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Scattering poles for connected sums of Euclidean space and Zoll manifolds

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ABSTRACT. – We prove lower bounds on the number of resonances in small neighbourhoods of the real line for connected sum of Euclidean space and Zoll manifold. The obtained estimate in dimension 3

$$\#\{\lambda \text{ resonance}: \Im\lambda \leq \alpha(|\Re\lambda| + 1)^{-\beta}, |\Re\lambda| \leq r\} > C_{\alpha,\beta} r^3$$

has the highest asymptotic order, compatible with the global upper bound.

RÉSUMÉ. – Nous prouvons une borne inférieure pour le nombre de résonances dans un petit voisinage de l'axe réel pour la somme connexe de l'espace euclidien et d'une variété de Zoll. Cette estimée (dans $\mathbb{R}^3$)

$$\#\{\lambda \text{ résonance}: \Im\lambda \leq \alpha(|\Re\lambda| + 1)^{-\beta}, |\Re\lambda| \leq r\} > C_{\alpha,\beta} r^3$$

a l'ordre asymptotique maximal compatible avec la borne supérieure globale.

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1. INTRODUCTION

The distribution of poles of the resolvent of the Laplacian in $\mathbb{R}^n$ with compactly supported perturbations (known also as scattering poles or resonances) has been studied intensively in the last decades. Lax-Phillips [13] determine the asymptotic distribution of the resonances on the imaginary line in the obstacle case. Many authors (*see* [15], [17], [21], [8], [23]) give global upper bounds on the number of resonances for diverse cases of scattering. On the other hand, it is important to provide lower bounds (or even existence results) for the number of resonances in fixed regions of the complex plane. Obviously, it is of special importance to study the influence of the dynamical structure of the perturbation on the number of resonances near the continuous spectrum (*see* e.g. the survey of M. Zworski [23]). Several contributions [7], [10], [11], [4], [5], [6], [16] clarify the situation around the Lax-Phillips conjecture – the existence of a sequence of resonances converging to the real line.

Here we provide the first (as far as we know) elliptic case of scattering by a compact perturbation in $\mathbb{R}^3$ where the order of the lower estimate on the number of resonances near the real line (*see* Theorem 3.1) coincides with the upper bound for all resonances. This phenomenon is observed on "metric" connected sums of Euclidean space and manifolds all of whose geodesics are periodic with the same prime period. We show that the special dynamical properties of such a metric (and topological) perturbation imply that a very substantial part of the resonances lie between the real line and a curve, converging (polynomially) to $\mathbb{R}$.

The class of compact perturbations of the Laplacian is obtained by cutting off a geodesic ball of a fixed radius from $\mathbb{R}^3$ and from a 3-dimensional Zoll manifold M (*see* Section 2) and by gluing the resulting spheres together. Scattering on the so obtained Riemannian manifold in the case when M is the 3-sphere with radius R was studied by Sjöstrand and Zworski in [18], Example 3 (*see* also [23]). They proved that if R is sufficiently large then one can estimate the number of the resonances between some logarithmic curve and the real line in the following way:

$$\sum_{|\Re\lambda|\leq r;\ |\Im\lambda|<(2\pi Rk-\varepsilon)^{-1}\ln|\lambda|} e^{-(2\pi Rk-\varepsilon)|\Im\lambda|} > C_R r^3 - o_{R,k,\varepsilon}(r^3),$$

where $\varepsilon > 0$, $k \in N$ are arbitrary and the summation is over all scattering poles λ. It follows in particular that the upper bound $(Cr^3 + C)$ for the number of all poles in $\{|z| < r\}$ obtained by Vodev in [20], [21] (*see* also [8]) is sharp. Moreover the above estimate was used by Petkov and Vodev

[16] who proved the Lax-Phillips conjecture in this case (they prove the existence of a sequence of resonances converging to the real line).

Our main result (*see* Theorem 3.1) is that for perturbations by Zoll manifolds with sufficiently long geodesics (with length $2\pi R$, $R > 0$ – sufficiently large), the number of scattering poles in small polynomial neighborhoods of the real axis

$$E_{\alpha,\beta}(r) = \{z \in \mathbb{C} : 0 \leq \Im z \leq \alpha(|\Re z| + 1)^{-\beta}, \ |\Re z| \leq r\}, \ \alpha, \beta > 0$$

is greater than $C_{\alpha,\beta}r^3$. Thus, the counting function for the resonances in $E_{\alpha,\beta}(r)$ has the maximal possible asymptotic order in view of the global upper bound. In particular we give two topologically different examples of perturbations for which the above result holds.

Our proof follows a method which originates in the work of Sjöstrand and Zworski [18] and it is based on a careful study of the singularities of the trace distribution. We give a lower estimate for the quantity $|\widehat{\varphi_q u_R}(\lambda)|$ uniformly on λ and q (here u_R is the wave trace for the perturbation and the support of $\varphi_q \in C_c^\infty$ is concentrated around $2\pi qR$, $q \to \infty$). The specific form of the perturbation combined with results from [18] lead to the conclusion that we must estimate only $|\widehat{\varphi_q v_R}(\lambda)|$, where v_R is the wave trace on the compact manifold. We pay special attention to obtain the explicit dependence of the remainder terms with respect to q and to do this we use the fact that on Zoll manifolds the wave trace is an approximately periodic distribution. We count the resonances near the real axis using a method similar to those in [4].

2. PERTURBATION

Let M be a compact smooth 3-dimensional manifold with a Riemannian metric g_R such that all geodesics of (M, g_R) are simple closed of length $2\pi R$. By a slight abuse of general usage we call such an object a Zoll manifold. For each odd dimension there are two *topologically different* rank one symmetric spaces. They are the classical examples of manifolds whose geodesics are all periodic with the same minimal period. Explicitly we have the sphere $S^n(R)$ with radius R in the Euclidean $(n+1)$-dimensional space and the real projective space $\mathbb{RP}^n(R)$, which is obtained as factor $S^n(2R)/\mathbb{Z}_2$ (note that the radius of the sphere is chosen to be $2R$, so that the length of the closed geodesics on $\mathbb{RP}^n(R)$ to be $2\pi R$). The eigenvalues of the Laplacian on these manifolds and their multiplicities are (*see* [1]) for $S^3(R)$:

$$\lambda_{k,R} = (k^2 + 2k)/R^2,\ k = 0, 1, 2, \ldots,$$
$$N_{k,R} = (k + 1)^2,\ k = 0, 1, 2, \ldots$$

for $\mathbb{RP}^3(R)$:

$$\lambda_{k,R} = (k^2 + k)/R^2,\ k = 0, 1, 2, \ldots,$$
$$N_{k,R} = (2k + 1)^2,\ k = 0, 1, 2, \ldots$$

Denote by α the Maslov index along the closed geodesics on M. The square roots

$$\mu_{k,R} = \sqrt{\lambda_{k,R} - \alpha/(4R)}$$

of the shifted eigenvalues $\lambda_{k,R}$ (repeated according to their multiplicity) of the Laplacian on a general Zoll manifold cluster around an arithmetic progression in the following manner [9], Section 29.2 (*see* also [22]):

(H) *There exist positive constants* j_R, k_R *and* C_R *such that:*

$$\{\mu_{j,R},\ j > j_R\} \subset \bigcup_{k=k_R+1}^{\infty} \{x \in \mathbb{R} :\ |x - k/R| \leq C_R/k\},$$

and for $c_R > c > 0$ *in any interval*

$$I_k = \{x \in \mathbb{R} :\ |x - k/R| \leq C_R/k\}$$

for $k > k_R$ *there exist exactly* $c_R k^2 + \mathcal{O}_R(k)$ *elements* $\mu_{j,R}$.

Since Condition (H) holds for a large class of perturbations of the Laplacian (in particular – Stark effect) [9], [22] the following proofs hold also for such perturbations.

Remark. – Note that in [3] it is proved that property (H) of the spectrum characterizes Zoll manifolds.

Let $\pi : T^*M \to M$ be the cotangent bundle of M and let us denote by UM_R the bundle of unit spheres in T^*M. We shall use the canonical Riemannian metric g_1 on UM_R as defined in [2], Ch. 1, M.

Let $s > 0$ and $x \in M$. Denote by $B_R(x, s)$ the geodesic ball of radius s around x in (M, g_R) and by $B(0, s)$ the geodesic ball of radius s around 0 in the Euclidean space $\mathbb{R}^3$. In what follows we fix $x \in M$ and $s > 0$ and we assume that $s < R$ and it is so small that $B_R(x, s)$ is contained in the domain of normal geodesic coordinates centered at x.

We need the following technical fact, where we use the volume corresponding to the metric g_1.

LEMMA 2.1. – *We have the estimate:*

$$\mathrm{vol}_{UM_R}(m \in UM_R : \ \pi \circ \Phi_t(m) \in B_R(x,s) \ for\ some\ t) = \mathcal{O}_r(R),$$

where Φ_t is the geodesic flow on UM_R.

We skip the easy proof.

Now we define the perturbed manifold (X, G_R). Let

$$W_R = B_R(x,s) \setminus B_R(x,s/2) \text{ and } W = B(0,s) \setminus B(0,s/2)\,.$$

Denote for each $p > 0$:

$$S_R(y,p) = \partial B_R(y,p) \text{ and } S(0,p) = \partial B(0,p)\,,$$

where $y \in M$. Obviously

$$\partial W_R = S_R(x,s) \cup S_R(x,s/2) \text{ and } \partial W = S(0,s) \cup S(0,s/2)\,.$$

We identify $W_R \subset M$ and $W \subset \mathbb{R}^3$ so that $S(0,s/2)$ is identified with $S_R(x,s)$ and $S(0,s)$ is identified with $S_R(x,s/2)$. Obviously this gives us a smooth (and even real analytic) structure on the connected sum X of M and $\mathbb{R}^3$. To define the Riemannian structure G_R on X we glue the metrics from (M, g_R) and $\mathbb{R}^3$ by using a partition of the unity subordinated to the open covering:

$$X = \left(M \setminus \overline{B_R(x,s/2)}\right) \cup \left(\mathbb{R}^3 \setminus \overline{B(0,s/2)}\right).$$

In the rest of this paper we shall study scattering on the Riemannian manifold (X, G_R).

3. SCATTERING POLES

Denote by H_R the Laplacian on the manifold (X, G_R) or the Laplacian with an additional perturbation supported on $M \setminus B_R(x,s)$ (considered as a part of (X, G_R)), which does not change the principal symbol. Obviously H_R is an admissible compact perturbation of the free Laplacian in $\mathbb{R}^3$ in the sense of Definition 1 from [8] (*see* also [21]). Let

$$R_\chi(z) = \chi R(z)\chi, \ z \in \{z \in \mathbb{C} : \ \Im z < 0\}$$

where $R(z) = (H_R - z^2)^{-1}$ and $\chi \in C_c^\infty$ is equal to 1 in a neighborhood of the perturbation.

The cutoff resolvent R_χ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ which we denote also by R_χ. The poles of $R_\chi(z)$ are called *resonances* or *scattering poles*.

Denote by Λ_R the set of all scattering poles. Then $\Lambda_R \subset \{\Im z > 0\}$ and since the perturbation is elliptic we have [21], [8]:

$$N_R(r) = \#\{\lambda \in \Lambda_R : \ |\lambda| \leq r\} \leq C_R r^3 + C_R\,, \tag{3.1}$$

where $C_R > 0$ and the poles are counted according to their multiplicities.

Motivated by the trace formula (of Poisson type) proved by Melrose [14] and in full generality by Sjöstrand-Zworski [19] we denote by $u_R(t)$ the distribution

$$u_R(t) = \begin{cases} \dfrac{1}{2} \sum\limits_{\lambda \in \Lambda_R} e^{i\lambda t}, \ t > 0\,; \\ \dfrac{1}{2} \sum\limits_{\lambda \in \Lambda_R} e^{-i\lambda t}, t < 0\,. \end{cases}$$

and introduce a counting function for $\alpha, \beta > 0$:

$$N_{\alpha,\beta}^R(r) = \#\{\lambda \in \Lambda_R : \ \Im\lambda \leq \alpha(|\Re\lambda| + 1)^{-\beta}, \ |\Re\lambda| \leq r\}\,.$$

In what follows we fix a function $\varphi(t) \in C_c^\infty(-1, 1)$, such that for $\lambda \in \mathbb{R}$:

$$0 \leq \varphi(t), \ \varphi(0) = (2\pi)^{-1}, \ \hat{\varphi}(-\lambda) = \hat{\varphi}(\lambda) \geq 0$$

and we pose for some $1 > \varepsilon > 0$ and $T > 0$:

$$\varphi_{\varepsilon,T}(t) = \varphi(\varepsilon^{-1}(t - 2\pi T))\,.$$

The reasoning from [18] combined with Lemma 2.1 gives us the following estimate for small $\varepsilon > 0$

$$|\widehat{\varphi_{\varepsilon,qR} u_R}(\lambda)| \geq |\widehat{\varphi_{\varepsilon,qR} v_R}(\lambda)| + \mathcal{O}(R\lambda^2) + \mathcal{O}_R(\lambda)\,, \tag{3.2}$$

where

$$v_R = \sum_{j=1}^{\infty} e^{i\mu_{j,R} t}, \ t > 0$$

and $\mu_{j,R}$ are the square roots of the shifted eigenvalues of the Laplacian on the Zoll manifold M, repeated according to their multiplicity. As was pointed out in Section 2 the numbers $\mu_{j,R}$ satisfy Condition (H).

The main result which we prove in this work is

THEOREM 3.1. – *Let* $0 < \alpha$ *and* $0 < \beta < 1$. *For all sufficiently large* $R > 0$ *we have the following lower bound on the number of resonances in* Λ_R *near the real axis:*

$$N^R_{\alpha,\beta}(r) > C_{R,\alpha,\beta} r^3 ,$$

where the positive constant $C_{R,\alpha,\beta}$ *is independent of* r.

Remark. – In the course of the proof of Theorem 3.1 we use results from [4] and [5] – all done in space dimension 3. For that reason, here we study only this case. Nevertheless, it is possible to extend the result for any odd dimension $n > 3$ and we expect that the corresponding estimate will be

$$N^R_{\alpha,\beta}(r) > C_{R,\alpha,\beta} r^n$$

with a constant $C_{R,\alpha,\beta} > 0$ independent of r.

4. ESTIMATE

In this section we prove Theorem 3.1.

First we shall estimate the remainder terms in the Poisson summation formula:

PROPOSITION 4.1. – *For any* $\lambda, R > 0$ *and* $\varepsilon > 0$ *sufficiently small we have:*

$$|\widehat{\varphi_{\varepsilon,qR} v_R}(\lambda)| > C_\varepsilon R^3 \lambda^2 - C_{R,\varepsilon} q \lambda ,$$

where $0 < C_\varepsilon, C_{R,\varepsilon}$ *do not depend on* λ *and* q *and* C_ε *do not depend on* R.

Proof. – We compare $v_R(t)$ with

$$v_0(t) = \sum_{j=0}^{\infty} (j+1)^2 e^{i(t/R)(j+1)} = \sum_{j=1}^{\infty} j^2 e^{i(t/R)j} ,$$

which is a periodic distribution and can be calculated. Namely, from the Poisson summation formula ([9], Section 7.2) we have

$$\sum_{j=-\infty}^{\infty} e^{i(t/R)j} = (2\pi R) \sum_{j=-\infty}^{\infty} \delta(t - 2j\pi R) .$$

Hence:

$$v_0(t) = -2\pi R^3 \sum_{j=-\infty}^{\infty} \delta''(t - 2j\pi R) - \sum_{j=0}^{\infty} j^2 e^{-itj/R}.$$

We pose

$$w(t) = \sum_{j=0}^{\infty} j^2 e^{-itj/R}$$

and prove the following

LEMMA 4.2. – *For any* $\lambda, \varepsilon, R > 0$ *we have the estimates:*

(i) $|\widehat{\varphi_{\varepsilon,qR} w}(\lambda)| \leq C_{\varepsilon,R}$;

(ii) $|[\widehat{\varphi_{\varepsilon,qR} v_R} - \widehat{\varphi_{\varepsilon,qR} v_0}](\lambda)| \leq C'_{\varepsilon,R} q\lambda$;

where the positive constants $C_{\varepsilon,R}, C'_{\varepsilon,R}$ *are independent of* λ *and* q.

Proof. – (i) From the Paley-Wiener estimate we get:

$$|\widehat{\varphi_{\varepsilon,qR}}(\xi)| \leq \varepsilon C_M (1 + |\varepsilon\xi|)^{-M}, \ \xi \in \mathbb{R} \text{ for all } M,$$

where $C_M > 0$ is independent of ε, q and ξ. Hence for $0 < \varepsilon < 1$ and $\lambda > 0$

$$\begin{aligned} |\widehat{\varphi_{\varepsilon,qR} w}(\lambda)| &\leq \sum_{j=1}^{\infty} j^2 |\widehat{\varphi_{\varepsilon,qR}}(\lambda + j/R)| \\ &\leq \sum_{j=1}^{\infty} \frac{C_4 j^2}{(1 + |\varepsilon(\lambda + j/R)|)^4} \leq C_\varepsilon \sum_{j=1}^{\infty} \frac{j^2}{(1 + \varepsilon j/R)^4}, \end{aligned}$$

where $C_\varepsilon > 0$ depends on φ, but it is independent of λ and q. Since the series in the right-hand side converges we get (i).

(ii) It will become clear from the reasoning below that in Condition (H) for Λ_R in Section 2 we may use the term k^2 instead of $c_R k^2 + \mathcal{O}_R(k)$. We have:

$$\begin{aligned} \widehat{\varphi_{\varepsilon,qR} v_R}(\lambda) - \widehat{\varphi_{\varepsilon,qR} v_0}(\lambda) &= \sum_{j=1}^{j_R} \widehat{\varphi_{\varepsilon,qR}}(\lambda - \mu_{j,R}) \\ &\quad - \sum_{j=1}^{k_R} \widehat{\varphi_{\varepsilon,qR}}(\lambda - \frac{j}{R}) + \sum_{k=k_R+1}^{\infty} A_{k,R,q,\varepsilon}(\lambda), \quad (4.1) \end{aligned}$$

where

$$A_{k,R,q,\varepsilon}(\lambda) = \int e^{-it\lambda}\varphi_{\varepsilon,qR}(t)\sum_{m=1}^{k^2}\left[e^{it\mu_{k,m}} - e^{itk/R}\right]dt$$

and $\mu_{k,1}, \ldots, \mu_{k,k^2}$ are the elements in the interval I_k (*see* Condition (H)). Now we estimate $A_{k,R,q,\varepsilon}(\lambda)$ using the obvious equality

$$e^{ix} - e^{iy} = 2i\sin\frac{x-y}{2}e^{i\frac{x+y}{2}}$$

and for $A_{k,R,q,\varepsilon}(\lambda)$ we obtain:

$$A_{k,R,q,\varepsilon}(\lambda) = 2i\sum_{m=1}^{k^2}\int e^{-it[\lambda-(\mu_{k,m}+k/R)/2]}\sin\frac{t(\mu_{k,m}-k/R)}{2}\varphi_{\varepsilon,qR}(t)\,dt\,.$$

The Paley-Wiener estimate gives for any integer $M > 0$:

$$\begin{aligned}|A_{k,R,q,\varepsilon}(\lambda)| &\leq \sum_{m=1}^{k^2}\frac{C\max\limits_{t\in\operatorname{supp}\varphi_{\varepsilon,qR}}\sum\limits_{j=0}^{M}\left|\partial_t^j\left(\sin\frac{t(\mu_{k,m}-k/R)}{2}\varphi_{\varepsilon,qR}(t)\right)\right|}{(1+|\lambda-(\mu_{k,m}+k/R)/2|)^M}\\ &\leq \sum_{m=1}^{k^2}\frac{C_{M,R,\varepsilon}qk^{-1}}{(1+|\lambda-(\mu_{k,m}+k/R)/2|)^M}\,.\end{aligned}$$

Then we get:

$$\sum_{k=k_R+1}^{\infty}|A_{k,R,q,\varepsilon}(\lambda)| \leq \sum_{k=k_R+1}^{\infty}\left(\sum_{m=1}^{k^2}\frac{C_{M,R,\varepsilon}qk^{-1}}{(1+|\lambda-(\mu_{k,m}+k/R)/2|)^M}\right),$$

where $C_{M,R,\varepsilon} > 0$ is independent of q, λ and k. Using the properties of $\mu_{k,m}$, described in Condition (H), it is easy to see that the term in the right-hand side is estimated from above by:

$$C_{M,R,\varepsilon}q\sum_{k=k_R+1}^{\infty}k^2\frac{k^{-1}}{(1+|\lambda-k/R|)^M} \leq C'_{M,R,\varepsilon}q\lambda^1\,,$$

where $C'_{M,\varepsilon,R} > 0$ is independent of q and λ.

Since the first two sums in (4.1) are rapidly decreasing with respect to λ, uniformly on q, the lemma is proved. $\diamondsuit$

We continue with the proof of the proposition. Applying Lemma 4.2 and the formula for $v_0(t)$ we get for small $\varepsilon > 0$:

$$\begin{aligned}|\widehat{\varphi_{\varepsilon,qR}v_R}(\lambda)| &\geq |\widehat{\varphi_{\varepsilon,qR}v_0}(\lambda)| - |\widehat{\varphi_{\varepsilon,qR}v_R}(\lambda) - \widehat{\varphi_{\varepsilon,qR}v_0}(\lambda)| \\ &> |\widehat{\varphi_{\varepsilon,qR}v_0}(\lambda)| - C_{\varepsilon,R}q\lambda \\ &> (2\pi)R^3\big|(\delta''(\cdot - 2q\pi R), e^{-it\lambda}\varphi_{\varepsilon,qR}(\cdot))\big| - C_{\varepsilon,R} - C'_{\varepsilon,R}q\lambda \\ &> R^3\lambda^2 C_\varepsilon - C_{\varepsilon,R} - C'_{\varepsilon,R}q\lambda\,.\end{aligned}$$

This proves the proposition. ◇

We need the counting function:

$$N_\rho^R(r) = \#\{\lambda \in \Lambda_R : \Im\lambda \leq \rho \ln|\lambda|,\ |\Re\lambda| \leq r\};\ \rho > 0\,.$$

Since the global upper bound for the poles in Λ_R is given in (3.1) and hence it is $\mathcal{O}(r^3)$ we have the following result, proved in [4] (Corollary 2.4 from [4]):

PROPOSITION 4.3. – *Let Λ_R be the set of all scattering poles. Then for any $r > 2$ we have the estimate:*

$$N^R_{5/(2\pi Rq)}(r) > C\int_1^{r/2} |\widehat{\varphi_{\varepsilon,qR}u}(\lambda)|\,d\lambda - C_\varepsilon\,, \tag{4.2}$$

where C and C_ε are positive constants independent of r and q.

Proof of Theorem 3.1. – Using (3.2) and Proposition 4.1 we obtain

$$\begin{aligned}|\widehat{\varphi_{\varepsilon,qR}u}(\lambda)| &> C_\varepsilon R^3\lambda^2 - C_{\varepsilon,R}q\lambda + \mathcal{O}(R\lambda^2) + \mathcal{O}_R(\lambda) \\ &= \lambda^2R^3\big[C_\varepsilon + \mathcal{O}(R^{-2})\big] + \mathcal{O}_R(\lambda) - C_{\varepsilon,R}\lambda q\,.\end{aligned}$$

Then if R is sufficiently large we get

$$|\widehat{\varphi_{\varepsilon,qR}u}(\lambda)| > C_{\varepsilon,R}\lambda^2 - C'_{\varepsilon,R}\lambda q\,, \tag{4.3}$$

where the positive constants $C_{\varepsilon,R}$ and $C'_{\varepsilon,R}$ are independent of λ and q.

Now we shall use a method similar to those from Section 4 from [4] to complete the proof of the theorem. From (4.2) and (4.3) we obtain:

$$N^R_{5/(2\pi Rq)}(r) > C_{\varepsilon,R}r^3 - C'_{\varepsilon,R}qr^2\,, \tag{4.4}$$

where $C_{\varepsilon,R}, C'_{\varepsilon,R} > 0$ are independent of r and q.

We fix $\alpha > 0$ and $1 > \beta > 0$ and following [4] we denote

$$G_{\alpha,\beta} = \{z \in \mathbb{C};\ \Im z \leq \alpha(|\Re z| + 1)^{-\beta}\}$$

and choose a constant $C_{\alpha,\beta,\beta'}$ in such way that

$$G_{\alpha,\beta,\beta'} = \{z \in \mathbb{C} :\ 0 \leq \Im z \leq C_{\alpha,\beta,\beta'} |\Re z|^{-\beta'} \ln |z|,\ |\Re z| \geq 1\} \subset G_{\alpha,\beta}\,,$$

where $0 < \beta < \beta' < 1$. We pose also:

$$S_\rho = \{z \in \mathbb{C} :\ \Im z = \rho \ln |z|,\ |\Re z| \geq 1\}\,;$$
$$S_{\alpha,\beta,\beta'} = \{z \in \mathbb{C} :\ \Im z = C_{\alpha,\beta,\beta'} |\Re z|^{-\beta'} \ln |z|,\ |\Re z| \geq 1\}\,.$$

Obviously

$$|\Re(S_{\alpha,\beta,\beta'} \cap S_{5/(2\pi q R)})| = C'_{\alpha,\beta,\beta',R} q^{1/\beta'}\,,$$

where $C'_{\alpha,\beta,\beta',R}$ is positive and independent of q. Hence from (4.4) we get:

$$N^R_{\alpha,\beta}(C'_{\alpha,\beta,\beta',R} q^{1/\beta'}) > N^R_{5/(2\pi q R)}(C'_{\alpha,\beta,\beta',R} q^{1/\beta'})$$
$$> C_{\alpha,\beta,\beta',R,\varepsilon} q^{3/\beta'} - C'_{\alpha,\beta,\beta',R,\varepsilon} q^{1+2/\beta'}\,.$$

This can be written as follows:

$$N^R_{\alpha,\beta}(r) > C_{\alpha,\beta,\beta',R,\varepsilon} r^3 - C'_{\alpha,\beta,\beta',R,\varepsilon} r^{2+\beta'} > C''_{\alpha,\beta,\beta',R,\varepsilon} r^3\,,$$

where we use that $\beta' < 1$.

Thus the theorem is proved. $\diamondsuit$

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