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Geometric modular action and transformation groups[1]

by

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ABSTRACT. – We study a weak form of geometric modular action, which is naturally associated with transformation groups of partially ordered sets and which provides these groups with projective representations. Under suitable conditions it is shown that these groups are implemented by point transformations of topological spaces serving as models for space-times, leading to groups which may be interpreted as symmetry groups of the space-times. As concrete examples, it is shown that the Poincaré group and the de Sitter group can be derived from this condition of geometric modular action. Further consequences and examples are discussed.

RÉSUMÉ. – Nous étudions une forme faible de l'action modulaire géométrique, naturellement associée avec les groupes de transformation d'ensembles partiellement ordonnés et conduisant à des représentations projectives de ces groupes.

Sous des conditions appropriées, nous montrons que ces groupes sont engendrés par des transformations ponctuelles d'espaces topologiques servant de modèles d'espace-temps, conduisant à des groupes pouvant s'interpréter comme des groupes de symmétrie de ces espace-temps.

[1] This paper is an extended version of talks given at the IAMP Congress Satellite Conference on the General Theory of Quantized Fields in Paris in the summer of 1994 and at the Symposium on Algebraic and Constructive Quantum Field Theory at the University of Göttingen in the summer of 1995.

A titre d'exemples concrets, les groupes de Poincaré et de Sitter peuvent être considérés comme dérivant d'une telle condition d'action modulaire géométrique. Plusieurs autres conséquences et exemples sont discutés.

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1. INTRODUCTION

The ground-breaking work of Bisognano and Wichmann [4] established that for quantum field theories satisfying the Wightman axioms the modular objects - both the automorphism group and the involution - associated by Tomita-Takesaki theory (*see* e.g. [7]) to the vacuum state and local algebras assigned to wedgelike spacetime regions in Minkowski space have geometrical interpretation. These early results opened up a number of fascinating lines of research for algebraic quantum field theory. One of these is the possibility that physically interesting states can be *characterized* by the geometric action of modular objects associated with suitably chosen local algebras - this approach was taken in [9], where it was shown how the vacuum state on Minkowski space can be characterized by the action of the modular objects associated with wedge algebras. A second line is the construction of nets of local algebras with the desired covariance properties given a state, a small number of algebras, and a suitable action of the associated modular objects. This was briefly addressed in [6], but the most complete result in this direction has been the construction of conformally invariant nets of local algebras in this manner [23]. Another closely related research program is the generation of unitary representations of spacetime symmetry groups by modular objects which implement elements of these symmetry groups. This course of study was opened up by Borchers [6], was carried further in [23], [9], [8], and is one major occupation of this paper.

As explained in our first paper on the subject [9], we are also interested in the *derivation* of spacetime symmetry groups, not just their representations. If the space-time is derivable from experience, then it should be possible in principle to determine it from the observables and the preparations of the theory. It must be emphasized that the modular objects of the Tomita-Takesaki theory are completely determined by the choice of state and algebra, and that in algebraic quantum theory the state corresponds operationally to the preparation of the system, while the algebras are generated by the observables of the system. We therefore view the modular objects as being (at least in principle) directly derivable from the operationally given quantities in an experiment. It is of conceptual interest to determine which theoretical quantities are deducible from observation, as opposed to being posited for convenience.

Though it would be desirable to derive the topology, dimension, etc. of space-time, we shall presume the space-time to be given at least as a topological set. With this assumption, to derive the space-time means to determine its metric structure from the given operational quantities. An important step towards this goal is the determination of the group of isometries of the space-time. From our point of view, if the net of observable algebras is covariant under the action of a unitary representation of some group of point transformations of the underlying topological space, then that group should be understood as the group of isometries of the metric structure to be imposed upon the topological space. We mention other papers [16], [24] in which the causal (*i.e.* conformal) structure of the space-time is derived from states and nets of algebras of observables.

For the ends just mentioned, we proposed in [9] a condition of geometrical modular action. Let $\mathfrak{R}$ be a suitable collection of open sets on a space-time $\mathcal{M}$ and $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{R}}$ be a net of C^*-algebras indexed by $\mathfrak{R}$, each of which is a subalgebra of the C^*-algebra $\mathcal{A}$. A state on $\mathcal{A}$ will be denoted by ω and the corresponding GNS representation of $\mathcal{A}$ will be signified by $(\mathcal{H}_\omega, \pi_\omega, \Omega)$. For each $\mathcal{O} \in \mathfrak{R}$ the von Neumann algebra $\pi_\omega(\mathcal{A}(\mathcal{O}))''$ will be denoted by $\mathcal{R}(\mathcal{O})$. We shall call a subcollection $\mathfrak{S} \subset \mathfrak{R}$ of regions whose intersections yield $\mathfrak{R}$, *i.e.* $\{\bigcap_{\mathcal{O}\in\hat{\mathfrak{S}}} \mathcal{O} \mid \hat{\mathfrak{S}} \subset \mathfrak{S}\} = \mathfrak{R}$, a generating family (for $\mathfrak{R}$).

Condition of Geometrical Modular Action

Given the structures indicated above, the pair $(\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}, \omega)$ satisfies the condition of geometrical modular action if the collection of algebras $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}$ is stable under the action of the modular conjugations $J_\mathcal{O}$ associated with $(\mathcal{R}(\mathcal{O}), \Omega)$, $\mathcal{O} \in \mathfrak{S}$. In other words, for every pair of regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{S}$ there is some region $\mathcal{O}_1 \circ \mathcal{O}_2 \in \mathfrak{S}$ such that

$$J_{\mathcal{O}_1} \mathcal{R}(\mathcal{O}_2) J_{\mathcal{O}_1} = \mathcal{R}(\mathcal{O}_1 \circ \mathcal{O}_2).$$

(Setting $\mathcal{R}(\cap_\iota \mathcal{O}_\iota) \equiv \cap_\iota \mathcal{R}(\mathcal{O}_\iota)$, the geometric action of the modular conjugations can then be extended to the algebras associated with arbitrary regions in $\mathfrak{R}$.)

This condition was motivated by examples in Minkowski space-time in which modular objects have a geometric action satisfying the above condition (*see* e.g. [4], [15]). We shall present subsequent examples in de Sitter space-time which also fulfill the requirements of this condition of geometric modular action. Characteristic of these known examples of geometric modular action are the following facts: most states ω do *not* yield modular objects with geometric action (so using this condition as a selection principle has teeth); even when a state is felicitously selected, not all pairs $(\mathcal{R}(\mathcal{O}), \Omega)$ have modular objects which act geometrically correctly; but sufficiently many regions $\mathcal{O}$ lead to geometrically acting modular objects so that the collection $\Im$ of such regions does indeed constitute a generating family for a topological base of the space-time.

This paper is an overview and announcement of results obtained in collaboration with Detlev Buchholz and Olaf Dreyer. We are obliged here to suppress most proofs. Readers interested in further details and references will need to refer to [10].

2. NETS OF VON NEUMANN ALGEBRAS AND TRANSFORMATION GROUPS

We begin with a more abstract setting of the above condition, since then the connection with transformation groups on partially ordered sets and projective representations of these groups emerges particularly clearly. We shall return to the original situation in the next section.

If $(I, \leq)$ is a directed set and the property of isotony holds, *i.e.* if for any $i_1, i_2 \in I$ such that $i_1 \leq i_2$ one has $\mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_2}$, *i.e.* $\mathcal{A}_{i_1}$ is a subalgebra of $\mathcal{A}_{i_2}$, then a collection of C^*-algebras $\{\mathcal{A}_i\}_{i \in I}$ indexed by I is said to be a net. However, for our purposes it will suffice that $(I, \leq)$ be only a partially ordered set and that $\{\mathcal{A}_i\}_{i \in I}$ satisfy isotony. We are therefore working with two partially ordered sets, $(I, \leq)$ and $(\{\mathcal{A}_i\}_{i \in I}, \subseteq)$, and we require that the assignment $i \mapsto \mathcal{A}_i$ be an order-preserving bijection (*i.e.* is an isomorphism in the structure class of partially ordered sets). According to our present view, any such assignment which is not an isomorphism in this sense involves some kind of physically unnecessary redundancy in the description. In algebraic quantum field theory I is usually a collection of open subsets of an appropriate space-time $\mathcal{M}$. In such a case the algebra $\mathcal{A}_i$ is interpreted as the C^*-algebra generated by all the observables measurable in the space-time region i.

If $\{\mathcal{A}_i\}_{i \in I}$ is a net, then the inductive limit $\mathcal{A}$ of $\{\mathcal{A}_i\}_{i \in I}$ exists and may be used as a reference algebra. However, even if $\{\mathcal{A}_i\}_{i \in I}$ is not a net, it is possible [11] to naturally embed the algebras $\mathcal{A}_i$ in a C^*-algebra $\mathcal{A}$ in such a way that the inclusion relations are preserved. A net automorphism is an automorphism α of $\mathcal{A}$ such that there exists an order-preserving bijection

$\hat{\alpha}$ on I with $\alpha(\mathcal{A}_i) = \mathcal{A}_{\hat{\alpha}(i)}$ for all $i \in I$. Symmetries, whether dynamical or otherwise, are determined by the basic operational quantities of the theory and are expressed in terms of the net of observable algebras as net automorphisms (or antiautomorphisms) [21].

Given a state ω on the algebra $\mathcal{A}$, one can, as above, consider the corresponding GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ and the von Neumann algebras $\mathcal{R}_i \equiv \pi_\omega(\mathcal{A}_i)''$. We extend the assumption of nonredundancy of indexing to these algebras, *i.e.* we assume also that the map $i \mapsto \mathcal{R}_i$ is an order-preserving bijection. If the GNS vector Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$, then from the modular theory of Tomita-Takesaki, we are presented with a collection $\{J_i\}_{i \in I}$ of modular involutions (and a collection $\{\Delta_i\}_{i \in I}$ of modular operators), directly derivable from the state and the algebras. This collection $\{J_i\}_{i \in I}$ of operators on $\mathcal{H}_\omega$ generates a group $\tilde{\mathcal{J}}$, which becomes a topological group in the strong operator topology on $\mathcal{B}(\mathcal{H}_\omega)$.

In the following we shall denote the adjoint action of J_i upon the elements of the net $\{\mathcal{R}_i\}_{i \in I}$ by $\mathrm{ad}J_i$, *i.e.* $\mathrm{ad}J_i(\mathcal{R}_j) \equiv J_i \mathcal{R}_j J_i$. The content of the Condition of Geometrical Modular Action in this setting is that each $\mathrm{ad}J_i$ leaves the set $\{\mathcal{R}_i\}_{i \in I}$ invariant, in other words that for $i, j \in I$ there exists an $i(j) \in I$ such that $J_i \mathcal{R}_j J_i = \mathcal{R}_{i(j)}$. The elements $\{\mathrm{ad}J_i\}_{i \in I}$ generate a group $\mathcal{J}$ acting on the set $\{\mathcal{R}_i\}_{i \in I}$. We also find that $\mathcal{J}$ induces a group $\mathcal{T}$ of transformations on the index set I. In particular, for each $i \in I$, we define $T_i : I \mapsto I$ by $T_i(j) \equiv i(j)$, and $\mathcal{T}$ is the group generated by the set $\{T_i \mid i \in I\}$. Given our assumptions, $\mathcal{J}$ and $\mathcal{T}$ are isomorphic groups. For the convenience of the reader, we summarize our standing assumptions.

Standing Assumptions

For the net $\{\mathcal{A}_i\}_{i \in I}$ and the state ω on $\mathcal{A}$ we assume

(i) $i \mapsto \mathcal{R}_i$ is an order-preserving bijection;

(ii) Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$;

(iii) each $\mathrm{ad}J_i$ leaves the set $\{\mathcal{R}_i\}_{i \in I}$ invariant.

We next collect the basic properties of the group $\mathcal{T}$ (and therefore also $\mathcal{J}$) in the following lemma.

LEMMA 2.1. – *The group $\mathcal{T}$ defined above has the following properties.*

(1) For each $i \in I$, $T_i^2 = E$, where E is the identity map on I.

(2) For every $T \in \mathcal{T}$ one has $TT_iT = T_{T(i)}$.

(3) If $T(i) = i$ for some $T \in \mathcal{T}$ and $i \in I$, then $TT_i = T_iT$.

(4) One has $T_i(i) = i$, for some $i \in I$, if and only if the algebra $\mathcal{R}_i$ is maximally abelian. If $\mathcal{T}$ acts transitively on I, then $T_i(i) = i$, for some $i \in I$ if and only if $T_i(i) = i$, for all $i \in I$. Moreover, if $T_i(i) = i$ for some $i \in I$, then i is an atom in $(I, \leq)$, i.e. if $j \in I$ and $j \leq i$, then $j = i$.

For index sets without atoms, such as the set $\mathcal{W}$ used later in this paper, Lemma 2.1 (4) implies that $\mathcal{R}_i$ must be nonabelian for every $i \in I$.

The next point to be made is that the Standing Assumptions entail that the groups $\mathcal{T}$ and $\mathcal{J}$ come provided with a projective representation. For an arbitrary $T \in \mathcal{T}$ there may be many ways of writing T as a product of the elementary $\{T_i \mid i \in I\}$; for each $T \in \mathcal{T}$ choose a minimal product $T = \Pi_{j=1}^{n(T)} T_{i_j}$ (*i.e.* a product such that any other product of elemental T_i's yielding T contains at least $n(T)$ factors). There may, of course, be more than one such minimal product; which choice one makes is irrelevant for our present purposes. Having made such a choice for each $T \in \mathcal{T}$, define $J(T) \equiv \Pi_{j=1}^{n(T)} J_{i_j}$.

THEOREM 2.2. – *The above construction provides an (anti)unitary projective representation of $\mathcal{T}$ on $\mathcal{H}_\omega$ with coefficients in an abelian subgroup $\mathcal{Z} \subset \mathcal{U}(\mathcal{H}_\omega)^2$ which commute elementwise with $\tilde{\mathcal{J}}$ on $\mathcal{H}_\omega$.*

We feel it is useful to elaborate further the relation between the groups $\tilde{\mathcal{J}}$ and $\mathcal{T}$ (or $\mathcal{J}$). An operator $S \in \tilde{\mathcal{J}}$ is said to be an internal symmetry of the net $\{\mathcal{R}_i\}_{i \in I}$, if $S\mathcal{R}_k S^{-1} = \mathcal{R}_k$ for all $k \in I$. Note that these internal symmetries generate a subgroup $\mathcal{S}$ in the center of $\tilde{\mathcal{J}}$.

LEMMA 2.3. – *The surjective map $\xi : \tilde{\mathcal{J}} \mapsto \mathcal{T}$ given by*

$$\xi(J_{i_1} \cdots J_{i_m}) = T_{i_1} \cdots T_{i_m}, \quad i_1, \ldots, i_m \in I, \quad m \in \mathbb{N},$$

is a group homomorphism. Its kernel is the internal symmetries group $\mathcal{S}$.

This lemma may be reformulated as the assertion that there exists a short exact sequence

$$1 \to \mathcal{S} \xrightarrow{\imath} \tilde{\mathcal{J}} \xrightarrow{\xi} \mathcal{T} \to 1,$$

where $\imath$ denotes the natural identification map. In other words, $\tilde{\mathcal{J}}$ is a central extension of the group $\mathcal{T}$ by $\mathcal{S}$, a situation for which the mathematics has reached a certain maturity. This will be particularly useful when we address in Section 8 the cohomological problem naturally suggested by Theorem 2.2.

[2] $\mathcal{U}(\mathcal{H}_\omega)$ is the group of (anti)unitary operators acting on $\mathcal{H}_\omega$.

We believe that it would be of interest to determine which groups can arise in this manner.[3] If the elements of the index set I can be identified with suitable subsets of a topological space, and the elements of these groups can be identified with point transformations on this topological space in some manner (two paths to this end are discussed below), then these groups would be natural candidates for spacetime symmetry groups. In any case, we shall show how under suitable circumstances the Poincaré group and the de Sitter group *can* be derived in this manner.

3. TRANSFORMATIONS OF SPACE-TIMES INDUCED BY NET AUTOMORPHISMS - THE APPROACH THROUGH THE SMALL

Although there are to be found in the literature conditions upon nets of algebras which are sufficient to derive a topological space upon which the nets may be viewed as being based [3], we feel that our present understanding of this topic is unsatisfactory. We therefore shall assume for the rest of this paper that the elements of the index set I of the net $\{\mathcal{R}_i\}_{i\in I}$ have already been identified with suitable subsets of some topological space. The details of this assumption will be spelled out below.

Hence the question we wish to address next is: when do net (anti)automorphisms of such nets induce point transformations on this underlying topological space? One approach initiated by Araki [2] and further developed by Keyl [16] has been to hypothesize that these net automorphisms can be interpreted as mapping neighborhood bases to neighborhood bases, yielding a natural candidate for a point transformation. We shall discuss this tack - through the small - first. It has the advantage of being quite general in nature - only point-set topology is involved in the hypotheses and proof of the theorem we state below. However, it does have the disadvantage of requiring control of the map on an entire base for the topology and not merely on a generating set for the topology. A second approach - through the large - will be presented in the next section and has advantages and disadvantages which are entirely complementary to those of the approach through the small.

The topological manifolds $\mathcal{M}$ of interest for models of physical space-times are Lorentzian manifolds, which are locally compact metrizable topological spaces. Since the results in this section require only point-set

[3] Explicit computation has shown that both finite and continuous groups can be obtained with appropriate choices of net and index set.

topology, we shall work in the class of locally compact second countable Hausdorff spaces, which includes all space-times of physical interest. The following theorem gives sufficient conditions for a net automorphism (in the guise of an isomorphism on the partially ordered index set) on a net of algebras indexed by a base for the topology of a space to induce an action, in fact a homeomorphism, on the underlying space.

THEOREM 3.1. – [16] *Let $\mathcal{M}$ be a locally compact second countable Hausdorff space and $\hat{\alpha}$ be an order-preserving bijection on $(\Re, \subseteq)$, where $\Re$ is a base for the topology on $\mathcal{M}$, which satisfies the additional condition*

$$\mathcal{O} \subset \bigcup_{\hat{\mathcal{O}} \in \hat{\Re}} \hat{\mathcal{O}} \quad if\ and\ only\ if \quad \hat{\alpha}(\mathcal{O}) \subset \bigcup_{\hat{\mathcal{O}} \in \hat{\Re}} \hat{\alpha}(\hat{\mathcal{O}}), \tag{3.1}$$

for all $\mathcal{O} \in \Re$ and all $\hat{\Re} \subset \Re$. Then there exists a homeomorphism $\tilde{\alpha} : \mathcal{M} \to \mathcal{M}$ such that the equality $\hat{\alpha}(\mathcal{O}) = \{\tilde{\alpha}(x) \mid x \in \mathcal{O}\}$ obtains for any $\mathcal{O} \in \Re$.

In terms of a net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\Im}$, indexed by a generating family $\Im$ of a basis $\Re$ for the topology of a space $\mathcal{M}$, the Standing Assumptions from the previous section take the form:

Standing Assumptions

For the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\Im}$ and the state ω on $\mathcal{A}$ we assume

(i) $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ is an order-preserving bijection;

(ii) Ω is cyclic and separating for each algebra $\mathcal{R}_{\mathcal{O}}$, $\mathcal{O} \in \Im$;

(iii) each $\mathrm{ad}J_{\mathcal{O}}$ leaves the set $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\Im}$ invariant.

To these assumptions we add the following, in order to be able to use the approach through the small to point transformations suggested by Theorem 3.1.

(iv) For each $\mathcal{O} \in \Im$ $\mathrm{ad}J_{\mathcal{O}}$ acting upon $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\Re}$ induces an order-preserving bijection $\hat{\alpha}_{\mathcal{O}}$ on the set $\Re$, where for any $\mathcal{O}_0 \in \Re$ one defines $\mathcal{R}(\mathcal{O}_0) \equiv \bigcap_{\substack{\mathcal{O}_0 \subset \mathcal{O} \\ \mathcal{O}\in\Im}} \mathcal{R}(\mathcal{O})$, and $\hat{\alpha}_{\mathcal{O}}$satisfies condition (3.1).

We now can state the following result, which is an immediate consequence of the preceding theorem.

THEOREM 3.2. – *Let $\Re$ be a base for the topology of the locally compact second countable Hausdorff space $\mathcal{M}$, $\Im$ be a generating family for $\Re$ and $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\Im}$ be a net of C^*-algebras. Let ω be a state on $\mathcal{A}$ for which in the corresponding GNS-representation the above Assumptions (i)-(iv) obtain. Then for every $J \in \{J_{\mathcal{O}} \mid \mathcal{O} \in \Im\}$ there exists a homeomorphism $\tilde{\alpha}_J : \mathcal{M} \mapsto \mathcal{M}$ such that $J\mathcal{R}(\mathcal{O})J = \mathcal{R}(\tilde{\alpha}_J(\mathcal{O}))$ for every $\mathcal{O} \in \Re$.*

Though, as indicated above, Assumption (iii) yields a map on the set $\Re$, it is far from clear that the map is an order-preserving bijection on $\Re$. This additional requirement plus the condition (3.1) is the content of Assumption (iv) and is the disadvantage of the approach through the small.

To illustrate the application of this theorem, let us consider the set $\mathcal{W}$ of all wedges in $\mathbb{R}^4$. With the "right wedge" W_R defined by $W_R \equiv \{x \in \mathbb{R}^4 \mid x_1 > |x_0|\}$, the set $\mathcal{W}$ denotes the set of all regions obtained by applying the elements of $\mathcal{P}_+^\uparrow$ to W_R. With the choices $\Im = \mathcal{W}$ and $\mathcal{M} = \mathbb{R}^4$ and Assumptions (i)-(iv), we have a group $\mathcal{G}$ of homeomorphisms of $\mathbb{R}^4$ mapping wedges onto wedges, *i.e.* leaving the set $\mathcal{W}$ invariant. We claim that it follows that $\mathcal{G}$ is a subgroup of the Poincaré group. In the proof of this claim we make use of well-known results of Alexandroff [1], Zeeman [25], Borchers and Hegerfeldt [5] and others to the effect that bijections on $\mathbb{R}^4$ leaving lightcones invariant must be elements of the extended Poincaré group, $\mathcal{DP}$, generated by the Poincaré group and the dilatation group. By suitably modifying their arguments, we can demonstrate the following proposition.

PROPOSITION 3.3. – *Let $\tilde{\alpha} : \mathbb{R}^n \mapsto \mathbb{R}^n$, $n \geq 3$, be a bijective map such that $\tilde{\alpha}$ maps every wedge in $\mathbb{R}^n$ onto some other wedge in $\mathbb{R}^n$. Then $\tilde{\alpha} = (\lambda\Lambda, x)$ for some $\lambda > 0$, $\Lambda \in \mathcal{L}$ and $x \in \mathbb{R}^n$, i.e. $\tilde{\alpha} \in \mathcal{DP}$.*

We therefore have for every $W \in \mathcal{W}$ an element $g_W = (\lambda_W \Lambda_W, x_W)$ of $\mathcal{DP}$ such that $J_W \mathcal{A}(\mathcal{O}) J_W = \mathcal{A}(\lambda_W \Lambda_W \mathcal{O} + x_W)$ for every $\mathcal{O} \in \Re$. But since $J_W^2 = 1$, Assumption (i) implies that $\lambda_W^2 \Lambda_W^2 + \lambda_W \Lambda_W x_W + x_W = 1 \in \mathcal{DP}$; in particular, one has $\lambda_W^2 \Lambda_W^2 = 1 \in \mathcal{DP}$. Hence $\lambda_W = 1$ for all $W \in \mathcal{W}$. We observe therefore that $\mathcal{G}$ is a subgroup of the Poincaré group itself.

4. TRANSFORMATIONS OF SPACE-TIMES INDUCED BY NET AUTOMORPHISMS - THE APPROACH THROUGH THE LARGE

With the specific choice $\Im = \mathcal{W}$ and $\mathcal{M} = \mathbb{R}^4$, one naturally has more structure available than that assumed in Theorem 3.1, which can be used to avoid making the additional Assumption (iv) in the previous section. In fact, one can define points as intersections of suitable edges of wedges, so that transformations of wedges could lead to point transformations. This analysis has been done, and we present only the results of a very lengthy argument.

In this setting we are supplied with a map $\hat{\alpha} : \mathcal{W} \mapsto \mathcal{W}$ for each $W \in \mathcal{W}$, and we assume that $\hat{\alpha}$ satisfies the conditions:

B1. $\hat{\alpha}^2$ is the identity mapping on $\mathcal{W}$.

B2. $\hat{\alpha}(W') = \hat{\alpha}(W)'$, for all $W \in \mathcal{W}$.[4]

B3. $W_1, W_2, W \in \mathcal{W}$ satisfy $W_1 \cap W_2 \subset W$ if and only if $\hat{\alpha}(W_1) \cap \hat{\alpha}(W_2) \subset \hat{\alpha}(W)$.

B4. $W_1, W_2 \in \mathcal{W}$ satisfy $W_1 \cap W_2 = \emptyset$ if and only if $\hat{\alpha}(W_1) \cap \hat{\alpha}(W_2) = \emptyset$.

These conditions can be derived from the following assumptions on the net: (1) $\mathcal{R}(W') = \mathcal{R}(W)'$ for all $W \in \mathcal{W}_0$, the set of all wedges with the origin in their edges, (2) $W_1 \cap W_2 \subset W$ if and only if $\mathcal{R}(W_1) \cap \mathcal{R}(W_2) \subset \mathcal{R}(W)$, and (3) $W_1 \cap W_2 = \emptyset$ if and only if $\mathcal{R}(W_1) \cap \mathcal{R}(W_2) = \mathbb{C}\mathbf{1}$.

We shall say that two wedges $W_1, W_2 \in \mathcal{W}$ are coherent if one is obtained from the other by a translation, or, equivalently, if there exists another wedge W_3 such that $W_1 \subset W_3$ and $W_2 \subset W_3$. Given an arbitrary wedge $W \in \mathcal{W}$ there exists exactly one wedge, denoted by $N(W)$, which is contained in $\mathcal{W}_0$ and is coherent with W. This yields the map $N : \mathcal{W} \mapsto \mathcal{W}_0$, with which we may define the map $\hat{\alpha}_0 : \mathcal{W}_0 \mapsto \mathcal{W}_0$ by the obvious $\hat{\alpha}_0 \equiv N \circ \hat{\alpha}$.

For any wedge W, let $K(W)$ denote its edge. We define the sets $M_x \equiv \{W \in \mathcal{W}_0 \mid x \in K(W)\}$ and $N_x \equiv \{W \in \mathcal{W}_0 \mid x \in W\}$ for every $x \in \mathbb{R}^4$ such that $x \cdot x = -1$. The set M_x determines a line in Minkowski space by $x\mathbb{R} = \bigcap_{W \in M_x} K(W)$. Similarly, the set N_x determines the half-line contained in every wedge in N_x. Given such sets M_x, N_x, without prior knowledge of x, x may be recovered through taking intersection and then picking out the element of the intersection which has Minkowski length -1. We denote that point by $Q(N_x, M_x)$. Letting $dS^3 \equiv \{x \in \mathbb{R}^4 \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 = -1\}$, which can be identified with three-dimensional de Sitter space, we shall think of $Q(N_x, M_x)$ as a point in dS^3.

PROPOSITION 4.1. – *Let $\hat{\alpha} : \mathcal{W} \mapsto \mathcal{W}$ be a map satisfying conditions* **B1-B4** *and $\hat{\alpha}_0 \equiv N \circ \hat{\alpha}$. Then the equation*

$$\phi_{\hat{\alpha}}(x) \equiv Q(\hat{\alpha}_0(N_x), \hat{\alpha}_0(M_x)), \quad x \in dS^3,$$

defines a map of three-dimensional de Sitter space onto itself.

An Alexandroff-like theorem is also available in de Sitter space:

THEOREM 4.2. – [18] *If $\phi : dS^n \mapsto dS^n$, $n \geq 3$, is a bijection such that all points of separation zero are mapped to points of separation zero (i.e.*

[4] W' denotes the interior of the set of all points in $\mathbb{R}^4$ which are spacelike with respect to W.

lightlike separated points are mapped to lightlike separated points), then there exists a Lorentz transformation Λ *of the surrounding Minkowski space* $\mathbb{R}^{n+1}$ *such that* $\phi(x) = \Lambda x$, *for all* $x \in dS^n$.

We show that, in fact, the map $\phi_{\hat{\alpha}}$ satisfies the hypotheses of this theorem, yielding:

THEOREM 4.3. – *Under the same hypotheses as Proposition 4.1, there exists a Lorentz transformation* Λ *of the enveloping space* $\mathbb{R}^4$ *such that* $\phi_{\hat{\alpha}}(x) = \Lambda x$ *for all* $x \in dS^3$. *Thus* $\hat{\alpha}_0(W) = \Lambda W$ *for all* $W \in \mathcal{W}_0$.

We use this fact to define yet another map $\hat{\alpha}_T : \mathcal{W} \mapsto \mathcal{W}$ by $\hat{\alpha}_T(W) \equiv \Lambda^{-1}\hat{\alpha}(W)$. Evidently, for every $W \in \mathcal{W}$ the wedges W and $\hat{\alpha}_T(W)$ are coherent. We show that, in fact, the map $\hat{\alpha}_T$ is implemented by a fixed translation in $\mathbb{R}^4$. We consider the set of wedges $C(W_R)$ which are coherent with the wedge W_R. Each element W_1 of this equivalence class is determined uniquely by a point $\pi_E(W_1) \equiv (a, b)$ in the (x_0, x_1)-plane E. This defines a map $\pi_E : C(W_R) \mapsto E$ and therewith a map $\hat{\alpha}_E : E \mapsto E$ given by $\hat{\alpha}_E(x) \equiv \pi_E(\hat{\alpha}_T(x + W_R))$. The plane E can be canonically identified with two-dimensional Minkowski space. It is shown that $\hat{\alpha}_E$ maps an arbitrary lightlike (resp. timelike, spacelike) line onto a parallel lightlike (resp. timelike, spacelike) line. Since $\hat{\alpha}_E$ maps lines onto lines, it must be an affine-linear map, and since the image of a line is always parallel to the original line, $\hat{\alpha}_E$ must be a translation. This leads to the following theorem.

THEOREM 4.4. – *Under the same hypotheses as Proposition 4.1, there exist a Lorentz transformation* Λ *and an element* $a \in \mathbb{R}^4$ *such that* $\hat{\alpha}(W) = \Lambda W + a$, *for all* $W \in \mathcal{W}$.

5. $\mathcal{G}$ CONTAINS THE PROPER ORTHOCHRONOUS POINCARÉ GROUP

So whether we obtain the point transformations on $\mathbb{R}^4$ through the large or through the small, in both cases we construct a subgroup $\mathcal{G}$ of the Poincaré group, which is related to the group $\mathcal{T}$ as follows: For each $T \in \mathcal{T}$ there exists an element $g \in \mathcal{G}$ such that $T(W) = gW \equiv \{gx \mid x \in W\}$, for all $W \in \mathcal{W}$. The question we consider next is, how large is the group $\mathcal{G} \subset \mathcal{P}$? In addressing this question, the difference between our approach and that of other papers concerning the relation between modular structures and representations of spacetime symmetries becomes particularly clear. The other authors explicitly or implicitly assume that the group $\mathcal{G}$ is already known, indeed, that it is equal to the desired symmetry group, whereas we have derived this group from the operational data. We are therefore obliged to demonstrate that $\mathcal{G}$ is sufficiently large to be of physical interest.

In this section we shall show that the group $\mathcal{G}$ contains the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$, under an additional assumption. In order to assure that the group $\mathcal{G}$ be large enough and with the foreknowledge that $\mathcal{P}_+^\uparrow$ acts transitively upon the set $\mathcal{W}$, we shall restrict our attention to the case where the group $\mathcal{J}$ acts transitively upon the set $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$, which implies that the groups $\mathcal{T}$ and $\mathcal{G}$ act transitively upon the set of wedges $\mathcal{W}$. The fact that the index set $(\mathcal{W}, \subseteq)$ has no atoms entails that the algebras $\mathcal{R}(W)$ are all nonabelian, as pointed out following Lemma 2.1.

We have seen that the group $\mathcal{G}$ is contained in $\mathcal{P}$, which itself has four connected components. In fact, the transitivity of the action of $\mathcal{G}$ upon the set $\mathcal{W}$, which implies that for every $W_1, W_2 \in \mathcal{W}$, there exists an element $g \in \mathcal{G}$ such that $g g_{W_1} g^{-1} = g_{W_2}$, entails the relation $g_{W_1} g_{W_2} = g_{W_1} g g_{W_1} g^{-1} = g_{W_1} g g_{W_1}^{-1} g^{-1}$ (since each g_W is an involution), and the right-hand side is a group commutator. In the Poincaré group such commutators are always contained in the identity component $\mathcal{P}_+^\uparrow$. Hence, for any wedges $W_1, W_2 \in \mathcal{W}$ we have for the product of the corresponding group elements $g_{W_1} g_{W_2} \in \mathcal{P}_+^\uparrow$. But this implies the following lemma.

LEMMA 5.1. – *The group $\mathcal{G}$ can have at most two connected components.*

Let $W \in \mathcal{W}$ be a wedge and let $\mathrm{Inv}(W) \equiv \{g \in \mathcal{G} \mid gW = W\}$ be the invariance subgroup of W in $\mathcal{G}$. The assumption that $\mathcal{G}$ acts transitively upon the set $\mathcal{W}$ entails that up to isomorphism, $\mathrm{Inv}(W)$ does not depend upon the choice of wedge W.

LEMMA 5.2. – $\mathrm{Inv}(W_1)$ *and* $\mathrm{Inv}(W_2)$ *are isomorphic for all* $W_1, W_2 \in \mathcal{W}$.

Consider the wedge $W_0 \equiv \{x \in \mathbb{R}^4 \mid x_3 > |x_0|\}$, and let $\mathrm{InvL}(W_0) \equiv \{\Lambda \in \mathcal{L} \mid \Lambda W_0 = W_0\}$ be its invariance subgroup in the full Lorentz group $\mathcal{L}$. The elements of $\mathrm{InvL}(W_0)$ given by the temporal reflection $T = \mathrm{diag}(-1,1,1,1) \in \mathcal{L}_-^\downarrow$, the spatial reflection $P_1 = \mathrm{diag}(1,-1,1,1) \in \mathcal{L}_-^\uparrow$, and their product $P_1 T = \mathrm{diag}(-1,-1,1,1) \in \mathcal{L}_+^\downarrow$ are distinguished, because all elements of $\mathrm{InvL}(W_0)$ can be obtained by multiplying $\mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$ by these involutions. As above, also the group $\mathrm{InvL}(W)$ does not depend (up to an isomorphism) on the choice of the wedge W. We point out that $\mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$ is an abelian group, whereas $\mathrm{InvL}(W_0)$ is not abelian, precisely because of the mentioned involutions.

We consider the homomorphism $\sigma : \mathcal{G} \mapsto \mathcal{L}$ which acts as $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^4 \supset \mathcal{G} \ni (\Lambda, a) \mapsto \Lambda \in \mathcal{L}$. Since $\mathcal{G}$ acts transitively on all wedges, it is clear that $G \equiv \sigma(\mathcal{G})$ must act transitively on the subset $\mathcal{W}_0 = \{\Lambda W_0 \mid \Lambda \in \mathcal{L}_+^\uparrow\} \subset \mathcal{W}$.

Thus for every $\Lambda \in \mathcal{L}_+^\uparrow$ there exists an element $g_\Lambda \in G$ such that $\Lambda W_0 = g_\Lambda W_0$. Since $G \subset \mathcal{L}$, one has $g_\Lambda^{-1}\Lambda \in \mathcal{L}$ and $g_\Lambda^{-1}\Lambda W_0 = W_0$. Hence, there exists an element $\tilde{\Lambda} \in \mathrm{InvL}(W_0)$ such that $g_\Lambda = \Lambda \cdot \tilde{\Lambda}$. Note that g_Λ and $\tilde{\Lambda}$ must be in the same connected component of the Lorentz group.

It is convenient to consider the following alternatives: (i) $G \cap \mathrm{InvL}(W_0) = \mathrm{Inv}(W_0)$ is nontrivial or (ii) $\mathrm{Inv}(W_0)$ is trivial. By Lemma 5.2, whichever of these cases holds for the wedge W_0 must also be true for all other wedges. We sketch the proof that case (i) implies the desired conclusion, which is the following proposition. We shall show that case (ii) is excluded by assumption.

PROPOSITION 5.3. – *The subgroup $G = \sigma(\mathcal{G})$ of the Lorentz group $\mathcal{L}$, which acts transitively upon the set $\mathcal{W}_0$ of wedges whose edges contain the origin of $\mathbb{R}^4$, must contain the proper orthochronous subgroup $\mathcal{L}_+^\uparrow$.*

We begin by considering case (i). If $\mathcal{G}$ has only one connected component, one must have $\mathcal{G} \subset \mathcal{P}_+^\uparrow$, which here means we may assume $G \subset \mathcal{L}_+^\uparrow$. In this case $\mathrm{Inv}(W)$ is abelian, which significantly simplifies the argument.

LEMMA 5.4. – *Let $G \subset \mathcal{L}_+^\uparrow$ be a subgroup acting transitively upon the set of wedges $\mathcal{W}_0$ containing the origin in their edges, and let $G \cap InvL(W_0) \neq \{1\}$. Then $G = \mathcal{L}_+^\uparrow$.*

Proof. – Let $g_1 \neq 1$ be an element of $G \cap \mathrm{InvL}(W_0)$. One concludes that (using the notation established above)

$$\Lambda g_1 \Lambda^{-1} = g_\Lambda \tilde{\Lambda}^{-1} g_1 \tilde{\Lambda} g_\Lambda^{-1} = g_\Lambda g_1 g_\Lambda^{-1} \in G,$$

for all $\Lambda \in \mathcal{L}_+^\uparrow$, since under the given assumptions, one must have $\tilde{\Lambda} \in \mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$, which is abelian, and $g_1 \in G \cap \mathrm{InvL}(W_0) \subset \mathcal{L}_+^\uparrow \cap \mathrm{InvL}(W_0)$. It follows that $G = \mathcal{L}_+^\uparrow$, since $\mathcal{L}_+^\uparrow$ is simple. □

We shall next assume that G has exactly two components $G_+ = G \cap \mathcal{L}_+^\uparrow$ and G_-, related by an involution I: $G_- = I \cdot G_+$, when we are in case (i). A category argument yields a neighborhood $V \subset \mathcal{L}_+^\uparrow$ of the identity such that $\{\Lambda g \Lambda^{-1} \mid \Lambda \in V\}$ is contained in G for some $1 \neq g \in G$. Then an argument about the possible subgroups of $\mathcal{L}_+^\uparrow$ this set can generate yields the following result.

LEMMA 5.5. – *Let $G \subset \mathcal{L}$ be a group which acts transitively upon the set $\mathcal{W}_0$ of wedges whose edges contain the origin of $\mathbb{R}^4$, which has two components related by an involution, and which satisfies $G \cap InvL(W_0) \neq \{1\}$. Then G contains $\mathcal{L}_+^\uparrow$.*

Together, Lemmas 5.1, 5.4 and 5.5 imply that if case (i) obtains, then we have the desired inclusion $\mathcal{L}_+^\uparrow \subset G$. Hence Prop. 5.3 is proven once we have established that case (ii) is impossible. Let us now assume that case (ii) obtains, *i.e.* the intersection $G \cap \mathrm{InvL}(W_0)$ is trivial. This entails that for every $\Lambda \in \mathcal{L}_+^\uparrow$ there exists exactly one $g_\Lambda \in G$ and a unique $\tilde{\Lambda}(\Lambda) \in \mathrm{InvL}(W_0)$ such that $g_\Lambda = \Lambda \cdot \tilde{\Lambda}$. Thus, under the given assumption we have a map from $\mathcal{L}_+^\uparrow$ to $\mathrm{InvL}(W_0)$ taking Λ to $\tilde{\Lambda}$, *viz.* $m : \mathcal{L}_+^\uparrow \mapsto \mathrm{InvL}(W_0)$, with $m(\Lambda) = \tilde{\Lambda}$. With $\rho : SL(2,\mathbb{C}) \mapsto \mathcal{L}_+^\uparrow$ the canonical restriction from the covering group, we have a map $M : SL(2,\mathbb{C}) \mapsto \mathrm{InvL}(W_0)$ given by $M \equiv m \circ \rho$. We shall need to know the induced action of the involutions T, P_1, TP_1 on $SL(2,\mathbb{C})$.

LEMMA 5.6. – *The space and time reflections (P and T) acting on four-dimensional Minkowski spacetime induce the same automorphic action upon $SL(2,\mathbb{C})$, given by $\pi(A) = A^{*-1}$, whereas the reflection of the 1-axis P_1 induces the action $\pi_1(A) = -R\overline{A}R^*$, where $R = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.*

The fact that $\mathcal{G}$ is a group yields the following functional equation for M.

$$M(A)M(B) = M(A\gamma_A(B)), \quad A, B \in SL(2,\mathbb{C}), \tag{5.1}$$

where γ_A is an automorphism of $SL(2,\mathbb{C})$ defined as follows. Given $A \in SL(2,\mathbb{C})$, $M(A)$ can be written uniquely as a product of one of the reflections T, P_1 or TP_1 and an element of $\mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$. The subgroup of $SL(2,\mathbb{C})$ corresponding to $\mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$ (after suitable choice of coordinates) is the maximally abelian subgroup $\mathcal{D}$ of matrices of the form $N_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{C}\setminus\{0\}$. Hence, a choice of $A \in SL(2,\mathbb{C})$ determines (up to a sign) an $N_\lambda \in \mathcal{D}$. With this in mind, Lemma 5.6 implies that the action of γ_A on $SL(2,\mathbb{C})$ is given by

$$\gamma_A(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) = \begin{pmatrix} \alpha & \lambda^2\beta \\ \lambda^{-2}\gamma & \delta \end{pmatrix}, \text{ if } M(A) \in \mathcal{L}_+^\uparrow, \tag{a}$$

$$\begin{pmatrix} \overline{\delta} & -\lambda^2\overline{\gamma} \\ -\lambda^{-2}\overline{\beta} & \overline{\alpha} \end{pmatrix}, \text{ if } M(A) \in \mathcal{L}_-^\downarrow, \tag{b}$$

$$\begin{pmatrix} \overline{\alpha} & -\lambda^2\overline{\beta} \\ -\lambda^{-2}\overline{\gamma} & \overline{\delta} \end{pmatrix}, \text{ if } M(A) \in \mathcal{L}_-^\uparrow, \tag{c}$$

$$\begin{pmatrix} \delta & \lambda^2\gamma \\ \lambda^{-2}\beta & \alpha \end{pmatrix}, \text{ if } M(A) \in \mathcal{L}_+^\downarrow, \tag{d}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{C}$.

We prove that the only solution of (5.1) is $M \equiv 1$ (in $\mathcal{L}_+^\uparrow$), which would contradict the fact that the map $m : \mathcal{L}_+^\uparrow \mapsto \mathrm{InvL}(W_0)$ is the identity map when restricted to $\mathrm{InvL}(W_0) \cap \mathcal{L}_+^\uparrow$. In other words, case (ii) is impossible, which would complete the proof of Proposition 5.3.

Let $\mathcal{O}_C$ be the subgroup of upper triangular matrices and $\mathcal{U}_C$ be the subgroup of lower triangular matrices in $SL(2, \mathbb{C})$. Note that in cases (a) and (c) γ_A leaves the sets $\mathcal{O}_C$ and $\mathcal{U}_C$ invariant, while in the other cases γ_A exchanges the two. Moreover, as long as γ_A is not the identity, one has in case (a) both $\{\gamma_A(X)X^{-1} \mid X \in \mathcal{U}_C\} = \mathcal{U}_C$ and $\{\gamma_A(X)X^{-1} \mid X \in \mathcal{O}_C\} = \mathcal{O}_C$. These observations and equation (5.1) entail that the map M is trivial at least on the triangular matrices in $SL(2, \mathbb{C})$ which are in case (a).

LEMMA 5.7. – *For any triangular matrix X in $SL(2, \mathbb{C})$ such that $M(X) \in \mathcal{L}_+^\uparrow$, one has $M(X) = 1$.*

Defining $\mathcal{E} \equiv \{A \in SL(2, \mathbb{C}) \mid M(A) = 1\}$, it is easy to verify that equation (5.1) entails that $\mathcal{E}$ is a group and that $M(AB) = M(B)$ for all $A \in \mathcal{E}$ and $B \in SL(2, \mathbb{C})$. Using these facts, the functional equation (5.1), and Lemma 5.7, we show that, in fact, M is trivial on the entire set of triangular matrices.

LEMMA 5.8. – *For any triangular matrix X in $SL(2, \mathbb{C})$, one has $M(X) = 1$.*

Let $\mathcal{E}_0$ denote the group $\mathcal{U}_C \cap \mathcal{E}$. Lemma 5.8 is proven by establishing (1) if $X \in \mathcal{U}_C \setminus \mathcal{E}_0$, then $\mathcal{E}_0 \cdot X \subset \mathcal{U}_C \setminus \mathcal{E}_0$ and (2) if $X, X' \in \mathcal{U}_C \setminus \mathcal{E}_0$, then $X'X^{-1} \in \mathcal{E}_0$. Hence, for each $X \in \mathcal{U}_C \setminus \mathcal{E}_0$ one has the disjoint decomposition $\mathcal{U}_C = \mathcal{E}_0 \cup \mathcal{E}_0 \cdot X$. But for each $X \in \mathcal{U}_C$ there exists an element $Y \in \mathcal{U}_C$ such that $Y^2 = X$. If $Y \in \mathcal{E}_0$, then so is X, since $\mathcal{E}_0$ is a group. Thus for $X \in \mathcal{U}_C \setminus \mathcal{E}_0$ one must have $Y \notin \mathcal{E}_0$. On the other hand, if $Y \in \mathcal{E}_0 \cdot X$, then $YX^{-1} \in \mathcal{E}_0$, so that $1 = Y \cdot YX^{-1} \in Y \cdot \mathcal{E}_0$, which implies $Y^{-1} \in \mathcal{E}_0$. But then, once again, one must have $Y \in \mathcal{E}_0$. This is a contradiction unless the set $\mathcal{U}_C \setminus \mathcal{E}_0$ is empty. A similar argument is employed for elements of $\mathcal{O}_C$. But $\mathcal{O}_C \cup \mathcal{U}_C$ generates all of $SL(2, \mathbb{C})$. By the group property of $\mathcal{E}$, we may therefore conclude from Lemma 5.8 that M is trivial on $SL(2, \mathbb{C})$.

The sketch of the proof of Prop. 5.3 is therewith completed. The last step to be taken in this section is to show that under the stated assumptions, our group $\mathcal{G}$ also contains the translations and therefore contains the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$. This is done by using the semidirect product structure of $\mathcal{P}$ and the cocycle relations which arise from the fact that $\mathcal{G}$ is a group. We summarize:

THEOREM 5.9. – *If $\mathcal{G}$ acts transitively upon the set $\mathcal{W}$ of wedges, then it must contain the proper orthochronous subgroup $\mathcal{P}_+^\uparrow$.*

6. TOWARDS AN ALGEBRAIC PCT THEOREM

In this section we shall present detailed information about the generating involutions g_W of our group $\mathcal{G}$. For each $W \in \mathcal{W}$ there is a corresponding $g_W \in \mathcal{G}$, and we have seen that $g_W = (\Lambda_W, a_W) \in \mathcal{P}$ for some element Λ_W of the Lorentz group and some element a_W of the translation group. We wish to determine the values of a_W and Λ_W.

By definition of the group element g_W, we have the following relation.

$$\mathcal{A}(W)' = J_W \mathcal{A}(W) J_W = \mathcal{A}(g_W W) = \mathcal{A}(\Lambda_W W + a_W), \quad W \in \mathcal{W}. \tag{6.1}$$

In addition, for any element (Λ, a) of the invariance group $\text{InvP}(W) \equiv \{(\Lambda, a) \in \mathcal{P} \mid \Lambda W + a = W\}$ one has $U(\Lambda, a))\mathcal{A}(W)U(\Lambda, a))^{-1} = \mathcal{A}(W)$, which also implies that $U(\Lambda, a))\mathcal{A}(W)'U(\Lambda, a))^{-1} = \mathcal{A}(W)'$. Hence it follows from (6.1) that $U(\Lambda, a))\mathcal{A}(\Lambda_W W + a_W)U(\Lambda, a))^{-1} = \mathcal{A}(\Lambda_W W + a_W)$, which implies

$$\mathcal{A}(\Lambda\Lambda_W W + \Lambda a_W + a) = \mathcal{A}(\Lambda_W W + a_W),$$

for any $(\Lambda, a) \in \text{InvP}(W)$. Standing Assumption (i) then implies that $(\Lambda, a)[\Lambda_W W + a_W] = \Lambda\Lambda_W W + \Lambda a_W + a = \Lambda_W W + a_W$ for any $(\Lambda, a) \in \text{InvP}(W)$. Since any element of the Poincaré group maps wedges onto wedges, and since the only wedges left invariant by all elements of $\text{InvP}(W)$ are W and W', it follows that the image $\Lambda_W W + a_W$ of W under g_W is either W or W'. Of course, if $g_W W = W$, then equation (6.1) implies that $\mathcal{A}(W)$ is abelian, which has already been excluded by the Standing Assumptions and the fact that $(\mathcal{W}, \subseteq)$ has no atoms. We may therefore conclude the next lemma.

LEMMA 6.1. – *Under the Standing Assumptions and with the requirement that $\mathcal{J}$ acts transitively upon the set $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$, one must have $g_W W = W'$, for each wedge region W.*

This leads us to the following proposition, which, for simplicity, we formulate in terms of the wedge $W_R = \{x \in \mathbb{R}^4 \mid x_1 > |x_0|\}$.

PROPOSITION 6.2. – *If the Standing Assumptions hold and $\mathcal{J}$ acts transitively upon the set $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$, then the element $g_{W_R} = (\Lambda_{W_R}, a_{W_R})$*

of the Poincaré group induced on $\mathbb{R}^4$ *by the action upon the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$ *via conjugation by the modular conjugation* J_{W_R} *corresponding to the right wedge algebra* $\mathcal{R}(W_R)$ *does not involve any translation (i.e.* $a_{W_R} = 0$*), satisfies* $\Lambda_{W_R}^2 = 1$*, and has the form* $\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$*, where* $L_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $L_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$*. Since all wedges are transforms of* W_R *under* $\mathcal{P}_+^\uparrow$*, these assertions are also true, with the obvious modifications, for the Lorentz element* Λ_W *corresponding to any wedge* $W \in \mathcal{W}$*. Moreover,* $\mathcal{G}$ *is exactly the proper Poincaré group* $\mathcal{P}_+$*.*

Note that by Prop. 6.2, one has $\mathcal{R}(W_R)' = J_{W_R}\mathcal{R}(W_R)J_{W_R} = \mathcal{R}(g_{W_R}W_R) = \mathcal{R}(W_R')$. Hence, by Theorem 5.6 and the fact that $\mathcal{P}_+^\uparrow$ acts transitively upon the set $\mathcal{W}$, we may conclude the next result.

COROLLARY 6.3. – *Under the same assumptions as in Proposition 6.2, the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$ *satisfies wedge duality and the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\Re}$ *satisfies Einstein locality.*

We see therefore that wedge duality and locality, which are normally postulated in the literature, are consequences in our approach.

7. CONTINUITY OF THE REPRESENTATION

We next provide natural conditions which imply that the projective representation of $\mathcal{G}$ whose existence is assured by Theorem 2.2 is strongly continuous. These conditions essentially involve a continuity property of the map $W \mapsto \mathcal{R}(W)$. Consider a collection of wedges $\{W_\epsilon\}_{\epsilon>0}$ indexed by a continuous path in wedge parameter space so that $W_\epsilon \to W$ as $\epsilon \to 0$. Let $I_\epsilon \equiv W_\epsilon \cap W$ and $A_\epsilon \equiv W_\epsilon \cup W$ denote the indicated intersection and union. Our net continuity assumption is that for any such collection $\{W_\epsilon\}_{\epsilon>0}$, one must have $\mathcal{R}(W) = (\bigcup_{\epsilon>0} \mathcal{R}(I_\epsilon))'' = \bigcap_{\epsilon>0} \mathcal{R}(A_\epsilon)$. This assumption is related to but not the same as net additivity. We begin with the following result which is crucial for our purposes.

PROPOSITION 7.1. – *Let* $\{W_\epsilon\}_{\epsilon>0}$ *be a collection of wedges indexed by a continuous path in wedge parameter space so that* $W_\epsilon \to W$ *as* $\epsilon \to 0$*. If the above-described net continuity condition holds, then the net* $\{J_{W_\epsilon}\}_{\epsilon>0}$ *converges strongly to* J_W *as* $\epsilon \to 0$*. In addition, the net* $\{\Delta_{W_\epsilon}^{it}\}_{\epsilon>0}$ *converges strongly to* Δ_W^{it} *as* $\epsilon \to 0$*.*

This continuity in the representation of the involutive generators of our group $\mathcal{G}$, plus the fact that these involutions include all $\mathcal{P}_+^\uparrow$-conjugates of the involution given in Proposition 6.2, implies that by making a specific choice of the projective representation (instead of an arbitrary one, as in Section 2), the following result can be proven.

THEOREM 7.2. – *With the Standing Assumptions,* $\mathcal{M} = \mathbb{R}^4$, $\mathfrak{I} = \mathcal{W}$, *the transitivity of the action of* $\mathcal{J}$ *on the set* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$, *and the net continuity condition mentioned at the beginning of this section, there exists a strongly continuous projective representation of the group* $\mathcal{P}_+^\uparrow$ *which acts geometrically correctly upon the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{I}}$.

We next address the question of whether this continuous projective representation can be lifted to a continuous representation.

8. COHOMOLOGY AND THE POINCARÉ GROUP

Assume that $\mathcal{P}_+^\uparrow \ni \lambda \mapsto U(\lambda) \in \mathcal{U}(\mathcal{H}_\omega)$[5] is a continuous projective representation of $\mathcal{P}_+^\uparrow$ by (anti)unitary operators on $\mathcal{H}$ constructed from products of modular involutions as in the previous section. By Theorem 2.2, the coefficients of this projective representation generate an abelian group[6] $\mathcal{Z} \subset \mathcal{U}(\mathcal{H}_\omega)$ which commutes with $\tilde{\mathcal{J}}$, and thus with each $U(\lambda)$. In particular, $\mathcal{Z}$ is a trivial $\mathcal{P}_+^\uparrow$-module. The theorem we prove is the following[7].

THEOREM 8.1. – *Under the above assumptions, there exists a continuous unitary representation of the covering group* $ISL(2,\mathbb{C})$ *of the Poincaré group* $\mathcal{P}_+^\uparrow$ *which acts geometrically correctly on the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{R}}$. *Moreover, there exists a continuous unitary representation of* $\mathcal{P}_+^\uparrow$ *(acting geometrically correctly on the net) if there is no element of order 2 in the group* $\mathcal{Z}$. *If there is an element of order 2 in* $\mathcal{Z}$, *then there is a cohomological obstruction which may prevent the projective representation of* $\mathcal{P}_+^\uparrow$ *from being lifted to a homomorphism.*

We are interested for physical reasons in the continuity of the representations, but we find ourselves obliged to use the Borel cohomology on locally compact groups initiated by Mackey [19] and fully defined and extended by Moore (*see* e.g. [20]), since the computational situation for

[5] Note that $\mathcal{U}(\mathcal{H}_\omega)$, provided with the strong (or weak) operator topology, is a complete, metrizable, second countable topological group.

[6] From the remark made earlier, it follows that also $\mathcal{Z}$ is a complete, metrizable, second countable topological group.

[7] A related theorem with different assumptions and proof may be found in [8].

continuous cohomologies seems to be exceedingly complicated. Fortunately, this is sufficient for our purposes.

Let G be a group and A be an abelian group. A central extension of G by A is a triple $(\tilde{G}, \phi, \iota)$ with $\tilde{G}$ a group, ι an injective homomorphism from A to $\tilde{G}$ satisfying $\iota(A) \subset \mathrm{center}(\tilde{G})$ and ϕ a homomorphism from $\tilde{G}$ onto G satisfying $\mathrm{kernel}(\phi) = \iota(A)$. In other words, the sequence

$$\{1\} \longrightarrow A \xrightarrow{\iota} \tilde{G} \xrightarrow{\phi} G \longrightarrow \{1\} \tag{8.1}$$

is exact, with $\{1\}$ denoting the trivial group. Since we wish to keep track of continuity, the homomorphism ι is required to be a homeomorphism onto a closed subgroup of $\tilde{G}$ and ϕ must be continuous and open (so that $\tilde{G}/i(A) \simeq \tilde{G}/\mathrm{kernel}(\phi) \simeq G$).

For connected semisimple Lie groups, the standard universal covering group coincides with the universal covering group in the sense of Moore, which itself coincides with the universal central extension [20]. Given a central extension (8.1) of G by A, assume that $\sigma : G \mapsto \tilde{G}$ is a section with $\sigma(1) = 1$, in other words it is a (Borel measurable) set map such that $\phi(\sigma(g)) = g$ for all $g \in G$. The function $\gamma(\sigma) = \gamma : G \times G \mapsto \tilde{G}$ defined by $\gamma(g,h) \equiv \sigma(g)\sigma(h)\sigma(gh)^{-1}$ is a measure of the amount σ diverges from a homomorphism, and, of course, the associativity in $\tilde{G}$ implies that γ is a 2-cocycle. Note that because $\phi(\gamma(g,h)) = 1$, γ actually takes values in the subgroup A. Let $Z^2(G,A)$ denote the set of all such (Borel measurable) 2-cocycles (which turns out to be an abelian group). Let $B^2(G,A)$ denote the A-valued coboundaries, *i.e.* the subgroup of $Z^2(G,A)$ consisting of functions $\gamma : G \times G \mapsto A$ for which there exists a (Borel measurable) $\beta : G \mapsto A$ such that $\gamma(g,h) = \beta(g)\beta(h)\beta(gh)^{-1}$ for all $g,h \in G$. The quotient group $Z^2(G,A)/B^2(G,A)$ is precisely the second cohomology group $H^2(G,A)$. One therefore sees that if $H^2(G,A) = \{1\}$, then every (A-valued) projective representation σ of G in $\tilde{G}$ determines a 2-cocycle γ which is actually a 2-coboundary. Thus, by defining $\tilde{\sigma} \equiv \beta(g)^{-1}\sigma(g)$, a straightforward calculation shows that $\tilde{\sigma} : G \mapsto \tilde{G}$ is a (Borel measurable) homomorphism, *i.e.* a representation, as desired. And if $H^2(G,A)$ is nontrivial, then it is possible to start with a section σ for which there exists no β for which $\beta^{-1}\sigma$ yields a homomorphism. In this case, the question would have to be settled for a given section individually.

We first present the relevant cohomological result for the universal covering groups, which are simply connected. $SL(2,\mathbb{C})$, resp. $ISL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$, is the covering group of $\mathcal{L}_+^\uparrow$, resp. $\mathcal{P}_+^\uparrow$.

LEMMA 8.2. – *The second cohomology groups $H^2(SL(2,\mathbb{C}),\mathcal{Z})$ and $H^2(ISL(2,\mathbb{C}),\mathcal{Z})$ are trivial.*

The situation is naturally different for the non-simply connected Lorentz and Poincaré groups.

LEMMA 8.3. – *The second cohomology groups* $H^2(\mathcal{L}_+^\uparrow, \mathcal{Z})$ *and* $H^2(\mathcal{P}_+^\uparrow, \mathcal{Z})$ *are trivial if and only if* $\mathcal{Z}$ *does not contain an element of order 2.*

Now, if $H^2(\mathcal{L}_+^\uparrow, \mathcal{Z})$ is trivial, then arguing as before, a given section determines a 2-cocycle which is actually a coboundary. Thus, one does indeed obtain a (unitary) representation of the Lorentz group. But in Moore's cohomology, the cochains are only Borel measurable on the group, *i.e.* although the original section σ is continuous, the function β may only be Borel measurable, so that $\tilde{\sigma} \equiv \beta^{-1}\sigma$ may be only Borel measurable. However, the following result, which was attributed to Mackey in [26], closes this gap.

LEMMA 8.4. – *If* G *and* A *are locally compact second countable groups and* $h : G \mapsto A$ *is a Borel measurable homomorphism, then* h *is continuous.*

9. THE SPECTRUM CONDITION AND ALGEBRAIC PCT AND SPIN & STATISTICS THEOREMS

Let W be a wedge containing the origin of $\mathbb{R}^4$ in its edge, and let $V(\mathbb{R}^4)$ be the representation of the translation group obtained above. Borchers [6] has isolated a geometric condition for the modular automorphism group Δ_W^{it} associated with the pair $(\mathcal{R}(W), \Omega)$ which is intimately connected to the spectrum condition (the spectrum of $V(\mathbb{R}^4)$ is contained in the closed forward lightcone).

Geometric Action of the Modular Automorphism Group

For every positive lightlike vector $e^{(0)}$ *such that* $W + e^{(0)} \subset W$, *the following relation holds:*

$$\Delta_W^{it} V(e^{(0)}) \Delta_W^{-it} = V(e^{-2\pi t} e^{(0)}), \quad \forall t \in \mathbb{R}. \tag{9.1}$$

This condition is equivalent to the representation $V(\mathbb{R}^4)$ satisfying the spectrum condition[8].

PROPOSITION 9.1. – [9] *Let* $V(\mathbb{R}^4)$ *be a representation of the translation group obtained in the previous section. Then* $V(\mathbb{R}^4)$ *satisfies the relativistic*

[8] We cite the result as proven in [9]; it was motivated by an analogous theorem proven under slightly more restrictive conditions by Wiesbrock [22]. The proof of the implication that the spectrum condition implies (9.1) is due to Borchers [6].

spectrum condition if and only if relation (9.1) holds for some wedge $W \in \mathcal{W}_0$.

Note that because we have a representation of $\mathcal{P}_+^\uparrow$ which acts geometrically correctly upon the net and which leaves the state invariant, if (9.1) holds for one $W \in \mathcal{W}_0$, it must hold for all $W \in \mathcal{W}_0$. In our work the only role played by the modular automorphism groups $\mathrm{ad}\Delta_W^{it}$ is to characterize algebraically the spectrum condition via Proposition 9.1.

With this additional condition, algebraic PCT and Spin & Statistics theorems can be proven. If the Standing Assumptions and the assumption that $\mathcal{J}$ acts transitively upon the set $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$ are supplemented with relation (9.1) and the net continuity condition from Section 7[9],then the conclusions of Proposition 6.2 and the other results established above imply that all of the hypotheses made in [17] in order to derive the PCT and Spin & Statistics theorems are fulfilled. We refer the reader to [17] for details.

In the papers [8], [14], among others, the starting point of the analysis was the assumption of what is called *modular covariance.* If $W \in \mathcal{W}$ is a wedge region and $\{v(t)\}_{t\in\mathbb{R}}$ is the one-parameter subgroup of Lorentz boosts leaving the wedge W invariant, then modular covariance is said to hold if

$$\Delta_W^{it}\mathcal{R}(\mathcal{O})\Delta_W^{-it} = \mathcal{R}(v(t)\mathcal{O}), \quad \text{for all} \quad t \in \mathbb{R}, \quad \mathcal{O} \in \mathcal{K} \cup \mathcal{W},$$

in other words, if the modular automorphism group associated to the algebra for the wedge W implements the mentioned boost subgroup. There are some further variations of this condition in the literature (*see*, e.g [13]), but they have in common the requirement that one already is given an action of the Lorentz group on the space-time.

The condition (9.1) on the modular automorphism groups is truly an additional hypothesis. To make this clear, we present one of our examples of a net satisfying our condition of geometric modular action (also in the stronger form of the first paper [9]) and all of the other assumptions made in this paper except (9.1). Hence this example violates the spectrum condition. In addition, the modular groups associated to wedge algebras do not coincide with the representation of the Lorentz boosts, i.e. modular covariance *fails* in this example. It follows that the assumption of modular covariance excludes some Poincaré covariant nets. It is, however, still an open question whether modular covariance can fail in the presence of the spectrum condition (in this setting, where Lorentz covariance is assured).

[9] plus Kuckert's technical postulate that the internal symmetry group is compact

Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\Re}$ be the standard net of von Neumann algebras for the (hermitian, scalar) free field with mass $m > 0$ on the corresponding Fock space $\mathcal{F}$ and let α_λ be the automorphic action of the Poincaré group on this net. Let Θ be the PCT-operator on $\mathcal{F}$ for this net and $\theta \in \mathcal{L}$ be its induced action on $\mathbb{R}^4$. For each $\mathcal{O} \in \Re$ define $\mathcal{B}(\mathcal{O}) = \mathcal{A}(-\mathcal{O}) = \Theta\mathcal{A}(\mathcal{O})\Theta$. Let $\hat{\mathcal{A}}(\mathcal{O}) \equiv \mathcal{A}(\mathcal{O}) \otimes \mathcal{B}(\mathcal{O})$ act on $\mathcal{F} \otimes \mathcal{F}$. The net $\{\hat{\mathcal{A}}(\mathcal{O})\}_{\mathcal{O}\in\Re}$ is clearly local. One verifies that $\hat{\alpha}_\lambda \equiv \alpha_\lambda \otimes \beta_\lambda$, with $\beta_\lambda \equiv \alpha_{\theta\lambda\theta}$, $\lambda \in \mathcal{P}_+^\uparrow$, defines an automorphic local action on $\{\hat{\mathcal{A}}(\mathcal{O})\}_{\mathcal{O}\in\Re}$. With $U(\lambda)\mathcal{A}(\mathcal{O})U(\lambda)^{-1} = \mathcal{A}(\mathcal{O}_\lambda)$ the unitary implementation of α_λ on $\mathcal{F}$, $V(\lambda) \equiv \Theta U(\lambda)\Theta$ implements the action of β_λ. With $U(t) = e^{itH}$ implementing the time translations, one has $V(t) = \Theta e^{itH}\Theta = e^{-itH}$. So $\{V(\lambda)\}$ violates the spectrum condition, as does $\{\hat{U}(\lambda) \equiv U(\lambda) \otimes V(\lambda)\}$, which implements $\hat{\alpha}_\lambda$.

By the results of Bisognano and Wichmann [4], applicable to the free field, one knows that for the wedge $W = W_R$, the modular structure for $(\mathcal{A}(W), \Omega)$ is given by $J_W = \Theta_1$ ($\Theta_1 = \Theta U_1(\pi)$, where $U_1(\pi)$ implements the rotation R_1 of π about the 1-axis) and $\Delta_W^{it} = B_1(t)$, where $B_1(t) \in U(\mathcal{P}_+^\uparrow)$ implements the Lorentz boosts $v_1(t)$ in the x^1-direction. The corresponding modular objects for $(\mathcal{B}(W), \Omega)$ are given by ${}_{\mathcal{B}}J_W = \Theta\Theta_1\Theta = \Theta_1$ and ${}_{\mathcal{B}}\Delta_W^{it} = \Theta B_1(t)\Theta = B_1(-t)$. It follows that the modular structure for $(\hat{\mathcal{A}}(W), \Omega \otimes \Omega)$ is given by $\hat{J}_W = \Theta_1 \otimes \Theta_1$ and $\hat{\Delta}_W^{it} = B_1(t) \otimes B_1(-t)$. One checks that $\hat{J}_W\hat{\mathcal{A}}(\mathcal{O})\hat{J}_W = \hat{\mathcal{A}}(-\mathcal{O}_{R_1})$, so the condition of geometric modular action is satisfied. Turning to the modular groups, one sees $\hat{\Delta}_W^{it}\hat{\mathcal{A}}(\mathcal{O})\hat{\Delta}_W^{-it} = \mathcal{A}(\mathcal{O}_{v_1(t)}) \otimes \mathcal{B}(\mathcal{O}_{v_1(-t)}) \neq \hat{\mathcal{A}}(\mathcal{O}_{v_1(t)})$. Hence, modular covariance fails.

We wish to make a few comments about the uniqueness of the representation of $\mathcal{P}_+^\uparrow$ which has been obtained above.

PROPOSITION 9.2. – [9] *Let $U(\mathbb{R}^4)$ and $V(\mathbb{R}^4)$ be two continuous unitary representations of the translations on $\mathcal{H}_\omega$ which act geometrically correctly on the net $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\Re}$*[10]*, leave Ω invariant, and satisfy the spectrum condition. Then $U(x) = V(x)$ for every $x \in \mathbb{R}^4$.*

For the representation of the entire Poincaré group, the best result seems to be that of [8], which asserts that if the distal split property holds, then the representation of $\mathcal{P}_+^\uparrow$ is also unique.

[10] Assumed here to satisfy locality.

10. GEOMETRIC MODULAR ACTION AND THE DE SITTER GROUP

All of the arguments and results presented above concerning the Poincaré group in four dimensions are also true *mutatis mutandis* for the de Sitter group in three dimensions. We mention the following result as an example. We let $\overline{\mathcal{W}}_0 \equiv \{W \cap dS^3 \mid W \in \mathcal{W}_0\}$.

THEOREM 10.1. – *If with the choices* $\mathfrak{S} = \overline{\mathcal{W}}_0$ *and* $\mathcal{M} = dS^3$, *the Standing Assumptions, conditions* **B1-B4** *with* $\mathcal{W}$ *replaced by* $\overline{\mathcal{W}}_0$, *and the transitive action of* $\mathcal{J}$ *on* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}$ *all hold, then there exists a projective (anti)unitary representation of the identity component of the de Sitter group in three dimensions which acts geometrically correctly on the net* $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}$.

The results from Section 6 onwards also have their counterparts, but we shall not list them here. To demonstrate that this theorem is not vacuous, we recall an example due to Fredenhagen [12]. Consider once again the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{W}}$ from the previous section on $\mathbb{R}^{n+1}$, with $n = 3, 4$. The results of Bisognano and Wichmann [4] entail that all the hypotheses of the previous sections hold for this net in the vacuum state. For each region $\mathcal{O} \in \overline{\mathcal{W}}_0$, we define $\mathcal{R}(\mathcal{O}) \equiv \bigvee_{\lambda 0} \mathcal{A}(\lambda\mathcal{O})$. This net is covariant under the de Sitter group, and the assumptions in Theorem 10.1 are satisfied by this net in the vacuum state.

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