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On the dynamical meaning of spectral dimensions

by

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ABSTRACT. – Dynamical Localization theory has drawn attention to general spectral conditions which make quantum wave packet diffusion possible, and it was found that dimensional properties of the Local Density of States play a crucial role in that connection. In this paper an abstract result in this vein is presented. Time averaging over the trajectory of a wavepacket up to time T defines a statistical operator (density matrix). The corresponding entropy increases with time proportional to log T, and the coefficient of proportionality is the Hausdorff dimension of the Local Density of States, at least if the latter has good scaling properties. In more general cases, we give spectral upper and lower bounds for the increase of entropy. © Elsevier, Paris

Key words: Quantum chaos, dynamical loalisation, fractal, Hansdorff dimensions, local density of states.

RÉSUMÉ. – La théorie de la localisation dynamique a permis de souligner les conditions sous lesquelles la diffusion des paquets d'ondes quantiques est possible et on a montré que les propriétés dimensionnelles de la densité d'état locale y jouent un rôle crucial. Cet article fournit un résultat abstrait dans ce contexte. La moyenne temporelle sur une trajectoire du paquet d'onde jusqu'à l'instant T définit un opérateur statistique (matrice dénsité). L'entropie correspondante croît proportionnellement à Log(T) et le coefficient de proportionnalité est la dimension de Hausdorff de la densité d'état locale, au moins si cette dernière suit une loi d'échelle convenable. Dans le cas général, nous donnons des bornes spectrales supérieures et inférieures sur l'accroissement de cette entropie. © Elsevier, Paris

1. INTRODUCTION

B.V.Chirikov is one of the initiators of Quantum Chaos: the research area centered about the basic question, which of the distinctive marks of chaotic classical systems survive in the quantal domain. His attention was mainly focused on dynamical, directly observable aspects, and on deterministic diffusion in particular, which is probably the most concrete manifestation of chaos in classical hamiltonian systems. The Kicked Rotor – a quantum version of the Standard Map, to which Chirikov's name is tightly associated – revealed that quantization tends to suppress classical diffusion [1], and brought into light the phenomenon of Dynamical Localization, so called in view of its formal similarity to Anderson localization [2]. From the mathematical viewpoint, this discovery brought the quantum dynamics of chaotic maps into the realm of mathematical localization theory, thus significantly enlarging the scope of the latter ; from the physical viewpoint, it established a bridge to Solid State Physics, which led to identify the Localization Length as a fundamental characteristic scale of quantum dynamics, in the presence of a classical chaotic diffusion. The quasi-classical estimate for the localization length found by Chirikov and co-workers [3] is based on a heuristic argument – the so-called Siberian argument [4] – which is in fact a relation between dynamical and spectral properties, built upon the Heisenberg relation. Crudely handwaving though it may appear to a mathematical eye, the Siberian argument has a depth, the exploration of which has led to nontrivial mathematical results. Properly reformulated, it shows that quantum one-dimensional unbounded diffusion is only possible, if the spectrum (whether of energy or of quasi-energy) is singular; where by diffusion we loosely mean any type of sub-ballistic propagation, with the spread of the wavepacket over the relevant domain (position or momentum) increasing with some power of time less than 1.

Attention was thus drawn to the dynamical implications of singular spectra [5], and of singular continuous spectra in particular; for, although pure point spectra can also give rise to unbounded growth of expectation values of observables, due to non-uniform localization of eigenfunctions, they do not lead to any unbounded spread if the latter is measured by "intrinsic" quantities such as inverse participation ratios or the like.

Spectral analysis of the kicked rotor reveals a qualitative scenario somewhat similar to the one appearing with quasi-periodic Schrödinger operators, such as the Harper (or almost-Mathieu) operator. Its spectral type sensitively depends on the arithmetic nature of an incommensuration parameter linked to the kicking period and to the Planck constant. For "typical" irrational values there is evidence of pure-point spectrum; on the other hand, for rational values absolute continuity of the spectrum is proven. It follows that for a set of "not too irrational" values, presumably of zero measure, but nevertheless of the 2nd Baire category, there is still a continuous spectrum [6], which is suspected, but not proven, to be singular. This issue is closely connected to localization, for, if the latter could be proven for a dense set of irrationals, then purely singular continuity of the spectrum on a 2nd category parameter set would follow from Simon's Wonderland Theorem [7].

Generally speaking, there are two kinds of problems associated with singular continuous spectra. The first one is physical: what is their physical relevance in general, and in Quantum Chaos in particular. The second is mathematical, and includes the analysis of their dynamical implications.

Concerning the physical relevance of singular continuous spectra, and their connection to classical chaos, the situation is still unclear. Such spectra have been proven to occur in models, which are directly relevant to mesoscopic physics: e.g., electrons in quasi crystals, and cristalline electrons in magnetic fields. Still, physicists often tend to repress them as mathematical curiosities, encouraged in that by their instability: they easily collapse into pure point spectra under tiny perturbations. Such arguments can be reversed, because point spectra which lie "infinitely close" to singular continuous ones may be expected to have highly nontrivial dynamical properties ; it looks likely that, prior to entering the final localized state, the wavepacket dynamics will display features that may be better understood by assimilating the spectrum to a "fractal", much in the same way that fractal analysis of certain sets which are *not* fractal is still instructive on not too small geometric scales.

How does the diffusion which is produced by a singular continuous spectrum compare with classical chaotic diffusion – when the latter is present in the classical limit? This question cannot be posed for the best known examples of singular spectra such as the Harper model, because those have integrable classical limits. For this reason the Kicked Harper model was invented [8], which is intermediate between the Harper and the Kicked Rotor model, sharing quasi-periodicity with the former and classical chaoticity with the latter. It turns out that the time scale over which quantum diffusion mimics classical chaotic diffusion is distinct from the one where quantum "fractal" diffusion becomes manifest, with qualitative and quantitative differencies from the former [9]. On the grounds of such findings it appears that the two types of diffusion bear scarcely any relation to each other. This may reflect a qualitative difference between their underlying dynamical mechanisms: whereas quasi-classical diffusion proceeds by excitation of all states around the initial one, "fractal" diffusion is brought about by a coherent chain of quasi-resonant transitions (however, the reader should be warned that there are certain risks of over-simplification in this qualitative picture). In any case, the role of classical chaos in the parametric "band dynamics" which eventually leads to multifractal spectra has been recently demonstrated on numerical results [10].

From the mathematical standpoint, some exact results have been proven, which connect asymptotic dynamics to spectral quantities related to "multifractality" of the spectrum (or, more precisely, of the LDOS, Local Density of States). These results consist in estimates for the power-law decay of correlations [11], and in lower bounds for the spread of wave packets [4, 12, 13]. These results rest on asymptotic estimates for Fourier transforms of fractal measures [4, 14]. The problem of finding *upper* bounds (or possibly exact estimates) for the asymptotic spread of wavepackets is still open (non-rigorous approaches have been implemented, though [15, 18, 19]).

Upper bounds appear to require more detailed information concerning the specific structure of the Hamiltonian, and its (generalized) eigenfunctions. Improved lower estimates exploiting both spectral and eigenfunction-related information have been obtained, heuristically [27] and rigorously [16] – the latter on a special model, which has the striking peculiarity of displaying ballistic propagation, even with LDOS of arbitrarily small positive Hausdorff dimension. In spite of such findings, the search for purely spectral bounds is not yet doomed to failure, at least within the class of discrete Schrödinger operators; for in that class there is a somewhat rigid connection between spectral measures and eigenfunctions, so that information about the latter is certainly encoded in the LDOS itself [17]. An additional problem with this class of operators is the role of dimensional properties of the *global* Density of States (DOS), for which there are contrasting indications. On the one hand, the DOS can be smooth even in the presence of localization; on the other, in some quasi-periodic cases with fractal spectra there are numerical indications [18] that transport is tightly determined by DOS, though numerical analysis shows significant differences between the multifractal structures of DOS and LDOS even in such cases. [20].

In summary, no exact one-to-one relation has been as yet established between spectral dimensions and asymptotic properties of the dynamics; with the only exception of the "correlation dimension", which is known to rule the decay of correlations [11]. In this paper an abstract result is proven, which identifies the information dimension with the coefficient of logarithmic growth of a dynamically defined entropy, thus providing that dimension with a direct dynamical meaning. Indications will also be given, that upper bounds should be sought in terms of fractal (box-counting), rather than Hausdorff, dimensions; an abstract example will in fact be given, of a zero-dimensional LDOS with fractal dimension 1, which leads to ballistic propagation.

It is also worth mentioning that the dynamical role of dimensional properties of singular continuous spectra may be an interesting issue in a purely classical context, too. Classical dynamical systems which have a singular continuous spectrum (in the orthocomplement of constants) are known long since to make up a quite large subset in the class of measure-preserving transformations [21] ; they often lie close to the bottom of the ergodic hierarchy, as they may display weak mixing as maximal ergodic property. Beyond that, not much is known about their dynamical properties, and about the role of spectral dimensions in particular. It is in fact difficult to find concrete examples, in which "transport" can be meaningfully investigated. Certain substitution systems are rigorously known to have a singular continuous spectral component [22], but, to the best of the present author's knowledge, no explicit example of a classical dynamical system with a purely singular continuous spectrum (in the orthocomplement of constants) is rigorously known. Good candidates are certain polygonal billiards [23]; another system, which can be assigned to this class on the grounds of numerical evidence, is the "driven spin model" [25], which is a classical dynamical system that also admits of a quantum interpretation. This model is a member in a class of skew-products for which singular continuity of the spectrum is established as a generic property [22]. Both for the case of billiards, and of driven spins, a multifractal analysis of spectral measures has been numerically implemented, and results have been dynamically interpreted [24, 25].

In closing this Introduction, it may be necessary to underline that it is far from comprehensive on some of the general issues touched in it, which go beyond the study of dynamical implications of dimensional properties of spectra.

This study has found motivations , among others, from the search for a quantum counterpart of deterministic diffusion; in contributing this paper it is a pleasure to acknowledge Boris Chirikov's tutorial explanations of the Siberian argument, to which the results presented below can be ultimately traced back.

2. ENTROPY OF TIME-AVERAGING, AND ITS GROWTH

Consider discrete-time evolution of a quantum system with states in a separable Hilbert space $\mathcal{H}$: the state at time $t \in Z$ is $\psi(t) = U^t\psi(0)$, where U is a fixed unitary operator. While the discrete-time formulation used here does not set serious restrictions on the elaborations below, most of which also apply to a continuous time dynamics, it has the advantage of including quantum maps (which can usually be pictured as one-cycle propagators of periodically driven systems).

Time averages of an observable A following the evolution of a given initial state $\psi \equiv \psi(0)$ are defined by

$$\langle A\rangle_T = \frac{1}{T}\sum_{s=0}^{T-1}\langle\psi(s)|A|\psi(s)\rangle$$

and are in fact statistical averages, $\langle A\rangle_T = Tr(\rho(T)A)$, with the density matrix

$$\rho(T) = \frac{1}{T}\sum_{s=0}^{T-1}|\psi(s)\rangle\langle\psi(s)| \tag{1}$$

These density matrices are finite rank, positive operators, and to everyone of them is associated the entropy

$$\mathcal{S}(\rho(T)) = Tr(\rho(T)\ln\rho(T)^{-1}) \tag{2}$$

which is a measure for the size of the statistical ensemble defined by the states of the system between times 0 and $T-1$; in the following it will be denoted $\mathcal{S}(\psi, T)$. Our aim here is to estimate the asymptotic growth of $\mathcal{S}(\psi, T)$ with T .

The basic tool in getting upper estimates will be the following well-known result. For $x \in [0,1]$ define $\Theta(x) = -x\ln x$ if $x \neq 0$, $\Theta(0) = 0$; then an elementary convexity argument shows that:

LEMMA. – *For any statistical operator* ρ, *and for any Hilbert base* $B = \{\varphi_n\}_{n\in Z}$,

$$\mathcal{S}(\rho) \leq \sum_{n\in Z}\Theta(\langle\varphi_n|\rho|\varphi_n\rangle) \tag{3}$$

With the statistical operator defined in eqn.(1), $\langle\varphi_n|\rho|\varphi_n\rangle \equiv p_n(T)$ is just the averaged probability of finding the system in state φ_n between times 0 and $T-1$. The rhs of (3) is the Shannon entropy of the probability distribution $p_n(T)$, which in the following will be denoted $S(\psi, B, T)$. Since the rank of (3) is at most T, the base B can be chosen so that the sum over B in (3) contains at most T nonzero terms; then well-known properties of the Shannon entropy yield the bound $\mathcal{S}\ \ln T$, which is exact, e.g., if U has a Lebesgue spectrum in $[0, 2\pi]$, because in that case a base B can be found, so that U acts as a shift over B. On the opposite extreme, if U has a pure point spectrum, then using in (3) an eigenbase of U we immediately find that $\mathcal{S}(\psi, T)$ remains bounded at all times. Thus we see that entropy cannot increase with time faster than $\ln T$, and that its actual increase is related to the degree of continuity of the spectrum.

This qualitative indication will now be turned into an exact result, which calls appropriate dimensions into play, as a measure of the "degree of continuity". We first review their definitions.

The spectral measure of ψ (also called Local Density of States at ψ) is the unique measure $d\mu$ on $[0, 2\pi]$ such that, at all times t,

$$\langle\psi|U^t|\psi\rangle = \int_0^{2\pi} e^{it\lambda} d\mu(\lambda)$$

The dependence of this measure on the state ψ will be left understood in the sequel. Various dimensions of the Hausdorff or multifractal type can be assigned to the measure $d\mu$; the ones we shall use are the upper and lower Hausdorff dimensions $\dim_H^{\pm}(\mu)$, and the fractal dimension $\dim_F(\mu)$, which are defined as follows.

$\dim_H^-(\mu)$ is the supremum of the set of values $\alpha \in [0, 1]$ such that $\mu(A) = 0$ for all Borel sets $A \subseteq [0, 2\pi]$ which have Hausdorff dimension smaller than α.

$\dim_H^+(\mu)$ is the infimum of the set of values $\alpha \in [0, 1]$ such that there is a set $A \subseteq [0, 2\pi]$ of Hausdorff dimension α, with $\mu(A) = \mu([0, 2\pi])$.

If the upper and lower dimensions coincide, then the measure is said to have exact Hausdorff dimension, given by their common value. Note that measures which have both a point and a continuous part do not fall in this class, if the dimension of the continuous part of the measure is positive.

Finally, the fractal dimension is defined as

$$\dim_F(\mu) = \sup_{0<\epsilon<1} \inf_K \{\dim_F(K) : \ K \ compact\ s.t.\ \mu(K) > 1-\epsilon\} \quad (4)$$

where $\dim_F(K)$ is the fractal (box-counting) dimension of K. In general, $\dim_H^-(\mu) \le \dim_H^+(\mu) \le \dim_F(\mu)$, but the three dimensions may coincide;

such is the case, e.g., when the measure has, $\mu-$almost everywhere in $[0, 2\pi]$, a well-defined, constant scaling exponent [26]. In that case their common value is the same as the information dimension $D(\mu)$. For such "exactly scaling" measures the behavior of entropy is particularly simple:

THEOREM 1. – *If the spectral measure is exactly scaling, with dimension D then $\mathcal{S}(\psi, T) \sim D \ln T$ asymptotically as $T \to \infty$.*

This is the central result of this paper: it will be obtained via Propositions 1-5 below. Ineq. (3) suggests that upper bounds to entropy growth can be obtained by choosing a suitable base B in the cyclic subspace of ψ, and then analyzing the growth with time of the Shannon entropy $S(\psi, T, B)$ of the distribution $p_n(T)$ over the base B.

The latter is a measure of the "width" of the distribution, and in order to estimate its growth we will estimate the growth of $n_\epsilon = n_\epsilon(T, B)$, defined as the smallest integer such that the total probability assigned by $p_n(T)$ to states $|n| \leq n_\epsilon$ be larger than $1 - \epsilon^2$. To this end we shall use the following technical tool:

PROPOSITION 1. – *Let B be any base in the cyclic subspace of ψ, K_ϵ a compact set in $[0, 2\pi]$ such that $\mu(K_\epsilon) > 1 - \frac{\epsilon^2}{8}$, and N an integer > 1. Given a partition of $[0, 2\pi]$ in intervals $I_j = [2\pi j N^{-1}, 2\pi(j+1)N^{-1}]$, $(j = 0, ..., N-1)$, let*

$$W_B(n, N) = \sum_{I_j \cap K_\epsilon \neq \emptyset} \left| \langle \psi | E_{I_j} \varphi_n \rangle \right|^2$$

where E_{I_j} are spectral projectors associated with the intervals I_j. Define $\nu_B(\epsilon, N)$ as the smallest integer ν such that $\sum_{|n| > \nu} W_B(n, N) < \epsilon$. Then there are numerical constants c_1, c_2 so that:

$$n_\epsilon(T, B) \leq \nu_B(c_2 \epsilon^3, N)$$

for all $N > c_1 \epsilon^{-1} T$.

Proof. – Note that $\sum_{n \in Z} W_B(n, N) \leq 1$, so $\nu_B(\epsilon, N)$ is a meaningful quantity. First, we use the Spectral theorem to identify the cyclic subspace of ψ with $L^2([0, 2\pi], d\mu)$, in which $\psi(t)$ is represented by the function $e^{i\lambda t}$ of $\lambda \in [0, 2\pi]$. Then we define stepwise approximations to $\psi(t)$ for $0 \leq t < T$:

$$\psi_{K_\epsilon, N}(t) = \sum_{I_j \cap K_\epsilon \neq \emptyset} e^{2\pi i j t N^{-1}} \chi_{I_j}(\lambda)$$

It is immediately seen that

$$\|\psi(t) - \psi_{K_\epsilon,N}(t)\|^2 \leq \frac{\epsilon^2}{8} + \frac{4\pi^2 t^2}{N^2}$$

which can be kept $\leq \frac{\epsilon^2}{4}$ at all times from 0 to T, by choosing $N > c_1 T \epsilon^{-1}$, with c_1 a numerical factor. Then,

$$\begin{aligned}\sum_{|n|>n_0} p_n(T) &\leq \frac{\epsilon^2}{2} + \frac{2}{T}\sum_{s=0}^{N-1}\sum_{|n|>n_0} |\langle\varphi_n|\psi_{K_\epsilon,N}(s)\rangle|^2 \\ &\leq \frac{\epsilon^2}{2} + \frac{1}{2c_2\epsilon}\sum_{|n|>n_0}\sum_{I_j\cap K_\epsilon\neq\emptyset} |\langle\varphi_n|\chi_{I_j}\psi\rangle\rangle|^2 \\ &= \frac{\epsilon^2}{2} + \frac{1}{2c_2\epsilon}\sum_{|n|>n_0} W_B(n,N)\end{aligned}$$

which will be less than ϵ^2 if $n_0 \geq \nu(c_2\epsilon^3, N)$ (note that in the 1st inequality the upper limit of the sum over s has been changed from $T-1$ to $N-1$, which is certainly larger). □

Remark. – Viewed as functions of n at fixed λ in the spectrum of U, the functions $\varphi_n(\lambda)$ are (generalized) eigenfunctions of U. Thus proposition (1) establishes a connection between dynamics and structure of eigenfunctions.

If the measure is purely continuous, then there is a particular base B_F in the cyclic subspace of ψ, which allows for optimal control on the growth of entropy . The vectors of B_F are represented by functions $\varphi_n \in L^2([0,2\pi], d\mu)$ defined as follows:

$$\varphi_n(\lambda) = e^{2\pi i n F(\lambda)} \tag{5}$$

where $F(\lambda) = \mu([0,\lambda])$ is the distribution function of the spectral measure. The φ_n ($n \in Z$) are a complete orthonormal set because they are the image of the Fourier base $\{e^{2\pi i n x}\}_{n\in Z}$ in $L^2([0,1], dx)$ under the isomorphism which is established (when $d\mu$ is continuous) between $L^2([0,2\pi], d\mu)$ and $L^2([0,1], dx)$ by $\lambda \to F(\lambda)$.

PROPOSITION 2. – *If $d\mu$ is purely continuous then there is a numerical constant c_5 so that, for any $d > \dim_F(\mu)$, $n_\epsilon(T, B_F) \leq c_5\epsilon^{-4}T^d$ for all sufficiently large T.* .

Proof. – We use Proposition 1. Observing that

$$|\langle E_{I_j}\varphi_n|\psi\rangle|^2 = \left|\int_{I_j} e^{2\pi i n F(\lambda)} dF(\lambda)\right|^2 = \frac{\sin^2 \pi n \mu(I_j)}{\pi^2 n^2} \tag{6}$$

we obtain

$$\sum_{|n|>n_0} W_{B_F}(n,N) \leq c_3 \frac{\sharp(K_\epsilon, N)}{n_0}$$

where $\sharp(K_\epsilon, N)$ is the number of intervals I_j which overlap K_ϵ. We now choose K_ϵ so that its box-counting dimension be smaller than $\dim_F(\mu)$, which is made possible by the very definition (4) of the latter quantity. If we use dyadic partitions, $N = 2^M$, then, for any $d > \dim_F(\mu)$,

$$\sharp(K_\epsilon, N) < 2^{Md}$$

for all sufficiently large M, so $\nu(\epsilon, 2^M) < c_4\epsilon^{-1}2^{Md}$. Finally, defining M by $2^{M-1} \leq c_1 T\epsilon^{-1} < 2^M$, proposition 1 says that $n_\epsilon(T) < c_5\epsilon^{-4}T^d$ for all sufficiently large T. □

PROPOSITION 3. – *If* $d\mu$ *is purely continuous, then* $\limsup_{T\to\infty} \frac{S(\psi, B_F, T)}{\ln T} \leq \dim_F(\mu)$.

Proof. – Since $S(\psi, B_F, T)$ does not depend on the labeling of the base vectors, at given time T let us rearrange them in such a way that the probability supported by a vector is monotonically non- increasing with the label of the vector, thus obtaining a base $\overline{B}_F$; then clearly $n_\epsilon(T, \overline{B}_F) \leq n_\epsilon(T, B_F)$, so Proposition 2 also holds for $\overline{B}_F$. Therefore, the distribution $\overline{p}_n(T)$ over $\overline{B}_F$ obeys

$$\sum_{m>n} \overline{p}_m(T) \leq c_6 T^{\frac{d}{2}} n^{-\frac{1}{2}}$$

Monotonicity of $\overline{p}_n(T)$ then implies

$$\overline{p}_n(T) \leq c_7 T^{\frac{d}{2}} n^{-\frac{3}{2}} \tag{7}$$

Let $\overline{n}_\epsilon$ the smallest integer larger than $c_5\epsilon^{-4}T^d$, so that the total probability on states of $\overline{B}_F$ beyond $\overline{n}_\epsilon$ is less than ϵ^2 by construction. For $n > \overline{n}_\epsilon$ the rhs of (7) is certainly smaller than e^{-1} for small enough ϵ, so we can use monotonicity of $\Theta(x)$ for $x \in (0, e^{-1})$ to the effect that:

$$\begin{aligned}\sum_n \Theta(\overline{p}_n(T)) &= \left(\sum_{n\leq\overline{n}_\epsilon} + \sum_{n>\overline{n}_\epsilon}\right)\Theta(\overline{p}_n(T)) \\ &\leq \ln\overline{n}_\epsilon - \ln(1-\epsilon^2) + \sum_{n>\overline{n}_\epsilon}\Theta(c_7 T^{\frac{d}{2}} n^{-\frac{3}{2}}) \\ &\leq (1 + c_8\epsilon^2)\ln\overline{n}_\epsilon + O\left(\epsilon^2 \ln\frac{1}{\epsilon}\right)\end{aligned}$$

where the last term is only dependent on ϵ. Proposition 2 follows immediately from the definition of $\overline{n}_\epsilon$, because $S(\psi, B_F, T) = S(\psi, \overline{B}_F, T)$, and ϵ is arbitrary as well as $d > \dim_F(\mu)$. □

Let us extend Proposition 3 to measures having a point component, $d\mu = d\mu_p + d\mu_c$. Let $\mathcal{H}_p$, $\mathcal{H}_c$ be the pure point and the continuous subspace of the evolution operator U, and $P = \mu_p([0, 2\pi])$ the squared norm of the projection of ψ on $\mathcal{H}_p$. Then we can write $\psi = \sqrt{P}\psi_p + \sqrt{1-P}\psi_c$, with $\psi_p \in \mathcal{H}_p, \psi_c \in \mathcal{H}_c$ and $||\psi_p|| = ||\psi_c|| = 1$.

PROPOSITION 4. – *If $d\mu_c$ denotes the continuous component of the spectral measure, and P is the squared norm of the pure point component of ψ, then*

$$\limsup_{T\to\infty} \frac{S(\psi, T)}{\ln T} \leq (1 - P) \dim_F(\mu_c) \tag{8}$$

Proof. – Let us choose the base $B = B_p \cup B_c$, where B_p is an eigenbase of U in $\mathcal{H}_p$ and B_c is the base (5) associated with $d\mu_c$ in $\mathcal{H}_c$. The vectors ψ_c, ψ_p evolve, under repeated applications of U, independently of each other,with spectral measures $d\mu_p, d\mu_c$ respectively; their distributions over B are disjoint, so one immediately finds that that:

$$\begin{aligned} S(\psi, B, t) = PS(\psi_p, B_p, T) + (1 - P)S(\psi_c, B_c, T) \\ + P \ln \frac{1}{P} + (1 - P) \ln \frac{1}{1 - P} \end{aligned} \tag{9}$$

The 1st term in the rhs is constant in time, and the second is estimated by proposition 2. □

The following proposition sets a lower bound to entropy.

PROPOSITION 5. – $\liminf_{T\to\infty} \frac{S(\psi,T)}{\ln T} \geq dim_H^-(\mu)$.

Proof. – This proposition is an immediate consequence, not of published results, but of their proofs.

Let B be an eigenbase of $\rho(T)$, so that (3) becomes an equality. Such a base consists of at most T vectors in the subspace spanned by $\psi(0), ..., \psi(T)$, plus any orthonormal set spanning the orthogonal complement of that subspace. A lower bound to $S(\psi, B, T)$ is established as follows. Given $\epsilon \in (0, 1)$ let $m_\epsilon(T)$ be the smallest number of base vectors which are needed to support more than $1 - \epsilon$ of the distribution $p_n(T)$. In the Appendix we prove the following elementary estimate, which holds if $m_\epsilon > 3$:

$$S(\psi, B, T) \geq \epsilon \ln \frac{1}{\epsilon} + \epsilon \ln(m_\epsilon(T) - 3) \tag{10}$$

General lower bounds on wave packet propagation[26] entail that, for any small $\eta > 0$, $m_\epsilon(T) > c_\epsilon T^{\dim_H^-(\mu)-\eta}$ at all sufficiently large T. Therefore, unless $dim_H^-(\mu) = 0$ (in which case Proposition 4 is obviously true), $m_\epsilon(T)$ will be definitively larger than 3, we can insert the lower bound on $m_\epsilon(T)$ into (10), and thus obtain the desired result, because η, ϵ are arbitrary. □

Thus theorem 1 finally emerges, as a consequence of propositions 1-5 for the special case that the spectral measure is exactly scaling (proposition 4 being necessary to that end only in case of zero-dimensional measures).

Remarks:

1. For measures having a point component, the lower bound 0 given by proposition 5 is not optimal: one can prove that $\dim^-(\mu)$ can be replaced by $(1 - P)\dim_H^-(\mu_c)$.
2. The dynamical entropy defined by (2) is not related to quantum analogs of the Kolmogorov-Sinai entropy, as it is only determined by 2-point time correlations . In fact,

$$\mathcal{S}(\psi, T) = \sum_i \Theta(p_i) \tag{11}$$

where p_i are eigenvalues of $\rho(T)$. It is easily seen that $p_i = T^{-1}r_i$, where r_i are eigenvalues of the $T \times T$ autocorrelation (Toeplitz) matrix $R_{st} = \langle\psi(s)|\psi(t)\rangle = \hat{\mu}(t - s)$ (the hat denoting Fourier transform). In this light, Theorem 1 also applies to L^2- stationary stochastic processes, as a statement relating the asymptotic distribution of their autocorrelation eigenvalues to scaling properties of the power spectrum. It may result in a convenient method for numerically estimating the dimension of the power spectrum.
3. The apparition of the fractal, rather than the Hausdorff, dimension in the upper bound of Propositions 3,4 is not an artifact of the proof. In the Appendix an abstract example is given of a spectral measure which has zero Hausdorff dimension and fractal dimension 1, for which $\limsup_{T\to\infty} \frac{S(\psi, B_F, T)}{\ln T} = 1$. This "frequently ballistic" behavior is also detected in the growth of moments of the probability distribution $p_n(T)$, and it is controlled by the fractal dimension.

3. APPENDIX

1. Proof of ineq. (10):

Denote $S(T) = S(\psi, T, B)$ and define $\mathcal{A}_{\epsilon,T}$ as the set of labels $n \in Z$ such that $\ln \frac{1}{p_n(T)} > S(T)/\epsilon$. From

$$S(T) \geq \sum_{n \in \mathcal{A}_{\epsilon,T}} \Theta(p_n(T)) \geq \epsilon^{-1} S(T) \sum_{n \in \mathcal{A}_{\epsilon,T}} p_n(T) \tag{12}$$

it follows that the complement $\mathcal{B}_{\epsilon,T}$ of $\mathcal{A}_{\epsilon,T}$ supports more than $1 - \epsilon$ of the distribution $p_n(T)$; hence, it cannot consist of less than $m_\epsilon(T)$ elements. Now define $\overline{\mathcal{B}}_{\epsilon,T}$ as the set of those elements of $\mathcal{B}_{\epsilon,T}$ which have $p_n(T) < e^{-1}$. There cannot be less than $m_\epsilon - 3$ such elements, therefore

$$S(T) \geq \sum_{n \in \overline{\mathcal{B}}_{\epsilon,T}} \Theta(p_n(T)) \geq (m_\epsilon - 3)\epsilon^{-1} S(T) e^{-S(T)/\epsilon} \tag{13}$$

because $\Theta(x)$ is increasing in $(0, e^{-1})$; (10) immediately follows. □

2. An example of a non-exactly scaling, continuous measure, and dynamical consequences theoreof

The construction below is a special case in a class of measures taken from ref.[28]. Write $\lambda \in [0, 2\pi]$ as $2\pi x$ with $x \in [0,1]$, and let $a_n(x) \in \{0,1\}$ be the binary digits of x. A well-known construction allows to define measures on $[0, 2\pi]$ as images of cylinder-set measures on $\{0,1\}^{\mathbf{N}}$, which are in turn constructed by assigning the distribution of the random variables $a_n(x)$. Let us consider the particular measure $d\mu$ in $[0, 2\pi]$ which is obtained when the a_n's are independent random variables distributed as follows: $a_n(x) = 0$ with probability 1 if $k! \leq n < (k+1)!$ with k even, $a_n(x) = 0$ with probability $\frac{1}{2}$ if $k! \leq n < (k+1)!$ with k odd.

For integer k, consider the dyadic partition of $[0, 2\pi]$ in $2^{k!}$ intervals of equal size Δ_k. It is easily seen that these intervals have either measure 0 or measure μ_k, given by:

$$\mu_k = \prod_{j=1}^{r} \left(\frac{1}{2}\right)^{(2j-1)(2j-1)!} \qquad for \quad even \quad k = 2r;$$

$$\mu_k = \frac{1}{2}\mu_{k-1} \quad for \quad odd \quad k \tag{14}$$

whence it follows that

$$\lim_{r\to\infty}\frac{\ln\mu_{2r+1}}{\ln\Delta_{2r+1}}=0;\quad \lim_{r\to\infty}\frac{\ln\mu_{2r}}{\ln\Delta_{2r}}=1. \tag{15}$$

Then $\dim_H(\mu) = 0$. In fact, for λ in the support of $d\mu$, let $I_\delta(\lambda) = (\lambda-\delta, \lambda+\delta)$. If $\delta_k \equiv 2\Delta_k$, then, for all k, $I_{\delta_k}(\lambda)$ contains the full dyadic interval of size Δ_k, and measure μ_k, which contains λ, so that:

$$\alpha(\lambda)\equiv\liminf_{\delta\to 0}\frac{\ln\mu(I_\delta(\lambda))}{\ln 2\delta}\leq\liminf_{r\to\infty}\frac{\ln\mu(I_{\delta_{2r+1}}(\lambda))}{\ln 2\delta_{2r+1}}$$
$$\leq\lim_{r\to\infty}\frac{\ln\mu_{2r+1}}{\ln 4\Delta_{2r+1}}=0$$

(note that $\ln 4\Delta_{2r+1}$ is negative at large r). Therefore the Hausdorff dimension of $d\mu$ is zero, because it coincides with the essential supremum with respect to $d\mu$ of the scaling exponent $\alpha(\lambda)$[26].

On the other hand, the fractal dimension (4) of $d\mu$ is 1. In fact, if a compact K has $\mu(K) > 1-\epsilon$, then a covering of K with dyadic intervals of the $(2r)!$-th generation requires at least $\sharp_K \geq (1-\epsilon)\mu_{2r}^{-1}$ intervals, so

$$\dim_F(K)\geq\lim_{r\to\infty}\frac{\ln\mu_{2r}}{\ln\Delta_{2r}}=1.$$

Let us explore how does a wavepacket with the just defined spectral measure $d\mu$ spread over the base B_F defined by (10); specifically, we shall use (5) to estimate the growth of $S(\psi, B_F, T)$ from below.

Given $\epsilon \in (0,1)$, let us choose $\epsilon_1 < 1$ such that $1-\epsilon^2-\frac{\epsilon_1^2}{2} > 0$. Then, going back to the proof of Proposition 1, and using the same notations, we have that, for any finite set $\mathcal{F}$ of indices,

$$\sum_{n\in\mathcal{F}}p_n(T)\leq\frac{\epsilon_1^2}{2}+\frac{1}{2c_2\epsilon_1}\sum_{n\in\mathcal{F}}W_{B_F}(n,N) \tag{16}$$

if $N > c_1 T\epsilon_1^{-1}$. For all integer r, define $T_r^{(\epsilon_1)}$ as the largest integer less or equal to $\epsilon_1 c_1^{-1}2^{(2r)!}$, so that (16) holds with $T = T_r^{(\epsilon_1)}$ and $N = N_r = 2^{(2r)!}$ for all values of r. Since all the intervals in the partition have either measure 0 or measure μ_{2r}, from (6) we get $W_{B_F}(n, N_r) \leq \mu_{2r}$. Substituting this into (16) we find that, in order that the total probability at time $T_r^{(\epsilon_1)}$ on states $n \in \mathcal{F}$ be larger than $1-\epsilon^2$, $\mathcal{F}$ has to be chosen such that

$$\sharp(\mathcal{F}) > 2c_2\left(1-\epsilon^2-\frac{\epsilon_1^2}{2}\right)\epsilon_1\mu_{2r}^{-1}$$

Therefore, since $m_\epsilon(T)$ in (10) is the smallest number of base vectors which support more than $1 - \epsilon^2$ of the distribution $p_n(T)$,

$$\limsup_{T\to\infty} \frac{\ln m_\epsilon(T)}{\ln T} \geq \limsup_{r\to\infty} \frac{\ln m_\epsilon(T_r^{(\epsilon_1)})}{\ln T_r^{(\epsilon_1)}}$$
$$\geq \limsup_{r\to\infty} \frac{\ln \mu_{2r}^{-1}}{\ln T_r^{(\epsilon_1)}} = \lim_{r\to\infty} \frac{\ln \mu_{2r}}{\ln \Delta_{2r}} = 1$$

Estimate (10) and Proposition 2 now yield $\limsup_{T\to\infty} \frac{S(\psi,B_F,T)}{\ln T} = 1$. Much in the same way one finds that the growth of $M_q(T)$ (the q-th moment of the probability distribution $p_n(T)$) follows $\limsup_{T\to\infty} \frac{\ln M_q(T)}{\ln T} \geq q$ (note however that in the present case $M_q(t) < \infty$ only if $q < 1$).

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