivative introduced by Weyl and du Plessis [dP; Scho].

When latin indices with subindices (e.g. $i_1, ..., i_r$) appear in the sequel, we assume that symmetrization has been carried out over the indices. If $T$ is any tensor with covariant rank $r$ ($r \geq 2$), then we denote the trace-free part of $T$ by $TS(T)$. For a symmetric tensor $T_{i_1...i_r}$ with $r \geq 2$ we write

$$\overset{1}{T}_{i_i...i_r} \underset{*}{=} \overset{2}{T}_{i_1...i_r} \quad \text{iff} \quad TS(\overset{1}{T}_{i_1...i_r} - \overset{2}{T}_{i_1...i_r}) = 0.$$

In [Wü5; GüW2] the following was proved:

LEMMA 2.1. – *If $T_{i_1...i_{k-s}}$ is a symmetric, conformally invariant tensor with covariant rank $(k-s)$ and weight $\omega$, then*

$$X^{\gamma}(\overset{c}{\triangledown}_{i_1}...\overset{c}{\triangledown}_{i_s} T_{i_{s+1}...i_k}) \underset{*}{=} s(2\omega + s - 2k + 1)\delta^{\gamma}_{i_1} \overset{c}{\triangledown}_{i_2}...\overset{c}{\triangledown}_{i_s} T_{i_{s+1}...i_k}.$$

DEFINITION 2.3. – *A conformally invariant tensor $T$ is called trivial if $T$ is generated by*

$$\{g_{ab}, g^{ab}, C_{abcd}\}. \tag{2.12}$$

Let $\mathcal{S}_r(\omega, n)$ be the set of all nontrivial conformal, symmetric, trace-free tensors of $\mathcal{R}$ with weight $\omega$ and covariant rank $r$.

LEMMA 2.2. – *If a monomial in $\mathcal{S}_r(\omega, n)$ contains $\alpha$ factors $\{\overset{c}{\triangledown}_{i_1}...\overset{c}{\triangledown}_{i_{k-2}} C_{i_{k-1}abi_k}\}$ and $q$ operators $\overset{c}{\triangledown}$, then*

$$r - 2\omega = 2\alpha + q \quad \text{and} \quad \omega \leq 0. \tag{2.13}$$

The number $\alpha$ is called $C$-order of a monomial of $T \in \mathcal{S}_r(\omega, n)$.

LEMMA 2.3. – *A tensor in $\mathcal{S}_r(\omega, n)$ contains a monomial with $\alpha = 1$, if and only if*

$$r = 2, \ n \text{ is even and } \omega = \frac{n-2}{2}.$$

Let $\mathcal{S}_r^{(\alpha)}(\omega, n)$ be the subset of those elements of $\mathcal{S}_r(\omega, n)$ whose monomials have the $C$-order $\alpha$. The Propositions 2.1, 2.2, Corollary 2.1, Definition 2.2 and Lemma 2.1 are very useful for the construction of nontrivial conformally invariant tensors. In [GüW2] generating systems are constructed for $\mathcal{S}_r(\omega, n)$ in the cases $(r, \omega) = (2, -1)$, $(1, -2)$, $(0, -3)$, $(2, -2)$, $(4, -1)$. In particular the following results were proved (see [Wü1, GüW2, Gü]):

PROPOSITION 2.3. – *If $S_r(-1, 4) \in \mathcal{S}_r(-1, 4)$ then one has*

$$S_r(-1,4) = 0 \ \ for \ \ r = 0, 1, 3, S_2(-1,4) = \alpha B, \ S_4(-1,4) = \sum_{m=1}^{3} \beta_m W^{(m)},$$

*where $B$ is the Bach tensor,*

$$W^{(1)}_{i_1\ldots i_4} := TS[\nabla^a C^b_{.i_1 i_2.}{}^c \nabla_a C_{bi_3 i_4 c} + 16 \nabla_u C^u_{.i_1 i_2.}{}^a \nabla_\nu C^\nu_{.i_3 i_4 a} + 4 C^a_{.i_1 i_2.}{}^b \{2 \nabla_a \nabla_u C^u_{.i_3 i_4 b} - C_{ai_3 i_4.}{}^c L_{bc}\}], \tag{2.14}$$

$$W^{(2)}_{i_1\ldots i_4} := TS[2 \nabla_{i_1} C^a_{.i_2 i_3.}{}^b \nabla_u C^u_{.abi_4} + 2 \nabla_u C^u_{.i_1 i_2 a} \nabla_\nu C^\nu_{.i_3 i_4.}{}^a - C^a_{.i_1 i_2.}{}^b \{2 \nabla_{i_3} \nabla_u C^u_{.abi_4} - C^c_{.abi_3} L_{ci_4}\}] \tag{2.15}$$

$$W^{(3)}_{i_1\ldots i_4} := TS[C^{abcd} C_{ai_1 i_2 d} C_{bi_3 i_4 c}]$$

and $\alpha$, $\beta_1$, $\beta_2$, $\beta_3$ $\in \mathbb{R}$.

## 3. GENERATING SYSTEMS OF $\mathcal{S}_6(-1, n)$ FOR $n \geq 4$

For a conformal tensor $T_k$ the Ricci-identity (2.10) reduces to

$$\overset{c}{\nabla}_{[a} \overset{c}{\nabla}_{b]} T_c = -\frac{1}{2} C_{abc.}{}^k T_k. \tag{3.1}$$

Furthermore, we use in the sequel the Bianchi identity

$$\nabla_{[a} C_{bc]ij} + \frac{1}{n-3} [g_{j[a|} \nabla_u C^u{}_{i|bc]} - g_{i[a|} \nabla_u C^u{}_{j|bc]}] = 0. \tag{3.2}$$

Because of Lemma 2.3 and (2.13) the sets $\mathcal{S}_6^{(1)}(-1, n)$ and $\mathcal{S}_6^{(4)}(-1, n)$ are empty. Thus it follows from (3.1) that

$$\mathcal{S}_6(-1, n) = \mathcal{S}_6^{(2)}(-1, n) \cup \mathcal{S}_6^{(3)}(-1, n). \tag{3.3}$$

Firstly, consider the case $\alpha = 2$: Then on account of (2.13) $q = 4$. Because of (3.1) and (3.2) a tensor of $\mathcal{S}_6^{(2)}(-1, n)$ has to be a linear combination of the following linearly independent tensors [3]

$$\overset{1}{C}{}^{(2)}_{i_1\ldots i_6} = TS[C_{abci_1} \overset{c}{\nabla}_{i_2} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{i_5} C^{abc}_{\ldots\ i_6}]$$

$$\overset{2}{C}{}^{(2)}_{i_1\ldots i_6} = TS[C^a_{.i_1 i_2.}{}^b \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{i_5} \overset{c}{\nabla}_u C^u_{.abi_6}]$$

$$\overset{3}{C}{}^{(2)}_{i_1\ldots i_6} = TS[C^a_{.i_1 i_2.}{}^b \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_a \overset{c}{\nabla}_u C^u_{.i_5 i_6 b}]$$

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[3] One can show the linear independence by means of the methods developed in [GüW2].

$$\overset{4}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}{}^a C^{b}_{.i_1i_2.}{}^{c}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_a C_{bi_5i_6c}]$$

$$\overset{5}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1} C_{abci_2}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5} C^{abc}_{\ldots}{}_{i_6}]$$

$$\overset{6}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1} C_{ai_2i_3b}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5}\overset{c}{\nabla}_{u} C^{uab}_{\ldots}{}_{i_6}]$$

$$\overset{7}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1} C^{a}_{.i_2i_3.}{}^{b}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{a}\overset{c}{\nabla}_{u} C^{u}_{.i_5i_6b}]$$

$$\overset{8}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{u} C^{uab}_{\ldots}{}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4} C_{ai_5i_6b}]$$

$$\overset{9}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{u} C^{u}_{.i_1i_2.}{}^{a}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{\nu} C^{\nu}_{.i_5i_6a}]$$

$$\overset{10}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^{a} C^{b}_{.i_2i_3.}{}^{c}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{a} C_{bi_5i_6c}]$$

$$\overset{11}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C_{abci_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5} C^{abc}_{\ldots}{}_{i_6}]$$

$$\overset{12}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{i_5}\overset{c}{\nabla}_{u} C^{u}_{.abi_6}]$$

$$\overset{13}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{a}\overset{c}{\nabla}_{u} C^{u}_{.i_5i_6b}]$$

$$\overset{14}{C}{}^{(2)}_{i_1\ldots i_6}=TS[\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{u} C^{u}_{.i_2i_3.}{}^{a}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{v} C^{v}_{.i_5i_6a}]. \tag{3.4}$$

We put

$$\overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{a} C_{bi_3i_4c}\overset{c}{\nabla}_{i_5} C^{abc}_{\ldots}{}_{i_6} \qquad \overset{2}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{a} C_{bi_3i_4c}\overset{c}{\nabla}{}^{a} C^{b}_{.i_5i_6.}{}^{c}$$

$$\overset{3}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{u} C^{u}_{.abi_3}\overset{c}{\nabla}_{i_4} C^{a}_{.i_5i_6.}{}^{b} \qquad \overset{4}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{a}\overset{c}{\nabla}_{u} C^{u}_{.i_2i_3b}\overset{c}{\nabla}_{i_4} C^{a}_{.i_5i_6.}{}^{b}$$

$$\overset{5}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{u} C^{u}_{.i_3i_4.}{}^{a}\overset{c}{\nabla}_{\nu} C^{\nu}_{.i_5i_6a} \qquad \overset{6}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{i_3} C_{ai_4i_5b}\overset{c}{\nabla}_{u} C^{uab}_{\ldots}{}_{i_6}$$

$$\overset{7}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C^{\gamma}_{.abi_3}\overset{c}{\nabla}_{i_4} C^{a}_{.i_5i_6.}{}^{b} \qquad \overset{8}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{a} C^{\gamma}_{.i_2i_3b}\overset{c}{\nabla}_{i_4} C^{a}_{.i_5i_6.}{}^{b}$$

$$\overset{9}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C_{ai_3i_4b}\overset{c}{\nabla}_{i_5} C^{\gamma ab}_{\ldots i_6} \qquad \overset{10}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C_{ai_3i_4b}\overset{c}{\nabla}{}^{a} C^{\gamma}_{.i_5i_6.}{}^{b}$$

$$\overset{11}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} C^{\gamma}_{.i_3i_4a}\overset{c}{\nabla}_{u} C^{u}_{.i_5i_6.}{}^{a} \qquad \overset{12}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{u} C^{u}_{.i_2i_3a}\overset{c}{\nabla}_{i_4} C^{\gamma}_{.i_5i_6.}{}^{a}$$

$$\overset{13}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1} C^{a}_{.i_2i_3.}{}^{b}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5}\overset{c}{\nabla}_{u} C^{u}_{.abi_6} \qquad \overset{14}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1} C^{a}_{.i_2i_3.}{}^{b}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{a}\overset{c}{\nabla}_{u} C^{u}_{.i_5i_6b}$$

$$\overset{15}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=C^{\gamma}_{.i_1i_2.}{}^{a}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{u} C^{u}_{.i_5i_6a} \qquad \overset{16}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=C_{ai_1i_2b}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}{}^{a} C^{b}_{.i_5i_6.}{}^{\gamma}$$

$$\overset{17}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=C^{\gamma}_{.abi_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4} C^{a}_{.i_5i_6.}{}^{b} \qquad \overset{18}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=\delta^{\gamma}_{i_1} C_{abci_2}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5} C^{abc}_{\ldots}{}_{i_6}$$

$$\overset{19}{Q}{}^{(2)\gamma}_{i_1\ldots i_6}=C_{ai_1i_2b}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5} C^{\gamma ab}_{\ldots}{}_{i_6} \tag{3.5}$$

Using (3.1), (3.2) and the transformation formulae (2.4), (2.5), (2.8), we obtain

$$\begin{bmatrix} X^{\gamma}(\overset{1}{C}{}^{(2)}_{i_1\ldots i_6}) \\ \cdot \\ \cdot \\ \cdot \\ X^{\gamma}(\overset{14}{C}{}^{(2)}_{i_1\ldots i_6}) \end{bmatrix} \underset{*}{=} A_6^{(2)}(-1,n) \begin{bmatrix} \overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_6} \\ \cdot \\ \cdot \\ \cdot \\ \overset{19}{Q}{}^{(2)\gamma}_{i_1\ldots i_6} \end{bmatrix},$$

where

$$A_6^{(2)}(-1,n) =$$

$$\begin{vmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&-24&8\\
0&0&0&0&0&0&0&0&0&0&0&0&-18&0&3&0&0&0&a\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-14&-6&a&0&0&0\\
-2&-12&4/a&0&0&-\frac{6}{a}&4&2&-4&-6&2\frac{c}{a}&\frac{2}{a}&-\frac{4}{a}&0&2\frac{b}{a}&-4&2&-1&-2\\
-30&0&30/a&0&0&0&6&0&-6&0&0&0&0&0&0&0&2&-3&-2\\
0&0&-10&0&0&0&a&0&0&0&0&2&-4&0&-1&0&0&0&0\\
0&0&1&-6&0&0&0&a&0&0&0&-5&0&-4&1&0&0&0&0\\
0&0&0&0&0&-18&0&0&0&0&-3&0&0&0&0&0&a&0&0\\
0&0&0&0&-12&0&0&0&0&0&a&0&0&0&a&0&0&0&0\\
-4&-10&-6/a&0&0&\frac{2}{a}&-6&-10&6&2&\frac{2}{a}&2\frac{b+c}{a}&0&0&0&0&0&0&0\\
-32&0&32/a&0&0&0&-8&0&8&0&0&0&0&0&0&0&0&0&0\\
0&0&-10&0&0&-4&0&0&a&0&1&-2&0&0&0&0&0&0&0\\
0&0&0&-10&0&1&0&0&0&a&-4&2&0&0&0&0&0&0&0\\
0&0&0&0&-10&0&0&0&0&0&0&2a&0&0&0&0&0&0&0
\end{vmatrix}$$

with $a = n-3$, $b = n-4$, $c = n-5$.

For $n > 4$, rank $A_6{}^{(2)}(-1,n) = 12$. If $n = 4$, one gets more identities concerning the tensors (3.4) and (3.5) (see [RW]; [Thi]). We obtain the following result:

PROPOSITION 3.1. – *(i) If $n > 4$ then $\mathcal{S}_6{}^{(2)}(-1,n)$ is generated by the tensors*

$$\begin{aligned}
\overset{1}{S}{}^{(2)}_6(-1,n) =\,& 7(n-3)\overset{1}{C}{}^{(2)}_{i_1\ldots i_6} - 228\overset{2}{C}{}^{(2)}_{i_1\ldots i_6} - 180\overset{3}{C}{}^{(2)}_{i_1\ldots i_6} \\
& - 45(n-3)\overset{4}{C}{}^{(2)}_{i_1\ldots i_6} - 41(n-3)\overset{5}{C}{}^{(2)}_{i_1\ldots i_6} \\
& + 1026\overset{6}{C}{}^{(2)}_{i_1\ldots i_6} + 630\overset{7}{C}{}^{(2)}_{i_1\ldots i_6} + 228\overset{8}{C}{}^{(2)}_{i_1\ldots i_6} \\
& + 90\frac{(n-4)}{(n-3)}\overset{9}{C}{}^{(2)}_{i_1\ldots i_6} + 54(n-3)\overset{10}{C}{}^{(2)}_{i_1\ldots i_6} \\
& + \frac{69}{2}(n-3)\overset{11}{C}{}^{(2)}_{i_1\ldots i_6} - 1026\overset{12}{C}{}^{(2)}_{i_1\ldots i_6} - 378\overset{13}{C}{}^{(2)}_{i_1\ldots i_6} \\
& - 108\frac{(n-4)}{(n-3)}\overset{14}{C}{}^{(2)}_{i_1\ldots i_6}
\end{aligned}$$

$$\begin{aligned}\overset{2}{S}{}_6^{(2)}(-1,n) =& -2(n-3)\overset{1}{C}{}^{(2)}_{i_1\dots i_6} + 48\overset{2}{C}{}^{(2)}_{i_1\dots i_6} + 16(n-3)\overset{5}{C}{}^{(2)}_{i_1\dots i_6} \\ & - 216\overset{6}{C}{}^{(2)}_{i_1\dots i_6} - 48\overset{8}{C}{}^{(2)}_{i_1\dots i_6} - \frac{360}{(n-3)}\overset{9}{C}{}^{(2)}_{i_1\dots i_6} \\ & - 15(n-3)\overset{11}{C}{}^{(2)}_{i_1\dots i_6} + 216\overset{12}{C}{}^{(2)}_{i_1\dots i_6} + \frac{432}{(n-3)}\overset{14}{C}{}^{(2)}_{i_1\dots i_6}.\end{aligned}$$

(ii) $\mathcal{S}_6{}^{(2)}(-1,4)$ is generated by the tensors

$$\begin{aligned}\overset{1}{S}{}_6^{(2)}(-1,4) =& -228\overset{2}{C}{}^{(2)}_{i_1\dots i_6} - 180\overset{3}{C}{}^{(2)}_{i_1\dots i_6} - 45\overset{4}{C}{}^{(2)}_{i_1\dots i_6} + 1026\overset{6}{C}{}^{(2)}_{i_1\dots i_6} \\ & + 630\overset{7}{C}{}^{(2)}_{i_1\dots i_6} + 228\overset{8}{C}{}^{(2)}_{i_1\dots i_6} + 54\overset{10}{C}{}^{(2)}_{i_1\dots i_6} - 1026\overset{12}{C}{}^{(2)}_{i_1\dots i_6} \\ & - 378\overset{13}{C}{}^{(2)}_{i_1\dots i_6}\end{aligned}$$

$$\begin{aligned}\overset{2}{S}{}_6^{(2)}(-1,4) =& 48\overset{2}{C}{}^{(2)}_{i_1\dots i_6} - 216\overset{6}{C}{}^{(2)}_{i_1\dots i_6} - 48\overset{8}{C}{}^{(2)}_{i_1\dots i_6} - 360\overset{9}{C}{}^{(2)}_{i_1\dots i_6} \\ & + 216\overset{12}{C}{}^{(2)}_{i_1\dots i_6} + 432\overset{14}{C}{}^{(2)}_{i_1\dots i_6}\end{aligned}$$

Now let be $\alpha = 3$: Then on account of (2.13) it is $q = 2$. Because of (3.1) and (3.2) a tensor of $\mathcal{S}_6{}^{(3)}(-1,n)$ has to be a linear combination of the following linearly independent tensors [3)]

$$\begin{aligned}
\overset{1}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C_{acbd}C^{a}{}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}C^{c}{}_{.i_5i_6.}{}^{d}] \\
\overset{2}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C_{acdi_1}C_{b..i_2}^{\;\;cd}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}C^{a}{}_{.i_5i_6.}{}^{b}] \\
\overset{3}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C_{acdi_1}C_{b..i_2}^{\;\;dc}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}_{i_4}C^{a}{}_{.i_5i_6.}{}^{b}] \\
\overset{4}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{acdi_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;cd}] \\
\overset{5}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{acdi_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;dc}] \\
\overset{6}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{acdi_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{b}C^{c}{}_{.i_5i_6.}{}^{d}] \\
\overset{7}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C^{c}{}_{.i_3i_4.}{}^{d}\overset{c}{\nabla}_{i_5}\overset{c}{\nabla}_{i_6}C_{acbd}] \\
\overset{8}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C^{c}{}_{.i_3i_4.}{}^{d}\overset{c}{\nabla}_{a}\overset{c}{\nabla}_{b}C_{ci_5i_6d}] \\
\overset{9}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{ai_3i_4.}{}^{c}\overset{c}{\nabla}_{i_5}\overset{c}{\nabla}_{u}C^{u}{}_{.bci_6}] \\
\overset{10}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{cabi_3}\overset{c}{\nabla}_{i_4}\overset{c}{\nabla}_{u}C^{u}{}_{.i_5i_6.}{}^{c}] \\
\overset{11}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C^{a}{}_{.i_1i_2.}{}^{b}C_{ai_3i_4c}\overset{c}{\nabla}_{b}\overset{c}{\nabla}_{u}C^{u}{}_{.i_5i_6.}{}^{c}] \\
\overset{12}{C}{}^{(3)}_{i_1\dots i_6} &= TS[C_{acbd}\overset{c}{\nabla}_{i_1}C^{a}{}_{.i_2i_3.}{}^{b}\overset{c}{\nabla}_{i_4}C^{c}{}_{.i_5i_6.}{}^{d}]
\end{aligned}$$

$$\overset{13}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C_{acdi_1}\overset{c}{\nabla}_{i_2}C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;cd}]$$
$$\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C_{acdi_1}\overset{c}{\nabla}_{i_2}C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;dc}]$$
$$\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C_{acdi_1}\overset{c}{\nabla}_{i_2}C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{b}C^{c}_{.i_5i_6.}{}^{d}]$$
$$\overset{16}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}C_{acdi_4}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;cd}]$$
$$\overset{17}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}C_{acdi_4}\overset{c}{\nabla}_{i_5}C_{b..i_6}^{\;\;dc}]$$
$$\overset{18}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}C_{acdi_4}\overset{c}{\nabla}_{b}C^{c}_{.i_5i_6.}{}^{d}]$$

$$\overset{19}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{a}C_{ci_3i_4d}\overset{c}{\nabla}_{b}C^{c}_{.i_5i_6.}{}^{d}]$$
$$\overset{20}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}C^{c}_{.i_4i_5.}{}^{d}\overset{c}{\nabla}_{i_6}C_{acbd}]$$
$$\overset{21}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{i_3}C^{c}_{.i_4i_5.}{}^{d}\overset{c}{\nabla}_{b}C_{cadi_6}]$$
$$\overset{22}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C_{cabi_1}\overset{c}{\nabla}_{i_2}C^{a}_{.i_3i_4.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.i_5i_6.}{}^{c}]$$
$$\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.i_3i_4.}{}^{c}\overset{c}{\nabla}_{i_5}C_{cabi_6}]$$
$$\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.i_3i_4.}{}^{c}\overset{c}{\nabla}_{a}C_{bi_5i_6c}]$$
$$\overset{25}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.aci_3}\overset{c}{\nabla}_{i_4}C_{bi_5i_6.}{}^{c}]$$
$$\overset{26}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.cai_3}\overset{c}{\nabla}_{i_4}C_{bi_5i_6.}{}^{c}]$$
$$\overset{27}{C}{}^{(3)}_{i_1\ldots i_6} = TS[C^{a}_{.i_1i_2.}{}^{b}\overset{c}{\nabla}_{u}C^{u}_{.i_3i_4a}\overset{c}{\nabla}_{\nu}C^{\nu}_{.i_5i_6b}]. \tag{3.6}$$

We set

$$\overset{1}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1}C^{a}_{.i_2i_3.}{}^{b}C_{ai_4i_5.}{}^{c}\overset{c}{\nabla}_{u}C^{u}_{.bci_6} \qquad \overset{2}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1}C^{a}_{.i_2i_3.}{}^{b}C_{cabi_4}\overset{c}{\nabla}_{u}C^{u}_{.i_5i_6.}{}^{c}$$
$$\overset{3}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1i_2.}{}^{b}C^{\gamma}_{.i_3i_4a}\overset{c}{\nabla}_{u}C^{u}_{.i_5i_6b} \qquad \overset{4}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1i_2.}{}^{b}C_{ai_3i_4b}\overset{c}{\nabla}_{u}C^{u}_{.i_5i_6.}{}^{\gamma}$$
$$\overset{5}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1}C_{a..i_2}^{\;\;cd}C_{bcdi_3}\overset{c}{\nabla}_{i_4}C^{a}_{.i_5i_6.}{}^{b} \qquad \overset{6}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1}C_{a..i_2}^{\;\;cd}C_{bi_3cd}\overset{c}{\nabla}_{i_4}C^{a}_{.i_5i_6.}{}^{b}$$
$$\overset{7}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1i_2.}{}^{b}C^{\gamma}_{.i_3a.}{}^{c}\overset{c}{\nabla}_{i_4}C_{bi_5i_6c} \qquad \overset{8}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1i_2.}{}^{b}C^{\gamma}_{.aci_3}\overset{c}{\nabla}_{i_4}C_{bi_5i_6.}{}^{c}$$

$$
\begin{aligned}
&\overset{9}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1 i_2.}{}^{b} C_{cabi_3} \overset{c}{\nabla}_{i_4} C^{\gamma}_{.i_5 i_6.}{}^{c}
&&\overset{10}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C_{abci_1} C^{\gamma}_{.i_2 i_3.}{}^{a} \overset{c}{\nabla}_{i_4} C^{b}_{.i_5 i_6.}{}^{c} \\
&\overset{11}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3.}{}^{b} C_{acbd} \overset{c}{\nabla}_{i_4} C^{c}_{.i_5 i_6.}{}^{d}
&&\overset{12}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3.}{}^{b} C^{c}_{.i_4 i_5.}{}^{d} \overset{c}{\nabla}_{c} C_{dabi_6} \\
&\overset{13}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1 i_2.}{}^{b} C_{ai_3 i_4 c} \overset{c}{\nabla}_{i_5} C^{\gamma\ c}_{.b.\ i_6}
&&\overset{14}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1 i_2.}{}^{b} C_{ai_3 i_4 c} \overset{c}{\nabla}_{b} C^{\gamma}_{.i_5 i_6.}{}^{c} \\
&\overset{15}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1 i_2.}{}^{b} C^{\gamma}_{.i_3 i_4.}{}^{c} \overset{c}{\nabla}_{c} C_{ai_5 i_6 b}
&&\overset{16}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = C^{a}_{.i_1 i_2.}{}^{b} C^{\gamma}_{.i_3 i_4.}{}^{c} \overset{c}{\nabla}_{a} C_{bi_5 i_6 c} \\
&\overset{17}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3.}{}^{b} C_{a..i_4}^{\ cd} \overset{c}{\nabla}_{b} C_{ci_5 i_6 d}
&&\overset{18}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3.}{}^{b} C_{a..i_4}^{\ cd} \overset{c}{\nabla}_{i_5} C_{bdci_6} \\
&\overset{19}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} = \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3.}{}^{b} C_{a..i_4}^{\ cd} \overset{c}{\nabla}_{i_5} C_{bcdi_6}.
\end{aligned}
\tag{3.7}
$$

Using (3.1), (3.2) and the transformation formulae (2.4), (2.5), (2.8), we obtain

$$
\begin{bmatrix} X^{\gamma}(\overset{1}{C}{}^{(3)}_{i_1\ldots i_6}) \\ \cdot \\ \cdot \\ \cdot \\ X^{\gamma}(\overset{27}{C}{}^{(3)}_{i_1\ldots i_6})^{\gamma} \end{bmatrix}
\underset{*}{=} A^{(3)}_{6}(-1,n)
\begin{bmatrix} \overset{1}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} \\ \cdot \\ \cdot \\ \cdot \\ \overset{19}{Q}{}^{(3)\gamma}_{i_1\ldots i_6} \end{bmatrix},
$$

where

$$
A^{(3)}_{6}(-1,n) =
$$

$$
\left[\begin{array}{ccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & -10 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -3 & 0 & 0 & 0 & -1 & 0 & 0 & -5 & -1 & -1 \\
12/a & 0 & -8/a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 8 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 4 & -6 & -4 & 0 & 0 & 0 \\
-4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/a & 0 & -3 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -4 \\
0 & 0 & -1/a & 0 & -3 & -3 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & -2 & -1 & 2 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 \\
0 & 0 & 2/a & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 2 & 0 & 0 & -6 \\
0 & 0 & 2/a & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & -6 & 0 \\
0 & 0 & 3/a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 3 & -3 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & 4 & -4 & 0 & 0 \\
8/a & 0 & 2/a & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & 8 & 0 & 0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 3 & -1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & -a & 0 & 0 & 0 \\
0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
-4 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & -1 & 0 & 0 & 0 & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

with $a = n - 3$.

It is rank $A_6^{(3)}(-1, n) = 17$ if $n > 4$. For $n = 4$ there are more identities concerning the tensors (3.6) and (3.7) (see [Thi]). So we get

PROPOSITION 3.2. – (i) *If $n > 4$, then $\mathcal{S}_6^{(3)}(-1, n)$ is generated by the tensors*

$$\overset{1}{S}{}_6^{(3)}(-1,n) = -346\overset{1}{C}{}^{(3)}_{i_1\ldots i_6} + 36\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 234\overset{3}{C}{}^{(3)}_{i_1\ldots i_6} + 140\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} - 310\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 240\overset{6}{C}{}^{(3)}_{i_1\ldots i_6} - 56\overset{7}{C}{}^{(3)}_{i_1\ldots i_6} + 52\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{280}{n-3}\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} + \frac{448}{n-3}\overset{10}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{448}{n-3}\overset{11}{C}{}^{(3)}_{i_1\ldots i_6} + 405\overset{12}{C}{}^{(3)}_{i_1\ldots i_6} - 220\overset{13}{C}{}^{(3)}_{i_1\ldots i_6} + 680\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} + 300\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} + 220\overset{21}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{560}{n-3}\overset{22}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{1030}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}$$

$$\overset{2}{S}{}_6^{(3)}(-1,n) = -64\overset{1}{C}{}^{(3)}_{i_1\ldots i_6} - 4\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 40\overset{3}{C}{}^{(3)}_{i_1\ldots i_6} + 4\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} - 56\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 32\overset{6}{C}{}^{(3)}_{i_1\ldots i_6} - 12\overset{7}{C}{}^{(3)}_{i_1\ldots i_6} + 4\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{8}{n-3}\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} + \frac{48}{n-3}\overset{10}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{48}{n-3}\overset{11}{C}{}^{(3)}_{i_1\ldots i_6} + 75\overset{12}{C}{}^{(3)}_{i_1\ldots i_6} + 120\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} + 40\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} + 20\overset{20}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{60}{n-3}\overset{22}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{130}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}$$

$$\overset{3}{S}{}_6^{(3)}(-1,n) = 3(n-3)\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 3(n-3)\overset{3}{C}{}^{(3)}_{i_1\ldots i_6} + 5(n-3)\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} - 5(n-3)\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 10\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} - 10(n-3)\overset{13}{C}{}^{(3)}_{i_1\ldots i_6} + 10(n-3)\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} + 10\overset{26}{C}{}^{(3)}_{i_1\ldots i_6} + \frac{10}{n-3}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}$$

$$\overset{4}{S}{}_6^{(3)}(-1,n) = +20(n-3)\overset{1}{C}{}^{(3)}_{i_1\ldots i_6} + 120(n-3)\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 120(n-3)\overset{3}{C}{}^{(3)}_{i_1\ldots i_6} + 232(n-3)\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} - 168(n-3)\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 96(n-3)\overset{6}{C}{}^{(3)}_{i_1\ldots i_6} + 4(n-3)\overset{7}{C}{}^{(3)}_{i_1\ldots i_6} + 12(n-3)\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} - 464\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} + 144\overset{10}{C}{}^{(3)}_{i_1\ldots i_6} - 144\overset{11}{C}{}^{(3)}_{i_1\ldots i_6} - 25(n-3)\overset{12}{C}{}^{(3)}_{i_1\ldots i_6} - 440(n-3)\overset{13}{C}{}^{(3)}_{i_1\ldots i_6} + 360(n-3)\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} + 120(n-3)\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} - 180\overset{22}{C}{}^{(3)}_{i_1\ldots i_6} + 440\overset{25}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{170}{n-3}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}$$

$$\begin{aligned}\overset{5}{S}{}^{(3)}_{6}(-1,n) = &- 596\overset{1}{C}{}^{(3)}_{i_1\dots i_6} - 78\overset{2}{C}{}^{(3)}_{i_1\dots i_6} - 186\overset{3}{C}{}^{(3)}_{i_1\dots i_6} - 76\overset{4}{C}{}^{(3)}_{i_1\dots i_6}\\ &- 256\overset{5}{C}{}^{(3)}_{i_1\dots i_6} - 272\overset{6}{C}{}^{(3)}_{i_1\dots i_6} + 4\overset{7}{C}{}^{(3)}_{i_1\dots i_6} + 12\overset{8}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ \frac{152}{n-3}\overset{9}{C}{}^{(3)}_{i_1\dots i_6} - \frac{560}{n-3}\overset{10}{C}{}^{(3)}_{i_1\dots i_6} + \frac{560}{n-3}\overset{11}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 745\overset{12}{C}{}^{(3)}_{i_1\dots i_6} + 220\overset{13}{C}{}^{(3)}_{i_1\dots i_6} + 580\overset{14}{C}{}^{(3)}_{i_1\dots i_6} + 120\overset{15}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 220\overset{19}{C}{}^{(3)}_{i_1\dots i_6} + \frac{700}{n-3}\overset{22}{C}{}^{(3)}_{i_1\dots i_6} + \frac{1810}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\dots i_6}\end{aligned}$$

$$\begin{aligned}\overset{6}{S}{}^{(3)}_{6}(-1,n) = &- 636\overset{1}{C}{}^{(3)}_{i_1\dots i_6} - 98\overset{2}{C}{}^{(3)}_{i_1\dots i_6} - 386\overset{3}{C}{}^{(3)}_{i_1\dots i_6} - 100\overset{4}{C}{}^{(3)}_{i_1\dots i_6}\\ &- 580\overset{5}{C}{}^{(3)}_{i_1\dots i_6} - 520\overset{6}{C}{}^{(3)}_{i_1\dots i_6} - 4\overset{7}{C}{}^{(3)}_{i_1\dots i_6} - 12\overset{8}{C}{}^{(3)}_{i_1\dots i_6}\\ &- \frac{240}{n-3}\overset{9}{C}{}^{(3)}_{i_1\dots i_6} + \frac{912}{n-3}\overset{10}{C}{}^{(3)}_{i_1\dots i_6} - \frac{912}{n-3}\overset{11}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 795\overset{12}{C}{}^{(3)}_{i_1\dots i_6} + 220\overset{13}{C}{}^{(3)}_{i_1\dots i_6} + 1180\overset{14}{C}{}^{(3)}_{i_1\dots i_6} + 320\overset{15}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 440\overset{18}{C}{}^{(3)}_{i_1\dots i_6} - \frac{1140}{n-3}\overset{22}{C}{}^{(3)}_{i_1\dots i_6} - \frac{2250}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\dots i_6}\end{aligned}$$

$$\begin{aligned}\overset{7}{S}{}^{(3)}_{6}(-1,n) = &- 68\overset{1}{C}{}^{(3)}_{i_1\dots i_6} - 12\overset{2}{C}{}^{(3)}_{i_1\dots i_6} - 36\overset{3}{C}{}^{(3)}_{i_1\dots i_6} + 12\overset{4}{C}{}^{(3)}_{i_1\dots i_6} - 88\overset{5}{C}{}^{(3)}_{i_1\dots i_6}\\ &- 96\overset{6}{C}{}^{(3)}_{i_1\dots i_6} + 4\overset{7}{C}{}^{(3)}_{i_1\dots i_6} + 12\overset{8}{C}{}^{(3)}_{i_1\dots i_6} - \frac{24}{n-3}\overset{9}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ \frac{144}{n-3}\overset{10}{C}{}^{(3)}_{i_1\dots i_6} - \frac{144}{n-3}\overset{11}{C}{}^{(3)}_{i_1\dots i_6} + 85\overset{12}{C}{}^{(3)}_{i_1\dots i_6} + 80\overset{14}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 120\overset{15}{C}{}^{(3)}_{i_1\dots i_6} + 80\overset{17}{C}{}^{(3)}_{i_1\dots i_6} - \frac{180}{n-3}\overset{22}{C}{}^{(3)}_{i_1\dots i_6} - \frac{390}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\dots i_6}\end{aligned}$$

$$\begin{aligned}\overset{8}{S}{}^{(3)}_{6}(-1,n) = &- 68\overset{1}{C}{}^{(3)}_{i_1\dots i_6} + 72\overset{2}{C}{}^{(3)}_{i_1\dots i_6} - 120\overset{3}{C}{}^{(3)}_{i_1\dots i_6} + 92\overset{4}{C}{}^{(3)}_{i_1\dots i_6}\\ &- 168\overset{5}{C}{}^{(3)}_{i_1\dots i_6} - 96\overset{6}{C}{}^{(3)}_{i_1\dots i_6} + 4\overset{7}{C}{}^{(3)}_{i_1\dots i_6} + 12\overset{8}{C}{}^{(3)}_{i_1\dots i_6}\\ &- \frac{24}{n-3}\overset{9}{C}{}^{(3)}_{i_1\dots i_6} + \frac{144}{n-3}\overset{10}{C}{}^{(3)}_{i_1\dots i_6} - \frac{144}{n-3}\overset{11}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 85\overset{12}{C}{}^{(3)}_{i_1\dots i_6} - 280\overset{13}{C}{}^{(3)}_{i_1\dots i_6} + 360\overset{14}{C}{}^{(3)}_{i_1\dots i_6} + 120\overset{15}{C}{}^{(3)}_{i_1\dots i_6}\\ &+ 80\overset{16}{C}{}^{(3)}_{i_1\dots i_6} - \frac{180}{n-3}\overset{22}{C}{}^{(3)}_{i_1\dots i_6} - \frac{390}{(n-3)^2}\overset{27}{C}{}^{(3)}_{i_1\dots i_6}\end{aligned}$$

$$\begin{aligned}\overset{9}{S}{}^{(3)}_6(-1,n) =&20(n-3)\overset{1}{C}{}^{(3)}_{i_1\ldots i_6} - (n-3)\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 10(n-3)\overset{3}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ (n-3)\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} - 14(n-3)\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 8(n-3)\overset{6}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 4(n-3)\overset{7}{C}{}^{(3)}_{i_1\ldots i_6} + 12(n-3)\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} - 2\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} + 166\overset{10}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 56\overset{11}{C}{}^{(3)}_{i_1\ldots i_6} - 25(n-3)\overset{12}{C}{}^{(3)}_{i_1\ldots i_6} + 30(n-3)\overset{14}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 10(n-3)\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} - 180(n-3)\overset{22}{C}{}^{(3)}_{i_1\ldots i_6} + 110\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{60}{n-3}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}\end{aligned}$$

$$\begin{aligned}\overset{10}{S}{}^{(3)}_6(-1,n) =&332(n-3)\overset{1}{C}{}^{(3)}_{i_1\ldots i_6} + 12(n-3)\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} + 120(n-3)\overset{3}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 12(n-3)\overset{4}{C}{}^{(3)}_{i_1\ldots i_6} + 168(n-3)\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} + 96(n-3)\overset{6}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 4(n-3)\overset{7}{C}{}^{(3)}_{i_1\ldots i_6} - 12(n-3)\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} + 24\overset{9}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 296\overset{10}{C}{}^{(3)}_{i_1\ldots i_6} + 144\overset{11}{C}{}^{(3)}_{i_1\ldots i_6} - 415(n-3)\overset{12}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 360(n-3)\overset{14}{C}{}^{(3)}_{i_1\ldots i_6} - 120(n-3)\overset{15}{C}{}^{(3)}_{i_1\ldots i_6} - 700(n-3)\overset{22}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 440\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} - \frac{270}{n-3}\overset{27}{C}{}^{(3)}_{i_1\ldots i_6}.\end{aligned}$$

(ii) $\mathcal{S}_6{}^{(3)}(-1,4)$ is generated by the tensors

$$\begin{aligned}\overset{1}{S}{}^{(3)}_6(-1,4) =&8\overset{2}{C}{}^{(3)}_{i_1\ldots i_6} - 7\overset{16}{C}{}^{(3)}_{i_1\ldots i_6} + 104\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} + 10\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} - 218\overset{25}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 72\overset{27}{C}{}^{(3)}_{i_1\ldots i_6} - 10\overset{28}{C}{}^{(3)}_{i_1\ldots i_6}\end{aligned}$$

$$\begin{aligned}\overset{2}{S}{}^{(3)}_6(-1,4) =&4\overset{5}{C}{}^{(3)}_{i_1\ldots i_6} - 5\overset{16}{C}{}^{(3)}_{i_1\ldots i_6} - 40\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} - 10\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} + 90\overset{25}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 40\overset{27}{C}{}^{(3)}_{i_1\ldots i_6} + 10\overset{28}{C}{}^{(3)}_{i_1\ldots i_6}\end{aligned}$$

$$\begin{aligned}\overset{3}{S}{}^{(3)}_6(-1,4) =&32\overset{9}{C}{}^{(3)}_{i_1\ldots i_6} - 3\overset{16}{C}{}^{(3)}_{i_1\ldots i_6} - 24\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} - 14\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} + 30\overset{25}{C}{}^{(3)}_{i_1\ldots i_6}\\ &- 56\overset{27}{C}{}^{(3)}_{i_1\ldots i_6} + 14\overset{28}{C}{}^{(3)}_{i_1\ldots i_6}\end{aligned}$$

$$\begin{aligned}\overset{4}{S}{}^{(3)}_6(-1,4) =&8\overset{8}{C}{}^{(3)}_{i_1\ldots i_6} - 7\overset{16}{C}{}^{(3)}_{i_1\ldots i_6} + 24\overset{23}{C}{}^{(3)}_{i_1\ldots i_6} + 82\overset{24}{C}{}^{(3)}_{i_1\ldots i_6} - 34\overset{25}{C}{}^{(3)}_{i_1\ldots i_6}\\ &+ 184\overset{27}{C}{}^{(3)}_{i_1\ldots i_6} - 2\overset{28}{C}{}^{(3)}_{i_1\ldots i_6}\end{aligned}$$

$$\overset{5}{S}{}_6^{(3)}(-1,4) = \overset{14}{C}{}^{(3)}_{i_1\dots i_6} + 4\overset{23}{C}{}^{(3)}_{i_1\dots i_6} + 2\overset{24}{C}{}^{(3)}_{i_1\dots i_6} - 10\overset{25}{C}{}^{(3)}_{i_1\dots i_6} + 6\overset{27}{C}{}^{(3)}_{i_1\dots i_6} - 2\overset{28}{C}{}^{(3)}_{i_1\dots i_6}$$

$$\overset{6}{S}{}_6^{(3)}(-1,4) = C^{a}_{.i_1 i_2 .}{}^{b} C_{a i_3 i_4 b} \overset{c}{\nabla}_c \overset{c}{\nabla}_u C^{u}_{.i_5 i_6 .}{}^{c}.$$

We summarize our results in the following

THEOREM 3.1. – (i) If $n > 4$, $\mathcal{S}_6(-1,n)$ is generated by

$$\{\overset{1}{S}{}_6^{(2)}(-1,n),\ \overset{2}{S}{}_6^{(2)}(-1,n),\ \overset{1}{S}{}_6^{(3)}(-1,n), \dots,\ \overset{10}{S}{}_6^{(3)}(-1,n)\}.$$

(ii) $\mathcal{S}_6(-1,4)$ is generated by [4]

$$\{\overset{1}{S}{}_6^{(2)}(-1,4),\ \overset{2}{S}{}_6^{(2)}(-1,4),\ \overset{1}{S}{}_6^{(3)}(-1,4), \dots,\ \overset{6}{S}{}_6^{(3)}(-1,4)\}.$$

## 4. ADDITION CONFORMAL TENSORS FOR $n = 4$

In a four dimensional pseudo Riemannian manifold $(M, g)$ with a smooth metric of the signature (+- - -) the *moments* of some conformally invariant field equations are symmetric, trace-free conformal tensors (see Section 6 and [Gü; Wü5]). In case of odd order these moments also depend on the Levi-Civita pseudo tensor $e_{\mathrm{abcd}}$. Thus for a general construction of such conformal tensors we have to extend the tensor algebra $\mathcal{R}$ (see 1.1) by the pseudo tensor $e_{\mathrm{abcd}}$. Let $\mathcal{R}^*$ be that tensor algebra which is generated by the tensors (1.1) and $e_{\mathrm{abcd}}$. Then all the results of Section 2 remain valid with exception of Proposition 2.2 $(\gamma)$ where the tensors (2.7) have to be extended by $e_{\mathrm{abcd}}$.

The Bianchi identity for the dual Weyl tensor

$$^*C_{abcd} := \frac{1}{2} e_{ab}{}^{ef} C_{efcd} \tag{4.1}$$

has the form

$$\nabla_{[a}{}^*C_{bc]ij} + g_{j[a|}\nabla_u{}^*C^{u}_{.i|bc]} - g_{i[a|}\nabla_u{}^*C^{u}_{.j|bc]} = 0. \tag{4.2}$$

Let $\mathcal{S}_r{}^*(\omega)$ be the set of all nontrivial conformal, symmetric, trace-free tensors of $\mathcal{R}^*$ with weight $\omega$ and covariant rank $r$. Let $\mathcal{S}_r{}^{*(\alpha)}(\omega)$ be the subset of those elements of $\mathcal{S}_r{}^*(\omega)$ whose monomials have the order $\alpha$.

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[4] This generating system for $\mathcal{S}_6(-1,4)$ is the same as the one derived in [GeW2] and has already been used in [Wü5], in detail $S^{k}\,{}^{(\nu)}_{6}(-1,4)$ is equal to $S_6^{(\nu,k)}(-1,4)$ of [Wü5], where $(k,\nu) \in \{(2,1), (2,2), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$.

In [Wü1] it was proved

PROPOSITION 4.1. – *The sets* $\mathcal{S}_1{}^*(-1)$, $\mathcal{S}_3{}^*(-1)$ *are empty.*

Now we consider *the case* $r = 5$:

Because of $\overset{c}{\nabla}_a \overset{c}{\nabla}_b {}^*C^{a}_{.i_1 i_2 .}{}^{b} = 0$ (see [KNT]) and Lemma (2.3) $\mathcal{S}_r{}^{*(1)}(-1)$ is empty and we have

$$\mathcal{S}_5{}^*(-1) = \mathcal{S}_5{}^{*(2)}(-1) \cup \mathcal{S}_5{}^{*(3)}(-1). \tag{4.3}$$

Firstly, let $\alpha = 2$: Then on account of (2.13) it is $q = 3$.

Under consideration of (4.2),

$$\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_{r-1}} {}^*C^{abc}_{\dots\ i_r} \overset{c}{\nabla}_{i_{r+1}} \dots \overset{c}{\nabla}_{i_{s-1}} C_{abci_5} \underset{*}{=} 0 \quad (r, s = 1, 2, 3, \dots) \tag{4.4}$$

and of some more identities with respect to the derivatives of $C\dots$ and ${}^*C\dots$ which one can show by means of the spinor formalism (see [Thi; GeW1]) a tensor of $\mathcal{S}_5{}^{*(2)}(-1)$ has to be a linear combination of the following linearly independent [3)] monomials ([GeW1; Ge])

$$\overset{1}{C}{}^{(2)}_{i_1\dots i_5} = TS[C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5}]$$

$$\overset{2}{C}{}^{(2)}_{i_1\dots i_5} = TS[C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}]$$

$$\overset{3}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5}]$$

$$\overset{4}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}]$$

$$\overset{5}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{u} C^{u}_{.abi_1} \overset{c}{\nabla}_{i_2} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5 .}{}^{b}]$$

$$\overset{6}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{u} C^{u}_{.i_1 i_2 a} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{\nu} C^{\nu}_{.i_4 i_5 .}{}^{a}]$$

$$\overset{7}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}{}^{d} C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{d} C_{ai_4 i_5 b}]$$

$${}^*\overset{1}{C}{}^{(2)}_{i_1\dots i_5} = TS[{}^*C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5}]$$

$${}^*\overset{2}{C}{}^{(2)}_{i_1\dots i_5} = TS[{}^*C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}]$$

$${}^*\overset{3}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{i_1} {}^*C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5}]$$

$${}^*\overset{4}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{i_1} {}^*C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}]$$

$${}^*\overset{5}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{u} {}^*C^{u}_{.abi_1} \overset{c}{\nabla}_{i_2} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5 .}{}^{b}]$$

$${}^*\overset{6}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}_{u} {}^*C^{u}_{.i_1 i_2 a} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{v} C^{v}_{.i_4 i_5 .}{}^{a}]$$

$${}^*\overset{7}{C}{}^{(2)}_{i_1\dots i_5} = TS[\overset{c}{\nabla}{}^{d}\, {}^*C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{d} C_{ai_4 i_5 b}].$$

We put

$$\overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5} \qquad \overset{2}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \delta^{\gamma}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}$$

$$\overset{3}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{a} C^{\gamma}_{.i_4 i_5 b} \qquad \overset{4}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := C^{\gamma}_{.abi_1} \overset{c}{\nabla}_{i_2} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5 .}{}^{b}$$

$$\overset{5}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := C^{\gamma}_{.i_1 i_2 .}{}^{a} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{u} C^{c}_{.i_4 i_5 a.} \qquad \overset{6}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \delta^{\gamma}_{i_1} \overset{c}{\nabla}{}^{d} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{d} C_{a i_4 i_5 b}$$

$$\overset{7}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \delta^{\gamma}_{i_1} \overset{c}{\nabla}_{i_2} C^{a}_{.i_3 i_4 .}{}^{b} \overset{c}{\nabla}_{u} C^{u}_{.abi_5} \qquad \overset{8}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \delta^{\gamma}_{i_1} \overset{c}{\nabla}_{u} C^{u}_{.i_2 i_3 a} \overset{c}{\nabla}_{v} C^{v}_{.i_4 i_5 .}{}^{a}$$

$$\overset{9}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \overset{c}{\nabla}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{i_4} C^{\gamma}_{.abi_5} \qquad \overset{10}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \overset{c}{\nabla}_{i_1} C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} C^{\gamma}_{.i_4 i_5 b}$$

$$\overset{11}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} := \overset{c}{\nabla}_{i_1} C^{\gamma}_{.i_2 i_3 .}{}^{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 a}$$

$${}^{*}\overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = \delta^{\gamma}_{i_1} {}^{*}C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{i_4} \overset{c}{\nabla}_{u} C^{u}_{.abi_5} \qquad {}^{*}\overset{2}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = \delta^{\gamma}_{i_1} {}^{*}C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 b}$$

$${}^{*}\overset{3}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = {}^{*}C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{a} C^{\gamma}_{.i_4 i_5 b} \qquad {}^{*}\overset{4}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = {}^{*}C^{a}_{.i_1 i_2 .}{}^{b} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{i_4} C^{\gamma}_{.abi_5}$$

$${}^{*}\overset{5}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = {}^{*}C^{\gamma}_{.i_1 i_2 .}{}^{a} \overset{c}{\nabla}_{i_3} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 a} \qquad {}^{*}\overset{6}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = \delta^{\gamma}_{i_1} \overset{c}{\nabla}_{i_2} {}^{*}C^{a}_{.i_3 i_4 .}{}^{b} \overset{c}{\nabla}_{u} C^{u}_{.abi_5}$$

$${}^{*}\overset{7}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = \overset{c}{\nabla}_{i_1} {}^{*}C^{a}_{.i_2 i_3 .}{}^{b} \overset{c}{\nabla}_{a} C^{\gamma}_{.i_4 i_5 b} \qquad {}^{*}\overset{8}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} = \overset{c}{\nabla}_{i_1} {}^{*}C^{\gamma}_{.i_2 i_3 .}{}^{a} \overset{c}{\nabla}_{u} C^{u}_{.i_4 i_5 a}.$$

Using the transformation formulae (2.4), (2.5), (2.8), one gets

$$\begin{bmatrix} X^{\gamma}(\overset{1}{C}{}^{(2)}_{i_1\ldots i_5}) \\ \cdot \\ \cdot \\ \cdot \\ X^{\gamma}(\overset{7}{C}{}^{(2)}_{i_1\ldots i_5}) \\ X^{\gamma}({}^{*}\overset{1}{C}{}^{(2)}_{i_1\ldots i_5}) \\ \cdot \\ \cdot \\ \cdot \\ X^{\gamma}({}^{*}\overset{7}{C}{}^{(2)}_{i_1\ldots i_5})^{\gamma} \end{bmatrix} = A_5^{*(2)}(-1) \begin{bmatrix} \overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} \\ \cdot \\ \cdot \\ \cdot \\ \overset{11}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} \\ {}^{*}\overset{1}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} \\ \cdot \\ \cdot \\ \cdot \\ {}^{*}\overset{8}{Q}{}^{(2)\gamma}_{i_1\ldots i_5} \end{bmatrix}$$

with

$$A_5^{*(2)}(-1) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and

$$A = \begin{bmatrix} -10 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 1 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & -1 & 0 & -4 & 0 & 1 & 0 & 1 \\ 0 & -4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & -10 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & 0 & 1 \\ -4 & 0 & -4 & 0 & 0 & -5 & -4 & 0 & 0 & -4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -10 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 1 & 0 & -5 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & -1 & -4 & 0 & 1 \\ 0 & -4 & 0 & 0 & 1 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 10 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -4 & 0 & -4 & 0 & 0 & 6 & 6 & 2 \end{bmatrix}$$

Because of rank $A = 7$, rank $B = 6$ we obtain

PROPOSITION 4.2. – $\mathcal{S}_5{}^{*(2)}(-1)$ *is generated by the tensor*

$$S_5^{(2)}(-1) = 4\,{}^*\overset{2}{C}{}^{(2)}_{i_1\ldots i_5} - 6\,{}^*\overset{4}{C}{}^{(2)}_{i_1\ldots i_5} + 26\,{}^*\overset{6}{C}{}^{(2)}_{i_1\ldots i_5} + {}^*\overset{7}{C}{}^{(2)}_{i_1\ldots i_5}. \qquad (4.5)$$

Finally, in the case $\alpha = 3$ we have $q = 1$ because of (2.13). A tensor of $\mathcal{S}_5{}^{*(3)}(-1)$ has to be a linear combination of the following linearly independent [3] monomials

$$\begin{aligned}
\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[C^{a}_{.i_1 i_2.}{}^{b} C_{a i_3 i_4.}{}^{d} \overset{c}{\nabla}_u C^{u}_{.bdi_5}] \\
\overset{2}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[C_{a..i_1}^{\;cd} C_{bcdi_2} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5.}{}^{b}] \\
\overset{3}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[C_{a..i_1}^{\;cd} C_{bi_2cd} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5.}{}^{b}] \\
{}^*\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[{}^*C^{a}_{.i_1 i_2.}{}^{b} C_{a i_3 i_4.}{}^{d} \overset{c}{\nabla}_u C^{u}_{.bdi_5}] \\
{}^*\overset{2}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[C_{a..i_1}^{\;cd} C_{bcdi_2} \overset{c}{\nabla}_{i_3} {}^*C^{a}_{.i_4 i_5.}{}^{b}] \\
{}^*\overset{3}{C}{}^{(3)}_{i_1\ldots i_5} &= TS[{}^*C_{a..i_1}^{\;cd} C_{bi_2cd} \overset{c}{\nabla}_{i_3} C^{a}_{.i_4 i_5.}{}^{b}]
\end{aligned}$$

After using the transformation formulae (2.4), (2.5), (2.8) and consideration of further identities [Thi], we obtain

$$X^{\gamma}(\overset{1}{C}{}^{(3)}_{i_1\ldots i_5}) \underset{*}{=} C^{a}_{.i_1 i_2.}\,{}^{b}C_{a i_3 i_4.}\,{}^{c}C^{\gamma}_{.bci_5} =: Q^{(3)\gamma}_{i_1\ldots i_5}$$

$$X^{\gamma}(\overset{2}{C}{}^{(3)}_{i_1\ldots i_5}) \underset{*}{=} -12 Q^{(3)\gamma}_{i_1\ldots i_5}$$

$$X^{\gamma}(\overset{3}{C}{}^{(3)}_{i_1\ldots i_5}) \underset{*}{=} 8 Q^{(3)\gamma}_{i_1\ldots i_5}$$

$$X^{\gamma}({}^{*}\overset{1}{C}{}^{(3)}_{i_1\ldots i_5}) = {}^{*}C^{a}_{.i_1 i_2.}\,{}^{b}C_{a i_3 i_4.}\,{}^{c}C^{\gamma}_{.bci_5} =:{}^{*}Q^{(3)\gamma}_{i_1\ldots i_5}$$

$$X^{\gamma}({}^{*}\overset{2}{C}{}^{(3)}_{i_1\ldots i_5}) \underset{*}{=} -12\,{}^{*}Q^{(3)\gamma}_{i_1\ldots i_5}$$

$$X^{\gamma}({}^{*}\overset{3}{C}{}^{(3)}_{i_1\ldots i_5}) \underset{*}{=} 8\,{}^{*}Q^{(3)\gamma}_{i_1\ldots i_5}.$$

Thus we get

PROPOSITION 4.3. – *$\mathcal{S}_5{}^{*(3)}(-1)$ is generated by the tensors*

$$\overset{1}{S}{}^{(3)}_{5}(-1) = 12\,{}^{*}\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} +{}^{*}\,\overset{2}{C}{}^{(3)}_{i_1\ldots i_5} \tag{4.6}$$

$$\overset{2}{S}{}^{(3)}_{5}(-1) = 8\,{}^{*}\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} -{}^{*}\,\overset{3}{C}{}^{(3)}_{i_1\ldots i_5} \tag{4.7}$$

$$\overset{3}{S}{}^{(3)}_{5}(-1) = 12\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} + \overset{2}{C}{}^{(3)}_{i_1\ldots i_5} \tag{4.8}$$

$$\overset{4}{S}{}^{(3)}_{5}(-1) = 8\overset{1}{C}{}^{(3)}_{i_1\ldots i_5} - \overset{3}{C}{}^{(3)}_{i_1\ldots i_5}. \tag{4.9}$$

It follows from the Propositions 4.1, 4.2, 4.3.

THEOREM 4.1. – *$\mathcal{S}_5{}^{*}(-1)$ is generated by the tensors* [5]

$$S^{(2)}_{5}(-1), \overset{1}{S}{}^{(3)}_{5}(-1), \ldots, \overset{4}{S}{}^{(3)}_{5}(-1).$$

[5] This generating system for $\mathcal{S}^{*}_{5}(-1)$ is the same as the one derived in [GeW1] and has already been used in [Wü5], in detail we have

$$\{S^{(2)}_{5}(-1), \overset{1}{S}{}^{(3)}_{5}(-1), \ldots, \overset{4}{S}{}^{(3)}_{5}(-1)\} \equiv \{S^{(2)}_{5}(-1), S^{(3,1)}_{5}(-1), \ldots, S^{(3,4)}_{5}(-1)\}_{[\text{Wü5}]}.$$

## 5. THE CASE $n = 3$

If $n = 3$, the Weyl tensor vanishes. However, the tensor

$$S_{abc} := \overset{c}{\nabla}_{[c} L_{b]a} \tag{5.1}$$

is a conformal tensor of the weight 0 [Sz]. It is $S_{abc} = 0$ iff $(M, g)$ is locally conformal flat. One can show easily the following properties of $S_{abc}$ [Scho,Ge]:

$$S^{a}_{.ab} = S^{a}_{.ba} = S_{ab.}{}^{b} = 0 \tag{5.2}$$

$$\nabla_{[i|} S_{a|bc]} = 0,\ \nabla^{u} S_{uab} = 0,\ \nabla^{u} S_{[ab]u} = 0 \tag{5.3}$$

$$S_{i[ab} g_{c|j|} g_{d]k} = 0. \tag{5.4}$$

LEMMA 5.1. – *For the tensor $S_{abc}$ it holds the identity*

$$T^{j}_{.abcd} := \delta^{j}_{[a} S_{b]cd} - g_{b[c|} S_{a|d].}{}^{j} + g_{a[c|} S_{b|d].}{}^{j} \equiv 0. \tag{5.5}$$

*Proof.* – The tensor $T^{j}_{.abcd}$ satisfies the conditions

$$T^{j}_{.abcd} = -T^{j}_{.bacd} = -T^{j}_{.abdc}$$

and $T^{j}_{.akc.}{}^{k} = 0$. Consequently we have $T^{j}_{.abcd} = 0$. (see [Lo]).

COROLLARY 5.1. – *It is*

$$\nabla_{[a} S_{b]cd} = g_{b[c|} \nabla_{u} S_{a|d].}{}^{u} - g_{a[c|} \nabla_{u} S_{b|d].}{}^{u}. \tag{5.6}$$

All results of Section 2 with exception of Proposition 2.2 where one has to replace the tensors (2.7) by

$$g^{ab},\ g_{ab},\ \overset{c}{\nabla}_{i_1} \ldots \overset{c}{\nabla}_{i_r} S_{i_{r+1}ab} \qquad r = 0, 1, 2, \ldots \tag{5.7}$$

remain valid. In particular we have

$$X^{\gamma}(S_{abc}) = 0,\ \overset{c}{\nabla}_{i} S_{abc} = \nabla_{i} S_{abc} \tag{5.8}$$

$$X^{\gamma}(\overset{c}{\nabla}_{i} S_{abc}) = -3\delta^{\gamma}_{i} S_{abc} - \delta^{\gamma}_{a} S_{ibc} - \delta^{\gamma}_{b} S_{aic} - \delta^{\gamma}_{c} S_{abi} + g_{ia} S^{\gamma}_{.bc} + g_{ib} S_{a.c}^{\ \gamma} + g_{ic} S_{ab.}{}^{\gamma} \tag{5.9}$$

$$X^{\gamma}(\overset{c}{\nabla}{}^{u} S_{abu}) = -2 S_{(ab).}{}^{\gamma} \tag{5.10}$$

$$\begin{aligned} X^{\gamma}(\overset{c}{\nabla}_{i} \overset{c}{\nabla}_{j} S_{abc}) = &- 8\delta^{\gamma}_{(i} \nabla_{j)} \quad S_{abc} + g_{ij} \nabla^{\gamma} S_{abc} - 2\delta^{\gamma}_{a} \nabla_{(i} S_{j)bc} \\ &- 2\delta^{\gamma}_{b} \nabla_{(i|} S_{a|j)c} - 2\delta^{\gamma}_{c} \nabla_{(i|} S_{ab|j)} \\ &+ 2 g_{a(i} \nabla_{j)} S^{\gamma}_{.bc} + g_{b(i} \nabla_{j)} S_{a.c}^{\ \gamma} + g_{c(i} \nabla_{j)} S_{ab.}{}^{\gamma}\ ; \end{aligned} \tag{5.11}$$

The tensors of $\mathcal{S}_r{}^{(\alpha)}(\omega, 3)$ are representable as a sum of monomials containing $\alpha$ factors $\{\overset{c}{\nabla}_{i_1}...\overset{c}{\nabla}_{i_k} S_{i_{k+1}ab}\}$ and $q := r - 2\omega - 2\alpha - 2$ operators $\overset{c}{\nabla}$ (with respect to $S_{abc}$).

In the following we construct generating systems for $\mathcal{S}_r{}^{(2)}(\omega, 3)$ in the cases $(r, \omega) = (0, -4), (1, -3), (2, -3), (3, -2), (4, -2)$.

Firstly, let be $r = 0$:

Because of (5.2),..., (5.4) and (5.6) a tensor of $\mathcal{S}_0{}^{(2)}(-4, 3)$ has to be a linear combination of the following monomials:

$$\overset{1}{S} = \nabla_a S_{bcd} \nabla^a S^{bcd}_{...}$$

$$\overset{2}{S} = \nabla_a S_{bcd} \nabla^b S^{acd}_{...}$$

$$\overset{3}{S} = S^{abc}_{...} \overset{c}{\nabla}{}^d \overset{c}{\nabla}_d S_{abc}.$$

Using the transformation formulae (5.8),..., (5.10) and the identifies (5.2),..., (5.4), (5.6) one obtains

$$\begin{bmatrix} X^\gamma(\overset{1}{S}) \\ X^\gamma(\overset{2}{S}) \\ X^\gamma(\overset{3}{S}) \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -2 & -8 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} \overset{1}{T}{}^\gamma \\ \overset{2}{T}{}^\gamma \end{bmatrix}.$$

with the transformation terms

$$\overset{1}{T}{}^\gamma = S_{abc} \nabla^\gamma S^{abc}$$

$$\overset{2}{T}{}^\gamma = S_{abc} \nabla^a S^{\gamma bc}.$$

The transformation matrix has the rank 2 from where we get

PROPOSITION 5.1. – *$\mathcal{S}_0{}^{(2)}(-4, 3)$ is generated by the tensor*

$$S_0^{(2)}(-4, 3) = -12\overset{1}{S} + 11\overset{2}{S} + 10\overset{3}{S}.$$

Now, let be $r = 2$:

Considering (5.2),..., (5.6) the following linear independent terms are left

$$\overset{1}{S}_{i_1 i_2} = TS[\nabla_{i_1} S^{abc}_{...} \nabla_{i_2} S_{abc}] \quad \overset{2}{S}_{i_1 i_2} = TS[\nabla^a S_{i_1 i_2 .}{}^b \nabla^u S_{abu}]$$

$$\overset{3}{S}_{i_1 i_2} = TS[\nabla_{i_1} S^{abc}_{...} \nabla_a S_{i_2 bc}] \quad \overset{4}{S}_{i_1 i_2} = TS[S^{abc}_{...} \overset{c}{\nabla}_{i_1} \overset{c}{\nabla}_{i_2} S_{abc}]$$

$$\overset{5}{S}_{i_1 i_2} = TS[S^{abc}_{...} \overset{c}{\nabla}_{i_1} \overset{c}{\nabla}_a S_{i_2 bc}] \quad \overset{6}{S}_{i_1 i_2} = TS[S_{i_1}{}^{bc}_{..} \overset{c}{\nabla}{}^a \overset{c}{\nabla}_a S_{i_2 bc}].$$

With due regard to the identities (5.2),..., (5.6), to those following out of them and the transformation formulae (5.8),..., (5.11) we obtain after a conformal transformation

$$\begin{bmatrix} X^{\gamma}(\overset{1}{S}_{i_1i_2}) \\ \cdot \\ \cdot \\ \cdot \\ X^{\gamma}(\overset{6}{S}_{i_1i_2}) \end{bmatrix} \underset{*}{=} \begin{bmatrix} -2 & -4 & -8 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{5}{2} & 1 & -\frac{3}{2} & -1 & -4 \\ 1 & -3 & -6 & -3 & -2 & 0 \\ -12 & 4 & 8 & 0 & 0 & 0 \\ -6 & 0 & 12 & 3 & -6 & 0 \\ 0 & 2 & 0 & -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} \overset{1}{T}{}^{\gamma}{}_{i_1i_2} \\ \cdot \\ \cdot \\ \cdot \\ \overset{6}{T}{}^{\gamma}{}_{i_1i_2} \end{bmatrix}$$

with the transformation terms

$$\begin{aligned} \overset{1}{T}{}^{\gamma}{}_{i_1i_2} &= \delta^{\gamma}_{i_1} S_{abc}\nabla_{i_2} S^{abc}_{\ \ ...} & \overset{2}{T}{}^{\gamma}{}_{i_1i_2} &= S_{i_1ab}\nabla_{i_2} S^{\gamma ab}_{\ \ ...} \\ \overset{3}{T}{}^{\gamma}{}_{i_1i_2} &= S_{abi_1}\nabla_{i_2} S^{ab\gamma}_{\ \ ...} & \overset{4}{T}{}^{\gamma}{}_{i_1i_2} &= S_{i_1ab}\nabla^{\gamma} S_{i_2..}^{\ \ ab} \\ \overset{5}{T}{}^{\gamma}{}_{i_1i_2} &= \delta^{\gamma}_{i_1} S_{abc}\nabla^{a} S_{i_2..}^{\ \ bc} & \overset{6}{T}{}^{\gamma}{}_{i_1i_2} &= S_{ab.}^{\ \ \gamma}\nabla^{a} S_{i_1i_2.}^{\ \ \ \ b}. \end{aligned}$$

The rank of the transformation matrix is 5 from where we get

PROPOSITION 5.2. – *$\mathcal{S}_2{}^{(2)}(-3,3)$ is generated by the tensor*

$$S_2^{(2)}(-3,3) = 18\overset{1}{S}_{i_1i_2} + 12\overset{2}{S}_{i_1i_2} - 18\overset{3}{S}_{i_1i_2} - 9\overset{4}{S}_{i_1i_2} + 8\overset{5}{S}_{i_1i_2} + 12\overset{6}{S}_{i_1i_2}.$$

In an analogous manner we obtain

PROPOSITION 5.3. – *$\mathcal{S}_4{}^{(2)}(-2,3)$ is generated by the tensor*

$$\begin{aligned} \overset{1}{S}_4(-2,3) &= -8\overset{1}{S}_{i_1...i_4} - 2\overset{2}{S}_{i_1...i_4} - 5\overset{3}{S}_{i_1...i_4} + 2\overset{4}{S}_{i_1...i_4} + 4\overset{6}{S}_{i_1...i_4} \\ \overset{2}{S}_4(-2,3) &= -12\overset{1}{S}_{i_1...i_4} - 7\overset{2}{S}_{i_1...i_4} - 12\overset{3}{S}_{i_1...i_4} + 2\overset{5}{S}_{i_1...i_4} + 12\overset{6}{S}_{i_1...i_4}, \end{aligned}$$

*where*

$$\begin{aligned} \overset{1}{S}_{i_1...i_4} &= TS[\nabla_{i_1} S_{i_2i_3.}^{\ \ \ \ a}\nabla^{u} S_{i_4au}] & \overset{2}{S}_{i_1i_4} &= TS[\nabla_{i_1} S_{i_2..}^{\ \ ab}\nabla_{i_3} S_{i_4ab}] \\ \overset{3}{S}_{i_1...i_4} &= TS[\nabla^{a} S_{i_1i_2.}^{\ \ \ \ b}\nabla_{a} S_{i_3i_4b}] & \overset{4}{S}_{i_1...i_4} &= TS[S_{i_1i_2.}^{\ \ \ \ a}\overset{c}{\nabla}_{i_3}\overset{c}{\nabla}^{u} S_{i_4au}] \\ \overset{5}{S}_{i_1...i_4} &= TS[S_{i_1..}^{\ \ ab}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_{i_3} S_{i_4ab}] & \overset{6}{S}_{i_1...i_4} &= TS[S_{i_1i_2.}^{\ \ \ \ a}\overset{c}{\nabla}^{b}\overset{c}{\nabla}_{b} S_{i_3i_4a}]. \end{aligned}$$

PROPOSITION 5.4. – *The sets $\mathcal{S}_1{}^{(2)}(-3,3)$ and $\mathcal{S}_3{}^{(2)}(-2,3)$ are empty.*

*Remark* 5.1. – One can show the linear independence of the in this section occuring tensors either by means of the spinor formalism for three dimensional manifolds [Il] or with the help of suitably choosen test metrics.

## 6. APPLICATIONS ON SPACE-TIMES

1. *Huygen's principle for conformally invariant field equations*

Let $(M, g)$ be a space-time, *i.e.* a 4-manifold with a smooth metric of Lorentzian signature. The following conformally invariant field equations are considered ([Gü; Wü5; CMcL; McLS]:

$$\text{Scalar wave equation} \qquad g^{ab}\nabla_a\nabla_b u - \frac{1}{6}Ru = 0 \qquad \mathrm{E}_1$$

$$\text{Maxwell's equations} \qquad \nabla_{[a}F_{bc]} = 0, \quad \nabla_a F^{a}_{.b} = 0 \qquad \mathrm{E}_2$$

$$\text{Weyl's neutrino equation} \qquad \nabla^{A}{}_{\dot{X}}\varphi_A = 0, \qquad \mathrm{E}_3$$

where $F_{ab}dx^a dx^b$ is the Maxwell 2-form, $\varphi$ a valence 1-spinor and $\nabla_{A\dot{X}}$ the covariant derivative on spinors. For any of the equations $E_1$-$E_3$ Huygens' principle (in the sense of Hadamard's "minor premise") is valid if and only if the tail term with respect to $E_\sigma$, $\sigma = 1, 2, 3$ vanishes ([Gü; Wü; CMcL]). Since the functional relationship between the tail terms and the metric is very complicated, the problem of the determination of all metrics for which any equation $E_\sigma$ satisfies Huygens' principle is not yet completely solved (see [Gü; Wü1-5; CMcL; McL; McLS; AMLW; RW]).

The usual method for solving this problem is the derivation and the exploitation of the moment equations ([Gü; Wü5])

$$I^{\sigma}_{i_1\ldots i_r} = 0 \qquad \sigma = 1, 2, 3 \qquad r = 0, 1, 2, ..., \qquad \mathrm{ME}^{\sigma}_{\mathrm{r}}$$

where the moments $I^{\sigma}_{r} = I^{\sigma}_{i_1\ldots i_r}$ are symmetric, trace-free, conformally invariant tensors of the weight $-1$. They are derived from the tail terms with respect to $E_\sigma$, $\sigma = 1, 2, 3$ by means of the conformal covariant derivative (2.6) ([Gü; Wü5]). If $g$ is analytic, we have the following relationship between the moments and the validity of Huygens' principle: The equation $E_\sigma$, $\sigma = 1, 2, 3$ satisfies Huygens' principle if and only if all corresponding moments vanish on $M$ ([Gü; Wü5]). Using the results on the theory of conformal tensors, in particular the results on generating systems of Sections 3 and 4, one obtains information about the general algebraic structure of the moments for $0 \leq r \leq 6$ ([Gü; Wü5]).

The following proposition was proved in ([Gü; Wü2,5]):

PROPOSITION 6.1. – *One has*

(i) $I_r^\sigma \equiv 0$ *if* $(r,\sigma) \in \mathcal{M} := \{(k,1) : k \,\text{odd}\} \cup \{(m,\sigma) : m \in \{0,1,3\}, \sigma \in \{1,2,3\}\}$,

(ii) $I_r^\sigma \in \mathcal{S}_r(-1,4)$ *if* $r$ *is even,*

(iii) $I_r^\sigma \in \mathcal{S}_r^*(-1)$ *if* $r$ *is odd.*

*The propositions 2.3, 6.1 and the Theorems 3.1 (ii), 4.1 imply*

PROPOSITION 6.2. – *There are real coefficients*

$$\alpha^{(\sigma)},\ \beta_k^{(\sigma)},\ \gamma^{(2,\sigma)},\ \gamma_l^{(3,\sigma)},\ \delta_m^{(2,\sigma)},\ \delta_p^{(3,\sigma)}$$

with

$$I_2^\sigma = \alpha^{(\sigma)} B, \qquad I_4^\sigma = \sum_{k=1}^{3} \beta_k^{(\sigma)} W^{(k)},$$

$$I_5^\sigma = \gamma^{(2,\sigma)} S_5^{(2)}(-1) + \sum_{l=1}^{2} \gamma_l^{(3,\sigma)} \overset{l}{S}{}_5^{(3)}(-1)$$

$$I_6^\sigma = \sum_{m=1}^{2} \delta_m^{(2,\sigma)} \overset{m}{S}{}_6^{(2)}(-1,4) + \sum_{p=1}^{6} \delta_p^{(3,\sigma)} \overset{p}{S}{}_6^{(3)}(-1,4) \qquad (\sigma = 1,2,3).$$

Detailed information about the coefficients of Proposition 6.2 are given in [Wü5]. The following proposition was proved in [Wü2,3,5; McL; CMcL; McLS; Gü]:

PROPOSITION 6.3. – *If* $g$ *is an Einstein metric, a central symmetric metric, a metric of Petrov type N or D, then it follows from the moment equations*

$$I_r^\sigma = 0, \qquad \sigma \in \{1,2,3\}, 0 \le r \le 6$$

that $g$ is conformally equivalent to a plane wave metric or to a flat metric [6].

2. *Conformal Einstein space-times*

The conformal tensors $B$ (see (2.11)), $W^2$ (see (2.15)) and $\overset{2}{S}{}_6^{(2)}(-1,4)$ (see Proposition 3.1(ii)) vanish in a special Einstein space-time, i.e. a space-time with $R_{ab} = 0$, if $n = 4$. Thus we have:

PROPOSITION 6.4. – *In the case* $n = 4$ *the conditions*

$$B = 0, \quad W^{(2)} = 0, \quad \overset{2}{S}{}_6^{(2)}(-1,4) = 0$$

[6] A conjecture is that the moment equations $I_r^\sigma = 0$ $(r = 2,4,5,6)$ are also sufficient for the validity of Huygens' principle for $E_\sigma$, $\sigma = 1,2,3$ and that these equations are fulfilled if and only if $g$ is conformally equivalent to a plane wave metric or to a flat metric [W2,5; CMcL].

*are necessary for $(M, g)$ to be conformally related to a special Einstein space-time.*

*Remark* 6.1. – H.W. Brinkmann [Bri; Scho] studied necessary and sufficient conditions for Riemannian spaces to be conformally related to Einstein spaces. However, since his arguments involved the existence and compatibility of differential equations, a constructive set of necessary and sufficient conditions is very difficult to infer. Kozamek, Newman, Tod [KNT] and the second author [Wü4] solved the problem in the physically interesting case $n = 4$ for all space-times excluding space-times of Petrov type $N$, using the conditions $B = 0$, $W^{(2)} = 0$. In the case of $N$-type the derivation of a constructive set of necessary and sufficient conditions for $(M, g)$ to be conformal to an Einstein space is much more difficult [KNT; Wü4].

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