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Solutions of semilinear Schrödinger equations in H^s

by

Hartmut PECHER

Fachbereich Mathematik, Bergische Universität – Gesamthochschule Wuppertal,
Gaußstr. 20 D-42097 Wuppertal Germany.
e-mail Hartmut.Pecher@math.uni-wuppertal.de

ABSTRACT. – It is shown that the Cauchy problem for the nonlinear Schrödinger equation $iu_t - \Delta u = c|u|^\sigma u$ has a local solution in the class $C^0([0,T], H^{s,2}(\mathbf{R}^n))$ if the initial value belongs to $H^{s,2}(\mathbf{R}^n)$, where $1 < s < n/2$, and $0 < \sigma < \frac{4}{n-2s}$ if $1 < s < 2$, $s-2 < \sigma < \frac{4}{n-2s}$ if $2 \leq s < 4$, and $s-3 < \sigma < \frac{4}{n-2s}$ if $s \geq 4$. If moreover $\sigma \geq \frac{4}{n}$ and $||\phi||_{H^{s,2}(\mathbf{R}^n)}$ is sufficiently small this solution is global.

RÉSUMÉ. – Il est démontré que le problème de Cauchy pour l'équation de Schrödinger non linéaire $iu_t - \Delta u = c|u|^\sigma u$ possède une solution locale dans l'espace $C^0([0,T], H^{s,2}(\mathbf{R}^n))$ si la valeur initiale appartient à $H^{s,2}(\mathbf{R}^n)$ sous condition que $1 < s < n/2$ et $0 < \sigma < \frac{4}{n-2s}$ si $1 < s < 2$, $s-2 < \sigma < \frac{4}{n-2s}$ si $2 \leq s < 4$ et $s-3 < \sigma < \frac{4}{n-2s}$ si $s \geq 4$. Si en plus $\sigma \geq \frac{4}{n}$ et si $||\phi||_{H^{s,2}(\mathbf{R}^n)}$ est suffisamment petite la solution est globale.

1. INTRODUCTION

Consider the Cauchy problem for semilinear Schrödinger equations in $\mathbf{R}^n$:

$$iu_t + Au = f(u), \quad A = -\Delta \tag{1}$$

$$u(0,x) = \phi(x) \tag{2}$$

where

$$f(u) = c|u|^{\sigma}u, \qquad c \in \mathbf{C} \tag{3}$$

$$\phi \in H^{s,2}(\mathbf{R}^n) \tag{4}$$

The corresponding integral equation reads as follows

$$u(t) = e^{itA}\phi + (Gf(u))(t) \tag{5}$$

where

$$(Gf(u))(t) := -i \int_0^t e^{i(t-\tau)A} f(u(\tau))\, d\tau \tag{6}$$

Large data global results can be found in J. Ginibre-G. Velo [4]. As a general reference see also the monograph by W. Strauss [12].

We are interested in solutions to (5) in the class $C^0(I, H^{s,2}(\mathbf{R}^n))$ where either $I = [0, T]$ (local problem) or $I = [0, +\infty)$ and $||\phi||_{H^{s,2}(\mathbf{R}^n)}$ small (small data global problem). It is well known that in the case $s \geq n/2$ and smooth f such solutions exist locally regardless of the value of σ. In this paper we concentrate on the range $0 < s < n/2$.

This case was already treated in detail by Th. Cazenave and F.B. Weissler in [2] which is the first main reference here.

Concerning the **local problem** two restrictions on the power σ were made by them

• an upper bound $\sigma \leq \frac{4}{n-2s}$ which is a consequence of embedding conditions for Sobolev spaces and

• a lower bound which ensures enough smoothness of the nonlinearity f at the origin. The precise assumption they had to impose is as follows: if $s \notin \mathbf{N}$ then $\sigma > [s]$, whereas if $s \in \mathbf{N}$ then $\sigma > s - 1$.

• In the special case $s = 2$ only $\sigma > 0$ was assumed.

• These lower bounds are unnecessary if σ is an even integer so that f is smooth.

Concerning the **global small data problem** they assumed in addition

• $\sigma = \frac{4}{n-2s}$.

The second main reference is the paper [3] by J. Ginibre, T. Ozawa and G. Velo who mainly consider the asymptotic behaviour of solutions of nonlinear Schrödinger equations but also give the following improvement concerning the problem of **local** H^s**-solutions**:

• If $s < 2$ the range $s - 1 < \sigma \leq \frac{4}{n-2s}$ is allowed.

Concerning the **global small data problem** assume in addition $\sigma \geq \frac{4}{n}$.

However it is clear that these bounds in the range $s > 1$ conflict with each other in many cases, namely excluding high dimensions for a given s.

The aim of the present paper now is to improve the lower bounds on σ.

The main results in section 3 show the **local solvability** in the space $C^0([0,T], H^{s,2}(\mathbf{R}^n))$ under the following assumptions (because the range $0 < s \leq 1$ is completely solved by [2], [3] we consider only s such that $1 < s < n/2$):

- If $1 < s < 2$ we need no lower bound on σ, *i.e.* $0 < \sigma < \frac{4}{n-2s}$.
- If $2 \leq s < 4$ we need $s-2 < \sigma < \frac{4}{n-2}$, and if $s \geq 4$: $s-3 < \sigma < \frac{4}{n-2s}$.

Moreover we have in all these cases the **global solvability** if in addition $\|\phi\|_{H^{s,2}(\mathbf{R}^n)}$ is sufficiently small and $\sigma \geq \frac{4}{n}$.

Thus the lower bound is less restrictive. Especially we are able to cover arbitrary dimensions n if $s < 2$. The limit case $\sigma = \frac{4}{n-2s}$ is not considered here (in contrast to [2], [3]) and would require probably special considerations.

The special (global large data) result for $s = 2$ which goes back to Kato [5] is the starting point for my paper now. However, we do not use any conservation law at all. Because our method will be to use Banach's fixed point theorem for (5) directly our aim is to minimize the number of derivatives lying on $f(u)$ in order to weaken the lower bound on σ . For this we use in principle the fact that the differential equation (1) allows to replace spatial derivatives by half the number of derivatives with respect to t and a certain nonlinear term. This idea is used for the inhomogeneous linear Schrödinger equation first (in section 2) by performing certain partial integrations in the inhomogeneous term. Because these calculations have to be carried out also for fractional values of s we are forced to use certain interpolation spaces with respect to x and t as well. For technical reasons we have to use a real interpolation method especially because a result on interpolation of intersection of spaces (*cf.* Lemma 2.2 below) could only be proven by this method. This means that we have to use Besov spaces–which are the real interpolation spaces of pairs of Sobolev spaces–instead of Sobolev spaces of fractional order in many places making things technically more complicated. On the other hand just these Besov spaces are more convenient to study the mapping properties of the nonlinear term. However I remark that due to well-known embeddings between Besov and Sobolev spaces in the final statement of the theorems it is possible to avoid these Besov spaces completely.

The basic fact which is used all the time as also in [2] are the by now standard estimates of type $L^q(0,T; L^r(\mathbf{R}^n))$ for the inhomogeneous linear Schrödinger equation (*see* also [14]). Technically we first apply the contraction mapping principle **in spaces of fractional time and space derivatives** (at least in the most critical range $1 < s < 2$) such as $B_2^{s/2,\gamma}(0,T;L^\rho(\mathbf{R}^n)) \cap L^\gamma(0,T;B_2^{s,\rho}(\mathbf{R}^n))$ (*see* (31) in Theorem 3.1) endowed with a non-natural metric d avoiding the use of too many derivatives again. *A posteriori* we then show that the solution belongs to $C^0([0,T],H^{s,2}(\mathbf{R}^n))$ (Theorem 3.2).

Corresponding results for semilinear wave equations which of course do not allow to replace space derivatives by a smaller number of time derivatives (as here) were given e.g. in [7] and [9].

The interpolation technique used in this paper could possibly be applied to other types of equations as well where spatial and time derivatives of different order appear.

The author was informed by the referee of a preprint by T. Kato [6] where related problems are discussed. Kato considers in [6] the Cauchy problem (1), (2), (4) but $f(u)$ not necessarily being homogeneous. New uniqueness results in various solution spaces of the type $L^\gamma(0,T;H^{s,\rho}(\mathbf{R}^n))$ are given, especially it follows from his results ([6], Corollary 2.3) that in all my cases ($1 < s < n/2$) the solution is unique in $C^0([0,T],H^{s,2}(\mathbf{R}^n))$. Moreover the local existence results of Cazenave-Weissler [2] are generalized to different spaces and more general nonlinearities. If however power nonlinearities like (3) are considered Kato also has to impose **upper and lower bounds** very similar to the conditions in [2] mentioned above to ensure the existence of a solution in $C^0([0,T],H^{s,2}(\mathbf{R}^n))$ which excludes e.g. for s slightly bigger than 1 any dimension $n \geq 7$ unless σ is an even integer (**in contrast to my assumptions**). Global existence in this space for small data is also proven in [6] provided the same upper and lower bounds for σ as for the local results hold (and not only in the critical case $\sigma = \frac{4}{n-2s}$ as in [2]) and provided $\sigma \geq 4/n$. This last condition coincides with my assumption, but again the other lower bound on σ causes the same problem (especially for $n \geq 7$) mentioned in the local case already. Because I only need weaker lower bounds many more cases are included.

In this paper $H^{k,p}$ denotes the Sobolev spaces and $B_q^{k,p}$ the Besov spaces (of fractional order k, too). As reference we use the monographs of H. Triebel [13] and J. Bergh-J. Löfström [1], where also different equivalent norms especially in the Besov spaces are introduced which are used as well as embedding theorems between Besov and Sobolev spaces.

I thank my colleague Michael Reeken for valuable discussions.

2. LINEAR ESTIMATES

In this section we consider the inhomogeneous linear Schrödinger equation

$$iu_t + Au = g, \qquad u(x,0) = \phi(x) \qquad \text{where} \quad A = -\Delta \tag{7}$$

The solution is given by

$$u(t) = e^{itA}\phi + (Gg)(t) \tag{8}$$

where

$$(Gg)(t) := -i\int_0^t e^{i(t-\tau)A} g(\tau)\, d\tau \tag{9}$$

The results in the following proposition are well-known.

DEFINITION 2.1 (*cf.* [2]). – *A pair* (γ, ρ) *is called admissible if* $2 \le \rho < \frac{2n}{n-2}$, $\frac{2}{\gamma} := n(\frac{1}{2} - \frac{1}{\rho})$.

PROPOSITION 2.1. – *Let* (γ, ρ) *be admissible. If* $n \ge 3$, $s \ge 0$ *the following estimates hold:*

$$||e^{itA}\phi||_{L^\gamma(\mathbf{R}, B_2^{s,\rho}(\mathbf{R}^n))} \le c||\phi||_{H^{s,2}(\mathbf{R}^n)} \tag{10}$$

$$\left\|\frac{d}{dt}(e^{itA}\phi)\right\|_{L^\gamma(\mathbf{R}, B_2^{s,\rho}(\mathbf{R}^n))} \le c||\phi||_{H^{s+2,2}(\mathbf{R}^n)} \tag{11}$$

Here L^∞ *can be replaced by* C^0 *and* $B_2^{s,\rho}$ *by* $H^{s,\rho}$.

Proof. – [2], Thm. 2.1 and 2.2.

PROPOSITION 2.2. – *If* $s \ge 0$, *the following estimates hold for admissible pairs* (γ, ρ), (q, r), $I := [0, T]$ *and c independent of T:*

$$||Gg||_{L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n))} \le c||g||_{L^{q'}(I, B_2^{s,r'}(\mathbf{R}^n))} \tag{12}$$

$$\left\|\frac{d}{dt}Gg\right\|_{L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n))} \le c\left(\left\|\frac{dg}{dt}\right\|_{L^{q'}(I, B_2^{s,r'}(\mathbf{R}^n))} + ||g(0)||_{H^{s,2}(\mathbf{R}^n)}\right) \tag{13}$$

$$\begin{aligned} &||Gg||_{L^\gamma(I, B_2^{s+2,\rho}(\mathbf{R}^n))} \\ &\quad \le c\left(\left\|\frac{dg}{dt}\right\|_{L^{q'}(I, B_2^{s,r'}(\mathbf{R}^n))} + ||g(0)||_{H^{s,2}(\mathbf{R}^n)} + ||g||_{L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n))}\right) \end{aligned} \tag{14}$$

$$\|Gg\|_{L^\gamma(I,B_2^{s+4,\rho}(\mathbf{R}^n))} + \left\|\frac{d}{dt}Gg\right\|_{L^\gamma(I,B_2^{s+2,\rho}(\mathbf{R}^n))} + \left\|\frac{d^2}{dt^2}Gg\right\|_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}$$
$$\leq c\left(\left\|\frac{d^2g}{dt^2}\right\|_{L^{q'}(I,B_2^{s,r'}(\mathbf{R}^n))} + \|g\|_{L^\gamma(I,B_2^{s+2,\rho}(\mathbf{R}^n))} + \left\|\frac{dg}{dt}\right\|_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}\right.$$
$$\left. + \|g(0)\|_{H^{s+2,2}(\mathbf{R}^n)} + \left\|\frac{dg}{dt}(0)\right\|_{H^{s,2}(\mathbf{R}^n)}\right) \tag{15}$$

Here again L^∞ can be replaced by C^0 and the Besov spaces by the corresponding Sobolev spaces.

Proof. – (12) [2], Thm. 2.2 (ii).

(13) By partial integration we get

$$\frac{d}{dt}Gg(t) = i\int_0^t \frac{d}{d\tau}\left[e^{i(t-\tau)A}g(\tau)\right]d\tau - i\int_0^t e^{i(t-\tau)A}\frac{dg}{dt}(\tau)\,d\tau - ig(t)$$
$$= -ie^{itA}g(0) + \left(G\frac{dg}{dt}\right)(t) \tag{16}$$

so that (12) and (10) give the desired estimate.

(14) Again integrating by parts gives

$$AGg(t) = \int_0^t \frac{d}{d\tau}\left[e^{i(t-\tau)A}g(\tau)\right]d\tau - \int_0^t e^{i(t-\tau)A}\frac{dg}{dt}(\tau)\,d\tau$$
$$= g(t) - e^{itA}g(0) - iG\frac{dg}{dt}(t) \tag{17}$$

Then we use (12) for g replaced by $\frac{dg}{dt}$, (10) and the fact that $A : \dot{B}_2^{s+2,\rho}(\mathbf{R}^n) \longrightarrow \dot{B}_2^{s,\rho}(\mathbf{R}^n)$ is an isomorphism ([2], Lemma 2.6).

(15) We use (17) twice, first multiplied by A and secondly for $\frac{dg}{dt}$:

$$A^2Gg(t) = Ag(t) - Ae^{itA}g(0) - iAG\frac{dg}{dt}(t)$$
$$= Ag(t) - Ae^{itA}g(0) - i\frac{dg}{dt}(t) + ie^{itA}\frac{dg}{dt}(0) - G\frac{d^2g}{dt^2}(t) \tag{18}$$

An application of (10) and (12) for $\frac{d^2g}{dt^2}$ gives the estimate for Gg.

By differentiation one has

$$\frac{d}{dt}Gg(t) = -ig(t) + iAGg(t) \tag{19}$$

Using the estimate just proven we have the bound for $\frac{d}{dt}Gg$.

Differentiation of (19) gives

$$\frac{d^2}{dt^2}Gg(t) = -i\frac{dg}{dt}(t) + iA\frac{d}{dt}Gg(t)$$

Again using the estimate of $\frac{d}{dt}Gg$ just proven the bound for $\frac{d^2}{dt^2}Gg$ follows which completes the proof.

PROPOSITION 2.3. – *Let $s \geq 0$, (γ, ρ) be an admissible pair, $I := [0, T]$. Then*

$$\begin{aligned} &\|Gg\|_{L^\gamma(I,B_2^{s+2,\rho}(\mathbf{R}^n))} + \|Gg\|_{H^{1,\gamma}(I,B_2^{s,\rho}(\mathbf{R}^n))} \\ &\quad \leq c(\|g\|_{H^{1,\gamma'}(I,B_2^{s,\rho'}(\mathbf{R}^n))} + \|g\|_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}) \end{aligned} \tag{21}$$

$$\begin{aligned} &\|Gg\|_{C^0(I,H^{s+2,2}(\mathbf{R}^n))} + \|Gg\|_{C^1(I,H^{s,2}(\mathbf{R}^n))} \\ &\quad \leq c(\|g\|_{H^{1,\gamma'}(I,B_2^{s,\rho'}(\mathbf{R}^n))} + \|g\|_{C^0(I,H^{s,2}(\mathbf{R}^n))}) \end{aligned} \tag{22}$$

Here c is independent of T, and the Besov spaces can be replaced by the corresponding Sobolev spaces.

Proof. – (22) If $(\gamma, \rho) = (\infty, 2)$ it is clear that (13) and (14) directly give the result.

(21) If $(\gamma, \rho) \neq (\infty, 2)$ then (13) and (14) show the claimed estimate if we show

$$\|g(0)\|_{H^{s,2}(\mathbf{R}^n)} \leq c\Big(\|g\|_{H^{1,\gamma'}(I,B_2^{s,\rho'}(\mathbf{R}^n))} + \|g\|_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}\Big) \tag{23}$$

This is a consequence of the trace theorem. More precisely we use [13], Thm. 1.8.2 with $A_0 = B_2^{s,\rho}(\mathbf{R}^n)$, $A_1 = B_2^{s,\rho'}(\mathbf{R}^n)$.

With the notation

$$\begin{aligned} T_0^1 := \{a \in A_0 + A_1 \,:\, &\exists g \in L^\gamma(\mathbf{R}^+, A_0) \\ &\text{with } g' \in L^{\gamma'}(\mathbf{R}^+, A_1) \text{ and } g(0) = a\} \end{aligned} \tag{24}$$

$$\|a\|_{T_0^1} := \inf_{g(0)=a} (\|g\|_{L^\gamma(\mathbf{R}^+,A_0)} + \|g'\|_{L^{\gamma'}(\mathbf{R}^+,A_1)}) \tag{25}$$

it holds $T_0^1 = (A_0, A_1)_{\Theta,p}$, where

$$\Theta = \frac{\frac{1}{\gamma}}{1 + \frac{1}{\gamma} - \frac{1}{\gamma'}} = \frac{1}{2} \qquad \text{and} \qquad \frac{1}{p} = \frac{1-\Theta}{\gamma} + \frac{\Theta}{\gamma'} = \frac{1}{2} \tag{26}$$

Thus

$$T_0^1 = \left(B_2^{s,\rho}(\mathbf{R}^n), B_2^{s,\rho'}(\mathbf{R}^n)\right)_{\frac{1}{2},2} = B_2^{s,2}(\mathbf{R}^n) = H^{s,2}(\mathbf{R}^n) \qquad (27)$$

by [1], Thm. 6.4.5. This gives (23) with I replaced by $\mathbf{R}^+$. But then it also holds for I, because the restriction operator from $H^{1,\gamma'}(\mathbf{R}^+, B_2^{s,\rho'}(\mathbf{R}^n))$ onto $H^{1,\gamma'}(I, B_2^{s,\rho'}(\mathbf{R}^n))$ is a retraction with a corresponding coretraction (extension). We refer to [13] here.

In the same way one can show

PROPOSITION 2.4. – *The estimate* (15) *is true without the term* $\|\frac{dg}{dt}(0)\|_{H^{s,2}(\mathbf{R}^n)}$ *in the special case* $(\gamma, \rho) = (q, r)$.

Next we want to give some interpolation results.

LEMMA 2.1. – *Let* $1 \leq \gamma < \infty$, $1 \leq \rho \leq \infty$, $0 < \Theta < 1$, $1 \leq q \leq \infty$. *Then*

$$(L^\gamma(\mathbf{R}, L^\rho(\mathbf{R}^n)), H^{1,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n)))_{\Theta,q} = B_q^{\Theta,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n)) \qquad (28)$$

The same identity holds for $\mathbf{R}$ *replaced by* I.

Proof. – It is well known that $\Lambda = \frac{d}{dt}$ is the infinitesimal generator of the translation group $G(t)u(\tau) = u(\tau + t)$ in $L^\gamma(\mathbf{R}, L^\rho(\mathbf{R}^n))$ with $D(\Lambda) = H^{1,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n))$. From [1], Thm. 6.7.3 and p. 160 we get $(L^\gamma(\mathbf{R}, L^\rho(\mathbf{R}^n)), D(\Lambda))_{\Theta,q} = B_q^{\Theta,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n))$ which is our claim. That the identity also holds for I is a consequence of the fact that the restriction operator from $B_q^{\Theta,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n))$ onto $B_q^{\Theta,\gamma}(I, L^\rho(\mathbf{R}^n))$ and from $H^{s,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n))$ onto $H^{s,\gamma}(I, L^\rho(\mathbf{R}^n))$ is a retraction with a corresponding coretraction (extension), *cf.* [13], Thm. 4.3.2.

As a direct consequence we get

PROPOSITION 2.5. – *Let* (γ, ρ) *be an admissible pair,* $0 \leq s < 2$. *We have*

$$\|e^{itA}\phi\|_{B_2^{s/2,\gamma}(\mathbf{R}, L^\rho(\mathbf{R}^n))} \leq c\|\phi\|_{H^{s,2}(\mathbf{R}^n)} \qquad (29)$$

Proof. – Use Lemma 2.1 and (11) and (11) with $s = 0$.

The next result concerns intersection of spaces.

LEMMA 2.2. – *Let* $0 < \Theta < 1$, $1 \leq w \leq \infty$, $1 < \gamma', \rho', q, r < \infty$ *or* $(q, r) = (\infty, 2)$. *Then*

$$\begin{aligned}
&(L^{\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n)), H^{1,\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n)) \cap L^q(\mathbf{R}, L^r(\mathbf{R}^n)))_{\Theta,w} \\
&\quad = (L^{\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n)), H^{1,\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n)))_{\Theta,w} \\
&\quad\quad \cap\ (L^{\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n)), L^q(\mathbf{R}, L^r(\mathbf{R}^n)))_{\Theta,w}
\end{aligned}$$

Proof. – We apply [10], Satz 1.1 and Korollar 1.1. With his notation we define $A_0 := L^{\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n))$, $A_1' := H^{1,\gamma'}(\mathbf{R}, L^{\rho'}(\mathbf{R}^n))$, $A_1'' := L^q(\mathbf{R}, L^r(\mathbf{R}^n))$, $A_1 := A_1' \cap A_1''$.

$$V_1'(t)a := F^{-1}(m(t,\tau)\hat{a}(\tau))$$
$$V_0'(t)a := F^{-1}(t(\tau^2+1)^{1/2}m(t,\tau)\hat{a}(\tau))$$

where $m(t,\tau) := \frac{1}{1+t(\tau^2+1)^{1/2}}$.

Here $\hat{}$ and F^{-1} denote the Fourier transform with respect to time and its inverse, $0 < t < \infty$. Following [10] we have to show

$$\|V_0'(t)a\|_{A_0} + t\|V_1'(t)\|_{A_1'} \leq cK(t,a,A_0,A_1'), \qquad a \in A_0 + A_1' \tag{30}$$

and

$$\|V_1'(t)a\|_{A_1''} \leq c\|a\|_{A_1''}, \qquad a \in A_1'' \tag{31}$$

Here K denotes the K-functional in real interpolation theory ([1], Chapter 3.1). In order to prove (30), let $a = \bar{a}_0(t) + \bar{a}_1(t)$ be an arbitrary decomposition with $\bar{a}_0(t) \in A_0, \bar{a}_1(t) \in A_1'$. We have by Lemma 2.3 below

$$\begin{aligned}&\|V_0'(t)a\|_{A_0}\\ &\quad\leq \|\bar{a}_0(t)\|_{A_0} + \|F^{-1}(m(t,\tau)\hat{\bar{a}}_0(t,\tau))\|_{A_0}\\ &\quad\quad + t\|F^{-1}(m(t,\tau)\hat{\bar{a}}_1(t,\tau))\|_{A_1'}\\ &\quad\leq c(\|\bar{a}_0(t)\|_{A_0} + t\|\bar{a}_1(t)\|_{A_1'})\end{aligned}$$

thus

$$\|V_0'(t)a\|_{A_0} \leq cK(t,a,A_0,A_1') \tag{32}$$

Moreover by Lemma 2.3 again

$$\begin{aligned}&t\|V_1'(t)\|_{A_1'}\\ &\quad\leq c(\|F^{-1}(t(\tau^2+1)^{1/2}m(t,\tau)\hat{\bar{a}}_0(t,\tau))\|_{A_0}\\ &\quad\quad + t\|F^{-1}(m(t,\tau)\hat{\bar{a}}_1(t,\tau))\|_{A_1'})\\ &\quad\leq c(\|F^{-1}((1-m(t,\tau))\hat{\bar{a}}_0(t,\tau))\|_{A_0} + t\|\bar{a}_1(t)\|_{A_1'})\\ &\quad\leq c(\|\bar{a}_0(t)\|_{A_0} + t\|\bar{a}_1(t)\|_{A_1'})\end{aligned}$$

thus

$$t\|V_1'(t)a\|_{A_1'} \leq cK(t,a,A_0,A_1') \tag{33}$$

Now (30) follows from (32) and (33), and (31) follows from Lemma 2.3.

LEMMA 2.3. – *Let* $1 < p,\ q < \infty$ *or* $p = 2,\ q = \infty,\ 0 < t < \infty,\ \tau \in \mathbf{R}$ $m(t,\tau) := \frac{1}{1+t(\tau^2+1)^{1/2}}$. *Then*

$$||F^{-1}(m(t,\tau)\hat{a}(\tau))||_{L^q(\mathbf{R},L^p(\mathbf{R}^n))} \leq c||a||_{L^q(\mathbf{R},L^p(\mathbf{R}^n))} \tag{34}$$

where c *is independent of* t.

Proof. – *a*) If $1 < p, q < \infty$ we use [11], Cor. 10. We need

$$|m(t,\tau)| + \int_{-\infty}^{+\infty} \left|\frac{\partial m}{\partial \tau}(t,\tau)\right| d\tau \leq c$$

which is easily checked.

b) If $p = 2,\ q = \infty$ we use [1], Lemma 6.15, and we need

$$||m(t,\cdot)||_{L^2(\mathbf{R})} \left|\left|\frac{\partial m}{\partial \tau}(t,\cdot)\right|\right|_{L^2(\mathbf{R})} \leq c \tag{36}$$

An elementary calculation shows $||m(t,\cdot)||^2_{L^2(\mathbf{R})} \leq \frac{c}{t(1+t)}$ and $\left|\left|\frac{\partial m}{\partial \tau}(t,\cdot)\right|\right|^2_{L^2(\mathbf{R})} \leq ct$ if $t \leq 1$ and $\leq \frac{c}{t^2}$ if $t \geq 1$, which completes the proof.

The interpolation results just obtained are now used to get

PROPOSITION 2.6. – *If* (γ,ρ) *is an admissible pair,* $(\gamma,\rho) \neq (\infty,2)$, $0 < \Theta < 1$, $\epsilon > 0$ *arbitrarily small, the following estimate holds with* c *independent of* I:

$$\begin{aligned}
&||Gg||_{L^\gamma(I,B_2^{2\Theta,\rho}(\mathbf{R}^n))} + ||Gg||_{B_2^{\Theta+\epsilon,\gamma}(I,L^\rho(\mathbf{R}^n))} \\
&\leq c\Big(||g||_{B_2^{\Theta+\epsilon,\gamma'}(I,L^{\rho'}(\mathbf{R}^n))} + ||g||_{L^{\tilde{\gamma}_\Theta+2\epsilon}(I,L^{\tilde{\rho}_\Theta}(\mathbf{R}^n)\cap L^{\tilde{\rho}_\Theta+4\epsilon}(\mathbf{R}^n))} \\
&\quad + ||g||_{L^{\tilde{\gamma}_\Theta}(I,L^{\tilde{\rho}_\Theta-\epsilon}(\mathbf{R}^n)\cap L^{\tilde{\rho}_\Theta+\epsilon}(\mathbf{R}^n))}\Big)
\end{aligned}$$

where $\frac{1}{\tilde{\gamma}_\Theta} = \frac{\Theta}{\gamma} + \frac{1-\Theta}{\gamma'}$, $\frac{1}{\tilde{\rho}_\Theta} = \frac{\Theta}{\rho} + \frac{1-\Theta}{\rho'}$.

Proof. – From (12) and (21) we have the following continuous mappings:

$$\begin{aligned}
G:&\quad L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n)) \longrightarrow L^\gamma(I,L^\rho(\mathbf{R}^n)) \\
G:&\quad H^{1,\gamma'}(I,L^{\rho'}(\mathbf{R}^n)) \cap L^\gamma(I,L^\rho(\mathbf{R}^n)) \longrightarrow L^\gamma(I,H^{2,\rho}(\mathbf{R}^n))
\end{aligned}$$

Thus Lemma 2.1 and Lemma 2.2 give

$$\begin{aligned}
G: B_\gamma^{\Theta+\epsilon,\gamma'}(I,L^{\rho'}(\mathbf{R}^n)) \cap \Big(L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n)),L^\gamma(I,L^\rho(\mathbf{R}^n))\Big)_{\Theta+\epsilon,2} \\
\longrightarrow \quad (L^\gamma(I,L^\rho(\mathbf{R}^n)),L^\gamma(I,H^{2,\rho}(\mathbf{R}^n)))_{\Theta+\epsilon,2}
\end{aligned} \tag{37}$$

Now we have by [13], Thm. 1.18.4 and Thm. 2.3.2 and [1], Thm. 3.4.1 and Thm. 6.2.4:

$$\begin{aligned}
&(L^\gamma(I, L^\rho(\mathbf{R}^n)), L^\gamma(I, H^{2,\rho}(\mathbf{R}^n)))_{\Theta+\epsilon,2} \\
&\quad \subset \big(L^\gamma(I, L^\rho(\mathbf{R}^n)), L^\gamma(I, H^{2,\rho}(\mathbf{R}^n))\big)_{\Theta+\epsilon,\gamma} \\
&\quad = L^\gamma(I, (L^\rho(\mathbf{R}^n), H^{2,\rho}(\mathbf{R}^n))_{\Theta+\epsilon,\gamma}) \\
&\quad = L^\gamma(I, B_\gamma^{2(\Theta+\epsilon),\rho}(\mathbf{R}^n)) \subset L^\gamma(I, B_2^{2\Theta,\rho}(\mathbf{R}^n))
\end{aligned} \tag{38}$$

Moreover using [1], Thm. 3.4.1 and [13], Thm. 1.18.4 we have

$$\begin{aligned}
&(L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)), L^\gamma(I, L^\rho(\mathbf{R}^n)))_{\Theta+\epsilon,2} \\
&\quad \supset \ (L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)), L^\gamma(I, L^\rho(\mathbf{R}^n)))_{\Theta+2\epsilon,\tilde{\gamma}_{\Theta+2\epsilon}} \\
&\quad \cap \ (L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)), L^\gamma(I, L^\rho(\mathbf{R}^n)))_{\Theta,\tilde{\gamma}_\Theta} \\
&\quad = L^{\tilde{\gamma}_{\Theta+2\epsilon}}(I, (L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta+2\epsilon,\tilde{\gamma}_{\Theta+2\epsilon}}) \\
&\quad \cap L^{\tilde{\gamma}_\Theta}(I, (L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta,\tilde{\gamma}_\Theta})
\end{aligned} \tag{39}$$

By [1], Thm. 3.4.1 (c) and Thm. 5.2.1 we conclude

$$\begin{aligned}
&(L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta+2\epsilon,\tilde{\gamma}_{\Theta+2\epsilon}} \\
&\supset \ (L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta,\tilde{\rho}_\Theta} \cap (L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta+4\epsilon,\tilde{\rho}_{\Theta+4\epsilon}} \\
&= L^{\tilde{\rho}_\Theta}(\mathbf{R}^n) \cap L^{\tilde{\rho}_{\Theta+4\epsilon}}(\mathbf{R}^n)
\end{aligned} \tag{40}$$

Similarly

$$(L^{\rho'}(\mathbf{R}^n), L^\rho(\mathbf{R}^n))_{\Theta,\tilde{\gamma}_\Theta} \supset L^{\tilde{\rho}_{\Theta-\epsilon}}(\mathbf{R}^n) \cap L^{\tilde{\rho}_{\Theta+\epsilon}}(\mathbf{R}^n) \tag{41}$$

From (37), (38), (39), (40), (41) we get the claimed estimate for $||Gg||_{L^\gamma(I, B_2^{2\Theta,\rho}(\mathbf{R}^n))}$. Similarly from (12) and Prop. 2.3

$$G : \ L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)) \to L^\gamma(I, L^\rho(\mathbf{R}^n))$$
$$G : \ H^{1,\gamma'}(I, L^{\rho'}(\mathbf{R}^n)) \cap L^\gamma(I, L^\rho(\mathbf{R}^n)) \to H^{1,\gamma}(I, L^\rho(\mathbf{R}^n))$$

One has $(L^\gamma(I, L^\rho(\mathbf{R}^n)), H^{1,\gamma}(I, L^\rho(\mathbf{R}^n)))_{\Theta+\epsilon,2} = B_2^{\Theta+\epsilon,\gamma}(I, L^\rho(\mathbf{R}^n))$ so that as before the claimed estimate for $||Gg||_{B_2^{\Theta+\epsilon,\gamma}(I, L^\rho(\mathbf{R}^n))}$ follows.

PROPOSITION 2.7. – *If $(\gamma, \rho) \neq (\infty, 2)$ is an admissible pair, $0 < \Theta < 1$, we have with c independent of I:*

$$\begin{aligned}
&||Gg||_{C^0(I, H^{2\Theta,2}(\mathbf{R}^n))} \\
&\quad \leq (||g||_{B_2^{\Theta+\epsilon,\gamma'}(I, L^{\rho'}(\mathbf{R}^n))} + ||g||_{L^{\gamma_{\Theta+\epsilon}}(I, L^{\rho_\Theta}(\mathbf{R}^n) \cap L^{\rho_{\Theta+2\epsilon}}(\mathbf{R}^n))})
\end{aligned} \tag{42}$$

where $\epsilon > 0$ is arbitrarily small, and $\frac{1}{\gamma_\Theta} = \frac{1-\Theta}{\gamma'}$, $\frac{1}{\rho_\Theta} = \frac{1-\Theta}{\rho'} + \frac{\Theta}{2}$.

Proof. – The proof is similar as in Prop. 2.6. We interpolate between the estimates (12) and (22):

$$G: L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)) \to L^\infty(I, L^2(\mathbf{R}^n))$$
$$G: H^{1,\gamma'}(I, L^{\rho'}(\mathbf{R}^n)) \cap L^\infty(I, L^2(\mathbf{R}^n)) \to L^\infty(I, H^{2,2}(\mathbf{R}^n))$$

As before we use Lemma 2.2 and

$$\begin{aligned}
&(L^\infty(I, L^2(\mathbf{R}^n)), L^\infty(I, H^{2,2}(\mathbf{R}^n)))_{\Theta+\epsilon,\infty} \\
&\quad = L^\infty(I, (L^2(\mathbf{R}^n), H^{2,2}(\mathbf{R}^n))_{\Theta+\epsilon,\infty}) \\
&\quad = L^\infty(I, B_\infty^{2(\Theta+\epsilon),2}(\mathbf{R}^n)) \subset L^\infty(I, B_2^{2\Theta,2}(\mathbf{R}^n)) = L^\infty(I, H^{2\Theta,2}(\mathbf{R}^n))
\end{aligned}$$

The first identity follows from [13], Thm. 1.5.3 and [8], p. 47. Similarly

$$\begin{aligned}
&(L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)), L^\infty(I, L^2(\mathbf{R}^n)))_{\Theta+\epsilon,\infty} \\
&\quad \supset (L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^n)), L^\infty(I, L^2(\mathbf{R}^n)))_{\Theta+\epsilon,\gamma_{\Theta+\epsilon}} \\
&\quad = L^{\gamma_{\Theta+\epsilon}}(I, (L^{\rho'}(\mathbf{R}^n), L^2(\mathbf{R}^n))_{\Theta+\epsilon,\gamma_{\Theta+\epsilon}}) \\
&\quad \supset L^{\gamma_{\Theta+\epsilon}}(I, L^{\rho_\Theta}(\mathbf{R}^n) \cap L^{\rho_{\Theta+2\epsilon}}(\mathbf{R}^n))
\end{aligned}$$

The claimed estimate now follows as in the preceding Proposition. The continuity of Gg can be easily seen by choosing a dense subset P of smooth functions, e.g. $C^\infty([0,T], C_0^\infty(\mathbf{R}^n))$. Now obviously $G : P \to C^0([0,T], H^{2\Theta,2}(\mathbf{R}^n))$ and the claimed estimate holds for $g \in P$, so that a limiting process shows $Gg \in C^0([0,T], H^{2\Theta,2}(\mathbf{R}^n))$.

3. THE NONLINEAR PROBLEM

We consider the integral equation ($A = -\Delta$):

$$u(t) = e^{itA}\phi + (Gf(u))(t) \tag{43}$$

where

$$(Gf(u))(t) := -i \int_0^t e^{i(t-\tau)A} f(u(\tau))\, d\tau \tag{44}$$

and

$$f(u) = c|u|^\sigma u, \qquad c \in \mathbf{C} \tag{45}$$

$$\phi \in H^{s,2}(\mathbf{R}^n) \tag{46}$$

First of all we construct local solutions of (43) for $s < 2$ by using the interpolation results of the previous section.

THEOREM 3.1. – *Let* $1 < s < 3/2$ *if* $n = 3$, $1 < s < 2$ *if* $n \geq 4$. *Moreover assume*

$$0 < \sigma < \frac{4}{n-2s} \tag{47}$$

1. *Let* (γ, ρ) *denote the special admissible pair*

$$\gamma = \frac{4(\sigma+2)}{\sigma(n-2s)}, \qquad \rho = \frac{\sigma+2}{1+\sigma s/n} \tag{48}$$

Then the integral equation (43) *with* (44), (45), (46) *has a unique local solution* $u \in Y$.

Here

$$Y := \{u \in X \,:\, u(0) = \phi\} \tag{49}$$

$$X := B_2^{s/2,\gamma}(I, L^\rho(\mathbf{R}^n)) \cap L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n)) \tag{50}$$

where $I := [0, T]$ *and* $T = T(\|\phi\|_{H^{s,2}(\mathbf{R}^n)})$.

2. *If the admissible pair* (γ, ρ) *is chosen as*

$$\gamma = \sigma + 2, \qquad \rho = \frac{2n(\sigma+2)}{n(\sigma+2)-4} \tag{51}$$

and if moreover

$$\sigma \geq 4/n \quad \textit{and } \|\phi\|_{H^{s,2}(\mathbf{R}^n)} \textit{ is sufficiently small} \tag{52}$$

then the integral equation (43) *with* (44), (45), (46) *has a unique global solution* $u \in Y$ *with* Y, X *defined by* (49), (50) *and* $I := [0, +\infty)$.

Proof. – We show that the mapping

$$(Su)(t) := e^{itA} + (Gf(u))(t) \tag{53}$$

is a contraction in a suitable ball in X endowed with the metric

$$d(u, v) := \|u - v\|_{L^\gamma(I, L^\rho(\mathbf{R}^n))} \tag{54}$$

Part 1. – In order to show that S maps X into itself we use Prop. 2.6 with (γ, ρ) as in (48), $g = f(u)$, and $\Theta = \frac{s}{2} - \epsilon$, and want to show

$$\|f(u)\|_{B_2^{\frac{s}{2},\gamma'}(I, L^{\rho'}(\mathbf{R}^n))} \leq cT^\kappa \|u\|^\sigma_{L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n))} \|u\|_{B_2^{\frac{s}{2},\gamma}(I, L^\rho(\mathbf{R}^n))} \tag{55}$$

with $\kappa > 0$.

Using one of the equivalent norms in Besov spaces ([13], Thm. 4.4.2(2)) we have with $I_\tau := \{t : t, t+\tau \in I\}$:

$$\begin{aligned} &\|f(u)\|_{B_2^{\frac{s}{2},\gamma'}(I,L^{\rho'}(\mathbf{R}^n))} \\ &\quad = \|f(u)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))} \\ &\quad + \left\{ \int_{-\infty}^{\infty} (|\tau|^{-\frac{s}{2}} \|f(u(t+\tau)) - f(u(t))\|_{L^{\gamma'}(I_\tau,L^{\rho'}(\mathbf{R}^n))})^2 \frac{d\tau}{|\tau|} \right\}^{1/2} \end{aligned} \tag{56}$$

Hölder's inequality gives

$$\begin{aligned} &\|f(u(t+\tau)) - f(u(t))\|_{L^{\rho'}(\mathbf{R}^n)} \\ &\quad \leq c(\|u(t+\tau)\|^{\sigma}_{L^{\frac{\rho\rho'\sigma}{\rho-\rho'}}(\mathbf{R}^n)} + \|u(t)\|^{\sigma}_{L^{\frac{\rho\rho'\sigma}{\rho-\rho'}}(\mathbf{R}^n)})\|u(t+\tau) - u(t)\|_{L^{\rho}(\mathbf{R}^n)} \end{aligned}$$

so that the second term in (56) is estimated by Hölder's inequality with exponents $\frac{\gamma}{\gamma-\gamma'}$ and $\frac{\gamma}{\gamma'}$ with respect to τ by $c\|u\|^{\sigma}_{L^{\frac{\sigma\gamma\gamma'}{\gamma-\gamma'}}(I,L^{\frac{\rho'\rho\sigma}{\rho-\rho'}}(\mathbf{R}^n))} \|u\|_{B_2^{\frac{s}{2}+\epsilon,\gamma}(I,L^{\rho}(\mathbf{R}^n))}$.

This can be estimated as required by (55) if the following conditions are satisfied:

$$\frac{\sigma\gamma\gamma'}{\gamma-\gamma'} < \gamma \tag{57}$$

and

$$B_2^{s,\rho}(\mathbf{R}^n) \subset L^{\frac{\rho'\rho\sigma}{\rho-\rho'}}(\mathbf{R}^n) \tag{58}$$

(57) is equivalent to $\sigma < \frac{4}{n-2s}$ (which was assumed) under the choice (48) of γ. (58) holds by [1], Thm. 6.5.1 provided $\frac{\rho-\rho'}{\rho'\rho\sigma} = \frac{1}{\rho} - \frac{s}{n}$ which is easily seen to be equivalent to our choice (48) of ρ.

Returning to (56) we remark that the estimate of $\|f(u)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))}$ is straightforward now:

$$\begin{aligned} \|f(u)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))} &\leq c\|u\|^{\sigma}_{L^{\frac{\sigma\gamma\gamma'}{\gamma-\gamma'}}\left(I,L^{\frac{\rho'\rho\sigma}{\rho-\rho'}}(\mathbf{R}^n)\right)} \|u\|_{L^{\gamma}(I,L^{\rho}(\mathbf{R}^n))} \\ &\leq cT^{\kappa}\|u\|^{\sigma}_{L^{\gamma}(I,B_2^{s,\rho}(\mathbf{R}^n))}\|u\|_{B_2^{s/2,\gamma}(I,L^{\rho}(\mathbf{R}^n))} \end{aligned}$$

Thus (55) is proved.

The next estimate we need in order to use Prop. 2.6 (using its notation also) is of the type ([1]):

$$\|f(u)\|_{L^{\tilde\gamma_\Theta}(I,L^{\tilde\rho_\Theta}(\mathbf{R}^n))} \le cT^\lambda \|u\|^{\sigma+1}_{B_2^{(s/2-\omega)\eta,\gamma}(I,B_2^{(1-\eta)s,\rho}(\mathbf{R}^n))} \tag{59}$$

where $0 < \eta < 1$, $\lambda > 0$, $\omega > 0$ small.

If (59) is fulfilled we use Lemma 4.1 below and $B_2^{s,\rho} \subset H^{s,\rho}$ $(\rho \ge 2)$ ([1], Thm. 6.4.4) and arrive at

$$\|f(u)\|_{L^{\tilde\gamma_\Theta}(I,L^{\tilde\rho_\Theta}(\mathbf{R}^n))} \le cT^\lambda \|u\|^{(\sigma+1)\eta}_{B_2^{s/2,\gamma}(I,L^\rho(\mathbf{R}^n))} \|u\|^{(\sigma+1)(1-\eta)}_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} \tag{60}$$

Proposition 2.6 and (55), (60) lead to

$$\|Gf(u)\|_X \le c(T^\kappa + T^\lambda)\|u\|_X^{\sigma+1} \tag{61}$$

Now we prove (59). We need the embeddings

$$\begin{aligned} B_2^{(s/2-\omega)\eta,\gamma}(I, B_2^{(1-\eta)s,\rho}(\mathbf{R}^n)) &\subset B_2^{(s/2-\omega)\eta,\gamma}(I, L^{\tilde\rho_\Theta(\sigma+1)}(\mathbf{R}^n)) \\ &\subset L^{\tilde\gamma_\Theta(\sigma+1)}(I, L^{\tilde\rho_\Theta(\sigma+1)}(\mathbf{R}^n)) \end{aligned}$$

These are true provided we have (62) and (63):

$$\frac{1}{\tilde\gamma_\Theta(\sigma+1)} > \frac{1}{\gamma} - \frac{s}{2}\eta \tag{62}$$

$$\frac{1}{\rho} > \frac{1}{\tilde\rho_\Theta(\sigma+1)} > \frac{1}{\rho} - \frac{(1-\eta)s}{n} \tag{63}$$

The last condition in (63) is equivalent to

$$1 - \eta > \frac{n}{s}\left(\frac{1}{\rho} - \frac{1}{\tilde\rho_\Theta(\sigma+1)}\right) = \frac{\sigma}{2(\sigma+2)(\sigma+1)}(2\sigma + (4+n) - 2s) \tag{64}$$

where the equality follows from elementary calculations using the definition (48) of ρ and $\tilde\rho_\Theta$ according to Prop. 2.6. The last expression lies between 0 and 1 if $0 < \sigma < \frac{4}{n-2s}$, so that η can be chosen such that (64) holds with almost equality, thus (63) is fulfilled.

Again an elementary calculation shows that (62) is equivalent to

$$1 - \eta < 1 - \frac{2}{s(\sigma+1)}\left[\frac{\sigma(n-2s)}{4(\sigma+2)}(\sigma + 2 - s) - \frac{2-s}{2}\right] \tag{65}$$

([1]) Here we ignore terms of order ϵ in the parameters which is possible because the conditions to be fulfilled are given by **strict** inequalities.

Thus (62) and (63) are compatible if the right hand side of (64) is smaller then the r.h.s. of (65). A lengthy calculation shows that this is the case iff $\sigma < \frac{4}{n-2s}$ which is exactly our assumption on σ so that (59) is proved.

Next we estimate the linear term $e^{itA}\phi$ in (53) by Prop. 2.5 and (10):

$$\|e^{itA}\phi\|_{B_2^{s/2,\gamma}(I,L^\rho(\mathbf{R}^n))} + \|e^{itA}\phi\|_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} \leq c\|\phi\|_{H^{s,2}(\mathbf{R}^n)} \tag{66}$$

From (53), (61) and (66) we conclude

$$\|Su\|_X \leq c\big(\|\phi\|_{H^{s,2}(\mathbf{R}^n)} + (T^\kappa + T^\lambda)\|u\|_X^{\sigma+1}\big) \tag{67}$$

Our next aim is the desired Lipschitz property of S with respect to the metric d defined in (54).

(12) and Hölder's inequality first with respect to space variables and exponents $\rho - 1$ and $\frac{\rho-1}{\rho-2}$ and then w.r. to time and exponents $\gamma - 1$ and $\frac{\gamma-1}{\gamma-2}$ gives

$$\begin{aligned}
&\|Gf(u) - Gf(v)\|_{L^\gamma(I,L^\rho(\mathbf{R}^n))}\\
&\quad \leq c\|f(u) - f(v)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))}\\
&\quad \leq c\Big(\||u|^\sigma(u-v)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))} + \||v|^\sigma(u-v)\|_{L^{\gamma'}(I,L^{\rho'}(\mathbf{R}^n))}\Big)\\
&\quad \leq c\left(\|u\|^\sigma_{L^{\frac{\sigma\gamma}{\gamma-2}}(I,L^{\frac{\sigma\rho}{\rho-2}}(\mathbf{R}^n))} + \|v\|^\sigma_{L^{\frac{\sigma\gamma}{\gamma-2}}(I,L^{\frac{\sigma\rho}{\rho-2}}(\mathbf{R}^n))}\right)\|u-v\|_{L^\gamma(I,L^\rho(\mathbf{R}^n))}\\
&\quad \leq cT^\beta\Big(\|u\|^\sigma_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} + \|v\|^\sigma_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}\Big)\|u-v\|_{L^\gamma(I,L^\rho(\mathbf{R}^n))}
\end{aligned} \tag{68}$$

The last estimate holds with $\beta > 0$ provided we have

$$\frac{\sigma\gamma}{\gamma-2} < \gamma \tag{69}$$

and

$$B_2^{s,\rho}(\mathbf{R}^n) \subset L^{\frac{\sigma\rho}{\rho-2}}(\mathbf{R}^n) \tag{70}$$

(69) is fulfilled by the definition (48) of γ iff $\sigma < \frac{4(\sigma+2)}{\sigma(n-2s)} - 2$ which is equivalent to our assumption $\sigma < \frac{4}{n-2s}$. (70) holds by Sobolev's embedding theorem and the embedding $B_2^{s,\rho}(\mathbf{R}^n) \subset H^{s,\rho}(\mathbf{R}^n)$ ([1], Thm. 6.4.4) if $\frac{\rho-2}{\sigma\rho} = \frac{1}{\rho} - \frac{s}{n}$ which is exactly our choice of ρ in (48).

From (68) we have

$$d(Su,Sv) \leq cT^\beta(\|u\|_X^\sigma + \|v\|_X^\sigma)d(u,v) \tag{71}$$

Standard arguments using (67), (71) and Banach's fixed point theorem complete the proof.

Part 2. – The proof is similar as in 1. but we need (55) with $\kappa = 0$. This requires equality in (57) which is fulfilled with our choice (51) of (γ, ρ).

(58) holds by [1], Thm. 6.5.1 provided $\frac{1}{\rho} \geq \frac{\rho-\rho'}{\rho'\rho\sigma} \geq \frac{1}{\rho} - \frac{s}{n}$ which is easily seen to be equivalent to $\frac{4}{n} \leq \sigma \leq \frac{4}{n-2s}$. Thus (55) with $\kappa = 0$ holds.

Next we need (59) with $\lambda = 0$. This would lead directly to (61) with $\kappa = \lambda = 0$. In order to check (59) we need (62) and (63) and moreover (because $I = [0, \infty)$ is unbounded now) $\frac{1}{\gamma} > \frac{1}{\tilde{\gamma}_\Theta(\sigma+1)}$. The latter however can easily be checked for any $\sigma > 0$. The last condition in (63) is now equivalent to

$$1 - \eta > \frac{n(\sigma+2)\sigma - 4(\sigma+1) + 4s - 4}{2s(\sigma+1)(\sigma+2)} \tag{72}$$

The last expression lies between 0 and 1 if $\frac{4}{n} \leq \sigma \leq \frac{4}{n-2s}$,so that η can be chosen s.th. (63) is fulfilled. An elementary calculation also shows that (62) is equivalent to

$$1 - \eta < 1 - \frac{\sigma}{(\sigma+1)(\sigma+2)} \tag{73}$$

Thus (62) and (63) are compatible if the r.h.s. of (72) is smaller than the r.h.s. of (73) which can be shown in our case $\sigma < \frac{4}{n-2s}$. Because (66) holds here, too we arrive at (67) with $\kappa = \lambda = 0$.

Concerning the Lipschitz property of S we get (68) with $\beta = 0$ because (69) holds with equality and (70) requires $\frac{1}{\rho} \geq \frac{\rho-2}{\sigma\rho} \geq \frac{1}{\rho} - \frac{s}{n}$ which by elementary calculations can be seen to be fulfilled if $\frac{4}{n} \leq \sigma \leq \frac{4}{n-2s}$. Thus (71) follows with $\beta = 0$.

The proof can be completed from (69) and (71) with $\kappa = \lambda = \beta = 0$ by standard arguments.

Now we are able to show *a posteriori* the desired regularity of the solution.

THEOREM 3.2. – *The solution* u *of Theorem* 3.1 *belongs to* $C^0(I, H^{s,2}(\mathbf{R}^n))$.

Proof.

1. Concerning part 1. of Thm. 3.1 we use Prop. 2.7 with $\Theta = \frac{s}{2} - \epsilon$ and γ, ρ as defined by (48) and have to estimate $\|f(u)\|_{B_2^{s/2,\gamma'}(I, L^{\rho'}(\mathbf{R}^n))}$ which has already been done in the previous theorem and (2) $\|f(u)\|_{L^{\tilde{\gamma}}(I, L^{\tilde{\rho}}(\mathbf{R}^n))}$

(2) We here again ignore terms of order ϵ in the parameters as before.

where $\frac{1}{\tilde{\gamma}} = \frac{1-s/2}{\gamma'}$, $\frac{1}{\tilde{\rho}} = \frac{1-s/2}{\rho'} + \frac{s}{4}$. We want to show an estimate analogeous to (59) and need the embeddings

$$B_2^{(s/2-\omega)\eta,\gamma}(I, B_2^{(1-\eta)s,\rho}(\mathbf{R}^n)) \subset B_2^{(s/2-\omega)\eta,\gamma}(I, L^{\tilde{\rho}(\sigma+1)}(\mathbf{R}^n)) \subset L^{\tilde{\gamma}(\sigma+1)}(I, L^{\tilde{\rho}(\sigma+1)}(\mathbf{R}^n))$$

These are true if the following two conditions are satisfied:

$$\frac{1}{\tilde{\gamma}(\sigma+1)} > \frac{1}{\gamma} - \frac{s}{2}\eta \tag{74}$$

$$\frac{1}{\rho} > \frac{1}{\tilde{\rho}(\sigma+1)} > \frac{1}{\rho} - \frac{(1-\eta)s}{n} \tag{75}$$

Using the definition (48) an elementary calculation shows that the second inequality in (75) is equivalent to

$$1 - \eta > \frac{\sigma}{4(\sigma+2)(\sigma+1)}(4\sigma + 8 + n - 2s) \tag{76}$$

The last expression lies between 0 and 1 s.th. η can be chosen s.th. almost equality holds here. (74) is equivalent to

$$1 - \eta < 1 - \frac{2}{s(\sigma+1)}\left[\frac{\sigma(n-2s)}{4(\sigma+2)}(\sigma + 2 - \frac{s}{2}) - \frac{2-s}{2}\right] \tag{77}$$

Thus (76), (77) are compatible if the r.h.s. of (76) is smaller than the r.h.s. of (77) which can be shown to be the case iff $\sigma < \frac{4}{n-2s}$, which is just our assumption on σ.

Combining the analogue of (59) just proven with Lemma 4.1 below completes the proof.

2. Concerning the global part of Thm. 3.1 we conclude as above but with (γ, ρ) defined by (51). The analogues of (76) and (77) are

$$1 - \eta > \frac{n\sigma^2 + (2n-4)\sigma + 2s - 8}{2s(\sigma+2)(\sigma+1)} \tag{78}$$

$$1 - \eta < \frac{\sigma+1}{\sigma+2} \tag{79}$$

These conditions are easily shown to be compatible if $\sigma < \frac{4}{n-2s}$. Moreover we need in (74) $\frac{1}{\gamma} > \frac{1}{\tilde{\gamma}(\sigma+1)}$ which is true and in (75) $\frac{1}{\rho} > \frac{1}{\tilde{\rho}(\sigma+1)}$ which is fulfilled if $\sigma \geq \frac{4}{n}$. The proof is completed as before.

4. HIGHER REGULARITY

If $2 < s < 4$ we use Prop. 2.3 and if $s \geq 4$ Prop. 2.4 without further interpolation. The case of non-integer s is the more complicated one because we have to use Besov spaces instead of Sobolev spaces.

THEOREM 4.1. – *Let* $s \notin \mathbf{N}$, $2 < s < 4$. *Assume*

$$s - 2 < \sigma < \frac{4}{n - 2s} \tag{80}$$

1. *Define* (γ, ρ) *as the special admissible pair*

$$\gamma = \frac{4(\sigma + 2)}{\sigma(n - 2s) + 2n\epsilon}, \qquad \rho = \frac{\sigma + 2}{1 + \frac{\sigma s}{n} - \epsilon} \tag{81}$$

where $\epsilon > 0$ *is sufficiently small* (i.e. (48) *slightly changed*).

$$X = H^{1,\gamma}(I, B_2^{s-2,\rho}(\mathbf{R}^n)) \cap L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n)) \tag{82}$$

$$Y = \{u \in X \,:\, u(0) = \phi\} \tag{83}$$

Then the integral equation (43) with (44), (45), (46) *has a unique local solution* $u \in Y$.

Here $I := [0, T]$ *and* $T = T(\|\phi\|_{H^{s,2}(\mathbf{R}^n)})$.

2. *If*

$$\gamma = \sigma + 2, \qquad \rho = \frac{2n(\sigma + 2)}{n(\sigma + 2) - 4} \tag{84}$$

and $\sigma \geq \frac{4}{n}$ *and* $\|\phi\|_{H^{s,2}(\mathbf{R}^n)}$ *is sufficiently small the integral equation* (43) *has a unique global solution* $u \in Y$ *with* Y, X *defined by* (82), (83) *and* $I = [0, +\infty)$.

Proof. Part 1. – We proceed as in Theorem 3.1. According to Prop. 2.3 we have to estimate

$$\|f(u)\|_{H^{1,\gamma'}(I, B_2^{s-2,\rho'}(\mathbf{R}^n))} \tag{85}$$

and

$$\|f(u)\|_{L^\gamma(I, B_2^{s-2,\rho}(\mathbf{R}^n))} \tag{86}$$

In order to treat (85) the most difficult term is $f'(u)u_t$. We use

$$\begin{aligned}
&\|f'(u)u_t\|_{B_2^{s-2,\rho'}(\mathbf{R}^n)} \\
&= \|f'(u)u_t\|_{L^{\rho'}(\mathbf{R}^n)} \\
&+ \sum_{|\alpha|=[s]-2} \left\{ \int_0^\infty \left(\tau^{-(s-[s])} \sup_{|h|\leq\tau} \|D^\alpha [f'(u(\cdot+h))u_t(\cdot+h) \right.\right. \\
&\qquad \left.\left. - f'(u(\cdot))u_t(\cdot)]\|_{L^{\rho'}(\mathbf{R}^n)} \right)^2 \frac{d\tau}{\tau} \right\}^{1/2} \qquad (87)
\end{aligned}$$

We have to treat the integral term by using the chain rule. Among the many terms which appear and can be treated in a similar manner the following one shows where the lower bound on σ comes in and also why the pair (γ, ρ) has to be changed slightly (which also excludes the limit case $\sigma = \frac{4}{n-2s}$):

$$\begin{aligned}
&\|[f^{([s]-1)}(u(\cdot+h)) - f^{([s]-1)}(u(\cdot))] D^{\alpha_1}u \cdots D^{\alpha_{[s]-2}}u\, u_t\|_{L^{\rho'}(\mathbf{R}^n)} \\
&\leq c\|u(\cdot+h) - u(\cdot)\|^{\sigma-([s]-2)}_{L^{p\rho'(\sigma-([s]-2))}} \\
&\quad \times \|D^{\alpha_1}u\|_{L^{p_1\rho'}} \cdots \|D^{\alpha_{[s]-2}}u\|_{L^{p_{[s]-2}\rho'}} \|u_t\|_{L^{p_0\rho'}} \qquad (88)
\end{aligned}$$

by Hölder's inequality with $\frac{1}{p} + \sum_{j=0}^{[s]-2} \frac{1}{p_j} = 1$, $|\alpha_j| = 1$ $(j = 1, \ldots, [s]-2)$. Here we assumed $s - 2 < \sigma < [s] - 1$, the case $\sigma \geq [s] - 1$ being easier because $f^{([s]-1)}$ being lipschitz in that case.

Returning to (87) we have the following bound from (88) of the corresponding integral term:

$$\begin{aligned}
&c\left\{ \int_0^\infty (\tau^{-\frac{s-[s]}{\sigma-([s]-2)}} \sup_{|h|\leq\tau} \|u(t+h) - u(t)\|_{L^{p\rho'(\sigma-([s]-2))}})^{2(\sigma-([s]-2))} \frac{d\tau}{\tau} \right\}^{1/2} \\
&\cdot \|D^{\alpha_1}u\|_{L^{p_1\rho'}} \cdots \|D^{\alpha_{[s]-2}}u\|_{L^{p_{[s]-2}\rho'}} \|u_t\|_{L^{p_0\rho'}} \qquad (89)
\end{aligned}$$

(Remark that $0 < \frac{s-[s]}{\sigma-([s]-2)} < 1$ if $s - 2 < \sigma < [s] - 1$!)

The integral here is bounded by $c\|u\|^{\sigma-([s]-2)}_{B^{\frac{s-[s]}{\sigma-([s]-2)}, p\rho'(\sigma-([s]-2))}_{2(\sigma-([s]-2))}}$. In order to get the desired bound for (89) by $c\|u\|^{\sigma}_{B_2^{s,\rho}(\mathbf{R}^n)} \|u_t\|_{B_2^{s-2,\rho}(\mathbf{R}^n)}$ the following embeddings are needed:

$$B_2^{s,\rho}(\mathbf{R}^n) \subset B^{\frac{s-[s]}{\sigma-([s]-2)}, p\rho'(\sigma-([s]-2))}_{2(\sigma-([s]-2))}(\mathbf{R}^n) \cap \bigcap_{j=1}^{[s]-2} H^{1,p_j\rho'}(\mathbf{R}^n) \qquad (90)$$

and

$$B_2^{s-2,\rho}(\mathbf{R}^n) \subset L^{p_0\rho'}(\mathbf{R}^n) \tag{91}$$

These are fulfilled if the following conditions hold ([13], Thm. 2.8.1 and Remark 1):

$$\frac{1}{\rho} \geq \frac{1}{p\rho'(\sigma - ([s]-2))} > \frac{1}{\rho} - \frac{s - \frac{s-[s]}{\sigma-([s]-2)}}{n} \quad \text{with } \textbf{strict} \text{ inequality}$$
$$\frac{1}{\rho} \geq \frac{1}{p_k\rho'} > \frac{1}{\rho} - \frac{s-1}{n} \qquad (k = 1, ..., [s]-2)$$
$$\frac{1}{\rho} \geq \frac{1}{p_0\rho'} > \frac{1}{\rho} - \frac{s-2}{n}$$

These conditions can be satisfied with suitably chosen p, p_k if

$$\begin{aligned} \frac{\sigma+1}{\rho} \geq \frac{1}{\rho'} > (\sigma - ([s]-2))\left(\frac{1}{\rho} - \frac{s - \frac{s-[s]}{\sigma-([s]-2)}}{n}\right) \\ + ([s]-2)\left(\frac{1}{\rho} - \frac{s-1}{n}\right) + \frac{1}{\rho} - \frac{s-2}{n} \end{aligned} \tag{92}$$

An elementary calculation shows that on the r.h.s. equality holds with (γ, ρ) defined by (48) and strict inequality with our choice (81). Because all the terms in (87) can be treated similarly (or easier) and lead to an equality equivalent to (92) we arrive at

$$\|f'(u)u_t\|_{B_2^{s-2,\rho'}(\mathbf{R}^n)} \leq c\|u\|^{\sigma}_{B_2^{s,\rho}(\mathbf{R}^n)}\|u_t\|_{B_2^{s-2,\rho}(\mathbf{R}^n)} \tag{93}$$

Thus Hölder's inequality with respect to t shows

$$\begin{aligned} &\|f'(u)u_t\|_{L^{\gamma'}(I,B_2^{s-2,\rho'}(\mathbf{R}^n))} \\ &\quad \leq c\|u\|^{\sigma}_{L^{(\sigma+1)\gamma'}(I,B_2^{s,\rho}(\mathbf{R}^n))}\|u_t\|_{L^{(\sigma+1)\gamma'}(I,B_2^{s-2,\rho}(\mathbf{R}^n))} \end{aligned} \tag{94}$$

With our choice (81) of γ one has $(\sigma+1)\gamma' < \gamma$ if $\sigma < \frac{4}{n-2s}$, *i.e.* (80), so that we have $\delta > 0$, s.th.

$$\|f'(u)u_t\|_{L^{\gamma'}(I,B_2^{s-2,\rho'}(\mathbf{R}^n))} \leq cT^{\delta}\|u\|_X^{\sigma+1} \tag{95}$$

A similar estimate holds for $\|f(u)\|_{L^{\gamma'}(I,B_2^{s-2,\rho'}(\mathbf{R}^n))}$.

It remains to consider (86). This is somehow easier because we have one derivative less compared to (85). We have

$$\|f(u)\|_{B_2^{s-2,\rho}(\mathbf{R}^n)} = \|f(u)\|_{L^\rho(\mathbf{R}^n)} + \sum_{|\alpha|=[s]-2} \left\{ \int_0^\infty \left(\tau^{-(s-[s])} \sup_{|h|\le\tau} \times \|D^\alpha[f(u(\cdot+h)) - f(u(\cdot))]\|_{L^\rho(\mathbf{R}^n)} \right)^2 \frac{d\tau}{\tau} \right\}^{1/2} \tag{96}$$

Because $f^{([s]-2)}$ is lipschitz in our case $\sigma > s-2$ all the terms appearing here after application of the chain rule can be treated essentially in the same manner.

A typical term with

$$\sum_{j=1}^{a} |\alpha_j| = [s]-2, \qquad a \le [s]-2 \tag{97}$$

is:

$$\begin{aligned} &\|f^{(a)}(u)D^{\alpha_1}u\cdots D^{\alpha_{a-1}}uD^{\alpha_a}(u(\cdot+h)-u(\cdot))\|_{L^\rho(\mathbf{R}^n)} \\ &\le c\|u\|^{\sigma-a+1}_{L^{\rho p(\sigma-a+1)}}\|D^{\alpha_1}u\|_{L^{\rho p_1}}\cdots\|D^{\alpha_{a-1}}u\|_{L^{\rho p_{a-1}}} \\ &\times \|D^{\alpha_a}(u(\cdot+h)-u(\cdot))\|_{L^{\rho p_a}} \end{aligned}$$

Here $\frac{1}{p} + \sum_{j=1}^{a} \frac{1}{p_j} = 1$.

Thus the corresponding term in (96) can be estimated by

$$\begin{aligned} &c\|u\|^{\sigma-a+1}_{L^{\rho p(\sigma-a+1)}}\|D^{\alpha_1}u\|_{L^{\rho p_1}}\cdots\|D^{\alpha_{a-1}}u\|_{L^{\rho p_{a-1}}}\|D^{\alpha_a}u\|_{B_2^{s-[s],\rho p_a}} \\ &\le c\|u\|^{\sigma+1}_{B_2^{s-2\Theta,\rho}} \end{aligned} \tag{98}$$

provided we have

$$B_2^{s-2\Theta,\rho}(\mathbf{R}^n) \subset L^{\rho p(\sigma-a+1)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a-1} H^{|\alpha_j|,\rho p_j}(\mathbf{R}^n) \cap B_2^{|\alpha_a|+s-[s],\rho p_a}(\mathbf{R}^n) \tag{99}$$

The embedding theorem ([13], Thm. 2.8.1) gives the sufficient conditions:

$$\begin{aligned} \frac{1}{\rho p(\sigma-a+1)} &> \frac{1}{\rho} - \frac{s-2\Theta}{n} \\ \frac{1}{\rho p_j} &> \frac{1}{\rho} - \frac{s-2\Theta+|\alpha_j|}{n} \quad (j=1,\ldots,a-1) \\ \frac{1}{\rho p_a} &> \frac{1}{\rho} - \frac{[s]-|\alpha_a|-2\Theta}{n} \end{aligned} \tag{100}$$

These can be satisfied with suitably chosen p, p_j if

$$\frac{1}{\rho} > (\sigma - a + 1)\left(\frac{1}{\rho} - \frac{s-2\Theta}{n}\right) + \sum_{j=1}^{a-1}\left(\frac{1}{\rho} - \frac{s-2\Theta+|\alpha_j|}{n}\right) + \frac{1}{\rho} - \frac{[s]-|\alpha_a|-2\Theta}{n} \tag{101}$$

Using (97) an elementary calculation shows that this is equivalent to

$$\Theta < \frac{2+\sigma(s-\frac{n}{\rho})}{2(\sigma+1)} \tag{102}$$

Because all the other terms in (96) can be estimated similarly and also lead to (102), we have if (102) is satisfied (with $\kappa > 0$):

$$\begin{aligned} \|f(u)\|_{L^\gamma(I,B_2^{s-2,\rho}(\mathbf{R}^n))} &\leq c\|u\|^{\sigma+1}_{L^{\gamma(\sigma+1)}(I,B_2^{s-2\Theta,\rho}\mathbf{R}^n))} \\ &\leq cT^\kappa\|u\|^{\sigma+1}_{B_2^{\Theta-\tilde{\delta},\gamma}(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n))} \\ &\leq cT^\kappa\|u\|^{\Theta(\sigma+1)}_{H^{1,\gamma}(I,B_2^{s-2,\rho}(\mathbf{R}^n))}\|u\|^{(1-\Theta)(\sigma+1)}_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} \\ &\leq cT^\kappa\|u\|^{\sigma+1}_X \end{aligned} \tag{103}$$

provided we have $B_2^{\Theta-\tilde{\delta},\gamma}(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n)) \subset L^{\gamma(\sigma+1)+\tilde{\delta}}(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n))$ ($\tilde{\delta}$ small) which requires

$$\frac{1}{\gamma(\sigma+1)} > \frac{1}{\gamma} - \Theta \tag{104}$$

In (103) we moreover interpolated by Lemma 4.1 below. Using the definition (81) of (γ,ρ) an elementary calculation again shows that (103), (104) are compatible with a suitably chosen $0 < \Theta < 1$ if $\sigma < \frac{4}{n-2s}$ which is our fundamental assumption (80) on σ.

The estimates for (85) and (86) just given imply by Prop. 2.3:

$$\|Gf(u)\|_X \leq c(T^\delta + T^\kappa)\|u\|^{\sigma+1}_X \tag{105}$$

The rest of the proof is exactly as in Theorem 3.1.

Part 2. – The global part is similar. One easily shows that (92) holds with our choice of (γ,ρ) if $\frac{4}{n} \leq \sigma < \frac{4}{n-2s}$. We get (95) with $\delta = 0$, because $(\sigma+1)\gamma' = \gamma$. Also (103) holds with $\kappa = 0$ if (104) is satisfied.

The conditions (102) and (104) are again compatible if $\sigma < \frac{4}{n-2s}$. We arrive at (105) with $\delta = \kappa = 0$. The proof is completed exactly as in Theorem 3.1.

In the next step we show *a posteriori* the desired regularity of u.

THEOREM 4.2. – *The solution u of Thm. 4.1 belongs to $C^0(I, H^{s,2}(\mathbf{R}^n)) \cap C^1(I, H^{s-2,2}(\mathbf{R}^n))$.*

Proof. Part 1. – Concerning the local solution by (22) it is sufficient to show the following estimate:

$$\|f(u)\|_{H^{1,\gamma'}(I,B_2^{s-2,\rho'}(\mathbf{R}^n))} + \|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))} \le cT^\lambda \|u\|_X^{\sigma+1} \quad (106)$$

with $\lambda \ge 0$ and (γ, ρ) as in (81).

The first term was already treated in the preceding proof.

Concerning the second term we use the well-known identity $H^{s-2,2} = B_2^{s-2,2}$ and estimate similarly as before (replacing ρ by 2 in (96)):

$$\begin{aligned} \|f(u)\|_{H^{s-2,2}(\mathbf{R}^n)} &= \|f(u)\|_{L^2(\mathbf{R}^n)} \\ &+ \sum_{|\alpha|=[s]-2} \left\{ \int_0^\infty \left(\tau^{-(s-[s])} \sup_{|h|\le\tau} \right.\right. \\ &\left.\left. \times \|D^\alpha[f(u(\cdot+h)) - f(u(\cdot))]\|_{L^2(\mathbf{R}^n)} \right)^2 \frac{d\tau}{\tau} \right\}^{1/2} \end{aligned} \quad (107)$$

In analogy to (99) and (100) we need here

$$B_2^{s-2\Theta,\rho}(\mathbf{R}^n) \subset L^{2p(\sigma-a+1)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a-1} H^{|\alpha_j|,2p_j}(\mathbf{R}^n) \cap B^{|\alpha_a|+s-[s],2p_a}(\mathbf{R}^n) \quad (108)$$

which is fulfilled if

$$\begin{aligned} \frac{1}{\rho} &> \frac{1}{2p(\sigma-a+1)} > \frac{1}{\rho} - \frac{s-2\Theta}{n} \\ \frac{1}{\rho} &> \frac{1}{2p_j} > \frac{1}{\rho} - \frac{s-2\Theta-|\alpha_j|}{n} \qquad (j = 1, ..., a-1) \\ \frac{1}{\rho} &> \frac{1}{2p_a} > \frac{1}{\rho} - \frac{[s]-|\alpha_a|-2\Theta}{n} \end{aligned}$$

The l.h.s. can be shown to be fulfilled and the r.h.s. leads to the following analogue of (102):

$$\Theta < \frac{2 + \sigma(s - \frac{n}{\rho}) + n(\frac{1}{2} - \frac{1}{\rho})}{2(\sigma + 1)} \tag{109}$$

If (109) is satisfied, we get (*cf.* (103)):

$$\begin{aligned} \|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))} &\le c\|u\|^{\sigma+1}_{L^\infty(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n))} \\ &\le c\|u\|^{\sigma+1}_{B_2^{\Theta-\epsilon,\gamma}(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n))} \\ &\le c\|u\|^{\Theta(\sigma+1)}_{H^{1,\gamma}(I,B_2^{s-2,\rho}(\mathbf{R}^n))}\|u\|^{(1-\Theta)(\sigma+1)}_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} \\ &\le c\|u\|^{\sigma+1}_X \end{aligned} \tag{110}$$

provided $B_2^{\Theta-\epsilon,\gamma}(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n)) \subset L^\infty(I,B_2^{s-2\Theta,\rho}(\mathbf{R}^n))$ which requires

$$\Theta > 1/\gamma \tag{111}$$

In (110) we also used Lemma 4.1 below.

Again it is elementary to check that (109) and (111) can be satisfied with $0 < \Theta < 1$ if $\sigma < \frac{4}{n-2}$. This completes the proof.

Part 2. – Concerning the global part we need (106) with $\lambda = 0$. The embedding (108) requires as before $\frac{2}{\rho}(\sigma + 1) > 1$ which is fulfilled if $\sigma \ge \frac{4}{n}$ and (109). The conditions (109) and (111) are again compatible if $\sigma < \frac{4}{n-2s}$, which completes the proof.

The following interpolation type lemma was used several times above.

LEMMA 4.1. – *If $s_1, s_2 \ge 0$, $0 < k \le 2$, $0 < \Theta < 1$, $\rho, \gamma \ge 1$ the following estimate holds (for $\epsilon > 0$ small):*

$$\begin{aligned} &\|u\|_{B_2^{k(\Theta-\epsilon),\gamma}(I,B_2^{\Theta s_1+(1-\Theta)s_2,\rho}(\mathbf{R}^n))} \\ &\quad \le c\,\|u\|^{\Theta}_{H^{k,\gamma}(I,B_2^{s_1,\rho}(\mathbf{R}^n))}\|u\|^{1-\Theta}_{L^\gamma(I,B_2^{s_2,\rho}(\mathbf{R}^n))} \end{aligned} \tag{112}$$

Here $H^{k,\gamma}$ can be replaced by $B_2^{k,\gamma}$ and also $(B_2^{s_1,\rho}, B_2^{s_2,\rho})$ by $(H^{s_1,\rho}, H^{s_2,\rho})$.

Proof. – Using $B_2^{\Theta s_1+(1-\Theta)s_2,\rho} = (B_2^{s_1,\rho}, B_2^{s_2,\rho})_{\Theta,2}$ and one of the equivalent definitions of Besov norms ([13], Thm. 4.4.2 (2)) we have

$$\begin{aligned}
&\|u\|_{B_2^{k(\Theta-\epsilon),\gamma}(I,B_2^{\Theta s_1+(1-\Theta)s_2,\rho}(\mathbf{R}^n))} \\
&\quad = \|u\|_{L^\gamma(I,B_2^{\Theta s_1+(1-\Theta)s_2,\rho}(\mathbf{R}^n))} \\
&\qquad + \left\{ \int_{-\infty}^{+\infty} (|\tau|^{-k(\Theta-\epsilon)} \|u(t+\tau) \right. \\
&\qquad\qquad \left. + u(t-\tau) - 2u(t)\|_{L^\gamma(I_\tau,B_2^{\Theta s_1+(1-\Theta)s_2,\rho}(\mathbf{R}^n))})^2 \frac{d\tau}{|\tau|} \right\}^{1/2} \\
&\quad \leq c\Bigg(\|u\|^\Theta_{L^\gamma(I,B_2^{s_1,\rho})} \|u\|^{1-\Theta}_{L^\gamma(I,B_2^{s_2,\rho})} \\
&\qquad + \left\{ \int_{-\infty}^{+\infty} (\tau^{-k(\Theta-\epsilon)} \|u(t+\tau) \right. \\
&\qquad\qquad \left. + u(t-\tau) - 2u(t)\|^\Theta_{L^\gamma(I_\tau,B_2^{s_1,\rho})} \|u\|^{1-\Theta}_{L^\gamma(I,B_2^{s_2,\rho})})^2 \frac{d\tau}{|\tau|} \right\}^{1/2} \Bigg) \\
&\quad \leq c(\|u\|^\Theta_{L^\gamma(I,B_2^{s_1,\rho})} \|u\|^{1-\Theta}_{L^\gamma(I,B_2^{s_2,\rho})} + \|u\|^\Theta_{B_{2\Theta}^{k-\frac{\epsilon k}{\Theta},\gamma}(I,B_2^{s_1,\rho})} \|u\|^{1-\Theta}_{L^\gamma(I,B_2^{s_2,\rho})})
\end{aligned}$$

The embedding $H^{k,\gamma} \subset B_{2\Theta}^{k-\frac{\epsilon k}{\Theta},\gamma}$ ([13], Thm. 2.3.2 and [1], Thm. 6.2.4) completes the proof.

Similarly using $B_2^{\Theta s_1+(1-\Theta)s_2,\rho} = (H^{s_1,\rho}, H^{s_2,\rho})_{\Theta,2}$ ([1], Thm. 6.2.4) we get the estimate with $(B_2^{s_1,\rho}, B_2^{s_2,\rho})$ replaced by $(H^{s_1,\rho}, H^{s_2,\rho})$.

THEOREM 4.3. – *Let $s \notin \mathbf{N}$, $s > 4$. Assume*

$$s - 3 < \sigma < \frac{4}{n-2s} \tag{113}$$

1. *Define (γ,ρ) as in* (81) *and*

$$X = H^{2,\gamma}(I, B_2^{s-4,\rho}(\mathbf{R}^n)) \cap H^{1,\gamma}(I, B_2^{s-2,\rho}(\mathbf{R}^n)) \cap L^\gamma(I, B_2^{s,\rho}(\mathbf{R}^n)) \tag{114}$$

$$Y = \{u \in X \,:\, u(0) = \phi\} \tag{115}$$

Then the integral equation (43) *with* (44), (45), (46) has a unique local solution $u \in Y$.

2. *If (γ,ρ) is defined by* (51), *$\sigma \geq \frac{4}{n}$ and $\|\phi\|_{H^{s,2}(\mathbf{R}^n)}$ sufficiently small the integral equation* (43) *has a unique global solution $u \in Y$ with Y, X defined by* (114), (115) *and $I = [0, +\infty)$.*

3. *If σ is an even integer* 1. *and* 2. *hold without the lower bounds on σ (remark that $\sigma \geq \frac{4}{n}$ is automatically fulfilled).*

Proof. – We proceed as in Theorem 3.1. According to (15) and Prop. 2.4 we have to estimate

$$\left\| \frac{d^2[f(u)]}{dt^2} \right\|_{L^{\gamma'}(I,B_2^{s-4,\rho'}(\mathbf{R}^n))} \tag{116}$$

$$\left\| \frac{d[f(u)]}{dt} \right\|_{L^{\gamma}(I,B_2^{s-4,\rho}(\mathbf{R}^n))} \tag{117}$$

$$\|f(u)\|_{L^{\gamma}(I,B_2^{s-2,\rho}(\mathbf{R}^n))} \tag{118}$$

$$\|f(\phi)\|_{H^{s-2,2}(\mathbf{R}^n)} \tag{119}$$

Since all the terms can be treated in essentially the same manner we concentrate on typical ones in each of the cases. Concerning (116) we consider the term

$$\left\| f^{([s]-2)}(u) D^{\alpha_1} u \cdots D^{\alpha_{[s]-4}} u \, u_t^2 \right\|_{L^{\gamma'}(I,B_2^{s-[s],\rho'}(\mathbf{R}^n))} \tag{120}$$

One here has to estimate e.g.

$$\begin{aligned} \Bigg\{ \int_0^{+\infty} \Big(\tau^{-(s-[s])} \sup_{|h|\leq\tau} \|[f^{([s]-2)}(u(\cdot+h)) - f^{([s]-2)}(u(\cdot))] \cdot \\ \cdot D^{\alpha_1} u \cdots D^{\alpha_{[s]-4}} u \, u_t^2 \|_{L^{\rho'}(\mathbf{R}^n)}\Big)^2 \frac{d\tau}{\tau} \Bigg\}^{1/2} \end{aligned} \tag{121}$$

where $\quad |\alpha_j| = 1 \quad (j = 1, ..., [s] - 2).$

If we have the most critical range $s - 3 < \sigma < [s] - 2$ the function $f^{([s]-2)}$ is no longer lipschitzcontinuous but only höldercontinuous unless σ is an even integer in which case we therefore need no lower bound on σ. Hölder's inequality with exponents p, p_j $(j = 1, .., [s] - 2)$ gives in this case

$$\begin{aligned} &\|[f^{([s]-2)}(u(\cdot+h)) - f^{([s]-2)}(u(\cdot))] D^{\alpha_1} u \cdots D^{\alpha_{[s]-4}} u \, u_t^2 \|_{L^{\rho'}(\mathbf{R}^n)} \\ &\leq c \, \|u(\cdot+h) - u(\cdot)\|^{\sigma-([s]-3)}_{L^{p\rho'(\sigma-([s]-3))}} \\ &\quad \times \|D^{\alpha_1} u\|_{L^{p_1\rho'}} \cdots \|D^{\alpha_{[s]-4}} u\|_{L^{p_{[s]-4}\rho'}} \\ &\quad \cdot \|u_t\|_{L^{p_{[s]-3}\rho'}} \|u_t\|_{L^{p_{[s]-2}\rho'}} \end{aligned} \tag{122}$$

Thus (121) is estimated by (similarly as before we remark $0 < \frac{s-[s]}{\sigma-([s]-3)} < 1$ if $s - 3 < \sigma < [s] - 2$):

$$\begin{aligned} &c\|u\|^{\sigma-([s]-3)}_{B_{2(\sigma-([s]-3))}^{\frac{s-[s]}{\sigma-([s]-3)}, p\rho'(\sigma-([s]-3))}(\mathbf{R}^n)} \|u\|_{H^{1,p_1\rho'}} \cdots \|u\|_{H^{1,p_{[s]-4}\rho'}} \\ &\times \|u_t\|_{L^{p_{[s]-3}\rho'}} \|u_t\|_{L^{p_{[s]-2}\rho'}} \end{aligned} \tag{123}$$

In order to get the desired bound by $c\|u\|^{\sigma-1}_{B_2^{s,\rho}(\mathbf{R}^n)}\|u_t\|^2_{B_2^{s-2,\rho}(\mathbf{R}^n)}$ we need the embeddings

$$B_2^{s,\rho}(\mathbf{R}^n) \subset B_{2(\sigma-([s]-3))}^{\frac{s-[s]}{\sigma-([s]-3)},p\rho'(\sigma-([s]-3))}(\mathbf{R}^n) \cap \bigcap_{j=1}^{[s]-4} H^{1,p_j\rho'}(\mathbf{R}^n) \tag{124}$$

and

$$B_2^{s-2,\rho}(\mathbf{R}^n) \subset L^{p_{[s]-3}\rho'}(\mathbf{R}^n) \cap L^{p_{[s]-2}\rho'}(\mathbf{R}^n) \tag{125}$$

These are fulfilled if ([13], Thm. 2.8.1):

$$\begin{aligned} \frac{1}{\rho} \geq \frac{1}{p\rho'(\sigma-([s]-3))} &> \frac{1}{\rho} - \frac{s - \frac{s-[s]}{\sigma-([s]-3)}}{n} \\ \frac{1}{\rho} \geq \frac{1}{p_j\rho'} &> \frac{1}{\rho} - \frac{s-1}{n} \qquad (j = 1, ..., [s]-4) \\ \frac{1}{\rho} \geq \frac{1}{p_j\rho'} &> \frac{1}{\rho} - \frac{s-2}{n} \qquad (j = [s]-3 \text{ or } [s]-2) \end{aligned}$$

These conditions are satisfied with suitably chosen p, p_j if

$$\begin{aligned} \frac{\sigma+1}{\rho} \geq \frac{1}{\rho'} > (\sigma-([s]-3))&\left(\frac{1}{\rho} - \frac{s-\frac{s-[s]}{\sigma-([s]-3)}}{n}\right) \\ &+ ([s]-2)\left(\frac{1}{\rho} - \frac{s-1}{n}\right) + \frac{2}{n} \end{aligned} \tag{126}$$

An elementary calculation again shows that this is true with our choice (81) or (51) of (γ, ρ) and our assumption $\sigma < \frac{4}{n-2}$ or $\frac{4}{n} \leq \sigma < \frac{4}{n-2}$ respectively. Similar estimates can be given for all the other terms in (116) so that we conclude with $\delta > 0$ in part 1. and $\delta = 0$ in part 2. of the theorem:

$$\begin{aligned} &\left\|\frac{d^2[f(u)]}{dt^2}\right\|_{L^{\gamma'}(I,B_2^{s-4,\rho'}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\sigma-1}_{L^{(\sigma+1)\gamma'}(I,B_2^{s,\rho}(\mathbf{R}^n))}\|u_t\|^2_{L^{(\sigma+1)\gamma'}(I,B_2^{s-2,\rho}(\mathbf{R}^n))} \\ &\leq cT^\delta\|u\|_X^{\sigma+1} \end{aligned} \tag{127}$$

because with our choice of γ and σ we have $(\sigma+1)\gamma' < \gamma$ and $(\sigma+1)\gamma' = \gamma$, respectively.

Next we estimate (117). Typically we have to treat

$$\|f^{(a+1)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}u\,u_t\|_{L^\gamma(I,B_2^{s-[s],\rho}(\mathbf{R}^n))} \tag{128}$$

where $\sum_{j=1}^a |\alpha_j| = [s]-4$, $a \leq [s]-4$.

This leads e.g. to (with Hölder exponents $p, p_j (j = 1, ..., a+1)$):

$$\Big\{\int_0^\infty (\tau^{-(s-[s])} \sup_{|h|\le\tau} \|f^{(a+1)}(u)$$
$$\times D^{\alpha_1}u \cdots D^{\alpha_a}u[u_t(\cdot+h) - u_t(\cdot)]\|_{L^\rho(\mathbf{R}^n)})^2 \frac{d\tau}{\tau}\Big\}^{1/2}$$
$$\le c\|u\|^{\sigma-a}_{L^{\rho p(\sigma-a)}} \|D^{\alpha_1}u\|_{L^{\rho p_1}} \cdots \|D^{\alpha_a}u\|_{L^{\rho p_a}} \|u_t\|_{B_2^{s-[s],\rho p_{a+1}}}$$
$$\le c\|u\|^{\sigma}_{B_2^{s-4\Theta,\rho}(\mathbf{R}^n)} \|u_t\|_{B^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n)} \tag{129}$$

(remark that this term disappears if σ is an even integer and $a > \sigma$) provided the following embeddings hold

$$B_2^{s-4\Theta,\rho}(\mathbf{R}^n) \subset L^{\rho p(\sigma-a)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a} H^{|\alpha_j|,\rho p_j}(\mathbf{R}^n) \tag{130}$$

and

$$B_2^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n) \subset B_2^{s-[s],\rho p_{a+1}}(\mathbf{R}^n) \tag{131}$$

These are fulfilled if ([13], Thm. 2.8.1):

$$\frac{1}{\rho} \ge \frac{1}{\rho p(\sigma-a)} > \frac{1}{\rho} - \frac{s-4\Theta}{n}$$
$$\frac{1}{\rho} \ge \frac{1}{\rho p_j} > \frac{1}{\rho} - \frac{s-4\Theta-|\alpha_j|}{n} \quad (j = 1, ..., a)$$
$$\frac{1}{\rho} \ge \frac{1}{\rho p_{a+1}} > \frac{1}{\rho} - \frac{[s]-2-2\tilde\Theta}{n}$$

A simple calculation shows that this requires

$$\sigma(s-4\Theta) + 2 - 2\tilde\Theta > \frac{n}{\rho}\sigma \tag{132}$$

If (132) is satisfied we conclude from (129) with $\frac{1}{\hat p} + \frac{1}{\hat q} = 1$:

$$\|f(u)\|_{L^\gamma(I,B_2^{s-[s],\rho}(\mathbf{R}^n))}$$
$$\le c\|u\|^{\sigma}_{L^{\gamma\sigma\hat p}(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))} \|u_t\|_{L^{\gamma\hat q}(I,B_2^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n))}$$
$$\le cT^\kappa \|u\|^{\sigma}_{B_2^{2(\Theta-\epsilon),\gamma}(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))} \|u_t\|_{B_2^{\tilde\Theta-\epsilon,\gamma}(I,B_2^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n))}$$
$$\le cT^\kappa \|u\|^{(1-\Theta)\sigma}_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))} \|u\|^{\Theta\sigma}_{H^{2,\gamma}(I,B_2^{s-4,\rho}(\mathbf{R}^n))} \|u_t\|^{1-\tilde\Theta}_{L^\gamma(I,B_2^{s-2,\rho}(\mathbf{R}^n))} \cdot$$
$$\cdot \|u_t\|^{\tilde\Theta}_{H^{1,\gamma}(I,B_2^{s-4,\rho}(\mathbf{R}^n))} \tag{133}$$

This holds with $\kappa = 0$ and $\kappa > 0$ small as well. Here we used Lemma 4.1 and the embeddings $B_2^{2(\Theta-\epsilon),\gamma}(I) \subset L^{\gamma\sigma\hat{p}+\tilde{\delta}}(I)$ ($\tilde{\delta} = 0$ or $\tilde{\delta} > 0$ small) and $B_2^{\tilde{\Theta}-\epsilon,\gamma}(I) \subset L^{\gamma\hat{q}}(I)$ which hold provided $\frac{1}{\gamma\sigma\hat{p}} > \frac{1}{\gamma} - 2\Theta$ and $\frac{1}{\gamma\hat{q}} > \frac{1}{\gamma} - \tilde{\Theta}$. These are satisfied with suitable $\hat{p}, \hat{q}$ if

$$2\sigma\Theta + \tilde{\Theta} > \frac{\sigma}{\gamma} \tag{134}$$

Now (132) and (134) are compatible with $0 < \Theta, \tilde{\Theta} < 1$ if $\frac{2\sigma}{\gamma} < 2+\sigma(s-\frac{n}{\rho})$ which can easily be seen to be satisfied with our choice (81) or (51) of (γ, ρ) and $\sigma < \frac{4}{n-2s}$.

Thus (117) is also estimated by $cT^{\kappa}\|u\|_X^{\sigma+1}$ with $\kappa = 0$ and $\kappa > 0$ small as well.

The term (118) was already treated in (103).

Finally in order to estimate (119) we use $H^{s-2,2} = B_2^{s-2,2}$. A typical term in the most critical case $s - 3 < \sigma < [s] - 2$ is:

$$\begin{aligned}
&\left\{\int_0^\infty \left(\tau^{-(s-[s])} \sup_{|h|\leq\tau} \| \, [f^{([s]-2)}(\phi(\cdot + h)) - f^{([s]-2)}(\phi(\cdot))] \cdot \right.\right.\\
&\qquad \left.\left. \cdot D^{\alpha_1}\phi \cdots D^{\alpha_{[s]-2}}\phi \right\|_{L^2(\mathbf{R}^n)}\right)^2 \frac{d\tau}{\tau}\Bigg\}^{1/2}\\
&\leq c\left\{\int_0^\infty (\tau^{-(s-[s])} \sup_{|h|\leq\tau} \|\phi(\cdot + h) - \phi(\cdot)\|^{\sigma-([s]-3)}_{L^{2p(\sigma-([s]-3))}})^2 \frac{d\tau}{\tau}\right\}^{1/2}\\
&\quad \times \prod_{j=1}^{[s]-2} \|D^{\alpha_j}\phi\|_{L^{2p_j}}\\
&\leq c\|\phi\|^{\sigma+1}_{H^{s,2}(\mathbf{R}^n)}
\end{aligned} \tag{135}$$

Here $\frac{1}{p} + \sum_{j=1}^{[s]-2} \frac{1}{p_j} = 1$ and the following embedding is used:

$$H^{s,2}(\mathbf{R}^n) \subset B_{2(\sigma-([s]-3))}^{\frac{s-[s]}{\sigma-([s]-3)},2p(\sigma-([s]-3))}(\mathbf{R}^n) \cap \bigcap_{j=1}^{[s]-2} H^{1,2p_j}(\mathbf{R}^n) \tag{136}$$

which is fulfilled if

$$\begin{aligned}
\frac{1}{2p(\sigma - ([s] - 3))} &> \frac{1}{2} - \frac{s - \frac{s-[s]}{\sigma-([s]-3)}}{n}\\
\frac{1}{2p_j} &> \frac{1}{2} - \frac{s-1}{n} \qquad (j = 1, ..., [s] - 2)
\end{aligned}$$

An easy calculation shows that with suitably chosen p, p_j these conditions hold if $\sigma < \frac{4}{n-2s}$.

The rest of the proof is exactly as before.

THEOREM 4.4. – *The solution u of Theorem 4.3 belongs to* $C^0(I, H^{s,2}(\mathbf{R}^n)) \cap C^1(I, H^{s-2,2}(\mathbf{R}^n)) \cap C^2(I, H^{s-4,2}(\mathbf{R}^n))$.

Proof. – According to (15) and Prop. 2.4 we have to estimate

$$\|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))} \tag{137}$$

$$\|f(u)\|_{H^{1,\infty}(I,H^{s-4,2}(\mathbf{R}^n))} \tag{138}$$

and

$$\|f(u)\|_{H^{2,\gamma'}(I,B_2^{s-4,\rho'}(\mathbf{R}^n))} \tag{139}$$

by $cT^\delta \|u\|_X^{\sigma+1}$ ($\delta \geq 0$ in the local case and $\delta = 0$ in the global case) with γ, ρ, X as in Theorem 4.3, and

$$\|f(\phi)\|_{H^{s-2,2}(\mathbf{R}^n)} \leq c\|\phi\|^{\sigma+1}_{H^{s,2}(\mathbf{R}^n)} \tag{140}$$

(139) and (140) are already treated in Theorem 4.3.

Concerning (137) we use $H^{s-2,2} = B_2^{s-2,2}$ and typically estimate (in the most critical case $s-3 < \sigma < [s]-2$) with $|\alpha_j| = 1\ (j = 1, ..., [s]-2)$ and Hölder exponents p, p_j:

$$\begin{aligned}
&\left\{ \int_0^\infty (\tau^{-(s-[s])} \sup_{|h|\leq\tau} \|[f^{([s]-2)}(u(\cdot+h)) - f^{([s]-2)}(u(\cdot))]\cdot \right.\\
&\qquad \left. \cdot D^{\alpha_1}u \cdots D^{\alpha_{[s]-2}}u\|_{L^2(\mathbf{R}^n)})^2 \frac{d\tau}{\tau} \right\}^{1/2}\\
&\leq c \left\{ \int_0^\infty \left(\tau^{-(s-[s])} \sup_{|h|\leq\tau} \|u(\cdot+h) - u(\cdot)\|^{\sigma-[s]+3}_{L^{2p(\sigma-[s]+3)}} \right)^2 \frac{d\tau}{\tau} \right\}^{1/2}\\
&\qquad \times \prod_{j=1}^{[s]-2} \|D^{\alpha_j}u\|_{L^{2p_j}}\\
&\leq \|u\|^{\sigma+1}_{B^{s-4\Theta,\rho}(\mathbf{R}^n)}
\end{aligned} \tag{141}$$

if the following embedding holds

$$B_2^{s-4\Theta,\rho}(\mathbf{R}^n) \subset B_{2(\sigma-[s]+3)}^{\frac{s-[s]}{\sigma-[s]+3}, 2\hat{p}(\sigma-[s]+3)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{[s]-2} H^{1,2p_j}(\mathbf{R}^n) \tag{142}$$

This is true if

$$\frac{1}{\rho} > \frac{1}{2p(\sigma - [s] + 3)} > \frac{1}{\rho} - \frac{s - 4\Theta - \frac{s-[s]}{\sigma-[s]+3}}{n}$$

$$\frac{1}{\rho} > \frac{1}{2p_j} > \frac{1}{\rho} - \frac{s - 4\Theta - 1}{n} \qquad (j = 1, ..., [s] - 2)$$

One easily checks that these inequalities hold with suitable p, p_j if

$$\sigma(s - 4\Theta) + 2 - 4\Theta > n\left(\frac{1}{\rho}(\sigma + 1) - \frac{1}{2}\right) \tag{143}$$

Moreover we need

$$\frac{\sigma + 1}{\rho} > \frac{1}{2} \tag{144}$$

which holds with ρ defined by (81) and also if ρ is given by (51) if $\sigma \geq \frac{4}{n}$.

If (143) is satisfied we conclude from (141)

$$\begin{aligned}\|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))} &\leq c\|u\|^{\sigma+1}_{L^\infty(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\sigma+1}_{B_2^{2(\Theta-\epsilon),\gamma}(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\Theta(\sigma+1)}_{H^{2,\gamma}(I,B_2^{s-4,\rho}(\mathbf{R}^n))}\|u\|^{(1-\Theta)(\sigma+1)}_{L^\gamma(I,B_2^{s,\rho}(\mathbf{R}^n))}\end{aligned}$$

by use of Lemma 4.1 provided we have $B_2^{2(\Theta-\epsilon),\gamma}(I) \subset L^\infty(I)$ which is true if we have

$$2\Theta > \frac{1}{\gamma} \tag{145}$$

With both of our choices of (γ, ρ) and σ (143) and (145) are compatible again.

The term (138) is treated like (117) with (γ, ρ) replaced by $(\infty, 2)$. Like there we get under the condition

$$\frac{1}{2} > \frac{1}{\rho}(\sigma + 1) - \frac{1}{n}[\sigma(s - 4\Theta) + 2 - 2\tilde{\Theta}] \tag{146}$$

the estimate

$$\begin{aligned}&\|f'(u)u_t\|_{L^\infty(I,H^{s-4,2}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\sigma}_{L^\infty(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))}\|u_t\|_{L^\infty(I,B_2^{s-2-2\tilde{\Theta},\rho}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\sigma}_{B_2^{2(\Theta-\epsilon),\gamma}(I,B_2^{s-4\Theta,\rho}(\mathbf{R}^n))}\|u_t\|_{B_2^{\tilde{\Theta}-\epsilon,\gamma}(I,B_2^{s-2-2\tilde{\Theta},\rho}(\mathbf{R}^n))} \\ &\leq c\|u\|^{\sigma+1}_X\end{aligned} \tag{147}$$

Here we used Lemma 4.1 and the embeddings $B_2^{2(\Theta-\epsilon),\gamma}(I) \subset L^\infty(I)$ and $B_2^{\tilde{\Theta}-\epsilon,\gamma}(I) \subset L^\infty(I)$ which hold if

$$2\Theta > 1/\gamma \qquad \text{and} \qquad \tilde{\Theta} > 1/\gamma \tag{148}$$

An elementary calculation again shows the compatibility of (146) and (148) with both of our choices of (γ, ρ) if $\sigma < \frac{4}{n-2s}$. Thus the desired estimate of (148) follows. This completes the proof.

It remains to consider the case of integer s. The proofs are somehow simpler because essentially we can use Sobolev instead of Besov spaces.

THEOREM 4.5. – *Let* $s \in \mathbf{N}$, $s \geq 2$. *Assume*

$$s - 2 < \sigma < \frac{4}{n-2s} \tag{149}$$

1. *If*

$$\gamma = \frac{4(\sigma+2)}{\sigma(n-2s)}, \qquad \rho = \frac{\sigma+2}{1+\sigma s/n} \tag{150}$$

$$X = H^{1,\gamma}(I, H^{s-2,\rho}(\mathbf{R}^n)) \cap L^\gamma(I, H^{s,\rho}(\mathbf{R}^n)) \tag{151}$$

$$Y = \{u \in X \,:\, u(0) = \phi\} \tag{152}$$

Then the integral equation (43) *with* (44), (45), (46) *has a unique local solution* $u \in Y$, $I = [0, T]$.

2. *If*

$$\gamma = \sigma + 2, \qquad \rho = \frac{2n(\sigma+2)}{n(\sigma+2)-4} \tag{153}$$

$\sigma \geq \frac{4}{n}$ *and* $||\phi||_{H^{s,2}(\mathbf{R}^n)}$ *sufficiently small the integral equation* (43) *has a unique global solution* $u \in Y$, *where* Y, X *are defined by* (151) *and* (152), $T = [0, +\infty)$.

3. *If* σ *is an even integer* 1. *and* 2. *hold without the lower bounds on* σ.

Proof. – We proceed as in Theorem 4.1 and according to Prop. 2.3 we have to estimate

$$||f(u)||_{H^{1,\gamma'}(I,H^{s-2,\rho'}(\mathbf{R}^n))} \tag{154}$$

and

$$||f(u)||_{L^\gamma(I,H^{s-2,\rho}(\mathbf{R}^n))} \tag{155}$$

Concerning (154) a typical term with $0 \leq a \leq s-2$, $\sum_{j=1}^{a} |\alpha_j| = a$, $\sum_{j=0}^{a} |\alpha_j| = s-2$ is estimated with Hölder exponents p, p_j $(j = 0, ..., a)$ as follows:

$$\begin{aligned} &\|f^{(a+1)}(u) D^{\alpha_1} u \cdots D^{\alpha_a} u D^{\alpha_0} u_t\|_{L^{\rho'}(\mathbf{R}^n)} \\ &\leq c \|u\|^{\sigma-a}_{L^{p\rho'(\sigma-a)}} \|D^{\alpha_1} u\|_{L^{p_1\rho'}} \cdots \|D^{\alpha_a} u\|_{L^{p_a\rho'}} \|D^{\alpha_0} u_t\|_{L^{p_0\rho'}} \\ &\leq c \|u\|^{\sigma}_{H^{s,\rho}(\mathbf{R}^n)} \|u_t\|_{H^{s-2,\rho}(\mathbf{R}^n)} \end{aligned}$$

(remark that this term disappears if $\sigma < a$ is an even integer) where the embeddings $H^{s,\rho}(\mathbf{R}^n) \subset L^{p\rho'(\sigma-a)}(\mathbf{R}^n) \cap \bigcap_{k=1}^{a} H^{|\alpha_k|, p_k\rho'}(\mathbf{R}^n)$ and $H^{s-2,\rho}(\mathbf{R}^n) \subset H^{|\alpha_0|, p_0\rho'}(\mathbf{R}^n)$ were used which hold provided

$$\begin{aligned} &\frac{1}{\rho} \geq \frac{1}{p\rho'(\sigma-a)} \geq \frac{1}{\rho} - \frac{s}{n} \\ &\frac{1}{\rho} \geq \frac{1}{p_j\rho'} \geq \frac{1}{\rho} - \frac{s-|\alpha_j|}{n} \quad (j = 1, ..., a) \\ &\frac{1}{\rho} \geq \frac{1}{p_0\rho'} \geq \frac{1}{\rho} - \frac{s-2-|\alpha_0|}{n} \end{aligned}$$

These conditions are satisfied with suitable p, p_j if

$$\frac{\sigma+1}{\rho} \geq \frac{1}{\rho'} \geq (\sigma-a)\left(\frac{1}{\rho} - \frac{s}{n}\right) + a\left(\frac{1}{\rho} - \frac{s}{n}\right) + \frac{1}{\rho} - \frac{s-2}{n} + \frac{1}{n}\sum_{j=0}^{a} |\alpha_j|$$

which is easily seen to be fulfilled under the definition (150) (with equality on the r.h.s.) and also with ρ defined by (51) if $\frac{4}{n} \leq \sigma \leq \frac{4}{n-2s}$.

Concerning (155) typically we have to estimate with a, α_j as before:

$$\begin{aligned} &\|f^{(a)}(u) D^{\alpha_1} u \cdots D^{\alpha_a} u\|_{L^{\rho}(\mathbf{R}^n)} \\ &\leq c \|u\|^{\sigma-a+1}_{L^{p\rho(\sigma-a+1)}} \|D^{\alpha_1} u\|_{L^{p_1\rho}} \cdots \|D^{\alpha_a} u\|_{L^{p_a\rho}} \\ &\leq c \|u\|^{\sigma+1}_{H^{s-2\Theta,\rho}(\mathbf{R}^n)} \end{aligned}$$

The needed embedding

$$H^{s-2\Theta,\rho}(\mathbf{R}^n) \subset L^{p\rho(\sigma-a+1)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a} H^{|\alpha_j|, \rho p_j}(\mathbf{R}^n)$$

holds if

$$\begin{aligned} &\frac{1}{p\rho(\sigma-a+1)} \geq \frac{1}{\rho} - \frac{s-2\Theta}{n} \\ &\frac{1}{p_j\rho} \geq \frac{1}{\rho} - \frac{s-2\Theta-|\alpha_j|}{n} \quad (j = 1, ..., a) \end{aligned}$$

which is easily seen to be satisfied if

$$\sigma(s-2\Theta)+2-2\Theta \geq n\sigma/\rho \tag{156}$$

Under this assumption we conclude by Lemma 4.1 (with $\kappa > 0$ and also with $\kappa = 0$):

$$\begin{aligned}&\|f(u)\|_{L^\gamma(I,H^{s-2,\rho}(\mathbf{R}^n))}\\ &\leq c\|u\|^{\sigma+1}_{L^{\gamma(\sigma+1)}(I,H^{s-2\Theta,\rho}(\mathbf{R}^n))}\\ &\leq cT^\kappa\|u\|^{\Theta(\sigma+1)}_{H^{1,\gamma}(I,H^{s-2,\rho}(\mathbf{R}^n))}\|u\|^{(1-\Theta)(\sigma+1)}_{L^\gamma(I,H^{s,\rho}(\mathbf{R}^n))} \leq cT^\kappa\|u\|^{\sigma+1}_X\end{aligned}$$

provided we have $H^{\Theta,\gamma}(I) \subset L^{\gamma(\sigma+1)+\epsilon}(I)$, *i.e.*

$$\frac{1}{\gamma(\sigma+1)} > \frac{1}{\gamma} - \Theta \tag{157}$$

As before (*see* (103) and (104)) (156), (157) are compatible under our assumptions. The rest of the proof is the same as in Theorem 4.1.

THEOREM 4.6. – *The solution of Theorem* 4.5 *belongs to* $C^0(I, H^{s,2}(\mathbf{R}^n)) \cap C^1(I, H^{s-2,2}(\mathbf{R}^n))$.

Proof. – Similarly as in the case of fractional s we have to estimate only $\|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))}$. The typical term $\|f^{(a)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}u\|_{L^2(\mathbf{R}^n)}$ with $0 \leq a \leq s-2$, $\sum_{j=1}^a |\alpha_j| = s-2$ is estimated with Hölder exponents p, p_j by $c\|u\|^{\sigma-a+1}_{L^{2p(\sigma-a+1)}}\|D^{\alpha_1}u\|_{L^{2p_1}}\cdots\|D^{\alpha_a}u\|_{L^{2p_a}} \leq c\|u\|^{\sigma+1}_{H^{s-2\Theta,\rho}(\mathbf{R}^n)}$ if $H^{s-2\Theta,\rho}(\mathbf{R}^n) \subset L^{2p(\sigma-a+1)}(\mathbf{R}^n) \cap \bigcap_{j=1}^a H^{|\alpha_j|,2p_j}(\mathbf{R}^n)$ which as in Theorem 4.2 leads to condition (109).

The rest of the proof proceeds exactly as there (replacing Besov by Sobolev spaces).

As before in the case of fractional s we can improve the lower bound on σ if $s \geq 4$.

THEOREM 4.7. – *Let* $s \in \mathbf{N}$, $s \geq 4$, *and assume*

$$s-3 < \sigma < \frac{4}{n-2s} \tag{158}$$

instead of (149).

Then the conclusion of Theorem 4.5 *is valid with*

$$X = H^{2,\gamma}(I, H^{s-4,\rho}(\mathbf{R}^n)) \cap H^{1,\gamma}(I, H^{s-2,\rho}(\mathbf{R}^n)) \cap L^\gamma(I, H^{s,\rho}(\mathbf{R}^n)).$$

Proof. – Similarly as in Theorem 4.3 we have to estimate (116)-(119) with Sobolev instead of Besov spaces. A typical term for (116) is

$\|f^{(a+2)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}uu_t^2\|_{L^{\rho'}(\mathbf{R}^n)}$ where $0\le a\le s-4$, $\sum_{j=1}^a|\alpha_j|=a$. Hölder's inequality with exponents p,p_j $(j=1,\dots,a+2)$ gives the estimate

$$\begin{aligned}&c\|u\|^{\sigma-a-1}_{L^{p\rho'(\sigma-a-1)}}\|D^{\alpha_1}u\|_{L^{p_1\rho'}}\cdots\|D^{\alpha_a}u\|_{L^{p_a\rho'}}\|u_t\|_{L^{p_{a+1}\rho'}}\|u_t\|_{L^{p_{a+2}\rho'}}\\&\le c\|u\|^{\sigma+1}_{H^{s,\rho}(\mathbf{R}^n)}\|u_t\|^2_{H^{s-2,\rho}(\mathbf{R}^n)}\end{aligned}$$

if the embeddings $H^{s,\rho}(\mathbf{R}^n)\subset L^{p\rho'(\sigma-a-1)}(\mathbf{R}^n)\cap\bigcap_{j=1}^a H^{|\alpha_j|,p_j\rho'}(\mathbf{R}^n)$ and $H^{s-2,\rho}(\mathbf{R}^n)\subset L^{p_{a+1}\rho'}(\mathbf{R}^n)\cap L^{p_{a+2}\rho'}(\mathbf{R}^n)$ hold. The embedding conditions can be shown to be fulfilled again under our assumptions on ρ. This gives the desired bound on (116) as there (*cf.* (127)).

The analogue of (117) with the typical term

$$\|f^{(a+1)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}u\,u_t\|_{L^\rho(\mathbf{R}^n)},$$

where $0\le a\le s-4$, $\sum_{j=1}^a|\alpha_j|=a$ is estimated by

$$\begin{aligned}&c\|u\|^{\sigma-a}_{L^{p\rho(\sigma-a)}}\|D^{\alpha_1}u\|_{L^{p_1\rho}}\cdots\|D^{\alpha_a}u\|_{L^{p_a\rho}}\|u_t\|_{L^{p_0\rho}}\\&\le c\|u\|^{\sigma}_{H^{s-4\Theta,\rho}(\mathbf{R}^n)}\|u_t\|_{H^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n)}\end{aligned}$$

if the embeddings $H^{s-4\Theta,\rho}(\mathbf{R}^n)\subset L^{p\rho(\sigma-a)}(\mathbf{R}^n)\cap\bigcap_{j=1}^a H^{|\alpha_j|,p_j\rho}(\mathbf{R}^n)$ and $H^{s-2-2\tilde\Theta,\rho}(\mathbf{R}^n)\subset L^{p_0\rho}(\mathbf{R}^n)$ hold.

The conditions to be fulfilled here are again given by (132) and (134) so that similarly as in the proof of Theorem 4.3 we get an estimate of (117) by $cT^\kappa\|u\|_X^{\sigma+1}$ with $\kappa>0$ and $\kappa=0$ as well.

Concerning (118) the typical term $\|f^{(a)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}u\|_{L^\rho(\mathbf{R}^n)}$ with $0\le a\le s-2$, $\sum_{j=1}^a|\alpha_j|=a$ can be estimated by $c\|u\|^{\sigma-a+1}_{L^{\rho p(\sigma-a+1)}}\|D^{\alpha_1}u\|_{L^{\rho p_1}}\cdots\|D^{\alpha_a}u\|_{L^{\rho p_a}}\le c\|u\|^{\sigma+1}_{H^{s-2\Theta,\rho}(\mathbf{R}^n)}$ under the condition $H^{s-2\Theta,\rho}(\mathbf{R}^n)\subset L^{\rho p(\sigma-a+1)}(\mathbf{R}^n)\cap\bigcap_{j=1}^a H^{|\alpha_j|,\rho p_j}(\mathbf{R}^n)$ which again leads to (102) so that (103) with $\kappa>0$ and $\kappa=0$ as well, *i.e.* the estimate for (118), follows as there.

Finally (119) and its typical term $\|f^{(a)}(\phi)D^{\alpha_1}\phi\cdots D^{\alpha_a}\phi\|_{L^2(\mathbf{R}^n)}$ with $a\le s-2$, $\sum_{j=1}^a|\alpha_j|=a$ can be estimated by $c\|\phi\|^{\sigma-a+1}_{L^{2p(\sigma-a+1)}}\|D^{\alpha_1}u\|_{L^{2p_1}}\cdots\|D^{\alpha_a}u\|_{L^{2p_a}}\le c\|\phi\|^{\sigma+1}_{H^{s,2}(\mathbf{R}^n)}$ if $H^{s,2}(\mathbf{R}^n)\subset L^{2p(\sigma-a+1)}(\mathbf{R}^n)\cap\bigcap_{j=1}^a H^{|\alpha_j|,2p_j}(\mathbf{R}^n)$ which can be shown to be fulfilled if $\sigma\le\frac{4}{n-2s}$. The rest of the proof proceeds as before.

THEOREM 4.8. – *The solution of Theorem* 4.7 *belongs to*

$$C^0(I,H^{s,2}(\mathbf{R}^n))\cap C^1(I,H^{s-2,2}(\mathbf{R}^n))\cap C^2(I,H^{s-4,2}(\mathbf{R}^n)).$$

Proof. – Similarly as in Theorem 4.4 it remains to estimate (137) and (138). In (137) a typical term is $\|f^{(a)}(u)D^{\alpha_1}u\cdots D^{\alpha_a}u\|_{L^2(\mathbf{R}^n)}$

with $0 \leq a \leq s-2$, $\sum_{j=1}^{a} |\alpha_j| = a$ which is estimated by $c\|u\|_{L^{2p(\sigma-a+1)}}^{\sigma-a+1} \|D^{\alpha_1}u\|_{L^{2p_1}} \cdots \|D^{\alpha_a}u\|_{L^{2p_a}} \leq c\|u\|_{H^{s-4\Theta,\rho}(\mathbf{R}^n)}^{\sigma+1}$ if the embedding $H^{s-4\Theta,\rho}(\mathbf{R}^n) \subset L^{2p(\sigma-a+1)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a} H^{|\alpha_j|,2p_j}$ holds which requires $\frac{1}{2p(\sigma-a+1)} \geq \frac{1}{\rho} - \frac{s-4\Theta}{n}$ and $\frac{1}{2p_j} \geq \frac{1}{\rho} - \frac{s-4\Theta-|\alpha_j|}{n}$ thus $\frac{\sigma+1}{\rho} \geq \frac{1}{2} \geq (\sigma+1)(\frac{1}{\rho} - \frac{s-4\Theta}{n}) + \frac{a}{n}$ which leads to (143) and (144) so that as in the proof of Theorem 4.4 the estimate $\|f(u)\|_{L^\infty(I,H^{s-2,2}(\mathbf{R}^n))} \leq c\|u\|_X^{\sigma+1}$ follows. Finally (138) has a typical term with $0 \leq a \leq s-4$, $\sum_{j=1}^{a} |\alpha_j| = a$:

$$\begin{aligned}
&\|f^{(a+1)}(u)D^{\alpha_1}u \cdots D^{\alpha_a}u\, u_t\|_{L^2(\mathbf{R}^n)} \\
&\leq c\|u\|_{L^{2p(\sigma-a)}}^{\sigma-a} \|D^{\alpha_1}u\|_{L^{2p_1}} \cdots \|D^{\alpha_a}u\|_{L^{2p_a}} \|u_t\|_{L^{2p_0}} \\
&\leq c\|u\|_{H^{s-4\Theta,\rho}(\mathbf{R}^n)}^{\sigma} \|u_t\|_{H^{s-2-2\tilde{\Theta},\rho}(\mathbf{R}^n)}
\end{aligned}$$

This estimate holds if $H^{s-4\Theta,\rho}(\mathbf{R}^n) \subset L^{2p(\sigma-a)}(\mathbf{R}^n) \cap \bigcap_{j=1}^{a} H^{|\alpha_j|,2p_j}(\mathbf{R}^n)$ and $H^{s-2-2\tilde{\Theta},\rho}(\mathbf{R}^n) \subset L^{2p_0}(\mathbf{R}^n)$ which holds if

$$\begin{aligned}
&\frac{1}{\rho} \geq \frac{1}{2p(\sigma-a)} \geq \frac{1}{\rho} - \frac{s-4\Theta}{n} \\
&\frac{1}{\rho} \geq \frac{1}{2p_j} \geq \frac{1}{\rho} - \frac{s-4\Theta-|\alpha_j|}{n} \qquad (j = 1, \ldots, a) \\
&\frac{1}{\rho} \geq \frac{1}{2p_0} \geq \frac{1}{\rho} - \frac{s-2-2\tilde{\Theta}}{n}
\end{aligned}$$

thus

$$\begin{aligned}
\frac{\sigma+1}{\rho} \geq \frac{1}{2} &\geq (\sigma-a)\left(\frac{1}{\rho} - \frac{s-4\Theta}{n}\right) \\
&\quad + a\left(\frac{1}{\rho} - \frac{s-4\Theta}{n}\right) + \frac{a}{n} + \frac{1}{\rho} - \frac{s-2-2\tilde{\Theta}}{n}
\end{aligned}$$

which leads to (144), (146) and (147). This completes the proof.

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