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Stability of the densities of invariant measures for piecewise affine expanding non-renormalizable maps

by

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ABSTRACT. – We consider piecewise affine expanding non-renormalizable interval maps, and prove the stability in L^1-norm of densities of their invariant measures. This based on the technique of studying of the Perron-Frobenius operators, developed in the paper of Baladi and Young.

Key words: Measure, map, non-renormalizable, stability, slope.

RÉSUMÉ. – On considère des applications affines par morceaux non-renormalisables, et on montre la stabilité des densités de mesure invariante pour la norme L^1. La démonstration est basée sur la technique d'opérateur de Perron-Frobenius, développée dans l'article de Baladi et Young.

1. INTRODUCTION

Dealing with a dynamical system, defined on the parameter space Π, an interesting question would be: how do the dynamical characteristics change when we move in Π. The case of piecewise affine maps on the interval is the first most natural candidate for questions of this type. Thus the object of this work will be piecewise affine expanding maps on the interval. In [3] the conjugacy classes were studied for such maps. It was shown that these classes are contained in a codimension 1 submanifold of the parameter space, in particular they have an empty interior. Another interesting question would be the structure of isentropes (*i.e.* the set of parameters for which the entropy is constant). For that, it is necessary to know how the densities of invariant measures (existence of which is known by Lasota-Yorke theorem [5]) depend on the parameters. Fortunately, there is an appropirite technique, developped in the paper by Baladi and Young [2]. There, small random pertubations of expanding maps on the interval are considered and the robustness of the their invariant densities and rates of mixing were proved. There are two subcases: the case of non-periodic critical points, and the case with periodic critical points. In the second case, sufficiently large slope is necessary to provide stability (more precisely, slope greater than 2). In our case of non-renormalizable piecewise affine maps on the interval, we show that usual expansion is enough, magically, the condition of being in the interior of non-renormalizability gives all indispensable! However, there exists an example in [2] (originally studied in [4]) of a map with a slope 2 everywhere, which is on the boundary of renormalizabilty, and which is NOT stochastically stable, not even stable in the sense of weak convergence [4] (for definitions of different kind of stability *see* [2]). In the rest, many given proofs are a verification of the corresponding proofs in [2], the real difference being in the case of periodic critical points.

2. DEFINITIONS AND MAIN RESULT

The continuous map $f : I \to I, I = [0,1]$ is *piecewise affine*, if there are points: $0 = a_0 < a_1 < ... < a_d = 1$ such that $f_{|[a_i a_{i+1}]}$ is affine and $Df_i Df_{i+1} < 0$. The points $a_0 ..., a_d$ are critical. With $J \subset I$ an interval and $n \geq 1$, the pair (J, n) is called a *renormalization*, if $f^n(J) \subset J, \bar{J} \neq I$ and the interiors $f^i(J), i = 0, ..., n-1$ are pairwise disjoint. A map which has a renormalization, is called *renormalizable*. The orbit $\bigcup f^i(J)$ is called a *cycle*. A cycle is *minimal* if $f^n_{|J}$ is non-renormalizable.

Let F_d be the family of d-modal piecewise affine maps. If $f, g \in F_d$, then the distance is defined by:

$$d(f,g) = \sup_{x \in [0,1]} \mid f(x) - g(x) \mid . \tag{1}$$

The map $f \in F_d$ is *eventually expanding* if there exists n such that the slope of n-th iteration of f is everywhere greater than 1. Let now $E_d \subset F_d$ be the space of d-modal piecewise affine eventually expanding maps, and NR the interior (with respect to the metric defined above) of the set of nonrenormalizable eventually expanding d-modal piecewise affine maps, $NR \subset E_d \subset F_d$. We know [7] that NR is non-empty, and that every eventually expanding piecewise affine map is non-renormalizable, or has finitely many minimal cycles [3]. Now, suppose f has periodic critical points a_i of period m_i. This would mean that the graphs of f^{m_i} touch the diagonal at a_i. If $f \in NR$, then a small enough perturbation (with respect to the metric (1)) will be still non-renormalizable, which gives the following condition (*see* Figure) on the slopes $s_{l,i}$ and $s_{r,i}$ of f^{m_i} to the right and to the left of the critical point a_i

$$1/ \mid s_{l,i} \mid +1/ \mid s_{r,i} \mid < 1 \tag{2}$$

We give some definitions: For $\phi : I \to C$ the total variation of ϕ on the interval $[a, b]$ is defined to be:

$$var_{[a,b]}\phi = \sup \Sigma_{i=0}^{n} \mid \phi(x_{i+1}) - \phi(x_i) \mid : \\ n \geq 1, \qquad a \leq x_0 < x_1 < ... < x_n \leq b \tag{3}$$

$\mid \phi \mid_1 := \int \mid \phi \mid$ denotes the L^1 norm of ϕ with respect to Lebesgue measure. Let $BV := \{\phi : I \to C : var\phi < \infty\}$. Sometimes one considers the Banach space $(BV, \|\|)$ where $\| \phi \| = var\phi + \mid \phi \mid_1$. In [2] the another norm was used: $\|\|_\gamma = \gamma var\phi + \mid \phi \mid_1$. $\gamma \leq 1$

Let L be the Perron-Frobenius operator, associated with f acting on $(BV, \|\|)$

$$L\phi(x) = \sum_{f(y)=x} \frac{\phi(y)}{\mid f'(y) \mid} \tag{4}$$

Let $\rho_{0,f}$ be an eigenfunction for eigenvalue 1, $\| \rho_{0,f} \|_1 = 1$. Then $\rho_{0,f}$ is the density of an invariant probability measure μ_0 for f. If $f \in NR \subset E_d$, it is known, that such measure exists [5] and is unique [3], and since f is topologically mixing [3] and $f \in E_d$, the absolutely continuous invariant measure is mixing as well [6]. Let $\rho_{0,g}$ be the corresponding density for g. The main result is

THEOREM 1. – *If $f, g \in F_d$ are two d-modal piecewise affine maps on the interval, $f \in NR$ is eventually expanding and is in the interior of non-renormalizable maps, and $d(f, g) \to 0$, then $\mid \rho_{0,f} - \rho_{0,g} \mid_1 \to 0$.*

3. PROOF

In the proof we follow the arguments in [2] of the proof of theorem 3 and 3'. We need to verify a sequence of lemmas from [2]. The first one is obvious:

LEMMA 1. – *For fixed* $n \geq 1$ *and* $\phi \in L^1$

$$| L_g^n \phi - L_f^n \phi |_1 \rightarrow 0 \text{ as } g \rightarrow f$$

The derivative of f and g is not well defined at the turning point. We put

$$| h'(a_i) | = \min\{| h'_+(a_i) |, | h'_-(a_i) |\} \tag{5}$$

where $h'_+(a_i) = \lim_{x \downarrow a_i} h'(x)$, $h'_-(a_i) = \lim_{x \uparrow a_i} h'(x)$, $h = f, g$
Let

$$\theta_f = \lim_{n \rightarrow \infty} \sup \left| \left(\frac{1}{(f^n)'} \right) \right|^{1/n}$$

We consider the partitions Z_g^n, Z_f^n of I into closed intervals of monotonicity of g^n and f^n correspondingly [*see* details in [2]]. If $d(f, g) < \epsilon_f(n) \exists$ a surjective map

$$\psi : \{\eta_g \in Z_g^n\} \rightarrow \{\eta_f \in Z_f^n\}$$

$$\text{and } 1 \leq \sharp \psi^{-1}(\eta_f) \leq 2^n$$

$\forall \eta_f \in Z_g^n \ \exists \eta_g \in \psi^{-1}(\eta_f)$ such that $f^n_{|\eta_f}$ and $g^n_{|\eta_g}$ have the same dynamics. Let $A = f^n(\eta_f) \cap g^n(\eta_g)$ and let $\eta_{G,f} = (f^n_{|\eta_f})^{-1}(A)$ and $\eta_{G,g} = (g^n_{|\eta_g})^{-1}(A)$. The intervals $\eta_{G,f}$ and $\eta_{G,g}$, corresponding to the same dynamics, are *associated* (terminology from [2]). Let $\eta_f = \eta_{G,f} \cup \eta_{B,f}$ where $\eta_{B,f}$ is *co-respondent* of $\eta_{G,f}$. In the case, when there are periodic critical points, or some critical points are mapped to another critical points, Z_g^n can have *non-admissible* intervals without associated intervals in Z_f^n (the map ψ is not injective). We define corespondent $\eta_{B,g}$ of $\eta_{G,g}$ to be $\eta_g \setminus \eta_{G,g}$ together with half of nonadmissible intervals immediately to the left and to the right of $\eta_{G,g}$. Let $B_f = \bigcup \eta_{B,f}$ and $B_g = \bigcup \eta_{B,g}$ the sets of "bad" intervals. For fixed n the measure of B_f and B_g goes to zero, as $d(f, g) \rightarrow 0$.

Let

$$M_i := \{k : k \geq 1, f^k(a_i) \in \{a_0, ..., a_d\}\}$$

and $M = \max_i M_i$. If f has periodic critical points $f^{m_i}(a_i) = a_i$, with m_i is minimal, and $s_{l,i}, s_{r,i}$ being the slopes to the left and to the right of f^{m_i}, we define:

$$\bar{\theta} = \max\left(\theta_f, sup_{a_i periodic} \left(\frac{1}{| s_{l,i} |} + \frac{1}{| s_{r,i} |} \right)^{1/m_i} \right),$$

It is essential that because of the condition (2), $\bar{\theta} < 1$. If f has no periodic critical points, (in which case $M < \infty$) we put $\bar{\theta} = \theta_f$. The basic ingredient in the proof of theorem 1 is the analog of lemma 9 in [2].

LEMMA 2. – *Let* $f \in NR$ *and* $\bar{\theta} < \Lambda^2 < 1$ *Then* $\exists C > 0$ *and* $N_0 \in Z^+$ *such that* $\forall n \geq N_0$ $\exists \epsilon(n) > 0$ *such that* $\forall g \in F_d$ *and* $d(f,g) < \epsilon < \epsilon(n)$ *we have*

$$\| L_f^n - L_g^n \|_{\Lambda^n} < C\Lambda^n$$

Proof. – For $\forall n$

$$\| L_f^n(\phi) - L_g^n(\phi) \| \leq \| L_g^n(\phi\chi_{B_g}) \| + \| L_f^n(\phi\chi_{B_f}) \| \\ + \| L^n\phi(\chi_{I\setminus B_f}) - L_g^n(\phi\chi_{I\setminus B_g}) \| \qquad (6)$$

Remark 1. – If $\theta_f < \tilde{\theta} < 1$ then $\exists$ n_0 such that $\sup\left|\frac{1}{(f^n)'}\right| \leq \tilde{\theta}^n$ for $\forall n \geq n_0$ $\exists n_1$ such that if $n \geq n_1$ and $\epsilon = \epsilon(n)$ is small enough, then $\sup\left|\frac{1}{(g^n)')}\right| \leq \tilde{\theta}^n$.

Remark 2. – If f, g are piecewise affine d-modal maps, then from the closeness of f and g in the uniform norm (1) follows that $| f' |$ and $| g' |$ are also close on the corresponding intervals of monotonicity.

Let's first estimate the "L^1" part of the norm. We have: for each subinterval of monotonicity $\eta \in B_f$

$$L_f^n(\phi\chi_\eta(x)) = \frac{\phi(y)}{| (f^n)'(y) |} \qquad \text{with} \quad x = f^n(y),\ x \in f^n\eta$$

then $| L_f^n(\phi\chi_\eta) |_1 \leq \int_\eta | \phi | \leq l(\eta)(var\phi + | \phi |_1)$. Summing over all intervals η we get

$$| L_f^n(\phi\chi_{B_f}) |_1 \leq c_{n,\epsilon}(var\phi + | \phi |_1) \qquad (7)$$

with $c_{n,\epsilon} \to O$ as $\epsilon \to 0$ $\forall n$ fixed, because the measure of the bad part B_f goes to zero. Similar estimates holds for g.

$$| L_g^n(\phi\chi_{B_g}) |_1 \leq c_{n,\epsilon}(var\phi + | \phi |_1) \qquad (8)$$

Now, let us estimate the "L^1" part of the last term in (6).

$$| L_f^n(\phi\chi_{I\setminus B_f}) - L_g^n(\phi\chi_{I\setminus B_g}) |_1 \leq 2\tilde{\theta}^n var\phi + c_{n,\epsilon} | \phi |_1 \qquad (9)$$

where $\theta < \tilde{\theta} < 1$, $c_{n,\epsilon} \to 0$ $\forall n$ fixed.

Indeed, let $x \in f^n(I \setminus B_f)$. By construction, of B_f, B_g $\exists\{\eta_1 ... \eta_N\} \subset I \setminus B_f$ and $\{\eta'_1 ... \eta'_N\} \subset I \setminus B_g$ such that $\forall \eta_i, \eta'_i$ are associated monotonicity intervals for f^n, g^n correspondingly, and $f^n(\eta_i) = g^n(\eta_i)$. Let

$$x \in f^n(\eta_{i_0}) = g^n(\eta'_{i_0})$$

Let y_f, y_g be the corresponding preimages

$$f^n(y_f) = x = g^n(y_g)$$

Then $d(y_f, y_g) = \tilde{c}_{n,\epsilon} \to 0$ as $\epsilon \to 0$ $\forall n$
and

$$L_f^n(\phi\chi_\eta)(x) = \frac{\mid \phi(y_f) \mid}{\mid (f^n)'(y_f) \mid} \leq (\mid \phi(y_g) \mid + var_{\eta \cup \eta'} \phi)(\frac{1}{\mid (g^n)'(y_g) \mid} + c_{n,\epsilon}) \tag{10}$$

To get (9), it suffices to use $1/\mid (g^n)'(y_g) \mid \leq \tilde{\theta}^n$. And, finally, the last, most difficult term, giving the variation over the whole interval:

$$var(L_f^n \phi - L_g^n \phi) \leq var L_f^n \phi + var L_g^n \phi \tag{11}$$

$$var L_h^n \phi \leq \tilde{\theta}^n var\phi + D \mid \phi \mid_1, h = f, g \tag{12}$$

Let η' be a monotonicity interval of f^n. We have:

$$var L_f^n(\phi\chi_{\eta'}) \leq var_{\eta'} \phi \sup_{\eta'} \frac{1}{(f^n)'} + var_{\eta'} \frac{1}{(f^n)'} \sup_{\eta'} \mid \phi \mid + 2 \sup_{\eta'} \mid \phi \mid$$

$$\sup_{\eta'} \frac{1}{(f^n)'}$$

$$= var_{\eta'} \phi \sup_{\eta'} \frac{1}{(f^n)'} + \sup_{\eta'} \mid \phi \mid (2 \sup_{\eta'} \frac{1}{\mid (f^n)' \mid} + var_{\eta'} \frac{1}{\mid (f^n)' \mid})$$

Let $\bar{\eta}$ be the smallest interval, containing η' and the associated interval $\eta \in Z_g^n$. Then

$$var L_f^n(\phi\chi_{\eta'}) \leq var_{\bar{\eta}} \phi \sup_{\eta'} \frac{1}{\mid (f^n)' \mid} + (var_{\bar{\eta}} \phi + \inf_{\bar{\eta}} \mid \phi \mid) \dot{4} \sup_{\eta'} \frac{1}{\mid (f^n)' \mid} \tag{13}$$

We used that, since f, g are piecewise linear one has:

$$var_{\eta'} \mid \frac{1}{(f^n)'} \mid \leq 2 \sup_{\eta'} \frac{1}{\mid (f^n)' \mid}$$

Correspondingly, for g

$$varL_g^n(\phi\chi_\eta) \leq 5var_{\bar{\eta}}\phi \sup_\eta \mid \frac{1}{(g^n)'} \mid +4 \inf_{\bar{\eta}} \mid \phi \mid \sup_\eta \mid \frac{1}{(g^n)'} \mid \quad (14)$$

We recall that η is the interval of monotonicity for g. Let $n_2 = \max\{n_1, n_0\}$, so for $n \geq n_2$ we have:

$$\sup_{\eta'} \mid \frac{1}{(f^n)'} \mid\leq \tilde{\theta}^n \quad \text{and} \quad \sup_\eta \mid \frac{1}{(g^n)'} \mid\leq \tilde{\theta}^n$$

Then for $n = n_2$, and $l(\bar{\eta})$ being the length of $\bar{\eta}$, we have:

$$\begin{aligned} varL_g^n(\phi\chi_\eta) &\leq 5var_{\bar{\eta}}\phi\tilde{\theta}^n + 4l(\bar{\eta}) \inf_{\bar{\eta}} \mid \phi \mid \frac{\sup_{\bar{\eta}} \mid \frac{1}{(g^n)'} \mid}{l(\bar{\eta})} \\ &\leq 5var_{\bar{\eta}}\phi\tilde{\theta}^n + 4Kl(\bar{\eta}) \inf_{\bar{\eta}} \mid \phi \mid \end{aligned} \quad (15)$$

where $K = \sup_\eta \frac{\sup_\eta \mid \frac{1}{(g^n)'} \mid}{l_{n_2}}$, l_{n_2} being equal to the infimum of the lengths of admissible intervals. When $\epsilon \to 0$, $l_{n_2} \to \inf l(\eta), \eta \in Z_f^{n_2}$. Also, $l(\bar{\eta}) \inf_{\bar{\eta}} \mid \phi \mid\leq \int_{\bar{\eta}} \mid \phi \mid$. Similar estimates holds for f.

Now, we have to sum (12) and (13) over all intervals of monotonicity for f^n and g^n correspondingly. Then the sum is divided into two parts: $\sum^*$ one is over all intervals corresponding to periodic critical points, and $\sum^{**}$ over the rest. For $n = n_2$, using (14) the second sum gives:

$$\sum^{**} varL_g^n\phi \leq 2^M[5\tilde{\theta}^n var\phi + 4K \mid \phi \mid_1] \quad (16)$$

The factor 2^M appears because good intervals are overcounted at most 2^M times, where $M < \infty$ for the case of non-periodic critical points (*see* [2]), and M is independent of the number of iterations n. Correspondingly, for the the first sum we have to sum over all non-admissible intervals, "generated" by the periodic critical points a_i, with slopes being products of the slopes to the left and to the right of a_i. Here, we suppose that n_2 is a common mupliple of the m_i larger than $\max\{n_0, n_1\}$.

$$\sum^{*} varL_g^n\phi \leq 5\sum_i \sum_{p+q=[n/m_i]} (1/\mid s_{l,i} \mid^p \cdot 1/\mid s_{r,i} \mid^q)[var\phi + \tilde{D} \mid \phi \mid_1] \quad (17)$$

$$\leq C_1\bar{\theta}^n var\phi + D^* \cdot \mid \phi \mid_1 \quad (18)$$

For n arbitrary we write $n = q \cdot n_2 + r$ with $r < n_2$. Then applying to (16), (17), (18) as in [2] a standard induction argument, we get the estimate (12)

($\tilde{\theta}$ would be slightly greater than the initial constant, still less than 1, we had to increase it several times). This finishes the proof of lemma 2.

In this way we proved an analog of dynamical lemma 9 in [2]. Now theorem 1 is a consequence of lemma 1 and 2 by the arguments literally the same as in [2] section 5E Pertubation Lemmas for Abstract Operators (*see* also Erratum).

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