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L. ACCARDI
V. MASTROPIETRO

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Stochastic bosonization for an interacting $d \geq 3$ Fermi system

by

L. ACCARDI and V. MASTROPIETRO

Centro V. Volterra e Dipartimento di Matematica,
Università di Roma Tor Vergata
via di Tor Vergata, 00133 – Roma
e-Mail: volterra@mat.utovrm.it

ABSTRACT. – The stochastic bosonization technique developed in a previous paper is applied to a self-interacting Fermi system. We prove that the evolution operator satisfies, in a proper limit, a quantum stochastic differential equation.

RÉSUMÉ. – On applique à un système de fermions en interaction la technique de bosonization stochastique développée dans un papier précédent. On prouve que l'opération d'évolution satisfait, dans une limite appropriée, une équation différentielle stochastique.

1. INTRODUCTION

In a previous paper [AcLuMa] the stochastic limit of a quadratic fermionic model interacting with an external filed was obtained. In the present paper, which has been motivated by a remark of a referee of the previous one, we show that similar ideas and techniques can be applied to deal with a truly interacting (quartic) fermionic model. Although the interaction introduces additional analytical and combinatorial difficulties, the final results are qualitatively similar to the quatratic case. Also in this case, for each time t, the evolution operator (given by the iterated series (1.2) below) converges, in the sense specified by Theorem (4.1), to a unitary operator U_t, which is the unique solution of the *quantum stochastic differential equation*:

$$dU_t = \left[idB_t^+ + idB_t - K dt\right] U_t$$

where K is a complex number whose form we explicitly determine. Also in this case the imaginary and real part of K coincide respectively with the ground state energy and the lifetime estimated by a second order perturbative computation (*cf.* [Ne]). Clearly the information encoded in the limit evolution operator goes much beyond these perturbative informations: for example it allows to compute transitions between arbitrary states and not only vacuum-to-vacuum. However we interpret this agreement of the stochastic limit approach with the usual perturbative techniques, in the cases when these are applicable, as an indication that the former approach gives a reasonably good insight about the behaviour of the system.

As in the case treated in [AcLuMa], we start from a fermionic theory and we find, after the stochastic limit, a purely bosonic theory (in this sense we speak of *bosonization*). Moreover, as in our previous paper, and in qualitative agreement with the existing literature on bosonization of Fermi systems in $d > 1$ (*cf.* for instance [HoMa]), the evolution operator is given by the exponential of a boson (*see* eq. (4.4) below which gives the explicit solution of the above stochastic differential equation).

Besides these similarities there are also substantial differences between the quadratic and the quartic case:

(i) The scaling needed to obtain a finite limit in the quartic case is not the usual van Hove scaling but a new one (*cf.* (1.3), (1.4)) which involves simultaneously both space and time, and which seems to appear for the first time in the present paper.

(ii) The quantum noise, driving the stochastic equation, is quite different in the two cases. In particular, in the quadratic case one started from a quadratic expression in the Fermi fields and found in the limit a single Bose field. In the quartic case one again finds a single Bose field and not, as one might naively expect, a quadratic expression in a Bose field. This is at odd with what happens in the usual exact bosonization, but it is a familiar feature of the stochastic limit: the limit quantum noise encodes the main characteristics of the interaction.

(iii) The present one is the first example of a stochastic limit of a purely self-interacting theory: there is no separation between *system* and *reservoir*.

The statement (iii) above has an interesting physical implication. One generally believes that the origins of stochasticity rely on an *reservoir (or environment)* transfering disorder to a *small* system. The present paper suggests a new, more sutble mechanism for this phenomenon, in which *a quantum system by self-interaction is its own environment and generates its own noise.* The usual picture, according to which the stochastic limit separates the slow time scale of the system from the fast time scale of the reservoir should be replaced here by a picture according to which the stochastic limit distinguishes two time scales *inside the quantum field itself*: the slow time scale (ordered motions) in which the variations are of order dt and the fast time scale (disordered motions) in which the variations occur at the rate of $(dt)^{\frac{1}{2}}$. In fact, given the above mentioned new scaling required by the self-interaction, one should speak of *space-time* scales. To individuate two physically meaningful space-time scales, which justify such an interpretation, is in our opinion an interesting open problem for the physical interpretation of the scaling introduced here.

Let us now describe more precisely the system we are going to consider. Let $\psi^{\varepsilon}_{x,\sigma}, \varepsilon = \pm 1$ be a *Fermi field* with periodic boundary conditions:

$$\psi^{\varepsilon}_{x,\sigma} = \frac{1}{L^{d/2}} \sum_k e^{i\varepsilon kx} a^{\varepsilon}_{k,\sigma}$$

where $\sigma = \pm\frac{1}{2}$ is the *spin* index, $k = \frac{2n\pi}{L}$, $n = (n_1, \ldots, n_d) \in Z^d$ and $\{a^{\varepsilon}_{k,\sigma}, a^{-\varepsilon'}_{k',\sigma'}\} = a^{\varepsilon}_{k,\sigma} a^{-\varepsilon'}_{k',\sigma'} + a^{-\varepsilon'}_{k',\sigma'}, a^{\varepsilon}_{k,\sigma} = \delta_{\varepsilon,\varepsilon'} \delta_{k,k'} \delta_{\sigma,\sigma'}$, and we use, for any operator X, the notation:

$$X^{\varepsilon} = \begin{cases} X & \text{if} \quad \varepsilon = -1 \\ X^{+} & \text{if} \quad \varepsilon = +1 \end{cases}$$

The Hamiltonian is:

$$\begin{aligned} H = H_0 + H_I = & \sum_{\sigma} \int_{\Lambda} d\vec{x} \psi^{+}_{x,\sigma} \left(\frac{\partial^2_x}{2m} - \mu \right) \psi^{-}_{x,\sigma} + \frac{1}{2} \sum_{\sigma} \lambda \\ & \times \int_{\Lambda} dx dy v(x-y) \psi^{+}_{x,\sigma} \psi^{-}_{x,\sigma} \psi^{+}_{y,-\sigma} \psi^{-}_{y,-\sigma} \end{aligned} \tag{1.1}$$

where $\Lambda \subset R^d$ is a square box of side L, $\mu = p_F^2/2m$ is the *chemical potential*, p_F is the *Fermi momentum* and m is the fermion mass.

The cutoff function $v(x)$ has the form

$$v(x) = \frac{1}{L^d} \sum_p u_p e^{-ipx}$$

and with this choice the above Hamiltonian is the standard model for the description of electrons in a metal, *see* for instance [Ne].

The free evolution is characterized by the following property:

$$\psi^{\varepsilon}_{x,t,\sigma} = e^{iH_0 t}\psi^{\varepsilon}_{x,\sigma}e^{-iH_0 t}$$
$$= \frac{1}{\sqrt{L^d}}\sum_k e^{i\varepsilon(kx+\left(\frac{k^2}{2m}-\mu\right)t)}a^{\varepsilon}_{k,\sigma} = \frac{1}{\sqrt{L^d}}\sum_k e^{i\varepsilon(kx+\varepsilon_k t)}a^{\varepsilon}_{k,\sigma}$$

and the Hamiltonian in interaction representation is

$$H_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}$$
$$= \frac{\lambda}{L^d}\frac{1}{2}\sum_{\sigma}\sum_{k_1,k_2,p} u_p a^+_{k_1+p,\sigma}a_{k_1,\sigma}a^+_{k_2-p,-\sigma}a_{k_2,-\sigma}e^{i\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)t}$$

In order to prevent divergencies it is convenient to *regularize* the interacting Hamiltonian in the following way:

$$H_I(t) = \frac{\lambda}{L^d}\sum_{\sigma}\sum_{k_1,k_2,p} u_p g_{k_1}g_{k_2}g_{k_1+p}g_{k_2-p}$$
$$: a^+_{k_1+p,\sigma}a_{k_1,\sigma}a^+_{k_2-p,-\sigma}a_{k_2,-\sigma} : e^{\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} + c.c.$$

where g_k is a cut-off function to be specified in the following and

$$: a^+_{k_1+p,\sigma}a_{k_1,\sigma}a^+_{k_2-p,-\sigma}a_{k_2,-\sigma} := a^+_{k_1+p,\sigma}a_{k_1,\sigma}a^+_{k_2-p,-\sigma}a_{k_2,-\sigma}$$
$$- \langle\phi_F, a^+_{k_1+p,\sigma}a_{k_1,\sigma}\phi_F\rangle\langle\phi_F, a^+_{k_2-p,-\sigma}a_{k_2,-\sigma}\phi_F\rangle$$

where $\phi_F = \prod_{|\vec{k}|\le p_F} a^{\varepsilon}_{\vec{k}}|0\rangle$ is the *ground state* of H_0.

The *evolution (wave) operator* at time T is defined in the usual way:

$$U_T = 1 + \sum_{n=1}^{\infty}(-i)^n\lambda^n\int_0^T dt_1\ldots\int_0^{t_{n-1}} dt_n H_I(t_1)\ldots H_I(t_n) \quad (1.2)$$

where the series converges in norm for each finite L.

Even a very small interaction can produce a relevant effect, if the time evolution T and the box side L are very large; this suggests to study the evolution operator in the limit $\lambda \to 0$, $L \to \infty$, $T \to \infty$. These three limit cannot be performed indispendently, otherwise one would obtain a trivial result. As for the *Friedrich-Van Hove* limit ([vH], [Da]) the right insight about how the limit has to be performed in order to avoid trivialities is given by the first non vanishing term of series obtained averaging eq. (1.2) over the ground state of H_0; one finds in fact that the only possibility so that this term is not trivial in the limit is to take the limits in the following way:

$$\lambda \to 0, \quad T \to \infty,\ L \to \infty \quad \lambda^2 T L^d \to \text{constant} \tag{1.3}$$

Notice that the limit eq. (1.3) is equivalent to the following *scaling limit*:

$$\lambda L^{d/2} = \lambda'; \quad \lambda' \to 0; \quad T \to \frac{T}{\lambda'^2}; \quad L \to \infty \tag{1.4}$$

which makes it formally similar to the Friedrichs-Van Hove limit in which $\lambda' = \lambda$. In our case the presence of the factor L^d is due to self-interaction.

In the following we shall always use the scaling (1.4) and therefore from now on we shall denote λ' simply by λ.

We shall prove in the following that the techniques of the *stochastic limit* of quantum systems, described in [AcLuMa] (*cf.* [AcLuVo] for a more detailed survey) can be applied to the present situation and lead to the usual results.

As usual in the stochastic limit approach, we start by considering the *collective operators*, which are averages (space-time averages, in our case) of the field operators. More precisely, we define:

$$B_{t,\lambda} = \lambda \int_0^{\frac{1}{\lambda^2}} dt_1 \frac{1}{L^{3d/2}} \frac{1}{2} \sum_\sigma \sum_{k_1,k_2,p} u_p g_{k_1} g_{k_2} g_{k_1+p} g_{k_2-p}$$

$$: a^+_{k_1+p,\sigma} a_{k_1,\sigma} a^+_{k_2-p,-\sigma} a_{k_2,-\sigma} : e^{i\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)t} \tag{1.5}$$

where $\varepsilon_k = \frac{k^2}{2m} - \mu$.

In Theorem (3.1) it is proved that the operators $B_{t,\lambda}, B^+_{t,\lambda}$ in the limit $\lambda \to 0,\ L \to \infty$ are boson gaussian fields in the sense that their joint correlations over the ground state converge in the limit to the corresponding joint correlations of a *boson gaussian field*, which we denote B_t, B^+_t; moreover, in this convergence, the ground state corresponds to a vector ψ in the limit space called *vacuum*.

The pair B^+_t, B_t is a *quantum Brownian motion, i.e.* a boson gaussian field over the 1-particle space $L^2(R_+, \mathcal{K})$, where $\mathcal{K}$ is the Hilbert space described at the end of section 2, with vacuum correlation functions given by:

$$\left.\begin{array}{ll} \langle\psi, B^+_t B_s \psi\rangle = C_1 \min(t,s) & \langle\psi, B_t B^+_s \psi\rangle = C_2 \min(t,s) \\ \langle\psi, B^+_t B^+_s \psi\rangle = C_3 \min(t,s) & \langle\psi, B_t B_s \psi\rangle = C_4 \min(t,s) \end{array}\right\} \tag{1.5}$$

where $C_i,\ i = 1, \ldots, 4$ depend only on the cut-off functions necessary to make the limit meaningful. Note that the collective operators eq. (1.5) live in the same Hilbert space where the original fermionic fields live, while the limiting field lives in a different Hilbert space.

Eventually, and this is our main result, we prove that the time rescaled evolution operator admits a limit U_t (*cf.* Theorem (4.1) for a precise statement) which satisfies a stochastic differential equation, driven by the quantum Brownian motion B_t, B_s^+ described above.

Our results hold for dimension $d \geq 3$ and, under special assumptions for the cut-off functions, for $d \geq 2$. Such dependence upon the dimension is not technical: it is well known, for instance by renormalization group methods (*see* [BeGa], [BeGaMa]), that the physical properties of a Fermi system with hamiltonian eq. (1.1) are completely different in $d = 1$ or $d = 3$.

2. THE COLLECTIVE OPERATORS

In this section we define more precisely our model and introduce some definitions and notations which will be useful in the following. Let be A^+ a representation of the CAR on $L^2(R^d)$ and for each $n = (n_1, \ldots, n_d) \in Z^d$ define

$$\Lambda_n = \{x = (x_1, \ldots, x_d) \in R^d \ : \ n_j \leq x_j < n_j + 1, \ j = 1 \ldots, d\}$$

Given $L > 0$, define $\Lambda_L = \{k = 2\pi n/L \ : \ n \in Z^d\}$ and, for each $k \in \Lambda_L$, define $a_k^\varepsilon = A^\varepsilon(\chi_{\Lambda_{Lk/2\pi}})$, $\varepsilon = \pm 1$.

The collective operators are defined as, if $\sigma = 0, 1$pipourj dfhjkl zehkq qjhsk

$$\begin{aligned} &B_\lambda^s(S_1, T_1, F, L) \\ &\quad = \frac{\lambda}{L^{3d/2}} \int_{S/\lambda^2}^{T/\lambda^2} \mathrm{d}t \sum_{k_1, k_2, p \subset \Lambda_L} \\ &\quad F^\sigma(k_1, k_2, p) e^{it\left(\varepsilon_{k_1+\sigma p} + \varepsilon_{k_2-\sigma p} - \varepsilon_{k_1+(1-\sigma)p} - \varepsilon_{k_2-(1-\sigma)p}\right)} \\ &\quad \frac{1}{2} \sum_\sigma \ : a_{k_1+\sigma p}^+ a_{k_1+(1-\sigma)p} a_{k_2-\sigma p}^+ a_{k_2-(1-\sigma)p} \ : \end{aligned} \tag{2.1}$$

with $F^0(k_1, k_2, p) = \overline{F}(k_1, k_2, p)$ and $F^1(k_1, k_2, p) = F(k_1, k_2, p)$.

We will study the collective operators in the stochastic limit eq. (1.4). In the limit we will see that such operators are defined in the non zero subspace K_0 of $L^2(\mathbf{R}^d) \otimes L^2(\mathbf{R}^d)$ with the property that, for any pair of vectors F, G is this subspace

$$\int_{\mathbf{R}} \mathrm{d}t \Bigg| \int_{\mathbf{R}^d} \mathrm{d}k_1 \int_{\mathbf{R}^d} \mathrm{d}k_2 \int_{\mathbf{R}^d} \mathrm{d}p$$
$$\{e^{-i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} \chi_{B_F}(k_1+p)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$
$$F(k_1,k_2,p)[G(k_2-p,\ k_1+p,p)+G(k_1+p,\ k_2-p,-p)$$
$$+\bar{G}(k_1,k_2,p)+\bar{G}(k_2,k_1,-p)]$$
$$+e^{i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} \chi_{B_F^c}(k_1+p)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F}(k_2)$$
$$\bar{F}(k_1,k_2,p)[\bar{G}(k_2-p,\ k_1+p,p)+\bar{G}(k_1+p,\ k_2-p,-p)$$
$$+G(k_1,k_2,p)+G(k_2,k_1,-p)]\Bigg| < +\infty \tag{2.2}$$

A possible choice for the cut-off function $F(k_1,k_2,p)$ is $u_p g_{k_1} g_{k_2}$ where u_p, g_k are such that $g_k = g_{|k|}$ and $u_p = u_{|p|}$ and vanishing at infinity faster than any power. It is possible in fact to check that, if $d \geq 3$, eq. (2.2) holds. Such a choice has a clear physical meaning: the function u_p is a cut-off of the momentum that the external field exchanges with the fermions while g_k is a *bandwidth cut-off* taking into account that the band structure in a metal forbids the electrons to have large momenta.

The proof of (2.2) is not completely trivial. By introducing more particular cut-off functions, the arguments drastically simplifies and one obtains a stronger result. Namely we consider the cut-off function $F(k,p)$ to be of the form $g_{k_1} g_{k_2} g_{k_1+p} g_{k_2-p} u_p$ where g_k is the sum of two C^∞ functions, the first, called $g_{1,k}$, with support in B_F and the second, called $g_{2,k}$, with support in B_F^c and decreasing faster then any power at infinity and both vanishing at $|k| = p_F$. Moreover we choose g_k real. Let us consider the first summand in eq. (2.2) (similar considerations hold of course for the other terms) which can be written then:

$$\int_{\mathbf{R}} \mathrm{d}t \Bigg| \int_{\mathbf{R}^d} \mathrm{d}k_1 \int_{\mathbf{R}^d} \mathrm{d}p$$
$$\times \int dk_2 e^{-i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} |u_p|^2 g_{1,k_1+p}^2 g_{2,k_2-p}^2 g_{2,k_1}^2 g_{1,k_2}^2 \Bigg|$$

and performing the change of variables $p' = pt$ we find:

$$\text{const} + \int_{|t|>1} \mathrm{d}t \frac{1}{t^d} \left| \int_{\mathbf{R}^d} \mathrm{d}p e^{-i(2p^2/t)} |u_{p/t}|^2 f_1(p,t) f_2(p,t) \right|$$

where:

$$f_1(p,t) = \int_{\mathbf{R}^d} \mathrm{d}k_1 e^{-i2k_1 p} g^2_{1,k_1+p/t} g^2_{2,k_1}$$
$$f_2(p,t) = \int_{\mathbf{R}^d} \mathrm{d}k_2 e^{-i2k_2 p} g^2_{2,k_2-p/t} g^2_{1,k_2}$$

Integrating by parts and noting that the integrand vanishes at the extrema of integration we obtain that for any integer $N|f_i(p,t)| \leq \frac{C_N}{1+p^N}$ where C_N is a suitable constant and $i = 1, 2$. The condition that g_k, u_k are vanishing at p_F has the only effect to smooth χ_{B_F} and $\chi_{B_F^c}$. Note that this choice of the cut-off is quite natural if the temperature is not 0. In this case in fact the χ functions are replaced by smooth C^∞ functions, which are the densities of the Fermi distributions. The functions g_k, g_{k+p} are band-width cut-off for the two fermions operators and u_p is the cut-off on the exhanged momentum. With these cut-offs our theory holds for $d \geq 2$.

We introduce finally, for further use, the following definition, assuming that g_k, u_p are real:

$$(u \otimes f | v \otimes g) := \int_{\mathbf{R}} \mathrm{d}t \int \mathrm{d}k_1 \mathrm{d}k_2 \int \mathrm{d}p g^2_{k_1} g^2_{k_2} g^2_{k_1+p} g^2_{k_2-p} v^2_p$$
$$[e^{-i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} \chi_{B_F}(k_1+p)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$
$$+e^{i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})t} \chi_{B_F^c}(k_1+p)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F}(k_2)\} \tag{2.3}$$

In the above assumptions the expression (2.3) defines a pre-scalar product on the test functions F, and the completion of the (quotient by the zero norm elements of the) space K_0 by this scalar product, denoted $\mathcal{K}$ is interpreted as the Hilbert space were the Brownian motion takes its values. This is a general feature of the stochastic limit.

In order to motivate the above definitions we compute the stochastic limit of the product of two collective operators.

In general we shall need in the following to compute expectation values of the form

$$\langle \phi_F, a^+_{k_1} a_{k_2} \cdot a^+_{k_3} a_{k_4} \cdot \ldots \cdot a^+_{k_{2n-1}} a_{k_{2n}} \phi_F \rangle$$
$$= \sum_{\pi} (-1)^{\nu(\pi)} \prod_{i,j} \delta_{i,\pi(j)} \langle \phi_F, a^{\varepsilon_i}_{k_i} a^{\varepsilon_j}_{k_j} \phi_F \rangle \tag{2.4}$$

where π is a permutation of the index set $\{1, \ldots, 2n\}$, $\nu(\pi)$ is the parity of the permutation and

$$\langle \phi_F, a_{k_1}^+ a_{k_2} \phi_F \rangle = \delta_{k_1,k_2} \chi_{B_F}(k_1) \quad \langle \phi_F, a_{k_1} a_{k_2}^+ \phi_F \rangle = \delta_{k_1,k_2} \chi_{B_F^c}(k_1)$$
$$\langle \phi_F, a_{k_1}^+ a_{k_2}^+ \phi_F \rangle = \langle \phi_F, a_{k_1} a_{k_2} \phi_F \rangle = 0$$

We study the ground state average of two collective operators in the limit (1.4); this defines the 2-points function of the limiting fields.

$$\lim_{\lambda \to 0} \lim_{L \to \infty} \langle \phi_F, B_\lambda^{\sigma(1)}(S_1, T_1, g, u, L) B_\lambda^{\sigma(2)}(S_2, T_2, g, u, L) \phi_F \rangle \tag{2.5}$$

Let us write eq. (2.5) if $\sigma(1) = \sigma(2) = 0$. We obtain

$$\begin{aligned} &\langle \phi_F, B_\lambda(S_1, T_1, g, u, L) B_\lambda(S_2, T_2, g, u, L) \phi_F \rangle \\ &= \frac{\lambda^2}{L^{3d}} \int_{\frac{S_1}{\lambda^2}}^{\frac{T_1}{\lambda^2}} dt_1 \int_{\frac{S_2}{\lambda^2}}^{\frac{T_2}{\lambda^2}} dt_2 \sum_{k_1, k_2, p} \sum_{k_1', k_2', p'} e^{it_1 \left(\varepsilon_{k_1+p} + \varepsilon_{k_2-p} - \varepsilon_{k_1} - \varepsilon_{k_2}\right)} \\ &e^{it_2 (\varepsilon_{k_1'+p'} + \varepsilon_{k_2'-p} - \varepsilon_{k_1'} - \varepsilon_{k_2'})} u_p u_{p'} g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2} g_{k_1'+p} g_{k_2'-p} g_{k_1'} g_{k_2'} \\ &\frac{1}{4} \sum_{\sigma, \sigma'} \langle \phi_F | \ : \ a_{k_1+p,\sigma}^+ a_{k_1,\sigma} a_{k_2-p,-\sigma}^+ a_{k_2,-\sigma} \\ &:: \ a_{k_1'+p',\sigma'}^+ a_{k_1',\sigma'}^+ a_{k_2'-p',-\sigma'}^+ a_{k_2',-\sigma} \ : \ |\phi_F\rangle \end{aligned} \tag{2.6}$$

But:

$$\begin{aligned} &\frac{1}{4} \sum_{\sigma, \sigma'} \langle \phi_F | \ : \ a_{k_1+p,\sigma}^+ a_{k_1,\sigma} a_{k_2-p,-\sigma}^+ a_{k_2,-\sigma} \\ &:: \ a_{k_1'+p',\sigma'}^+ a_{k_1',\sigma'}^+ a_{k_2'-p',-\sigma'}^+ a_{k_2',-\sigma} \ : \ |\phi_F\rangle \\ &= \frac{1}{2} (\delta_{k_2, k_1'+p'} \delta_{k_2-p, k_1'} \delta_{k_1, k_2'-p'} \delta_{k_1+p, k_2'} + \delta_{k_2-p, k_2'} \delta_{k_1+p, k_1'} \delta_{k_1, k_1'+p'} \delta_{k_2, k_2'-p'}) \\ &\chi_{B_F}(k_1 + p) \chi_{B_F^c}(k_1) \chi_{B_F}(k_2 - p) \chi_{B_F^c}(k_2) \end{aligned}$$

so that eq. (2.6) is given by:

$$\begin{aligned} &\frac{\lambda^2}{L^{3d}} \int_{\frac{S_1}{\lambda^2}}^{\frac{T_1}{\lambda^2}} dt_1 \int_{\frac{S_2}{\lambda^2}}^{\frac{T_2}{\lambda^2}} dt_2 \sum_{k_1, k_2, p} e^{i(t_1 - t_2)\left(\varepsilon_{k_1+p} + \varepsilon_{k_2-p} - \varepsilon_{k_1} - \varepsilon_{k_2}\right)} \\ &(u_p g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2})^2 \chi_{B_F}(k_1 + p) \chi_{B_F^c}(k_1) \chi_{B_F}(k_2 - p) \chi_{B_F^c}(k_2) \end{aligned}$$

In the limit $L \to \infty$, setting $t_2 - t_1 = \tau_2$, $t_1 = \frac{\tau_1}{\lambda^2}$ we obtain

$$\int_{S_1}^{T_1} d\tau_1 \int_{\frac{(S_2-\tau_1)}{\lambda^2}}^{\frac{(T_2-\tau_1)}{\lambda^2}} d\tau_2 \int dk_1 dk_2 dp$$

$$e^{-i\tau_2\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)}$$

$$(u_p g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2})^2 \chi_{B_F}(k_1+p)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$

$$\to {}_{\lambda\to 0}\langle \chi_{[S_1,T_1]}\chi_{[S_2,T_2]}\rangle (u\otimes g|u\otimes g)_{1,1} \qquad (2.7)$$

In the same way,

$$\langle \phi_F, B_\lambda(S_1,T_1,g,u,L)B_\lambda^+(S_2,T_2,g,u,L)\phi_F\rangle = {}_{L\to\infty}$$

$$\int_{\frac{(S_2-\tau_1)}{\lambda^2}}^{\frac{(T_2-\tau_1)}{\lambda^2}} d\tau_2 \int dk_1 dk_2 dp e^{-i\tau_2\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)}$$

$$(u_p g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2})^2 \chi_{B_F}(k_1+p)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$

$$\to {}_{\lambda\to 0}\langle \chi_{[S_1,T_1]}\chi_{[S_2,T_2]}\rangle (u\otimes g|u\otimes g)_{-1,1} \qquad (2.8)$$

$$\langle \phi_F, B_\lambda^+(S_1,T_1,g,u,L)B_\lambda(S_2,T_2,g,u,L)\phi_F\rangle = {}_{\lambda\to\phi,L\to\infty}$$

$$\int_{\frac{(S_2-\tau_1)}{\lambda^2}}^{\frac{(T_2-\tau_1)}{\lambda^2}} d\tau_2 \int dk_1 dk_2 dp e^{i\tau_2\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)}$$

$$(u_p g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2})^2 \chi_{B_F^c}(k_1+p)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F^c}(k_2)$$

$$\to {}_{\lambda\to 0}\langle \chi_{[S_1,T_1]}\chi_{[S_2,T_2]}\rangle (u\otimes g|u\otimes g)_{1,-1} \qquad (2.9)$$

$$\langle \phi_F, B_\lambda^+(S_1,T_1,g,u,L)B_\lambda^+(S_2,T_2,g,u,L)\phi_F\rangle = {}_{\lambda\to\phi,L\to\infty}$$

$$\int_{\frac{(S_2-\tau_1)}{\lambda^2}}^{\frac{(T_2-\tau_1)}{\lambda^2}} d\tau_2 \int dk_1 dk_2 dp e^{i\tau_2\left(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2}\right)}$$

$$(u_p g_{k_1+p} g_{k_2-p} g_{k_1} g_{k_2})^2 \chi_{B_F^c}(k_1+p)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F^c}(k_2)$$

$$\to {}_{\lambda\to 0}\langle \chi_{[S_1,T_1]}\chi_{[S_2,T_2]}\rangle (u\otimes g|u\otimes g)_{-1,-1} \qquad (2.10)$$

Note that

$$(u \otimes g|u \otimes g) = (u \otimes g|u \otimes g)_{1,1} + (u \otimes g|u \otimes g)_{1,-1} + (u \otimes g|u \otimes g)_{-1,1} + (u \otimes g|u \otimes g)_{-1,-1}$$

3. THE COLLECTIVE OPERATORS AS BOSON GAUSSIAN FIELDS

In this section we prove that the collective operators in the weak coupling limit are Boson Gaussian fields.

THEOREM 3.1. – *In the notations (2.1), (2.3), one has, for any* $N \in \mathbf{N}$, $S_1, T_1, \ldots, S_N, T_N \in \mathbf{R}, g_1 \otimes u_1, \ldots, g_N \otimes u_N \in K_0$, *with* $K_0 \subseteq L^2(\mathbf{R}^{2d})$ *defined above:*

$$\lim_{\lambda \to 0} \lim_{L \to \infty} \langle \phi_F, B_\lambda^{\sigma(1)}(L_1, T_1, g_1, u_1, L) \ldots B_\lambda^{\sigma(N)}(S_N, T_N, g_N, u_N, L)\phi_F \rangle = \langle \psi, B^{\sigma(1)}(S_1, T_1, g_1, u_1) \ldots B^{\sigma(N)}(S_N, T_N, g_N, u_N)\psi \rangle$$

where $\{B^{\#}, \psi\}$ *is the unique mean zero Boson Gaussian field with 2-point function given by:*

$$\langle \psi, B^{\varepsilon_1}(S_1, T_1, g_1, u_1) B^{\varepsilon_2}(S_2, T_2, g_2, u_2) = \langle \chi_{[S_1, T_1]}, \chi_{[S_2, T_2]} \rangle_{L^2(\mathbf{R})} (u_1 \otimes g_1 | u_2 \otimes g_2)_{\varepsilon_1, \varepsilon_2}$$

Remark. – The limiting fields defined by the above theorem are an example of a *quantum brownian motion.* This term is justified by the following consideration. Fixed the test functions u, g it is easy to check that the 1-parameter family $W_t(u \otimes g) = B^+(0, t, g, u) + B(0, t, g, u)$ is commutative, in the sense that the operators W_t commute for different t. The vacuum distribution of these fields is gaussian, with mean zero and variance $\min(s, t)(u \otimes g|u \otimes g)$. This means that $(W_t)_{t \geq 0}$ is Brownian motion with values in the Hilbert space $\mathcal{K}$. For a more detailed discussion, *see* [AcLuMa].

Proof. – We want to calculate

$$\lim_{\lambda\to 0}\lim_{L\to\infty}\langle\phi_F, B_\lambda^{\sigma_1}(S_1,T_1,g_1,v_1,L)\dots B_\lambda^{\sigma_n}(S_n,T_n,g_n,v_n,L)\phi_F\rangle$$
$$= \frac{\lambda^n}{L^{3nd/2}}\int_{S_1/\lambda^2}^{T_1/\lambda^2} \mathrm{d}t_1 \dots \int_{S_n/\lambda^2}^{T_n/\lambda^2} \mathrm{d}t_n \prod_{i=1}^n$$
$$\Big(\sum_{k_i',\hat{k}_i',p_i\in\Lambda_L} g_{k_i'} g_{\hat{k}_i'} u_{p_i} e^{it_i[\varepsilon_{k_i'+\sigma_i p_i}+\varepsilon_{\hat{k}_i'-\sigma_i p_i}-\varepsilon_{k_i'+(1-\sigma_i)p_i}-\varepsilon_{\hat{k}_i'+(1-\sigma_i)p_i}]}\Big)$$
$$\langle\phi_F, a^+_{k_1'+\sigma_1 p_1} a_{k_1'+(1-\sigma_1)p_1} a^+_{\hat{k}_1'-\sigma_1 p_1} a_{\hat{k}_1'-(1-\sigma_1)p_1} : \cdots$$
$$: a^+_{k_n'+\sigma_n p_n} a_{k_n'+(1-\sigma_n)p_n} a^+_{\hat{k}_n'-\sigma_n p_1} a_{\hat{k}_n'-(1-\sigma_n)p_1} : \phi_F\rangle \tag{3.1}$$

We perform the change of variables

$$\begin{aligned} &k_i' + \sigma_i p_i \to k_{2i-1}, \quad k_i' + (1-\sigma_i)p_i \to k_{2i}\,; \quad i = 1,\dots,n \\ &\hat{k}_i' + \sigma_i p_i \to k_{2i'-1}, \quad \hat{k}_i' + (1-\sigma_i)p_i \to k_{2i'}\,; \quad i = 1,\dots,n \end{aligned} \tag{3.2}$$

With notation (3.2), the creators are labeled by the *odd* indices; the annihilators by even indices. By eq. (2.4), each annihilator variable k_{2j} is equal to one (and only one) creator variable: this shall be denoted $k_{2\pi(j)-1}$. Thus, by definition of π : $k_{2j} = k_{2\pi(j)-1}$. With these notations, from (3.1), we obtain that, in the limit $L \to \infty$:

$$\lambda_n \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \dots \int_{S_n/\lambda^2}^{T_n/\lambda^2} dt_n \sum_\pi (-1)^{\nu(\pi)} \prod_{j=1}^n \int_d dk_{2j-1} dk_{2j'-1}$$
$$F(\{g\},\{u\},\{\chi\}) e^{it_j(\varepsilon_{k_{2j-1}}+\varepsilon_{k_{2j'-1}}-\varepsilon_{k_{2\pi(j)-1}}-\varepsilon_{k_{2\pi(j')-1}})} \tag{3.3}$$

where, due to momentum conservation,

$$F(\{g\},\{u\},\{\chi\}) = \prod_{j=1}^n \delta\big(k_{2j-1} + k_{2j'-1} - k_{2\pi(j)-1} - k_{2\pi(j')-1}\big)$$
$$\bar{F}(\{g\},\{u\},\{\chi\}$$

and $\bar{F}(\{g\},\{u\},\{\chi\})$ is a suitable function containing the cut-off function g and u and a product of the characteristic function χ.

To find the limit of (3.3), as $\lambda \to 0$, let us start considering the case in which n is even. Recall that we want to prove that, in the limit (4.9), the B^{σ}_{λ} tend to some boson operator B^{σ}. But each B^{σ}_{λ} is a sum (integral) of four fermionic operators $a^{+}_{k_{2j-1}} a_{k_{2j}} a^{+}_{k_{2j'-1}} a_{k_{2j'}}$, therefore *each such product should behave like a single object.* In order to prove this we have to show that, if in (3.3) the annihilator $a_{k_{2j}}$ produces a scalar product with the annihilator $a^{+}_{k_{2\pi(j)-1}}$ then the creator $a^{+}_{k_{2j-1}}$ must be paired with the annihilator $a_{k_{2\pi(j)}}$, the operator $a^{+}_{k_{2j'-1}}$ has to be paired with $a_{k_{2\pi(\pi'(j))-1}}$ and the operator $a_{k_{2j'}}$ has to be paired with $a^{+}_{k_{2\pi'(j)-1}}$. We shall prove that the terms for which this condition is not satisfied become negligible in the limit $\lambda \to 0$.

In order to evidentiate the negligible terms, it is convenient to rewrite the product (3.3) as a product over $n/2$ pairs (since, in the limit, each such pair shall define a scalar product in the bosonic one-particle space). To this goal, recall that, in (3.3) the permutation π of $\{1, \ldots, n\}$ is fixed and define inductively the subset $A_{\pi} = \{\nu_1, \ldots, \nu_{n/2}\} \subseteq \{1, \ldots, n\}$ as follows:

$$\nu_1 = 1$$
$$\nu_{j+1} = \min\{\{1, \ldots, n\} \backslash \{\nu_1, \pi(\nu_1), \ldots, \nu_j, \pi(\nu_j)\}$$

With these notations the expression (3.3) can be written:

$$\begin{aligned}
&\lambda^n \int_{S/\lambda^2}^{T/\lambda^2} \mathrm{d}t_1 \ldots \int_{S_n/\lambda^2}^{T_n/\lambda^2} dt_n \sum_{\pi} (-1)^{\nu(\pi)} \int d\{k\} F(\{g\}, \{u\}, \{\chi\}) \\
&\quad \times \prod_{j \in A_{\pi}} e^{it_j[\varepsilon_{k_{2j-1}} + \varepsilon_{k_{2j'-1}} - \varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(j')-1}}]} \\
&\quad e^{it_{\pi(j)}[\varepsilon_{k_{2\pi(j)-1}} + \varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} - \varepsilon_{k_{2\pi(\pi'(j))-1}}]} \\
&= \lambda^n \int_{S/\lambda^2}^{T/\lambda^2} \mathrm{d}t_1 \ldots \int_{S_n/\lambda^2}^{T_n/\lambda^2} dt_n \sum_{\pi} (-1)^{\nu(\pi)} d\{k\} F(\{g\}, \{u\}, \{\chi\}) \\
&\quad \times \prod_{j \in A_{\pi}} e^{i(t_j - t_{\pi(j)})[\varepsilon_{k_{2j-1}} + \varepsilon_{k_{2j'-1}} - \varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(j')-1}}]} \\
&\quad e^{it_{\pi(j)}[\varepsilon_{k_{2j-1}} + \varepsilon_{k_{2j'-1}} - \varepsilon_{k_{2\pi(j')-1}} + \varepsilon_{k_{2\pi'(j)-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} - \varepsilon_{k_{2\pi(\pi'(j))-1}}]}
\end{aligned} \tag{3.4}$$

with $\int d\{k\} = \int \prod_j dk_{2j-1} dk_{2j'-1}$.

Performing the change of variables $t_j - t_{\pi(j)} = \tau_j$ and $\lambda^2 t_{\pi(j)} = \tau_{\pi(j)}$ we obtain:

$$\sum_{\pi} (-1)^{\nu(\pi)} \prod_{j \in A_\pi} \int_{S_{\pi(j)}}^{T_{\pi(j)}} d\tau_{\pi(j)} \int_{(S_j - \tau_{\pi(j)})/\lambda^2}^{(T_j - \tau_{\pi(j)})/\lambda^2} d\tau_j$$
$$\times \int d\{k\} F(\{g\}, \{u\}, \{\chi\})$$
$$\times \prod_{j \in A_\pi} e^{i\tau_j [\varepsilon_{k_{2j-1}} + \varepsilon_{k_{2j'-1}} - \varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(j')-1}}]}$$
$$e^{i\frac{\tau_{\pi(j)}}{\lambda^2}[\varepsilon_{k_{2j-1}} + \varepsilon_{k_{2j'-1}} - \varepsilon_{k_{2\pi(j')-1}} + \varepsilon_{k_{2\pi'(j)-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} - \varepsilon_{k_{2\pi(\pi'(j))-1}}]}$$

In the limit $\lambda \to 0$ we distinguish two kinds of terms in the sum over π in eq. (3.4):

1) Let us consider a term in the sum over π in eq. (3.4) such that, for some $\bar{j}$, $k_{2\bar{j}-1} \neq k_{2\pi(\pi(\bar{j}))-1}$ or $k_{2\bar{j}'-1} \neq k_{2\pi(\pi'(\bar{j}))-1}$ or $k_{2\pi'(\bar{j}')-1} \neq k_{2\pi(\bar{j}')-1}$. Such a term is vanishing as $\lambda \to 0$ for the Riemann-Lebesgue lemma.

2) If $\forall \bar{j}$, $k_{2\bar{j}-1} = k_{2\pi(\pi(\bar{j}))-1}$ and $k_{2\bar{j}'-1} = k_{2\pi(\pi'(\bar{j}))-1}$ and $k_{2\pi'(\bar{j})-1} = k_{2\pi(\bar{j}')-1}$ than from (3.4) we obtain:

$$\lim_{\lambda \to 0, L \to \infty} \langle \phi_F, B_\lambda^{\sigma_1}(S_1, T_1, g_1, v_1, L) \ldots B_\lambda^{\sigma_n}(S_n, T_n, g_n, v_n, L) \phi_F \rangle$$
$$= \sum_p \prod_{i,j \in p} \langle \psi, B^{\sigma_i}(S_i, T_i, g_i, v_i) B^{\sigma_j}(S_j, T_j, g_j, v_j) \psi \rangle$$

where p is a pair partition of the indices $1, \ldots, n$.

A similar argument shows that if n is odd the left hand side of (4.9) vanishes in the $\lambda \to 0$, $L \to \infty$ limit.

4. LIMIT PROCESS

THEOREM 4.1. – *In the notations* (1.2), (2.1), (2.2), (2.3) *the limit*

$$\left.\begin{array}{l}\lim\limits_{\lambda \to 0} \lim\limits_{L \to \infty} \langle B_\lambda^{\sigma(1)}(S_1, T_1, f_1, u_1, L) \ldots B_\lambda^{\sigma(1)}(S_N, T_N, f_N, u_N, L) \phi_F, \\ \quad U_{t/\lambda^2}^{(\lambda, L)} B_\lambda^{\tau(1)}(S_1', T_1', f_1', u_1', L) \ldots B_\lambda^{\tau(1)}(S_N', T_N', f_N', u_N', L) \phi_F \rangle \end{array}\right\} \tag{4.1}$$

exists and is equal to

$$\left\langle \prod_{h=1}^{N} B^{\sigma(h)}(S_h, T_h, f_h, u_h)\psi, U_t \prod_{h=1}^{N'} B^{\tau(h)}(S'_h, T'_h, f'_h, u'_h)\psi \right\rangle \tag{4.2}$$

where $\sigma \in \{0,1\}^N$, $\tau \in \{0,1\}^{N'}$, *and* $B^{\#}(S,T,u,f)$ *is the quantum Brownian motion defined in Theorem* (4.1).

U_t *is the unique (unitary) solution of the Stochastic Differential Equation:*

$$U_t = 1 + \int_0^t \{idB_s^+(u \otimes g) + idB_s(u \otimes g) - (u \otimes g|u \otimes g)_{-}\mathrm{d}s\}U_s \tag{4.3}$$

where we use the notation:

$$dB_s^{\#}(u \otimes g) := dB^{\#}(0, s, u, g)$$

with $(u \otimes g|u \otimes g)_{-}$ *defined as* $(u \otimes g|u \otimes g)$ (*cf.* 2.3) *but with* $\int_{-\infty}^{+\infty}$ *replaced by* $\int_{-\infty}^{0}$.

Moreover the solution of (4.3) *is*

$$U_t = e^{i\left(B_t^+(u\otimes g) + B_t(u\otimes g) - \mathrm{Im}(u\otimes g|u\otimes g)_{-}\right)} \tag{4.4}$$

Remark. – The Ito correction term can be written as $(u \otimes g|u \otimes g)_{-} = iE_0 + \Gamma$ with E_0 and Γ real and $\Gamma > 0$. Using the well known distribution formula:

$$\int_{-\infty}^{0} e^{i\omega t}dt = iP\left(\frac{1}{\omega}\right) + \pi\delta(\omega)$$

where P denotes the principal part, we have that:

$$E_0 = P\int dk_1 dk_2 dp \frac{u_p^2 g_{k_2-p}^2 g_{k_1}^2 g_{k_1+p}^2}{\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_1+p} - \varepsilon_{k_2-p}}$$
$$[\chi_{B_F}(k_1+p_1)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$
$$-\chi_{B_F^c}(k_1+p_1)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F}(k_2)]$$

$$\Gamma = \int dk_1 dk_2 dp u_p^2 g_{k_2}^2 g_{k_2-p}^2 g_{k_1}^2 g_{k_1+p}^2 \delta(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_1+p} - \varepsilon_{k_2-p})$$
$$[\chi_{B_F}(k_1+p_1)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$
$$+\chi_{B_F^c}(k_1+p_1)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F}(k_2)]$$

If

$$\lim_{\lambda\to 0}\lim_{L\to\infty}\langle\phi_F, U_{t/\lambda^2}\phi_F\rangle = \langle\psi, U_t\psi\rangle$$

from (4.3) one obtains:

$$\frac{d}{dt}\langle\psi, U_t\psi\rangle = (iE_0 - \Gamma)\langle\psi, U_t\psi\rangle$$

so that E_0 and Γ are respectively the ground state energy shift and the lifetime of the ground state; they coincide with the correspondent quantities computated by a standard perturbation theory at the second order in λ (*see* for instance [Ne]).

Proof. – The proof of the Theorem 4.1 will be done in several steps. Let us define:

$$W_\lambda(S, T, g_1, u_1, L) = \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt B_\lambda(t, g_1, f_1, L)$$

and the operator Ψ_λ as:

$$\Psi_\lambda(g_1, u_1) = \sum_{n=1}^{\infty} \frac{(-i)^n \lambda^n}{n!} \left[\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt B_\lambda(t, g_1, u_1)\right]^n$$

We consider:

$$\langle\phi_F, \Psi_\lambda(g_1, u_1) U_{\frac{t}{\lambda^2}} \Psi_\lambda(g_2, u_2)\phi_F\rangle \tag{4.5}$$

Expanding eq. (4.5) with the iterative series one obtains:

$$\langle\phi_F, \Psi_\lambda(g_1, u_1)\Psi_\lambda(g_2, u_2)\phi_F\rangle + \sum_{n=1}^{\infty}(-i)^n I_n$$

$$I_n = \lambda^n \int_0^{\frac{t}{\lambda^2}} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\langle\phi_F, \Psi_\lambda(g_1, u_1) B_\lambda(t_1, g, u, L) \ldots B_\lambda(t_n, g, u, L)\Psi_\lambda(g_2, u_2)\phi_F\rangle$$

It is possible to prove, by an adaptation of the technique used in [Lu92], that $\sum_n (-i)^n I_n$ is a series absolutely convergent, uniformly in the pair (λ, t).

It is convenient to write $I_n = I_n^1 + I_n^2 + I_n^3$ where I_n^3 is given by the terms in which the fermionic operators belonging to the same B_λ^σ are *paired* with operators belonging to different B_λ^σ, and by the analysis of section 4 $\lim_{\lambda\to 0}\lim_{L\to\infty} I_n^3 = 0$ and:

$$I_n^1 = \sum_{\{m\}} \int_0^{\frac{t}{\lambda^2}} dt_1 \ldots \int_0^{t_{n-1}} dt_n$$

$$\times \prod_{i,j\in\{m\};i-j=1} \langle \phi_F, \lambda B_\lambda(t_i, g, u, L)\lambda B_\lambda(t_j, g, u, L)\phi_F\rangle$$

$$\times \prod_{i\in\{m_1\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \langle \phi_F, \lambda B_\lambda(t_i, g, u, L)\lambda B_\lambda(\tau, g_1, u_1, L)\phi_F\rangle$$

$$\times \prod_{i\in\{m_2\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \langle \phi_F, \lambda B_\lambda(t_i, g, u, L)\lambda B_\lambda(\tau, g_2\ u_2)\phi_F\rangle$$

$$\langle \phi_F, \Psi_\lambda(g_1, u_1)\Psi_\lambda(g_2, u_2), \phi_F\rangle$$

where $\{m\} \bigcup \{m_1\} \bigcup \{m_2\} = 1 \ldots n$ and:

$$\langle \phi_F, \lambda B_\lambda(t_1, g_1, u_1, L)\lambda B_\lambda(t_2, g_2, u_2, L)\phi_F\rangle$$

$$= \frac{\lambda^2}{L^{3d/2}} \sum_{k_1,k_2,p} g_{k_1}^2 g_{k_2}^2 g_{k_1+p}^2 g_{k_1-p}^2$$

$$[e^{-i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})(t_1-t_2)} \chi_{B_F}(k_1+p_1)\chi_{B_F^c}(k_1)\chi_{B_F}(k_2-p)\chi_{B_F^c}(k_2)$$

$$+e^{i(\varepsilon_{k_1+p}+\varepsilon_{k_2-p}-\varepsilon_{k_1}-\varepsilon_{k_2})(t_1-t_2)} \chi_{B_F^c}(k_1+p_1)\chi_{B_F}(k_1)\chi_{B_F^c}(k_2-p)\chi_{B_F}(k_2)$$

It is straightforward to check that I_n^1 converges to a different from zero value in the weak coupling limit. On the other hand by definition I_n^2 is given by:

$$I_n^2 = \int_0^{\frac{t}{\lambda^2}} dt_1 \ldots \int_0^{t_{n-1}} dt_n$$

$$\prod_{i,j\in\{\tilde{m}\}}^{*} \langle \phi_F, \lambda B_\lambda(t_i, g, u, L)\lambda B_\lambda(t_j, g, u, L)\phi_F\rangle$$

$$\prod_{\alpha=\pm 1} \prod_{k\in\{\tilde{m}_\alpha\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \langle \phi_F, \lambda B_\lambda(t_k, g, u, L)\lambda B_\lambda(\tau, g_\alpha, u_\alpha, L)\phi_F\rangle$$

$$\langle \phi_F, \Psi_\lambda(g_1, u_1)\Psi_\lambda(g_2, u_2)\phi_F\rangle$$

where $\prod^*$ means that at least a couple (i, j) is such that $j - i > 1$ and $\{\tilde{m}\} \bigcup \{\tilde{m}_1\} \bigcup \{m_2\} = 1..n$.

Performing the change of variables $\tau_j = \frac{t_j - t_i}{\lambda^2}$ we have that I_2 contains at least an integral of the form:

$$\int_0^{t_{i-1}} dt_i \int_{\frac{-t_j}{\lambda^2}}^{\frac{(t_{j-1}-t_i)}{\lambda^2}} d\tau_j \langle \phi_F, \lambda B_\lambda(t_i, g, u, L) \lambda B_\lambda(\lambda^2 \tau_j + t_i, g, u, L)\phi_F \rangle$$

with $t_{j-1} - t_i < 0$ and in the limit $\lambda \to 0$ the above expression is vanishing.

Deriving eq. (4.5) with respect to t we obtain:

$$\left\langle \phi_F, \Psi_\lambda(g_1, u_1) \frac{1}{\lambda} B_\lambda\left(\frac{t}{\lambda^2}, g, u\right) U_{\frac{t}{\lambda^2}} \Psi_\lambda(g_2, u_2)\phi_F \right\rangle \tag{4.6}$$

The above expression is an average over the ground state of a product of fermionic operators, and by eq. (2.4) it is given by a sum of terms in which each fermionic operator is *paired* in the sense of the preceding section with some other. We call I_1 the sum of the terms in which the fermionic operators in $B_\lambda\left(\frac{t}{\lambda^2}, g, u\right)$ are paired with the operators in $\Psi_\lambda(g_1, u_1)$, I_2 the analogue of I_3 with $\Psi_\lambda(g_2, u_2)$, I_3 the sum of terms in which the operators in $B_\lambda\left(\frac{t}{\lambda^2}, g, u\right)$ are paired with fermionic operators in $U_{\frac{t}{\lambda^2}}$; the other terms *i.e.* the terms in which the fermionic operators in $B_\lambda\left(\frac{t}{\lambda^2}\right)$ are paired with operators belonging to different B operators vanish in the $\lambda \to 0$, $L \to \infty$ limit, as follow from the computation of the preceding section.

Let us start considering the I_1 term. We need the following lemma:

LEMMA. – *It holds that:*

$$\begin{aligned} \lim_{\lambda \to 0, L \to \infty} \Big\langle \phi_F, & \lambda B_\lambda(S_1, T_1, g_1, u_1, L) \\ & \ldots \lambda B_\lambda(S_m, T_m, g_m, u_m, L) \frac{1}{\lambda} B_\lambda\left(\frac{t}{\lambda^2}, g, u, L\right)\phi_F \Big\rangle \\ & = \sum_{\tilde{i}} \chi_{S_{\tilde{i}}, T_{\tilde{i}}}(t)(g_{\tilde{i}} \otimes u_{\tilde{i}} | g \otimes u) \\ & \sum_\pi \prod_{i,j \in \pi/\tilde{i}} \langle \psi, B(T_i, S_i, g_i, u_i) B(T_j, S_j, g_j, u_j)\psi \rangle \end{aligned}$$

where $\pi/\tilde{i}$ is a partition in pairs of the indeces $1, \ldots, m$ without $\tilde{i}$ and $\sum_\pi$ is the sum over such partitions.

Proof. – The left hand side of the above equation is given, by eq. (2.4), by a sum of terms; noting that the terms in which the fermionic operators of the $B_\lambda\left(\frac{t}{\lambda^2}, g, u\right)$ operator are paired with fermionic operators with different times are vanishing in the limit $\lambda \to 0$, $L \to \infty$, the left hand side of the above equation is given by:

$$\lim_{\lambda\to 0}\sum_{\tilde{i}=1}^{m}\int_{S_{\tilde{i}}/\lambda^2}^{T_{\tilde{i}}/\lambda^2} dt_{\tilde{i}}\int dk_{\tilde{i}}dk_{\tilde{i}'}dp_{\tilde{i}}g^2_{k_{\tilde{i}}}g^2_{k_{\tilde{i}'}}g^2_{k_{\tilde{i}'}+p_{\tilde{i}}}g^2_{k_{\tilde{i}'}-p_{\tilde{i}}}$$
$$[e^{-i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}}+\varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}}-\varepsilon_{k_{\tilde{i}}}-\varepsilon_{k_{\tilde{i}'}})\left(\frac{t}{\lambda^2}-t_{\tilde{i}}\right)}$$
$$\chi_{B_F}(k_{\tilde{i}}+p_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}}-p_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}'})$$
$$+e^{i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}}+\varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}}-\varepsilon_{k_{\tilde{i}}}-\varepsilon_{k_{\tilde{i}'}})\left(\frac{t}{\lambda^2}-t_{\tilde{i}}\right)}$$
$$\chi_{B_F^c}(k_{\tilde{i}}+p_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}}-p_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}'})$$
$$\lambda^{m-1}\sum_{\pi}\prod_{i,j\in\pi/\tilde{i}}\int_{S_j/\lambda^2}^{T_j/\lambda^2} dt_j$$
$$\int_{S_i/\lambda^2}^{T_i/\lambda^2} dt_i\langle\phi_F, B_\lambda(t_j, g_j, u_j, L)B_\lambda(t_i, g_i, u_i, L)\phi_F\rangle$$
$$= \lim_{\lambda\to 0}\sum_{\tilde{i}=1}^{m}\int_{(S_{\tilde{i}}-t)/\lambda^2}^{(T_{\tilde{i}}-t)/\lambda^2} d\tau_{\tilde{i}}\int dk_{\tilde{i}}dk_{\tilde{i}'}dp_{\tilde{i}}g^2_{k_{\tilde{i}}}g^2_{k_{\tilde{i}'}}g^2_{k_{\tilde{i}'}+p_{\tilde{i}}}g^2_{k_{\tilde{i}'}-p_{\tilde{i}}}$$
$$[e^{-i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}}+\varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}}-\varepsilon_{k_{\tilde{i}}}-\varepsilon_{k_{\tilde{i}'}})\tau}\chi_{B_F}(k_{\tilde{i}}+p_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}}-p_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}'})$$
$$+e^{i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}}+\varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}}-\varepsilon_{k_{\tilde{i}}}-\varepsilon_{k_{\tilde{i}'}})\tau}\chi_{B_F^c}(k_{\tilde{i}}+p_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}})\chi_{B_F^c}(k_{\tilde{i}}-p_{\tilde{i}})\chi_{B_F}(k_{\tilde{i}'})$$
$$\lambda^{m-1}\sum_{\pi}\prod_{i,j\in\pi/\tilde{i}}\int_{S_j/\lambda^2}^{T_j/\lambda^2} dt_j$$
$$\int_{S_i/\lambda^2}^{T_i/\lambda^2} dt_i\langle\phi_F, B_\lambda(t_j,\, g_j, u_j, L)B_\lambda(t_i,\, g_i, u_i, L)\phi_F\rangle$$

where n_π/i is the subset of $1, ..i-1, i+i, ..m$ such that the lemma holds. By straightforward application of the above lemma we obtain:

$$\lim_{\lambda\to 0, L\to\infty} I_1 = (u_1\otimes g_1|u\otimes g)\chi_{[S_1,T_1]}(t)\langle\psi, \Psi_\lambda(g_1, u_1)U_t\Psi_\lambda(g_2, u_2)\psi\rangle$$
$$\lim_{\lambda\to 0, L\to\infty} I_2 = \langle u\otimes g|u_2\otimes g_2)\chi_{[S_2,T_2]}(t)\langle\psi, \Psi_\lambda(g_1, u_1)U_t\Psi_\lambda(g_2, u_2)\psi\rangle$$

In order to compute the I_3 term we need another lemma:

LEMMA. – *It holds that:*

$$\lim_{\lambda\to 0, L\to\infty} \sum_{n=0}^{\infty} \int_0^{\frac{t}{\lambda^2}} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\left\langle \phi_F, \lambda B_\lambda(t_1, g, u, L) \ldots \lambda B_\lambda(t_n, u, g, L)] \frac{1}{\lambda} B\left(\frac{t}{\lambda^2}, u, g, L\right) \phi_F \right\rangle$$

$$= (u \otimes g | u \otimes g)_- \langle \psi, u_t \psi \rangle$$

Proof. – The proof consists in showing that the terms in which $B_\lambda(t/\lambda^2, g, u, L)$ is paired with $B_\lambda(t_j, g_j, u_j, L)$ with $j \neq 1$ are vanishing in the limit $\lambda \to 0$, $L \to \infty$. The left hand side of the above expression can be written as:

$$\lim_{\lambda\to 0} \sum_{\tilde{i}=1}^{n} \int_0^{\frac{t}{\lambda^2}} dt_1 \ldots \int_{-t/\lambda^2}^{t_{\tilde{i}-1}-t/\lambda^2} d\tau_{\tilde{i}} \int_0^{t_{n-1}} dt_n$$

$$\times \int dk_{\tilde{i}} dk_{\tilde{i}'} dp_{\tilde{i}} g^2_{k_{\tilde{i}}} g^2_{k_{\tilde{i}'}} g^2_{k_{\tilde{i}'}+p_{\tilde{i}}} g^2_{k_{\tilde{i}'}-p_{\tilde{i}}}$$

$$[e^{-i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}} + \varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}} - \varepsilon_{k_{\tilde{i}}} - \varepsilon_{k_{\tilde{i}'}})\tilde{\tau}_i} \chi_{B_F}(k_{\tilde{i}} + p_{\tilde{i}}) \chi_{B_F^c}(k_{\tilde{i}}) \chi_{B_F}(k_{\tilde{i}} - p_{\tilde{i}}) \chi_{B_F^c}(k_{\tilde{i}'})$$

$$+ e^{i(\varepsilon_{k_{\tilde{i}}+p_{\tilde{i}}} + \varepsilon_{k_{\tilde{i}'}-p_{\tilde{i}'}} - \varepsilon_{k_{\tilde{i}}} - \varepsilon_{k_{\tilde{i}'}})\tilde{\tau}_i} \chi_{B_F^c}(k_{\tilde{i}} + p_{\tilde{i}}) \chi_{B_F}(k_{\tilde{i}}) \chi_{B_F^c}(k_{\tilde{i}} - p_{\tilde{i}}) \chi_{B_F}(k_{\tilde{i}'})$$

$$\lambda^{n-1} \sum_{\pi} \prod_{i,j \in \pi/\tilde{i}} \langle \phi_F, B_\lambda(t_j, g, u, L) B_\lambda(t_i, g, u, L) \phi_F \rangle$$

where $t_0 = t/\lambda^2$ and, if $\tilde{i} \neq 1$, than $t_{\tilde{i}-1} - t/\lambda^2 < 0$ and the above term is vanishing in the limit.

Using this lemma we have that:

$$\lim_{\lambda\to 0, L\to\infty} I_3 = (u \otimes g | u \otimes g)_- \langle \Psi_\lambda(g_1, u_1) U_t \Psi_\lambda(g_2, u_2) \rangle$$

and the theorem is proved.

The last part of the theorem is proved by a straightforward adaptation of [AcFriLu].

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