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Nonexistence of minimal blow-up solutions
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by

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ABSTRACT. - In this paper, we prove the existence of blow-up solutions
of Equation of the form $i u_t = -\Delta u - k(x) |u|^{4/N} u$ in $\mathbb{R}^N$ under some
conditions on $k(x)$. We then consider the problem to find minimal blow-up
solutions in $L^2$.

RÉSUMÉ. – On démontre l’existence de solutions explosives pour des
équations de la forme $i u_t = -\Delta u - k(x) |u|^{4/N} u$ dans $\mathbb{R}^N$, sous certaines
conditions sur $k(x)$. On considère ensuite le problème de trouver des
solutions singulières minimales dans $L^2$.

Mots clés : Schrödinger, critique, explosion, minimal, stabilité.

1. INTRODUCTION

In the present paper, we consider the nonhomogeneous nonlinear
Schrödinger equation with critical exponent

\begin{equation}
(1.1) \quad i \frac{\partial u}{\partial t} = -\Delta u - k(x) |u|^{4/N} u
\end{equation}

and

\begin{equation}
(1.2) \quad u(0, \cdot) = \phi(\cdot),
\end{equation}
where $\Delta$ is the Laplace operator on $\mathbb{R}^N$, $u : [0, T) \times \mathbb{R}^N \to \mathbb{C}$ and $\phi \in H^1(\mathbb{R}^N)$.

We assume in this paper that $k$ is a given $C^1$ function such that there are $k_1 > 0$, $k_2 > 0$ and $c > 0$ such that

$$(H.1) \quad \forall x \in \mathbb{R}^N, \quad k_1 \leq k(x) \leq k_2,$$

$$(H.2) \quad \forall x \in \mathbb{R}^N, \quad |\nabla k(x)| + |x \cdot \nabla k(x)| \leq c,$$

$$(H.3) \quad \text{there is } x_0 \in \mathbb{R}^N, \quad k(x_0) = k_2.$$

We say that $u(\cdot)$ is a solution of Eq. (1.1)-(1.2) on $[0, T)$ if $\forall t \in [0, T)$,

$$u(t) = S(t) \phi + i \int_0^t S(t-s) \{k(x) |u(s)|^{2/N} u(s)\} \, ds,$$

where $S(\cdot)$ is the group with infinitesimal generator $i \Delta$ and, for each $t$, $u(t)$ denotes the function $x \mapsto u(t, x)$.

It is easy to prove as in the homogeneous case:

$$(1.4) \quad k(x) \equiv k_0,$$

that Eq. (1.1)-(1.2) has a unique solution $u(t)$ in $H^1(\mathbb{R}^N)$ and there exists $T > 0$ such that, $\forall t \in [0, T)$, $u(t) \in H^1(\mathbb{R}^N)$ and either

$$T = +\infty,$$

or

$$T < +\infty \quad \text{and} \quad \lim_{t \to T^-} \|u(t)\|_{H^1} = +\infty,$$

where $\| \cdot \|_{H^1}$ is the usual norm on $H^1$, and $H^1$ is $H^1(\mathbb{R}^N)$ (see Ginibre and Velo [2], Kato [6]).

Furthermore, we have $\forall t \in [0, T)$,

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 \, dx$$

$$- \frac{1}{4} \int_{\mathbb{R}^N} k(x) |u(t, x)|^{\frac{4}{N} + 2} \, dx$$

$$= E(\phi).$$
In this paper we are interested in the study of singular solutions of Eq. (1.1)-(1.2). In the case where

\begin{equation}
    k(x) \neq k_0,
\end{equation}

there are no results available.

Let us first recall some results in the case where $k(x) \equiv k_0$. For such a nonlinearity, there is another identity which is the following.

Let $\phi \in \Sigma = H^1 \cap \{ |x| \phi \in L^2 \}$ then $\forall t < T$, $u(t) \in \Sigma$, and

\begin{equation}
    \frac{d}{dt} \int |x|^2 |u(t,x)|^2 \, dx = 4 \text{Im} \int x \cdot \overline{\nabla} u u \, dx,
\end{equation}

and

\begin{equation}
    \frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 \, dx = 16 E(\phi).
\end{equation}

From this identity, it follows easily that if

\begin{equation}
    E(\phi) < 0
\end{equation}

then

\[ T < +\infty \]

(see Zakharov, Sobolev, Synach [15] and Glassey [5]). Moreover blow-up solutions have three important properties.

(i) They are bounded from below in $L^2$ (Weinstein [18]). That is, let $Q_{k_0}$ be the unique radially symmetric solution of

\begin{equation}
    \Delta u + k_0 |u|^{\frac{4}{N}} u = u
\end{equation}

(see for existence Strauss, Berestycki, Lions, Peletier [1], [16], and for uniqueness Kwong [7]). If $u(t)$ is a blow-up solution then

\[ \| \phi \|_{L^2} \geq \| Q_{k_0} \|_{L^2}. \]

(ii) The set of minimal blow-up solutions is known (Merle [10], [11]).

Let $u(t)$ be a blow-up solution with minimal mass in $L^2$, \( \| \phi \|_{L^2} = \| Q_{k_0} \|_{L^2} \). There are then constants $\theta \in S^1$, $\omega > 0$, $x_0 \in \mathbb{R}^N$, $x_1 \in \mathbb{R}^N$, $\omega \in \mathbb{R}$, with $x_1 \neq 0$ and $x_1 \cdot x_0 \neq 0$.
$T > 0$ such that

\begin{equation}
(1.12) \quad u(t, x) = \left( \frac{\omega}{T - t} \right)^{\frac{n}{2}} \exp \left\{ i \left( \theta + \frac{|x - x_1|^2}{4(-T + t)} - \frac{\omega^2}{(-T + t)} \right) \right\} \times Q_{k_0} \left( \frac{\omega(x - x_1)}{T - t} - \omega x_0 \right).
\end{equation}

In [3], [4], we point out the importance of such solutions as limits of “stable” (from the numerical point of view) blow-up solutions for more complex equations which have (1.1) as a limit case (see Landam, Papanicolaou, C. and P. L. Sulem, Wang for numerical simulations [8], [14]).

(iii) At the blow-up time, there is a concentration phenomenon (Merle, Tsutsumi [12], Weinstein [19], Merle [9], Proposition A.3 in [4]). Indeed, let $u(t)$ be a blow-up solution of Eq. (1.1) and $T$ its blow-up time. There is then $x(t)$ for $t > T$ such that

\[
\forall R > 0, \quad \liminf_{t \to T} \| u(t) \|_{L^2(B(x(t), R))}^2 \geq \| Q_{k_0} \|_{L^2}^2,
\]

where $B(x, R)$ is the ball of radius $R$ and center $x$.

We first have the following result about existence of blow-up solutions.

**Theorem 1 (Existence and lower $L^2$-bound of blow-up solutions, concentration at the blow-up time).** - (i) **Lower $L^2$-bound:** Assume that $k$ satisfies (H.1)-(H.2). Let $\phi \in H^1$ be such that

\[
\| \phi \|_{L^2} < \| Q_{k_2} \|_{L^2}.
\]

Then $u(t)$ is globally defined in time.

(ii) **Existence of blow-up solutions:** Let $k$ satisfy (H.1)-(H.3). Assume in addition that $k$ satisfies (H.4) or (H.4)' where

(H.4) There is a $\rho_0 > 0$ such that

\[
(x - x_0) \cdot \nabla k(x) < 0 \quad \text{for} \ 0 < |x - x_0| < \rho_0
\]

and

(H.4)'

\[
\forall x, \quad (x - x_0) \cdot \nabla k(x) \leq 0,
\]

and $x_0$ is such that $k(x_0) = k_2$. Then there is $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, there is $\phi_\varepsilon \in H^1$ such that

\[
- \| \phi_\varepsilon \|_{L^2} = \| Q_{k_2} \|_{L^2} + \varepsilon,
\]
\[ u_\varepsilon(t) \text{ blows up in finite time where } u_\varepsilon(t) \text{ is the solution of Eq. (1.1)} \]

with initial data \( \phi_\varepsilon \). In addition, \( \varepsilon_0 = +\infty \) when \( k \) satisfies (H.4)'.

(iii) Concentration at the blow-up time: Let \( k \) satisfy (H.1)-(H.2), let \( u(t) \) be a blow-up solution of Eq. (1.1) and let \( T \) be its blow-up time. There is then \( x(t) \) for \( t < T \) such that

\[
\forall R > 0, \liminf_{t \to T} \| u(t) \|_{L^2(B(x(t), R))}^2 \geq \| Q_{k_2} \|_{L^2}^2.
\]

**Remark.** – In part (ii), assumption (H.4) or (H.4)' can be weaken (see section 3) and \( x_0 \) can be a local maximum. However, it is still an open problem to show existence of blow-up solutions in the case where there is no local maximum of \( k \).

Let us now consider \( k \) satisfying (H.1)-(H.3). The main question is whether there is or not \( L^2 \)-minimal blow-up solution: Is there a \( \phi \in H^1 \)

\[
\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2},
\]

\( u(t) \) blows-up in finite time where \( u(t) \) is the solution of (1.1)-(1.2).

These results related to \( L^2 \)-minimal blow-up solutions have a physical interest.

- In the case of existence of such a solution, we have a solution which blows up with minimal mass and is in some sense the limit point of numerically stable blow-up solution (see [8], [14]).

- In the case of nonexistence of such a solution, we obtain the existence of a space singularity which is in some sense, stable in time with respect to Eq. (1.1). We will call this kind of phenomenon a black hole (see Theorem 3).

**THEOREM 2** (\( L^2 \)-minimal blow-up solutions). – Consider \( k \) satisfying (H.1)-(H.2) and (H.5) where

\[
\text{(H.5)} \quad \exists \delta_0 > 0 \text{ and } R_0 > 0 \text{ such that for } |x| > R_0, \quad k(x) \leq k_2 - \delta_0 \quad \text{and } M = \{x; k(x) = k_2\} \text{ is finite.}
\]

(i) **Characterization:** Assume that \( \| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} \) and \( u(t) \) blows-up in finite time. There is then \( x_0 \in M \) such that

\[
- |u(t, x)|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0} \text{ in the distribution sense,}
- |x - x_0|^2 |u(t, x)|^2 \to 0 \text{ in } L^1, \text{ as } t \to T.
\]
(ii) Nonexistence result: Assume in addition that for \( x_0 \in M \), we have the following property:

\[(H.6) \quad \text{there is } \rho_0 \text{ and } \alpha_0 \in (0, 1) \text{ such that} \]
\[\nabla k(x) \cdot (x - x_0) \leq -|x - x_0|^{1+\alpha_0}, \quad \text{for } |x - x_0| \leq \rho_0.\]

There is then no blow-up solutions such that
\[\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2}.\]

Remark. – In the case where \( k(x) \equiv k_2 \) globally or \( k(x) \equiv k_2 \) for \( x \) near \( x_0 \), we are able to show the existence of minimal blow-up solution. Therefore, the existence of minimal blow-up solutions depends strongly on the form of the function \( k(x) \) near the points where \( k \) achieves its maximum. However, we do not know exactly the case of limiting behavior near \( x_0 \) (where \( x_0 \) is such that \( k(x_0) = k_2 \)) of \( k \) (between flatness near \( x_0 \) and assumption (H.5)) where there is nonexistence of minimal \( L^2 \) blow-up solutions.

We can in addition remark that in the elliptic situation in the case where \( k(x) \neq k_2 \) there is no solution of the equation
\[\Delta v + k(x) |v|^{\frac{N}{N-2}} v = \omega v\]
where \( \omega > 0 \) such that
\[\| v \|_{L^2} = \| Q_{k_2} \|_{L^2}.\]

**Theorem 3** (Stability in time of singularity). – Assume that \( x_0 \) is such that \( k(x_0) = k_2 \) and \( x_0 \) is a strict local maximum. Moreover, assume that there is no blow-up solution of Eq. (1.1)-(1.2) such that

\[\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2}.\]

Consider now a sequence \( \phi_n \in H^1 \) such that
- \( \| \phi_n \|_{L^2}^2 \to \| Q_{k_2} \|_{L^2}^2, \)
- \( |\phi_n(x)|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0} \) in the distribution sense,
- there is a \( c > 0 \) such that
\[E_{\varepsilon_n}(\phi_n) \leq c\]
where \( \varepsilon_n \to 0 \) as \( n \to +\infty \), \( \varepsilon_n > 0 \), \( q \in \left( \frac{4}{N+1}, 1 + \frac{4}{N-2} \right) \),
\[E_{\varepsilon}(u) = E(u) + \frac{\varepsilon}{q+1} \int |u|^{q+1}.\]
Then \( u_n(t) \), the solution of equation
\[iu_t = -\Delta u - k(x)|u|^{\frac{N}{N-2}} u + \varepsilon_n |u|^{q-1} u,\]
(1.13) \[u(0) = \phi_n,\]
(1.14)
is such that
- $u_n(t)$ is defined for all time,
- for all time $t > 0$,

$$\left| u_n(t, x) \right|^2 \rightarrow \| Q_{k_2} \|^2_{L^2} \delta_{x=x_0}$$

(1.15)

in the distribution sense as $n \rightarrow +\infty$,

and

$$\| u_n(t) \|_{L^2} \rightarrow \| Q_{k_2} \|_{L^2}, \text{ as } n \rightarrow +\infty.$$ (1.16)

Remark. – In this case, we say that $\| Q_{k_2} \|^2_{L^2} \delta_{x=x_0}$ is a singularity stable in time.

The plan of the paper is the following:
- In section two, we establish some conservation laws for solutions of (1.1) and derive some concentration properties at the blow-up time.
- In section three, we prove some blow-up results.
- Sections four and five are devoted to minimal blow-up solutions.
- Finally, in section six, we study the existence of black holes.

2. CONCENTRATION PROPERTIES OF BLOW-UP SOLUTIONS

In the first subsection, we give various identities satisfied by solutions of Eq. (1.1). We assume that $\phi \in \Sigma = H^1 \cap \{u; xu \in L^2\}$.

2.A. Conservation laws

Let us consider $u(t, x)$ solution of Eq. (1.1) and $T$ its blow-up time.

Proposition 2.1. – We have $\forall t \in [0, T)$,

$$\int |u(t, x)|^2 \, dx = \int |\phi(x)|^2 \, dx,$$

(2.1) (i)

$$E(u(t)) = E(\phi) \text{ where}$$

$$E(u) = \frac{1}{2} \int |\nabla u(x)|^2 \, dx - \frac{1}{4} \int k(x) |u(t, x)|^{\frac{4}{N}} + 2 \, dx,$$

(2.2) (ii)

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(2.3) (iii) \[ \frac{d}{dt} \int |x|^2 |u(t, x)|^2 dx = 4 \text{Im} \int \bar{u} \nabla u \cdot x, \]

(2.4) \[ \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 4 \left\{ 4 E(\phi) + \frac{1}{2N + 1} \int x \cdot \nabla k(x) |u(t, x)|^{\frac{2}{N}} dx \right\}. \]

Proof. – (i) and (ii) follow from direct calculation.

(iii) Let us show that

On the one hand,

On the other hand

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\[ \frac{d}{dt} \int \text{Im} \bar{u} \nabla u \cdot x = 4 E(\phi) \]

\[ + \frac{1}{2N + 1} \int x \cdot \nabla k(x) |u(t, x)|^{\frac{2}{N}} dx. \]

(2.6) \[ \frac{d}{dt} \text{Im} \int \bar{u} \nabla u \cdot x = \text{Im} \left\{ \int x \bar{u} \frac{\partial u}{\partial t} + \int x \frac{\partial \bar{u}}{\partial t} \nabla u \right\} \]

\[ = \text{Im} \left\{ 2 \int x \frac{\partial \bar{u}}{\partial t} \nabla u - N \int \frac{\partial \bar{u}}{\partial t} \right\}. \]

On the one hand,

(2.7) \[ N \text{Im} \int \bar{u} \frac{\partial u}{\partial t} = -N \text{Re} \int \bar{u} (\Delta u + k(x) |u|^{\frac{2}{N}} u) \]

\[ = -N \int k(x) |u|^{\frac{4}{N} + 2} + N \int |\nabla u|^2. \]

On the other hand

(2.8) \[ 2 \text{Im} \int x \frac{\partial \bar{u}}{\partial t} \nabla u \]

\[ = -2 \text{Re} \left\{ \int x \Delta u \nabla \bar{u} + \int x k(x) |u|^{\frac{2}{N}} u \nabla \bar{u} \right\} \]
From (2.6)-(2.8), (2.5) follows.

As in the case $k(x) = k_0$, let us derive some consequences of these conservation laws.

**Corollary 2.2.**

(i) \[
\frac{d}{dt} \int |\vec{x}| u(t, x) |^2 dx = 2 \text{Im} \int \bar{u} \nabla u,
\]

(ii) \[
\frac{d^2}{dt^2} \int |\vec{x}| u(t, x) |^2 dx = \frac{2}{N+1} \int \nabla |u(t, x) |^{\frac{4}{N}+2} dx.
\]

**Proof.** We have for all $x_0 \in \mathbb{R}^N$:

\[
\frac{d}{dt} \int |\vec{x} + \vec{x}_0|^2 |u(t, x) |^2 dx = 4 \text{Im} \int \bar{u} \nabla u \cdot (\vec{x} + \vec{x}_0).
\]

Therefore,

\[
\frac{d}{dt} \left\{ |x_0|^2 \int |u(t, x) |^2 dx + \int |x|^2 |u(t, x) |^2 dx + 2 \vec{x}_0 \cdot \int |\vec{x}| u(t, x) |^2 dx \right\}
\]

\[
= 4 \text{Im} \int \bar{u} \nabla u \cdot x + 4 \vec{x}_0 \text{Im} \int \bar{u} \nabla u,
\]

and from Proposition 2.1,

\[
(2.9) \quad \vec{x}_0 \cdot 2 \frac{d}{dt} \int |\vec{x}| u(t, x) |^2 dx = \vec{x}_0 \cdot 4 \text{Im} \int \bar{u} \nabla u.
\]

(i) follows from the fact that (2.9) is true for all $x_0 \in \mathbb{R}^N$. Proof of part (ii) is similar.
Let us write an energy type identity from Proposition 2.1 derived in the case $k(x) \equiv k_2$ by Anosov and rediscovered by Ginibre and Velo.

**Corollary 2.3.** We have

$$
\tilde{E}_t(u(t)) = \tilde{E}_0(u(0)) - \int_0^t \frac{s}{4N+2} \int x \cdot \nabla k |u(s,x)|^{\frac{N}{2}+2} \, dx \, ds
$$

$$
= \frac{1}{8} \int |x|^2 |\phi(x)|^2 \, dx
\quad - \int_0^t \frac{s}{4N+2} \int x \cdot \nabla k |u(s,x)|^{\frac{N}{2}+2} \, dx \, ds,
$$

where

$$
\tilde{E}_t(u) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{4N+2} \int k(tx) |v|^{\frac{N}{2}+2}
$$

with

$$
v = |t|^\frac{N}{2} e^{-i\frac{|x|^2 t}{4}} u(xt).
$$

**Proof.** Let $u \in \Sigma$,

(2.10)  \[
\tilde{E}_t(u) = \frac{1}{2} \int |t|^N \left| \left( -\frac{ixt}{2} + t \nabla \right) u(xt) \right|^2
\quad - \frac{1}{4N+2} t^2 \int k(tx) |t|^N |u(xt)|^{\frac{N}{2}+2} \, dx
\]

$$
= \frac{1}{2} \int \left| \left( -\frac{iy}{2} + t \nabla \right) u(y) \right|^2 \, dy
\quad - \frac{t^2}{4N+2} \int k(y) |u(y)|^{\frac{N}{2}+2} \, dy
\]

$$
= \frac{1}{2} \left\{ \frac{1}{4} \int |x|^2 |u(x)|^2 \, dx
\quad - t \text{Im} \int x \cdot \nabla u\bar{u} \right\} + t^2 E(u).
$$
Let us now consider \( \tilde{E}_t(u(t)) \)

\[
\frac{d}{dt} \tilde{E}_t(u(t)) = \frac{1}{2} \left\{ \frac{1}{4} \frac{d}{dt} \int |x|^2 |u(t, x)|^2 \, dx - \text{Im} \int x \cdot \nabla u(t) \bar{u}(t) \right\} \\
- \frac{t}{2} \int x \cdot \nabla u(t) \bar{u}(t) + 2tE(\phi).
\]

From Proposition 2.1, we have

\[
\frac{d}{dt} \tilde{E}_t(u(t)) = \frac{1}{2} \left\{ \text{Im} \int x \cdot \nabla u(t) \bar{u}(t) - \text{Im} \int x \cdot \nabla u(t) \bar{u}(t) \right\} \\
- \frac{t}{2} \left\{ 4E(\phi) + \frac{1}{2} \frac{1}{N+1} \int x \cdot \nabla k |u(t, x)|^{\frac{4}{N}+2} \, dx \right\} \\
+ 2tE(\phi) \\
= -\frac{t}{4} \frac{1}{N+2} \int x \cdot \nabla k |u(t, x)|^{\frac{4}{N}+2} \, dx,
\]

which concludes the proof of Corollary 2.3 and Section 2.A.

**2.B. Concentration properties of blow-up solutions of Eq. (1.1)**

In this section, we consider a blow-up solution of Eq. (1.1), \( u(t) \). Let \( T \) be its blow-up time. Assume that

\[ -0 < k_1 \equiv \inf_{x \in \mathbb{R}^N} k(x) \leq k_2 \equiv \sup_{x \in \mathbb{R}^N} k(x) < +\infty, \]
\[ -k \in C^1, \]
\[ -|\nabla k| \leq c_0. \]

We claim the following

**Proposition 2.4.** – There is \( x(t) \in \mathbb{R}^N \) such that for all \( R > 0, \)

\[
\lim_{t \to T} \inf_{t \to T} \|u(t)\|_{L^2(B(x(t), R))} \geq \|Q_{k_2}\|_{L^2},
\]

where \( Q_{k_2} \) is the unique positive radially symmetric solution of

\[
v = \Delta v + k_2 |v|^\frac{4}{N} v.
\]
Remark. – From scaling argument, we have \( Q_{k_2} = \frac{1}{k_2^{\frac{N}{4}}} Q \) where \( Q \) is the unique radially symmetric solution of (II, 1). In particular
\[
\| Q_{k_2} \|_{L^2} = \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{4}}}.
\]
In fact, we have a slightly more precise result.

**Proposition 2.5.** – There is \( x(t) \in \mathbb{R}^N \) such that for all \( R > 0 \),
\[
\liminf_{t \to T} \left\{ \frac{\| u(t) \|_{L^2(B(x(t), R))}}{\| Q_{k(x(t))} \|_{L^2}} \right\} \geq 1.
\]

Remark.
\[
\| Q_{k(x(t))} \|_{L^2} = \frac{\| Q \|_{L^2}}{[k(x(t))]^{\frac{N}{4}}} \geq \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{4}}}.
\]

Proof of Proposition 2.5 follows exactly the proof of Proposition 2.4 and will be omitted (it uses the fact that \( \forall R > 0, \)
\[
\sup_{|x-y| \leq R} \left| k \left( \frac{x}{\lambda(t)} \right) - k \left( \frac{y}{\lambda(t)} \right) \right| \leq c_0 \frac{|x-y|}{\lambda(t)} \leq \frac{Rc_0}{\lambda(t)} \to 0,
\]
where \( \lambda(t) = \| \nabla u(t) \|_{L^2} \).

**Sketch of proof of Proposition 2.4.** – It is a consequence of similar results in [18], [12], [9], [4]. Indeed, we have
\[
(2.11) \quad E_{k_2}(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 \left( \int \frac{k_2}{4} |u(t, x)|^{\frac{4}{N} + 2} dx \right)
\leq \frac{1}{2} \int |\nabla u(t, x)|^2 \left( \int \frac{k(x)}{4} |u(t, x)|^{\frac{4}{N} + 2} dx \right)
\leq E(u(t)) = E(\phi)
\]
and
\[
(2.12) \quad \| u(t) \|_{L^2} = \| \phi \|_{L^2}.
\]
Let us argue by contradiction. Assume there are \( R_0 > 0 \), \( \delta_0 > 0 \) and a sequence \( t_n \to T \) such that

\[
\sup_{x \in \mathbb{R}^N} \left\{ \int_{|x-y|<R_0} |u(t_n, x)|^2 \, dy \right\} \leq \| Q_{k_2} \|_{L^2}^2 - \delta_0.
\]

Then from results of [12], [19], [4], we have the existence of constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

(2.13) \hspace{1cm} \forall t_n, \quad -c_1 + c_2 \int |\nabla u(t_n, x)|^2 \, dx \leq E_{k_2}(u(t_n))

(see from example Proposition A.3 in [4]).

From (2.11), we deduce that \( \int |\nabla u(t_n, x)|^2 \, dx \leq c \) which contradicts that \( t_n \to T \). This concludes the proof of Proposition 2.4 and Theorem 1. (iii).

As a direct consequence of Proposition 2.4 and (2.12), we obtain

**Corollary 2.6.** (Lower bound for blow-up solutions). – Assume

\[
\| \phi \|_{L^2} < \| Q_{k_2} \|_{L^2} = \frac{\| Q \|_{L^2}}{k_2^N}.
\]

Then the solution \( u(t) \) is globally defined in time.

In fact, from the proof of Proposition 2.4, we have a useful corollary (see also [19]):

**Corollary 2.7.** – Let \( u_n \in H^1 \) be such that \( \| u_n \|_{L^2} \to \| Q_{k_2} \|_{L^2} \), \( \lambda_n = \| \nabla u_n \|_{L^2} \to +\infty \) as \( n \to +\infty \) and \( E(u_n) \leq c \) for a \( c > 0 \). There are sequences \( x_n \in \mathbb{R}^N \), \( \theta_n \in S^1 \) such that

\[
|u_n(x-x_n)|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0},
\]

and

\[
\lambda_n^{-\frac{N}{2}} e^{i\theta_n} u_n \left( \frac{x-x_n}{\lambda_n} \right) \to Q_{k_2} \text{ in } H^1.
\]

3. **BLOW-UP THEOREMS FOR SOLUTIONS OF EQ. (1.1)**

In the homogeneous case

(3.1) \hspace{1cm} k(x) \equiv k_0;
blow-up theorems are obtained using the virial identity

\begin{equation}
\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 16 E(\phi).
\end{equation}

(see [5], [15]). If \( E(\phi) < 0 \), then using the fact

\begin{equation}
\forall t, \int |x|^2 u(t, x)|^2 dx > 0
\end{equation}

and (3.2), we obtain a contradiction.

In the case where

\( k(x) \equiv k \)

such an identity is not true anymore (see (2.4)) and we have \( \forall x_0 \in \mathbb{R}^N \),

\begin{equation}
\frac{d^2}{dt^2} \int |x - x_0|^2 |u(t, x)|^2 dx = 16 E(\phi) + \frac{4}{2} + 1
\end{equation}

\( \times \int (x - x_0) \nabla k |u(t, x)|^{\frac{4}{N} + 2} dx. \)

Under some global or local conditions on the sign of

\( (x - x_0) \nabla k(x) \)

we are able to obtain some blow-up theorems for solutions of Eq. (1.1).

**Theorem 3.1** (Global condition on \( (x - x_0) \cdot \nabla k(x) \)). - Assume there is \( x_0 \in \mathbb{R}^N \) such that

\begin{equation}
\forall x \in \mathbb{R}^N, \quad (x - x_0) \cdot \nabla k(x) \leq 0
\end{equation}

so that \( x_0 \) is global maximum of \( k(x) \).

(i) Let \( \phi \in \Sigma \) be such that \( E(\phi) < 0 \). Then the solution \( u(t) \) of Eq. (1.1)

blows up in finite time.

For all \( \varepsilon > 0 \), there is \( \phi_\varepsilon \) such that

- \( \| \phi_\varepsilon \|_{L^2} = \| Q_{k_2} \|_{L^2} + \varepsilon \),

- \( u_\varepsilon(t) \) blows-up in finite time, where \( u_\varepsilon(t) \) is the solution of Eq. (1.1)

with initial data \( \phi_\varepsilon \).

**Theorem 3.2** (Local condition on \( (x - x_0) \cdot \nabla k(x) \)). - Assume there is

\( x_0 \in \mathbb{R}^N \) and \( \rho_0 > 0 \) such that

\begin{equation}
(x - x_0) \cdot \nabla k(x) < 0, \quad \text{for} \quad 0 < |x - x_0| < \rho_0,
\end{equation}

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so that $x_0$ is a local strict maximum of $k(x)$

$$k(x_0) \geq k(x) \quad \text{for } 0 < |x - x_0| < \rho_0.$$ 

There is $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists $\phi_\varepsilon \in \Sigma$ such that

- $\| \phi_\varepsilon \|_{L^2} = \| Q_{k(x_0)} \|_{L^2} + \varepsilon$,
- $u_\varepsilon(t)$ blows up in finite time where $u_\varepsilon(t)$ is the solution of Eq. (1.1) with initial data $\phi_\varepsilon$.

**Remark.** Theorem 3.2 implies Theorem 3.1 but the proof of Theorem 3.1 is completely elementary. Assumption (3.6) can be weaken and replaced by

$$(3.6)' \quad \{(x - x_0) \cdot \nabla k(x) \leq 0 \quad \text{for } 0 < |x - x_0| < \rho_0, 
(x - x_0) \cdot \nabla k(x) < 0 \quad \text{on } S,$$

where $S$ is a closed hypersurface included in $B(x_0, \rho_0)$ with $x_0$ in its interior.

In Theorem 3.1 or 3.2, we have to assume that $x_0$ is a local maximum. An open problem left in this direction is to obtain blow-up theorem in the case where there is no local maximum of $k$ in $\mathbb{R}^N$. For example, consider in $\mathbb{R}$ a function $k(x)$ such that

- $k' < 0$,
- $\lim_{x \to +\infty} k(x) = k_1 > 0$,
- $\lim_{x \to -\infty} k(x) = k_2 > 0$.

Is there a blow-up solution of Eq. (1.1)?

**Proof of Theorem 3.1.** The proof is completely elementary.

(i) Let $\phi \in \Sigma$ such that $E(\phi) < 0$. Consider $y(t) = \int |x - x_0|^2 |u(t, x)|^2 dx$ and assume by contradiction that $u(t)$ and $y(t)$ are defined for all time; we have $\forall t > 0, y''(t) \leq 16 E(\phi)$. Thus by integration

$$\forall t > 0, \quad y(t) \leq y(0) + ty'(0) + 8 t^2 E(\phi) = z(t).$$

Since $E(\phi) < 0, z(t) = 0$ for $t$ large which is contradiction. This concludes the proof of (i).

(ii) (3.5) implies directly that $x_0$ is a global maximum. Let $k_2 = k(x_0)$. For all $\varepsilon > 0$, consider for $\lambda > 0$, $w_{\varepsilon, \lambda} = (1 + \varepsilon) \frac{1}{\lambda^{\frac{N}{2}}} Q_{k_2} \left( \frac{x - x_0}{\lambda} \right)$.

$$\forall \lambda > 0,$$

$$(3.7) \quad \| w_{\varepsilon, \lambda} \|_{L^2} = (1 + \varepsilon) \| Q_{k_2} \|_{L^2}.$$
In addition,

\[
E (w_\epsilon, \lambda) = \frac{1}{2} \int |\nabla w_\epsilon, \lambda|^2 - \frac{1}{4} \int k(x) |w_\epsilon, \lambda|^\frac{4}{N} + 2
\]

\[
= E_{k_2} (w_\epsilon, \lambda) + \frac{1}{4} \int \frac{k(x_0) - k(x)}{w_\epsilon, \lambda} |w_\epsilon, \lambda|^\frac{4}{N} + 2
\]

where \( E_{k_2} (w) = \frac{1}{2} \int |\nabla w|^2 - \frac{1}{4} \int k_2 |w|^\frac{4}{N} + 2 \).

On the one hand, by scaling arguments

\[
E_{k_2} (w_\epsilon, \lambda) = (1 + \epsilon)^2 \frac{1}{\lambda^2} E_{k_2} (Q_{k_2})
\]

\[
+ ((1 + \epsilon)^2 - (1 + \epsilon)^{\frac{4}{N} + 2}) \frac{1}{\lambda^2} \int Q_{k_2}^\frac{4}{N} + 2.
\]

Since \( E_{k_2} (Q_{k_2}) = 0 \) (Pohazaev identity),

\[
\forall \lambda > 0, \quad E_{k_2} (w_\epsilon, \lambda) \leq - \frac{c(\epsilon)}{\lambda^2} \quad \text{where} \quad c(\epsilon) > 0.
\]

Since \( \forall x, \ Q_{k_2} (x) \leq c_0 e^{-c_1 |x|} \) and \( |\nabla k(x)| \leq c_0, \) for \( \lambda > 1, \)

\[
\int |k(x_0) - k(x) | \left| \begin{array}{c}
\frac{w_\epsilon, \lambda}{x} \end{array} \right|^\frac{4}{N} + 2
\]

\[
\leq c + c \int \frac{|x|}{\lambda^2 + N} e^{-c_1 |x|} \frac{1}{\lambda} d\lambda
\]

\[
\leq c + c \int \frac{|1|}{\lambda} e^{-c_1 |1|} \frac{1}{\lambda} d\lambda \leq c \left( 1 + \frac{1}{\lambda} \right).
\]

From (3.9)-(3.10) we derive that for \( \lambda \geq \lambda (\epsilon), \ E (w_\epsilon, \lambda) < 0 \) and for \( \epsilon > 0, \)
\( \phi_\epsilon = w_\epsilon, \lambda (\epsilon) \) satisfies the conclusions of Theorem 3.1. This concludes the proof of Theorem 3.1.

**Proof of Theorem 3.2.** – We remark that we had showed in the proof of Theorem 3.1 (ii) the following lemma.
LEMMA 3.3. - ∀ ∈ (0, 1), for all A(ε) > 0, there is a φ ∈ Σ such that
- \( \| φ_ε \|_{L^2} = \| Q_k(x_0) \|_{L^2} + ε, \)
- \( E(φ_ε) = -A(ε), \)
- \( \int |x|^2 |φ_ε|^2 \leq C, \) (where C is independent of ε and A(ε)),
- \( ∀ x ∈ \mathbb{R}^N, φ_ε(x) ∈ \mathbb{R}, \)
- \( \| \nabla φ_ε \|_{L^2} \underset{ε \to 0}{\to} +∞, \) and \( |φ_ε(x)|^2 \to \| Q_k(x_0) \|_{L^2}^2 \delta_{x=x_0}. \)

Proof. - It follows from the proof of Theorem 3.1 (ii) and direct computations.

We claim now for A(ε) sufficiently large as \( ε \to 0, \) the solution \( u_ε(t) \) associated with \( φ_ε \) blows up in finite time. We now assume that \( A(ε) \underset{ε \to 0}{\to} +∞. \) We argue by contradiction. We suppose that \( u_ε(t) \) is globally defined in time. The two key arguments of the proof are
- On one hand, the use of the geometry of \( k(x) \) near \( x_0 \) to control the evolution of the concentration point;
- On the other hand, the use of local virial identity as in [10], [11]. We proceed in three steps to obtain a contradiction.

Step 1. - Concentration properties of \( u_ε(t). \)

PROPOSITION 3.4 (Concentration in \( L^2 \) of \( u_ε(t) ). - For all ε' > 0, there is \( ε_0 \) such that, \( ∀ \epsilon ∈ (0, ε_0), ∀ t ≥ 0, \)

\[
∫_{|x−x₀|≤ε'} |u_ε(t, x)|^2 dx - ∫_{\mathbb{R}^N} Q_k^2(x_0)(x) dx < ε',
\]

and

\[
∫_{|x−x₀|≥ε'} |u_ε(t, x)|^2 dx ≤ ε'.
\]

Proof of Proposition 3.4. - One uses the fact that \( x_0 \) is a strict local maximum and some contraction lemma.

LEMMA 3.5. - Consider a sequence \( t_ε ∈ \mathbb{R}. \) We then have

\[
\| \nabla u_ε(t_ε) \|_{L^2} \underset{ε \to 0}{\to} +∞.
\]

Proof. - Indeed, by contradiction, assume there is a \( c > 0 \) such that for a sequence \( ε_n \to 0 \)

\[
\| \nabla u_{ε_n}(t_ε_n) \|_{L^2} \leq c.
\]

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Then by Sobolev imbeddings

\[ E(\phi_{\varepsilon_n}) = |E(u_{\varepsilon_n}(t_{\varepsilon_n}))| \leq \frac{1}{2} \| \nabla u_{\varepsilon_n}(t_{\varepsilon_n}) \|_{L^2}^2 + c \| u_{\varepsilon_n}(t_{\varepsilon_n}) \|_{L^{N+2}}^{N+2} \leq c \]

which contradicts the fact

\[ |E(\phi_{\varepsilon_n})| = A(\varepsilon_n) \to +\infty \text{ as } n \to +\infty. \]

Similarly with Proposition 2.5, we have the following lemma.

**Lemma 3.6.** Let \( u_n \) be such that for constants \( a, b, \)

\[(3.13) \quad \| u_n \|_{L^2} \leq a,\]

\[(3.14) \quad E(u_n) \leq b,\]

\[(3.15) \quad \| \nabla u_n \|_{L^2} \to +\infty \text{ as } n \to +\infty.\]

There is \( x_n \) such that for all \( R > 0 \)

\[ \liminf_{n \to +\infty} \left\{ \frac{\| u_n \|_{L^2(B(x_n, R))}}{\| Q_{k(x_n)} \|_{L^2}} \right\} \geq 1. \]

**Proof.** See Corollary 2.7.

Applying Lemma 3.6 with \( u_\varepsilon(t) \) \((a = 2 \| Q_{k(x_0)} \|_{L^2}, b = 0)\), we obtain the conclusion.

Indeed, consider \( \delta > 0 \) such that

\[(3.16) \quad \forall x, \quad \| Q_{k(x)} \|_{L^2}^2 \geq 2 \delta.\]

\[
\left( (3.16) \text{ is equivalent to, } \forall x, \quad \frac{\| Q \|_{L^2}^2}{k(x)^{\frac{N}{2}}} \geq 2 \delta \text{ or equivalently } \delta \leq \frac{\| Q \|_{L^2}^2}{2 k_2^{\frac{N}{2}}}. \right)
\]
Consider, for each $\epsilon > 0$, $\tilde{T}_\epsilon$ such that

\begin{equation}
\forall t \in [0, \tilde{T}_\epsilon), \quad \| u_\epsilon (t, x) \|_{L^2 (B (x_0, \frac{\epsilon \rho_1}{4}))}^2 \geq \| Q_k (x_0) \|_{L^2}^2 - \delta,
\end{equation}

(3.17)

and from (3.18)

\begin{equation}
\| u_\epsilon (\tilde{T}_\epsilon, x) \|_{L^2 (B (x_0, \frac{\epsilon \rho_1}{4}))}^2 = \| Q_k (x_0) \|_{L^2}^2 - \delta.
\end{equation}

(3.18)

From Lemma 3.3, for $\epsilon$ small enough, $\tilde{T}_\epsilon > 0$. Let us show that for $\epsilon < \epsilon_0$ (where $\epsilon_0 > 0$)

\begin{equation}
\tilde{T}_\epsilon = +\infty.
\end{equation}

(3.19)

Indeed, by contradiction, assume that for $\epsilon_n \to 0$

\begin{equation}
\tilde{T}_{\epsilon_n} = \tilde{T}_n < +\infty.
\end{equation}

(3.20)

Consider $u_n = u_{\epsilon_n} (\tilde{T}_n, x)$. $u_n$ satisfies (3.13)-(3.15), therefore from Lemma 3.6, there is $x_n$

\begin{equation}
\forall R, \quad \lim_{n \to +\infty} \inf \| u_n \|_{L^2 (B (x_n, R))}^2 \geq \lim_{n \to +\infty} \sup \| Q_k (x_n) \|_{L^2}^2.
\end{equation}

(3.21)

We chain for $n$ large

\begin{equation}
\| x_n - x_0 \| \leq \frac{\rho_0}{2}.
\end{equation}

(3.22)

Indeed if not

\begin{align*}
\liminf_{n \to +\infty} \| u_n \|_{L^2 (B (x_0, \frac{\epsilon \rho_1}{4}))}^2 &\geq \lim_{n \to +\infty} \| u_n \|_{L^2 (B (x_n, \frac{\epsilon \rho_1}{4}))}^2 \\
&\geq \limsup_{n \to +\infty} \| Q_k (x_0) \|_{L^2}^2 \geq 2 \delta
\end{align*}

and from (3.18)

\begin{equation}
\liminf_{n \to +\infty} \| u_n \|_{L^2}^2 \geq \| Q_k (x_n) \|_{L^2}^2 + \delta.
\end{equation}

(3.23)

Since $\| u_n \|_{L^2} = \| u_{\epsilon_n} (\tilde{T}_{\epsilon_n}, x) \|_{L^2} = \| \phi_{\epsilon_n} \|_{L^2} \to \| Q_k (x_0) \|_{L^2}$ as $n \to +\infty$, we obtain a contradiction. We then remark that

\begin{equation}
x_n \to x_0 \text{ as } n \to +\infty.
\end{equation}

(3.24)
Indeed, we have from (3.21)

\[(3.25) \quad \liminf_{n \to +\infty} \| \phi_{\epsilon_n} \|_{L^2}^2 = \liminf_{n \to +\infty} \| u_n \|_{L^2}^2 \geq \limsup_{n \to +\infty} \| Q_k(x_n) \|_{L^2}^2 \geq \limsup_{n \to +\infty} \frac{\| Q \|_{L^2}^2}{[k(x_n)]^{\frac{N}{2}}} \geq \| Q_k(x_0) \|_{L^2}^2 \limsup_{n \to +\infty} \left[ \frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}}.\]

From Lemma 3.3,

\[\| Q_k(x_0) \|_{L^2}^2 \geq \| Q_k(x_0) \|_{L^2}^2 \limsup_{n \to +\infty} \left[ \frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}} \]

or

\[\liminf_{n \to +\infty} k(x_n) \geq k(x_0),\]

which is equivalent from (3.6) and (3.22) to

\[x_n \to x_0 \quad \text{as} \quad n \to +\infty.\]

From (3.21)

\[(3.26) \quad \liminf_{n \to +\infty} \| u_{\epsilon_n}(\tilde{T}_{\epsilon_n}) \|_{L^2(x_0, \frac{\rho}{8})}^2 \geq \liminf_{n \to +\infty} \| u_{\epsilon_n}(\tilde{T}_{\epsilon_n}) \|_{L^2(x_n, \frac{\rho}{8})}^2 \geq \liminf_{n \to +\infty} \| u_n \|_{L^2(x_n, \frac{\rho}{8})}^2 \geq \| Q_k(x_0) \|_{L^2}^2\]

which is a contradiction with (3.18). Therefore there is \(\epsilon_0 > 0\) such that for \(0 < \epsilon < \epsilon_0\), \(\tilde{T}_\epsilon = +\infty\).

Let us conclude the proof of Proposition 3.4 by contradiction. We claim that (3.11), (3.12) follow from Lemma 3.3 and the conservation of mass. Assume there is \(t_{\epsilon_n}\) and \(\epsilon_n \to 0\), \(\epsilon' > 0\) such that

\[(3.27) \quad \left| \| Q_k(x_0) \|_{L^2}^2 - \int |x-x_0|\leq\epsilon' |u_n|^2 \right| \geq \epsilon',\]

where \(u_n = u_{\epsilon_n}(t_{\epsilon_n}, x)\).
As before, there is $x_n$ such that

\begin{equation}
(3.28) \quad \forall R, \quad \liminf_{n \to +\infty} \| u_n \|_{L^2(B(x_0, R))}^2 \geq \| Q_k(x_0) \|_{L^2}^2.
\end{equation}

We have from (3.17) and (3.19) by the same arguments than before that for $n$ large $|x_n - x_0| \leq \frac{\rho_0}{2}$ and then $x_n \to x_0$ as $n \to +\infty$.

In particular, from (3.28) as classical arguments,

\begin{equation}
(3.29) \quad \forall R, \quad \liminf_{n \to +\infty} \| u_n \|_{L^2(B(x_0, R))}^2 \geq \| Q_k(x_0) \|_{L^2}^2.
\end{equation}

Since

$$\| \phi_{\varepsilon_n} \|_{L^2} = \| u_{\varepsilon_n} \|_{L^2} = \| u_n \|_{L^2} \xrightarrow{n \to +\infty} \| Q_k(x_0) \|_{L^2},$$

we have

$$\lim_{n \to +\infty} (\| u_n \|_{L^2(B(x_0, R))}^2 - \| Q_k(x_0) \|_{L^2}^2) = 0,$$

which is a contradiction with (3.27). Thus Proposition 3.4 is proved.

**Remark.** – In the case where $x_0$ is a global maximum, we do not need to prove (3.19).

**Step 2.** – Energy estimates outside the concentration point.

Using local virial identity, we are able to prove the following proposition.

**Proposition 3.7.** – There are constants $0 < B_0 < \frac{\rho_0}{4}$, $c_1 > 0$ and $c_2 > 0$ independent of $\varepsilon$ such that $\forall t$,

$$- \left[ 8 E(\phi_{\varepsilon}) t^2 + \int_0^t (t-s) \frac{4}{(2 + \frac{N}{1})} \int_{|x-x_0| \leq 2B_0} (x-x_0) \times \nabla k(x) |u_{\varepsilon}(s, x)|^{\frac{4}{N}+2} dx ds \right]$$

$$\geq c_2 \int_0^t (t-s) \int_{|x-x_0| \geq \frac{\rho_0}{2}} |\nabla u_{\varepsilon}(s, x)|^2 dx ds - c_1.$$
LEMMA 3.8 (Local virial identity). – Consider $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$ with compact support.

\[
\frac{d}{dt} \int \psi(x) |u(t, x)|^2 = 2 \text{Im} \int \nabla \psi \nabla u \overline{u},
\]

(ii) $\frac{d^2}{dt^2} \int \psi(x) |u(t, x)|^2 = 2 \left\{ -\frac{2}{N(N + 1)} \int \Delta \psi k |u|^{\frac{4}{N} + 2}
+ 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i \overline{u} \partial_j \overline{u} - \frac{1}{2} \int |u|^2 \Delta^2 \psi
+ \frac{1}{N + 1} \int \nabla \psi \cdot \nabla k |u|^{\frac{4}{N} + 2} \ dx \right\}.

Proof. – It follows from similar calculation as in [10].

LEMMA 3.9. – Let $\rho(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $\rho \in L^\infty$ and $\nabla \rho \in L^\infty$. There is a $c_\rho > 0$ such that

\[
\forall u \in H^1, \quad \int |u(x)|^{\frac{4}{N} + 2} \rho^2(x) \ dx \leq c_\rho \left( \int u^2 \right)^{\frac{4}{N}} \times \left\{ \int |\nabla u|^2 \rho^2 + \int \nabla \rho^2 u^2 \right\}.
\]


We claim now that applying Lemma 3.8 to a suitable function $\psi(x)$, we obtain Proposition 3.7. Indeed, consider $\psi$ such that

(3.27) $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$ and $\psi(x) = \psi(|x|),$

(3.28) $\psi(x) < |x|^2$ for $|x| > \beta_0,$

$\psi(x) \equiv |x|^2$ for $|x| \leq \beta_0,$

$\frac{1}{2} |x|^2 \leq x \cdot \nabla \psi \leq 3 |x|^2$ for $|x| \leq 2 \beta_0,$

$\Delta \psi - 2 N \geq 0$ for $|x| \leq 2 \beta_0,$

(3.29) $\psi(x) \equiv c$ for $|x| \geq \frac{\rho_0}{2}.$
there are a constant $c_0$ and a function $g$ such that for

$$\beta_0 \leq |x| \leq \frac{\rho_0}{2}, \quad \forall a \in \mathbb{C}^N$$

$$
\left( \sum_i |a_i|^2 - \sum_{i,j} \frac{\partial_i \partial_j \psi}{2} a_i \bar{a}_j \right) \geq g(x) \left( \sum_i |a_i|^2 \right),
$$

where $g(x) \geq c_0$ for $|x| \geq 2 \beta_0$ and $g(x) \geq 0, \forall x$.

The existence of such a $\psi$ can be proved easily, and the proof is omitted.

We have then by Lemma 3.8 and Lemma 3.3, $\forall \varepsilon, \forall t > 0$,

$$\int \psi (x - x_0) |u_\varepsilon (t, x)|^2 \, dx$$

$$= \int \psi (x - x_0) |\phi_\varepsilon|^2$$

$$+ 2 t \text{Im} \int \nabla \psi \cdot \nabla \phi_\varepsilon \phi_\varepsilon$$

$$+ 2 \int_0^t (t - s) \left\{ - \frac{2}{N \left( \frac{2}{N} + 1 \right)} \int \Delta \psi \frac{1}{s} \left| u_\varepsilon (s) \right|^{\frac{4}{N} + 2}
$$

$$+ 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i u_\varepsilon (s) \overline{\partial_j u_\varepsilon (s)} - \frac{1}{2} \int \left| u_\varepsilon (s) \right|^2 \Delta^2 \psi$$

$$+ \frac{1}{\frac{2}{N} + 2} \int \nabla k \nabla \psi (x - x_0) \left| \frac{1}{s} \right|^{\frac{4}{N} + 2} \right\} ds.$$ 

From (3.31), the conservation of mass, Lemma 3.3 and (3.27)-(3.30), we obtain, $\forall \varepsilon, \forall t$

$$\left| \int_0^t (t - s) \left\{ - \frac{2}{N \left( \frac{2}{N} + 1 \right)} \int \Delta \psi \frac{1}{s} \left| u_\varepsilon (s) \right|^{\frac{4}{N} + 2}
$$

$$+ 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i u_\varepsilon (s) \overline{\partial_j u_\varepsilon (s)}$$

$$+ \frac{1}{\frac{2}{N} + 2} \int \nabla k \nabla \psi (x - x_0) \left| u_\varepsilon (s) \right|^{\frac{4}{N} + 2} \right\} ds \right|$$

$$\leq c_1 + c_1 t^2.$$
Thus
\begin{align}
(3.33) \quad & \int_t^s \left\{ 8 E(\phi_\epsilon) + \int_{|x-x_0| \geq \beta_0} -2 \frac{N}{N+1} \times (\Delta \psi - 2N) k |u_\epsilon|^{\frac{4}{N}+2} \\
& + 2 \left( \sum_{i,j} \partial_i \partial_j \psi \partial_i \mu_\epsilon \partial_j \bar{u}_\epsilon \right) - 4 \left( \sum_i |\partial_i u_\epsilon|^2 \right) \right\} \, dx \\
& + \frac{1}{2} \frac{N+2}{N+2} \int \nabla k \nabla \psi (x-x_0) \, |u_\epsilon|^{\frac{4}{N}+2} \, ds \leq c_1 + c_1 t^2,
\end{align}

or equivalently
\begin{align}
(3.34) \quad & \int_t^s 2 \int_{|x-x_0| \geq \beta_0} \left( \sum_i |\partial_i u|^2 \right) - \frac{1}{2} \left( \sum_{i,j} \partial_i \partial_j \psi \partial_i u \partial_j u \right) \\
& \leq c \left( |E(\phi_\epsilon)| t^2 + 1 + \int_0^T \int_{|x-x_0| \leq \frac{\beta_0}{N}} |u|^{\frac{4}{N}+2} \, dx \, ds \right) \\
& + \left| \int_0^T (T-t) \int_{|x-x_0| \leq \frac{\beta_0}{2}} \nabla k \cdot (x-x_0) \, |u|^{\frac{4}{N}+2} \, dx \, dt \right|.
\end{align}

In addition, from (3.6) and a compactness argument in \( \mathbb{R}^N \), we have
\begin{equation}
(3.35) \quad |(x-x_0) \cdot \nabla k| \geq c_0 > 0 \quad \text{for} \quad \beta_0 \leq |x-x_0| \leq \frac{\rho_0}{2}.
\end{equation}

Thus (3.34)-(3.35) yield Proposition 3.7.

**Step 3. – Conclusion of the proof.**

From Proposition 2.1, we have \( \forall \epsilon, \forall t > 0 \),
\begin{align}
\frac{d}{dt} \int |x|^2 |u_\epsilon(t, x)|^2 \, dx & = 4 \text{Im} \int \bar{u}_\epsilon \nabla u_\epsilon \cdot x, \\
\frac{d^2}{dt^2} \int |x|^2 |u_\epsilon(t, x)|^2 \, dx & = 4 \left\{ 4 E(\phi_\epsilon) + \frac{1}{N+1} \int (x-x_0) \nabla k |u_\epsilon|^{\frac{4}{N}+2} \right\}.
\end{align}
We integrate twice these identities and using Lemma 3.3 we obtain for \( t \),

\[
(3.36) \quad y_\varepsilon(t) = \int |x|^2 |u_\varepsilon(t, x)|^2 \, dx = 8E(\phi_\varepsilon)t^2 \\
+ \int |x|^2 |\phi_\varepsilon|^2 + \int_0^t (t-s) \frac{4}{2 + 1} \\
\times \int (x-x_0) \nabla k |u_\varepsilon(s)|^{\frac{4}{N}+2} \, dx \, ds
\]

\[
(3.37) \quad = 8E(\phi_\varepsilon)t^2 + \int |x|^2 |\phi_\varepsilon|^2 \\
+ \int_0^t (t-s) \frac{4}{2 + 1} \\
\times \int_{|x-x_0| \leq \rho_0} (x-x_0) \nabla k |u_\varepsilon(s)|^{\frac{4}{N}+2} \, dx \, ds \\
+ \int_0^t (t-s) \frac{4}{2 + 1} \\
\times \int_{|x-x_0| \geq \rho_0} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} \, dx \, ds.
\]

Let us estimate the last term.

**Lemma 3.10.** There is a constant \( c(\varepsilon) \) depending only on \( \varepsilon \) such that

(i) \( c(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \)

(ii) \( \forall t, \quad \left| \int_0^t (t-s) \frac{4}{2 + 1} \\
\times \int_{|x-x_0| \geq \rho_0} |x-x_0| \nabla k |u_\varepsilon|^{\frac{4}{N}+2} \, dx \, ds \right| \leq c(\varepsilon) \left| c_1 + 8E(\phi_\varepsilon)t^2 + \int_0^t (t-s) \frac{4}{2 + 1} \\
\times \int_{|x-x_0| \leq \rho_0} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} \, dx \, ds \right|. \)
**Proof.** We have

\[
\beta_\varepsilon (t) = \left\| \int_0^t (t - s) \frac{4}{N} + 1 \int_{|x-x_0| \geq \rho_0} \times (x-x_0) \nabla k |u_\varepsilon|^{\frac{N}{2} + 2} dx ds \right\|
\leq c \int_0^t (t - s) \int_{|x-x_0| \geq \rho_0} |u_\varepsilon|^{\frac{N}{2} + 2} dx ds
\leq c \int_0^t (t - s) \int \rho^2 (x) |u_\varepsilon|^{\frac{N}{2} + 2} dx ds,
\]

where \( \rho \) is a \( C^\infty \) function such that

- \( 0 \leq \rho \leq 1 \)
- \( \rho = 1 \) for \( |x-x_0| \geq \rho_0 \)
- \( \rho = 0 \) for \( |x-x_0| \leq \frac{\rho_0}{2} \).

Therefore from Lemma 3.9,

\[
\beta_\varepsilon (t) \leq c \int_0^t (t - s) \left\{ \left( \int_{|x-x_0| \geq \rho_0} |u_\varepsilon (s, x)|^2 dx \right)^{\frac{N}{2}} \right. \\
\left. \quad \left( \int |\nabla u_\varepsilon (s, x)|^2 \rho^2 (x) dx + \int \nabla \rho^2 (x) |u_\varepsilon (s, x)|^2 dx ds \right) \right\}
\leq c \int_0^t (t - s) \left\{ \left( \int_{|x-x_0| \geq \frac{\rho_0}{2}} |u_\varepsilon (s, x)|^2 dx \right)^{\frac{N}{2}} \right. \\
\left. \quad \left( \int |u_\varepsilon (s, x)|^2 + |\nabla u_\varepsilon (s, x)|^2 dx \right) \right\} ds.
\]

From Step 1, we have

\[
\beta_\varepsilon (t) \leq c (\varepsilon) \left\{ c_1 + t^2 + \int_0^t (t - s) \\
\times \int_{|x-x_0| \geq \frac{\rho_0}{2}} |\nabla u_\varepsilon (s, x)|^2 dx ds \right\}
\]
where
\[
c(\varepsilon) = \left( \sup_{t \in \mathbb{R}} \int_{|x-x_0| \geq \frac{\rho_0}{\varepsilon^2}} |u_{\varepsilon}(t, x)|^2 \right)^{\frac{2}{N}} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

From Proposition 3.7 and (3.40), we conclude the proof of Lemma 3.10.

Let \( \varepsilon_0 \) be such that
\[
c(\varepsilon) \leq \frac{1}{2}, \quad \text{for} \quad \varepsilon \leq \varepsilon_0.
\]

For \( \varepsilon \leq \varepsilon_0 \), \( \forall t \),
\[
y_{\varepsilon}(t) \leq \frac{1}{8} 8 E(\phi_{\varepsilon}) t^2 + c_1 - \frac{1}{2} E(\phi_{\varepsilon}) t^2
\]
\[
+ \frac{3}{2} \int_0^t (t-s) \frac{4}{2} \frac{4}{N+1} \left(x-x_0\right) \nabla k |u_{\varepsilon}|^{\frac{4}{N}+2} dx ds
\]
\[
+ \int_0^t (t-s) \frac{4}{2} \frac{4}{N+1} \left(x-x_0\right) \nabla k |u_{\varepsilon}|^{\frac{4}{N}+2} dx ds.
\]

Since \( (x-x_0) \nabla k \leq 0 \) on \( |x-x_0| \leq \rho_0 \), we have
\[
\forall t, \quad y_{\varepsilon}(t) \leq c_1 + \frac{1}{2} E(\phi_{\varepsilon}) t^2.
\]

Therefore, from the fact that \( y_{\varepsilon}(1) \geq 0 \), we obtain that for a \( c > 0 \),
\[
\forall \varepsilon \leq \varepsilon_0, \quad E(\phi_{\varepsilon}) \geq -c.
\]

This is a contradiction with Lemma 3.3 and the solution \( u_{\varepsilon}(t) \) for \( \varepsilon \leq \varepsilon_0 \) blows up in finite time. This concludes the proof of the Theorem 3.2 and Section 3.

4. PROPERTIES OF \( L^2 \)-MINIMAL BLOW-UP SOLUTIONS \((\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2})\)

In this section, we assume that \( k \) is \( C^1 \) and
\[
0 < k_1 \equiv \inf_{x \in \mathbb{R}^N} k(x) \leq k(x) \leq \sup_{x \in \mathbb{R}^N} k(x) \equiv k_2 < +\infty.
\]
Moreover, we assume compactness and nondegeneracy conditions on \( k(x) \), that is

\[(4.1)' \text{ There are } R_0 > 0, \ c_0 > 0 \text{ and } \delta_0 > 0 \text{ such that for } |x| \geq R_0, \]

\[ k(x) \leq k_2 - \delta_0, \quad |\nabla k(x)| \leq c_0, \]

and

\[(4.1)'' \text{ there are } x_1, \ldots, x_p \text{ such that } M = \{x; \ k(x) = k_2\} = \{x_1, \ldots, x_p\}. \]

In this section we are interested by qualitative properties satisfied by blow-up solutions such that

\[(4.2) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2} = \left\| \frac{Q}{k_2^{N/4}} \right\|_{L^2}. \]

We had seen in Section 2 that if

\[(4.3) \quad \|\phi\|_{L^2} < \|Q_{k_2}\|_{L^2} \]

then \( u(t) \) is globally defined.

Moreover under some compactness assumptions on \( k(x) \) in Section 3, we had seen that for all \( \varepsilon > 0 \), there is a blow-up solution with initial data \( \phi_{\varepsilon} \) such that

\[(4.4) \quad \|\phi_{\varepsilon}\|_{L^2} = \|Q_{k_2}\|_{L^2} + \varepsilon. \]

Therefore, if \( u(t) \), solution of Eq. (1.1) with initial data \( \phi \) satisfying (4.2), blows-up in finite time \( T < +\infty \), then \( u(t) \) is a minimal blow-up solution in \( L^2 \). Let \( u(t) \) be such a solution.

In the case \( k(x) \equiv k \), in [10], the following result has been proved: there is \( x_0 \in \mathbb{R}^N \) such that

\[
\begin{align*}
|u(t, x)|^2 & \rightarrow \|Q_{k_0}\|_{L^2} \delta_{x=x_0} \quad \text{as } t \rightarrow T, \\
|x - x_0|^2 |u|^2 & \rightarrow 0 \quad \text{in } L^1 \quad \text{as } t \rightarrow T.
\end{align*}
\]

Using variational arguments we prove the following in the case where \( k(x) \neq k \).
PROPOSITION 4.1. - Assume that \( \| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} \) and \( u(t) \) blows-up in finite time at \( T < +\infty \). We then have the existence of \( x_0 \) such that

\[
- |u(t, x)|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0} \text{ in the distribution sense as } t \to T,
\]

and

\[
- |x - x_0|^2 |u(t, x)|^2 \to 0 \text{ in } L^1 \text{ as } t \to T,
\]

and

\[
\nabla k(x_0) = 0, \quad k(x_0) = k_2.
\]

**Remark.** - It follows from Proposition 4.1 that we do not have ejection of mass in finite time with a minimal mass \( \| Q_{k_2} \|_{L^2} \). That is

\[
|u(t, x - x(t))|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=0} \quad \text{and} \quad |x(t)| \to +\infty \quad \text{as } t \to T.
\]

In the case where \( k(x) \) does not satisfy (4.1)' and there is a sequence \( x_n \) such that

- \( |x_n| \to +\infty \) as \( n \to +\infty \),
- \( k(x_n) \to k_2 \) as \( n \to +\infty \),

we still have the existence of \( x(t) \) such that

\[
|u(t, x + x(t))|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=0}.
\]

But we do not know whether \( x(t) \) is bounded or not.

**Remark.** - For a general initial data \( \| \phi \|_{L^2} > \| Q_{k_2} \|_{L^2} \), we don’t know whether the concentration point of the solution in \( L^2 \) at the blow-up time is a critical point of \( k(x) \) or not.

**Proof of Proposition 4.1.** - We establish the result in three steps. Let us consider \( u(t) \) solution of Eq. (1.1) with intial data \( \phi \in H^1 \) such that

\[
\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2}, \text{ and } u(t) \text{ blows up at } T < +\infty.
\]

**Step 1.** - **Variational estimates.**

We show that there is \( x(t) \) such that

\[
|u(t, x - x(t))|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=0}
\]

and

\[
\forall \delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T),
\]

\[
\int_{|x(t) - x| \geq \delta} |\nabla u(t, x)|^2 \, dx \leq c_\delta.
\]

**Step 2.** - **Localization of the concentration point.**
There is \( x_0 \in \mathbb{R}^N \) such that \( x(t) \to x_0 \) as \( t \to T \). Moreover \( k(x_0) = k_2 \) and \( \nabla k(x_0) = 0 \).

**Step 3.** Control of \( u(t, x) \) for \( x \) large and conclusion.

We then show that

\[
|x||u(t, x)| \in L^2 \quad \text{for all } t \in [0, T)
\]

and

\[
|x - x_0|^2 |u(t, x)|^2 \to 0 \quad \text{in } L^1 \quad \text{as } t \to T.
\]

**Step 1.** Variational estimates: Concentration and compactness outside the concentration point.

We show that there is \( x(t) \) such that

\[
|u(t, x + x(t))|^2 \to \|Q_{k_2}\|_{L^2} \delta_{x=0}
\]

and

\[
\forall \delta > 0, \text{ there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T), \quad \int_{|x(t) - x| \geq \delta} |\nabla u(t, x)|^2 \, dx \leq c_\delta.
\]

We claim this result as a consequence of the concentration properties (Section 2.B) and a crucial compactness lemma.

**Lemma 4.2** ([10], p. 433). - Let \( u_n \in H^1(\mathbb{R}^N) \) and \( R_0 > 0 \) such that for a \( c_0 \), we have \( E_{k_2}(u_n) \leq c_0 \),

- \( \int |u_n(x)|^2 \, dx \leq \int |Q_{k_2}(x)|^2 \, dx, \)

- \( \int |\nabla u_n(x)|^2 \, dx \to +\infty \text{ as } n \to +\infty, \)

- \( \int_{|x| \geq R_0} |u_n(x)|^2 \, dx \leq \varepsilon(N), \)

where \( \varepsilon(N) > 0 \) is depending only on \( N \). Then there is \( A > 0 \) depending only on \( R_0, c_0 \) such that

\[
\forall n, \quad \int_{|x| \geq 4R_0} |\nabla u_n(x)|^2 \, dx \leq A.
\]
Proof of (4.5)-(4.6). Let be \( x(t) \) defined in Section 2.B (Proposition 2.4). For all \( R > 0 \), we have

\[
\liminf_{t \to T} \| u(t) \|_{L^2(B(x(t), R))} \geq \| Q_{k_2} \|_{L^2}.
\]

Let

\[
v(t, x) = |u(t, x + x(t))|^2.
\]

(4.8) \( \| v(t, x) \|_{L^1} = \| u(t, x) \|_{L^2}^2 = \| \phi \|_{L^2}^2 = \| Q_{k_2} \|_{L^2}^2, \)

and from (4.7)

(4.9) \( \forall R > 0, \)

\[
\liminf_{t \to T} \int_{|x| < R} v(t, x) = \liminf_{t \to T} \int_{|x| < R} u(t, x + x(t))^2 \, dx \geq \| Q_{k_2} \|_{L^2}^2.
\]

Therefore from (4.8)-(4.9)

(4.10) \( v(t, x) \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=0} \text{ as } t \to T, \)

or equivalently

(4.11) \( |u(t, x + x(t))|^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=0} \text{ as } t \to T. \)

And, \( \forall R > 0 \)

(4.12) \( \int_{|x| > R} |u(t, x + x(t))|^2 \, dx \to 0 \text{ as } t \to T. \)

We now claim the following lemma.

**Lemma 4.3.** - (i) \( \forall t \in [0, T), \)

\[
- \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N} + 2} \, dx \leq \left( \frac{4}{N} + 2 \right) E(\phi)
\]

\[
- E_{k_2}(u(t)) \leq E(\phi).
\]
(ii) \( \forall \delta > 0, \) there is a \( c_\delta > 0 \) such that \( \forall t \in [0, T) \)
\[
\int_{|x-x(t)| \geq \delta} |\nabla u(t, x)|^2 \leq c_\delta.
\]

**Proof.** (i) Indeed \( \forall t \in [0, T), \)
\[
E(u(t)) = E(\phi).
\]
Therefore
\[
\begin{align*}
(4.13) \quad & \left\{ \frac{1}{2} \int |\nabla u(t, x)|^2 - \frac{1}{4} \int k_2 |u(t, x)|^{\frac{4}{N} + 2} dx \right\} \\
& + \frac{1}{4} \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N} + 2} dx = E(\phi).
\end{align*}
\]
Since \( \|u(t)\|_{L^2} = \|Q_{k_2}\|_{L^2} \), we have
\[
(4.14) \quad E_{k_2}(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 - \frac{1}{4} \int \frac{k_2}{N} + 2
\times \int k_2 |u(t, x)|^{\frac{4}{N} + 2} dx \geq 0
\]
and from (4.1),
\[
(4.15) \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N} + 2} dx \geq 0.
\]
From (4.13)-(4.15), we derive part (i) of the lemma.

(ii) Let \( \delta > 0. \) From (4.12), there is a \( t_\delta < T \) such that
\[
\forall t \in [t_\delta, T), \quad \int_{|x| > \frac{\delta}{4}} |u(t, x + x(t))|^2 dx < \varepsilon(N),
\]
where \( \varepsilon(N) \) is defined in Lemma 4.3.

From Lemma 4.2, we have the existence of \( A_\delta > 0 \) such that
\[
\forall t \in [t_\delta, T), \quad \int_{|x| > \delta} |\nabla u(t, x + x(t))|^2 dx \leq A_\delta.
\]
Since $\forall t \in [0, t_\delta]$,  
\[
\int_{|x| > \delta} |\nabla u(t, x + x(t))|^2 \, dx \leq \int |\nabla u(t, x)|^2 \, dx \leq c,
\]
we have the conclusion. This concludes the proof of Lemma 4.3 and of (4.5)-(4.6).

**Step 2. – Localisation of the concentration point.**

In this step we use strongly the assumptions $(4.1)'-(4.1)''$. Since  
\[
\int \bar{e} |u(t, x)|^2 \, dx \text{ as } t \to T \text{ can not be controlled as in the case } k(x) \equiv k_0,
\]
we cannot apply arguments such as in [10].

Let us show that there is $x_0$ such that

\[
(4.16) \quad x(t) \to x_0 \text{ as } t \to T,
\]

\[
(4.17) \quad k(x_0) = k_2 \quad \text{and} \quad \nabla k(x_0) = 0.
\]

**Proof of (4.16)-(4.17).**

**Lemma 4.4. – There is a constant $c_0 > 0$ such that**

\[
\forall t \in [0, T), \quad |x(t)| \leq c_0.
\]

**Proof.** – Indeed, from Lemma 4.3 and $(4.1)'$:

\[
(4.17) \quad \forall t, \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{N}{N-2}} \, dx \leq c,
\]

and

\[
\forall |x| \geq R_0, \quad k_2 - k(x) \geq \delta.
\]

Therefore

\[
\forall t, \quad \int_{|x| \geq R_0} \delta |u(t, x)|^{\frac{N}{N-2}} \, dx \leq c,
\]

and

\[
(4.18) \quad \forall t, \quad \int_{|x| \geq R_0} |u(t, x)|^{\frac{N}{N-2}} \, dx \leq \frac{c}{\delta}.
\]
Moreover, from (4.11) and Hölder inequality we have

\begin{equation}
\int_{|x-x(t)| \leq 1} |u(t, x)|^{\frac{N}{N+2}} \, dx \to +\infty \quad \text{as } t \to T.
\end{equation}

It follows from (4.18)-(4.19) that

\[ \limsup_{t \to T} |x(t)| \leq R_0 + 1, \]

and the conclusion follows.

**Lemma 4.5.** There is a \( x_0 \) such that

\[ x(t) \to x_0 \quad \text{as } t \to T \quad \text{and} \quad k(x_0) = k_2. \]

**Remark.** It follows directly from \( k(x_0) = k_2 = \max_{x \in \mathbb{R}^N} k(x) \) that

\[ \nabla k(x_0) = 0. \]

**Proof.** (i) We first remark that

\[ M(t) = \min_{i=1, \ldots, p} \{ |x(t) - x_i| \} \to 0 \quad \text{as } t \to T, \]

where \( x_1, \ldots, x_p \) are defined by (4.1)\( '' \). Indeed, by contradiction, assume that there are \( t_n \to T \) as \( n \to +\infty \) and \( \delta > 0 \) such that

\[ M(t_n) \geq \delta. \]

Compactness arguments in \( \mathbb{R}^N \) yield the existence of \( \alpha > 0 \) such that

\begin{equation}
\forall n, \quad \forall x \in B \left( x(t_n), \frac{\delta}{2} \right), \quad (k_2 - k(x)) \geq \alpha.
\end{equation}

Therefore from Lemma 4.3,

\[ \forall n, \quad \int_{|x-x(t_n)| \leq \frac{\delta}{2}} (k_2 - k(x)) |u(t_n, x)|^{\frac{N}{N+2}} \, dx \leq c_0 \]

and

\begin{equation}
\forall n, \quad \int_{|x-x(t_n)| \leq \frac{\delta}{2}} |u(t_n, x)|^{\frac{N}{N+2}} \, dx \leq c.
\end{equation}

(4.21) contradicts the fact that

\[ \int_{|x-x(t_n)| \leq \frac{\delta}{2}} |u(t_n, x)|^{\frac{N}{N+2}} \, dx \quad \text{as } n \to +\infty \]

\[ \to +\infty. \]
(from (4.11)). Therefore

\[ M(t) \to 0 \quad \text{as} \quad t \to T. \]

(ii) Let us show now that there is \( i \in \{1, \ldots, p\} \) such that

\[ x(t) \to x_i \quad \text{as} \quad t \to T. \]  

Let \( \delta = \frac{1}{4} \min_{i \neq j} \{|x_i - x_j|\} > 0 \) and \( \psi \in C^\infty \) such that

\[ -\psi(x) \equiv 1 \quad \text{for} \quad |x| < \delta, \]
\[ -0 \leq \psi(x) \leq 1, \]
\[ -\psi(x) \equiv 0 \quad \text{for} \quad |x| > 2\delta. \]

From Part (i) and Lemma 4.3 we have the existence of \( c > 0 \) such that

\[ \forall t \in [0, T), \quad \forall i = 1, \ldots, N, \]

\[ \int_{\delta < |x-x_i| < 2\delta} |\nabla u(t, x)|^2 \, dx \leq c. \]

We remark that \( \forall i = 1, \ldots, p \), there is \( e_i \) such that

\[ \int \psi(x-x_i) |u(t, x)|^2 \, dx \to e_i \quad \text{as} \quad t \to T. \]

Indeed from direct calculations and (4.23),

\[ \left| \frac{d}{dt} \int \psi(x-x_i) |u(t, x)|^2 \, dx \right| \]
\[ = \left| 4 \text{Im} \int \nabla \psi(x-x_i) u \nabla u \right| \]
\[ = \left| 4 \text{Im} \int_{\delta < |x-x_i| < 2\delta} \nabla \psi(x-x_i) u \nabla u \right| \]
\[ \leq c \left( \int_{\delta < |x-x_i| < 2\delta} |\nabla u|^2 \right)^{\frac{1}{2}} \leq c \]

and (4.24) follows.

Therefore, from (4.11)-(4.12) and (i), there is \( i_0 \in \{1, \ldots, p\} \) such that \( e_{i_0} = \|Q_{k_2}\|_{L^2} \) and

\[ x(t) \to x_{i_0} \quad \text{as} \quad t \to T. \]

This concludes the proof of Lemma 4.5 and (4.16)-(4.17).
Step 3. – Control of the solution at infinity and conclusion.
Let us show that $\phi \in \Sigma$, that is

(4.25) $|x| |\phi(x)| \in L^2,$

and

(4.26) $\int |x - x_0|^2 |u(t, x)|^2 \, dx \to 0 \quad \text{as} \quad t \to T.$

The proof will use the same type of argument than in [10].

We remark that from Lemmas 4.3 and 4.5, we have

(4.27) $\forall \delta > 0,$ there is a $c_\delta > 0$ such that $\forall t \in [0, T),$

$$\int_{|x - x_0| \geq \delta} |\nabla u(t, x)|^2 \, dx \leq c_\delta.$$

**Lemma 4.6.**

(i) $\int |x|^2 |\phi(x)|^2 < +\infty.$

There is a constant $c > 0$ such that

(ii) $\forall t \in [0, T), \int |x - x_0|^2 |u(t, x)|^2 \, dx \leq c.$

**Proof.** – Let us argue by contradiction. Suppose $\int |x|^2 |\phi(x)|^2 \, dx = +\infty.$

(i) Let us consider $\psi_A(x) = \tilde{\psi}_A(|x - x_0|)$ where

- $\tilde{\psi}_A(0) = 0,$
- $\tilde{\psi}_A'(r) = 0$ for $r \leq 1,$
- $\tilde{\psi}_A'(r) = r - 1$ for $1 \leq r \leq A,$
- $\tilde{\psi}_A'(r) = 2A - 1 - r$ for $A \leq r \leq 2A - 1$
- $\tilde{\psi}_A'(r) = 0$ for $r \geq 2A - 1.$
By direct calculations, we have for a $c > 0$,

(4.28) \[ \forall x, \quad \forall A \geq 1, \quad |\nabla \psi_A|^2 \leq c \psi_A + c, \]

(4.29) \[ 1 + \psi_4(x) \geq \frac{|x - x_0|^2}{4}, \quad \forall 1 \leq |x - x_0| \leq A, \]

(4.30) \[ \psi_A(x) \equiv c_A, \quad \text{for } |x - x_0| \geq 2A - 1. \]

Let $Y_A(t) = \int \psi_A(x) |u(t, x)|^2 \, dx$. We have for a $c > 0$,

(4.31) \[ \forall A \geq 1, \quad \forall t, \quad |Y_A'(t)| \leq c \sqrt{Y_A(t)} + 1, \]

(4.32) \[ Y_A(0) \to +\infty, \quad \text{as } A \to +\infty, \]

(4.33) \[ \forall A, \quad Y_A(t) \to 0 \quad \text{as } t \to T. \]

(4.33) follows from (4.11)-(4.12) and (4.30). (4.32) is a consequence of (4.29) and the fact that $\int |x|^2 \phi(x)^2 \, dx = +\infty$. (4.31) can be deduced from (4.28) and (4.27). Indeed,

\[
Y_A'(t) = 4 \Im \int \nabla \psi_A \, u \overline{u} \\
\leq c \left( \int_{|x-x_0| \geq 1} |\nabla u|^2 \right)^{1/2} \left( \int |\nabla \psi_A|^2 |u|^2 \right)^{1/2} \\
\leq c \left( \int \psi_A(x) |u|^2 + \int |u|^2 \right)^{1/2} \leq c (Y_A(t) + 1)^{1/2}.
\]

Integrating in time (4.31), we obtain

\[ \forall A, \quad \forall t \in [0, T), \quad |\sqrt{Y_A(0)} + 1 - \sqrt{Y_A(t)} + 1| \leq c. \]

Letting $t \to T$, we then have $\sqrt{Y_A(0)} + 1 \leq c$, which contradicts (4.32). Therefore

\[ \int |x|^2 |\phi(x)|^2 \, dx < +\infty. \]
(ii) Considering now
\[ \psi (x) = \tilde{\psi}(|x - x_0|) \]
where \( \tilde{\psi}(0) = 0, \tilde{\psi}'(r) = 0 \) for \( r \leq 1, \tilde{\psi}'(r) = r - 1 \) for \( r \geq 1 \). We obtain
- \( |Y'(t)| \leq c \sqrt{Y(t) + 1}, \)
- \( Y(0) < +\infty. \)

Therefore, there is a constant \( c > 0 \) such that
\[ \forall t \in [0, T), \ Y(t) \leq c, \]
and since \( 2 + \tilde{\psi}(r) \geq \frac{r^2}{4} \),
\[ \forall t \in [0, T), \ \int |x - x_0|^2 |u(t, x)|^2 \, dx \leq c. \]

**Lemma 4.7.**
\[ \lim_{t \to T} \int \frac{|x - x_0|^2 |u(t, x)|^2}{A} \, dx = 0. \]

The proof is the same than the one in [10] (Step 2, p. 442). Let us recall the key parts of the proof. From (4.11)-(4.12), we have \( \forall \, A \geq 0, \)
\[ \lim_{t \to T} \int_{|x - x_0| \leq A} |x - x_0|^2 |u(t, x)|^2 \, dx = 0. \]

The conclusion will follow from an uniform integrability property:
\[ \forall \varepsilon > 0, \text{ there is a } A_\varepsilon \text{ such that} \]
\[ \forall t \in [0, T), \ \int_{|x - x_0| \geq A_\varepsilon} |x - x_0|^2 |u(t, x)|^2 \, dx \leq \varepsilon. \]

**Proof of (4.35).** - Let us consider \( \psi \in C^4(\mathbb{R}^N, \mathbb{R}) \)
- \( \psi(x) = \psi(|x - x_0|), \)
- \( \psi(x) = 0 \) for \( |x - x_0| \leq 1, \)
- \( \frac{1}{2} |x|^2 \leq \psi(x) \leq |x|^2 \) for \( |x - x_0| \geq 2, \)
- there is \( c > 0 \) such that \( \forall x, \forall r \geq 0, \)
\[ |\nabla \psi(x)| \leq c |x - x_0| \text{ and } |\psi''(r)| + |\psi'''(r)| + |\psi''''(r)| \leq c, \]

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and

\[ \psi_A(x) = A^2 \psi\left(\frac{x}{A}\right). \]

Considering \( \frac{d}{dt} \int \psi_A(x)|u(t, x)|^2 \, dx \), we obtain the existence of \( \varepsilon(A) > 0 \) such that

\[ (4.36) \quad \sup_{t \in [0, T]} \int \psi_A(x)|u(t, x)|^2 \, dx \leq \varepsilon(A) \]

where \( \varepsilon(A) \to 0 \) as \( A \to +\infty \) (see proof below). The fact that \( \psi_A(x) \geq \frac{1}{2} |x - x_0|^2 \) for \( |x - x_0| \geq 2A \) implies (4.35) and the conclusion follows.

Proof of (4.36). – Let us define

\[ Y_A(t) = \int \psi_A(x)|u(t, x)|^2 \, dx. \]

We have

\[ |Y_A'(t)| = |2 \text{Im} \int \nabla \psi_A(x) u \bar{u}| \]

\[ \leq 2 \left| \int_{|x| \geq A} \nabla \psi_A u \bar{u} \right| \]

\[ \leq 2 \left( \int_{|x| \geq A} |\nabla \psi_A|^2 |u|^2 \right)^{1/2} \]

\[ \times \left( \int_{|x| \geq A} |\nabla u(t, x)|^2 \, dx \right)^{1/2}. \]

We can remark that \( \forall A \geq 1, \)

\[ \forall x, \quad |\nabla \psi_A|^2 \leq c \psi_A(x) + c. \]

Therefore from Lemma 4.6,

\[ |Y_A(t)| \leq c \left( \int_{|x| \geq A} \psi_A(x)|u|^2 + \int_{|x| \geq A} |u|^2 \right)^{1/2} \]

\[ \times \left( \int_{|x| \geq A} |\nabla u|^2 \right)^{1/2} \]

\[ \leq c \left( Y_A(t) + \frac{1}{A^2} \right)^{1/2} \left( \int_{|x| \geq A} |\nabla u|^2 \right)^{1/2}. \]
or equivalently

\begin{equation}
|Y_A' (t)| \leq Y_A (t) + \frac{1}{A^2} + c \int_{|x| \geq A} |\nabla u|^2.
\end{equation}

Since (4.27),

\[
\int_0^T \int_{|x| \geq A} |\nabla u (t, x)|^2 dx dt \leq c,
\]

the convergence dominated theorem yields

\[
\lim_{A \to +\infty} \int_0^T \int_{|x| \geq A} |\nabla u (t, x)|^2 dx dt = 0.
\]

Therefore by integration of (4.39),

\[
\lim_{A \to +\infty} \left\{ \sup_{t \in [0, T]} Y_A (t) \right\} \leq c \left\{ \lim_{A \to +\infty} \int_0^T \int_{|x| \geq A} |\nabla u (t, x)^2| dx dt \right\}
\]

\[
+ c \left\{ \lim_{A \to +\infty} Y_A (0) \right\} = 0,
\]

which concludes the proof of (4.36) and of Proposition 4.1.

5. NONEXISTENCE OF $L^2$-MINIMAL BLOW-UP SOLUTIONS

In this section, we discuss nonexistence and existence of $L^2$-minimal blow-up solutions.

Under some conditions on the function $k (x)$ at infinity, we saw in Section 4 that a blow-up solution such that

\begin{equation}
\| \phi \|_{L^2} = \| Q k_2 \|_{L^2}
\end{equation}

concentrates at the blow-up time at a point $x_0$ such that

\begin{equation}
k (x_0) = k_2, \quad \nabla k (x_0) = 0.
\end{equation}

In subsection 5.1, under some condition on the form of $k (x)$ for $x$ near $x_0$, we prove that such a solution does not exist. We briefly give the existence of such a solution in subsection 5.2 under some condition of flatness on $k (x)$ for $x$ near $x_0$. 

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5.1. Nonexistence of minimal blow-up solution

Let $x_0$ be such that $k(x_0) = k_2$ (in particular $\nabla k(x_0) = 0$). We assume for a $c_0 > 0$ that

$$\nabla k(x) \cdot (x - x_0) \leq -c_0 |x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0,$$

where $0 < \alpha_0 < 1$. It implies in particular

$$(5.3)x_0 \quad k(x_0) - k(x) \geq c|x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0;$$

(this condition does not allow $k(x)$ to be $C^2$ near $x_0$). We claim the following theorem.

**Theorem 5.1.** Assume that $k(x)$ satisfies $(5.3)x_0$. There is then no blow-up solution such that

$$|u(t, x)|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0} \text{ in the distribution sense as } t \to T$$

(where $T$ is the blow-up time).

This theorem has the following corollary:

**Corollary 5.2 (Nonexistence of $L^2$-minimal blow-up solutions).** Assume that $k$ satisfies $(4.1)$, $(4.1)'$, $(4.1)''$ and all $x_0$ such that $k(x_0) = k_2$ satisfies $(5.3)x_0$. There is no blow-up solutions such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$$

We remark that the corollary follows directly from Section 4 and Theorem 5.1. Let us prove Theorem 5.1.

**Proof of Theorem 5.1.** We argue by contradiction. Assume there is a $\phi \in H^1$ such that

$$(5.4) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2},$$

$u(t)$ blows-up in finite time $T$, and

$$(5.5) \quad |u(t, x)|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}.$$

A contradiction follows from asymptotic estimates on the solution and energy arguments.
LEMMA 5.3. (Energy estimates). – We have

(i) \[ E(\phi) \geq E_{k_2} (u(t)) \geq 0, \]

(ii) \[ E(\phi) \geq \frac{1}{N + 2} \int (k_2 - k(x)) |u(t, x)|^{\frac{N+2}{N}} dx \geq 0, \]

(iii) \[ E_{k_2} (u(t)) + \frac{1}{N + 2} \int (k_2 - k(x)) |u(t, x)|^{\frac{N+2}{N}} dx \leq E(\phi). \]

Proof. – Parts (i) and (ii) follow from \[ \| \phi \|_{L^2} = \| u(t) \|_{L^2} \leq \| Q_{k_2} \|_{L^2}, \]
Part iii), and the definition of \( k_2 \). The conservation of the energy yields (iii).

We claim that

\[
\int (k_2 - k(x)) |u(t, x)|^{\frac{N+2}{N}} dx \to +\infty
\]

which will be a contradiction with part (ii) of Lemma 5.3.

Proof of (5.6). – From (5.3)x₀, (5.6) is implied by

\[
\int_{|x-x_0| \leq \rho_0} |x-x_0|^{1+\alpha_0} |u(t, x)|^{\frac{N+2}{N}} dx \to +\infty \quad \text{for a } \rho_0 > 0.
\]

LEMMA 5.4. – We have the existence of \( x(t) \to x_0 \), \( \theta(t) \in \mathbb{R}^2 \), such that

\[
\frac{1}{\lambda(t)} e^{i\theta(t)} u\left(t, x(t) + \frac{x-x(t)}{\lambda(t)}\right) \to Q_{k_2} (x) \quad \text{in } H^1,
\]

where \( \lambda(t) = \| \nabla u(t) \|_{L^2} \to +\infty. \)

Proof. – See Corollary 2.7.

Therefore for \( t \) near \( T \)

\[
|x_0 - x(t)| < \frac{\rho_0}{2}
\]
and

\[
\int_{|x-x_0|\leq r_0} |x-x_0|^{1+\alpha_0} |u(t, x)|^{\frac{N}{2}+2} \, dx \\
\geq \int_{|y| \leq \frac{r_0}{t}} \left| \frac{y}{\lambda(t)} + (x(t) - x_0) \right|^{1+\alpha_0} \\
\times \left| u \left( t, \frac{y}{\lambda(t)} + x(t) \right) \right|^{\frac{N}{2}+2} \, dx \\
\geq \lambda(t)^2 \int_{|y| \leq 10} \frac{1}{\lambda(t)^{1+\alpha_0}} |y + (x(t) - x_0) \lambda(t)|^{1+\alpha_0} Q_{k_2}^{\frac{N}{2}+2}(y) \, dy \\
\geq c \left( \int_{|y| \leq 10} Q_{k_2}^{\frac{N}{2}+2}(y) \, dy \right) \frac{\lambda(t)^2}{\lambda(t)^{1+\alpha_0}} \\
\geq c \lambda(t)^{1-\alpha_0} \xrightarrow{t\to T} +\infty.
\]

This concludes the proof of (5.6). A contradiction follows and Theorem 5.1 is proved.

5.2. Existence of $L^2$-blow-up solution and open problems

Using the same method as [9′], that is a fixed point and compactness argument near the solution of the homogeneous Schrödinger equation

\[
u(t, x) = w(t)^N e^{i|w(x_0)|^2} \frac{|x-x_0|^2}{4t^2} Q_{k_2} \left( \frac{w(x-x_0)}{t} \right),
\]

we are able to prove the following proposition.

PROPOSITION 5.4 (Existence $L^2$-minimal blow-up solution under flatness condition). – Assume $k(x) \equiv k_2$ for $x$ near $x_0$. There is then a $L^2$-minimal blow-up solution $u(t)$ such that

\[
|u(t, x)|^2 \to \|Q_{k_2}\|_{L^2} \delta_{x=x_0} \quad \text{as} \quad t \to T
\]

(where $T$ is the blow-up time of $u(t)$).
Remark. – Section 5 leaves open the question of existence and nonexistence of $L^2$-minimal blow-up solution in the case where $k$ is a $C^2$ near $x_0$ and

$$c_1 \leq \left| \frac{D^2 k(x, x)}{|x - x_0|^{2+i}} \right| \leq c_2$$

for $i = 0, 1, \ldots$

In addition, knowing which $i$ (and eventually $c_1, c_2$) separates the cases of existence and nonexistence is an open question.

6. STABILITY OF SINGULARITY

In this section, we point out the relation between the nonexistence of minimal blow-up solutions and the existence of black holes. We define a black hole as a “space singularity stable in time with respect to initial data”. More precisely, assume that there is no minimal blow-up solution $Q(x_0) = k_2$ and $x_0$ is a strict local maximum. Then the singularity

(6.1) $|u|^2 = \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$

will be stable in some sense. That is,

THEOREM 6.1. – Consider a sequence of initial data $\phi_n$ in $H^1$ such that

(6.2) $\int |\phi_n|^2 \to \|Q_{k_2}\|_{L^2}^2$, as $n \to +\infty$,

(6.3) $|\phi_n(x)|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$, as $n \to +\infty$

in the distribution sense,

(6.4) there is a $c > 0$ such that $E_{\epsilon_n}(\phi_n) \leq c$,

where $E_{\epsilon_n}(u) = E(u) + \frac{\epsilon_n}{q+1} \int |u|^{q+1}$, $\epsilon_n > 0$, $\epsilon_n \to 0$, and

$$\frac{N+2}{N-2} > q > \frac{4}{N} + 1.$$ Let $u_n(t)$ be the solution of

(6.5) $\begin{cases} iu_t = -\Delta u - k(x)|u|^{\frac{N}{2}} u + \epsilon_n |u|^{q-1} u \\ u_n(0) = \phi_n. \end{cases}$

(6.6) For all time $t > 0$,

$$|u_n(t, x)|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$$ in the distribution sense as $n \to +\infty$. 

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Remark. – We have considered $u_n(t)$ solution of equation (6.5) to assure that $u_n(t)$ will be defined for all time. The same conclusions hold for solutions of equation (1.1) ($\varepsilon_n = 0$) on their maximum common time existence interval.

Remark. – In the case of nonexistence of minimal blow-up solution such that

$$E(\phi) \leq a,$$

if we assume $E(\phi_n) \to a$, same conclusion holds.

Remark. – It is an open problem to show that there is no black hole at a mass level different of $\|Q_{k_2}\|_{L^2}^2$. We conjecture there is none.

Proof of Theorem 6.1. – We do it in three steps.

Step 1. – Reduction.

We claim using concentration properties that Theorem 6.1 is implied by the following property

(6.7) $\forall t, \liminf_{n \to +\infty} \{ \inf_{s \in [0, t]} \| \nabla u_n(s) \|_{L^2} \} = +\infty.$

(6.7) implies (6.6). – Indeed, assume (6.7) and let us fix $t > 0$. From Corollary 2.7, there is a $x_n(s)$ such that

$$\|u_n(s, x - x_n(s))\|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x = x_0}$$

in the distribution sense uniformly in $s$, that is: $\forall \delta_1 > 0, \forall \delta_2 > 0$, for $n$ large

$$\sup_{s \in [0, t]} \int_{|x - x_n(s)| \geq \delta_2} |u_n(s, x)|^2 \, dx \leq \delta_1.$$

We remark that the energy identity

$$E_{\varepsilon_n}(u_n(t)) = E_{\varepsilon_n}(\phi_n) \leq c$$

implies $E(u_n(t)) \leq c - \frac{\varepsilon_n}{q + 1} \int |u_n(t)|^{q+1} \leq c.$

Moreover, direct continuity arguments on the solution (with respect to the initial data) show that we can choose for a fixed $n, x_n(\cdot): [0, t] \to \mathbb{R}^N$ continuous with respect to $s$.

We claim that

$$\lim_{n \to +\infty} \sup_{s \in [0, t]} |x_n(s) - x_0| = 0.$$
Indeed, by contradiction, assume there is \( \delta > 0 \) such that \( \forall n, \) there is \( s_n \in [0, t] \) such that

\[
\forall n, \quad |x_n(s_n) - x_0| \geq \delta.
\]

We remark from (5.2) that

\[
x_n(0) \to x_0 \quad \text{as } n \to +\infty.
\]

Since \( x_n(s) \) is a continuous function of \( s \), there is a sequence \( \tau_n \in [0, t] \) such that

\[
|x_n(\tau_n) - x_0| = \delta.
\]

From the fact that \( x_0 \) is a strict local maximum, taking \( \delta \) small enough, there is \( \varepsilon > 0 \) such that

\[
k(x_n(\tau_n)) \leq k_2 - \varepsilon_0.
\]

By similar arguments than in the proof of Proposition 2.5, we have in addition

\[
\liminf_{n \to +\infty} \frac{\|\phi_n\|_{L^2}}{\|Q_{k_2}(x_n(\tau_n))\|_{L^2}} \geq \liminf_{n \to +\infty} \frac{\|u_n(\tau_n)\|_{L^2}}{\|Q_k(x_n(\tau_n))\|_{L^2}} \geq \liminf_{n \to +\infty} \frac{\|u_n(\tau_n)\|_{L^2} (B(x_n(\tau_n)))}{\|Q_k(x_n(\tau_n))\|_{L^2}} \geq 1.
\]

Going to the limit in (6.13) as \( n \to +\infty \), we obtain

\[
\frac{\|Q\|_{L^2}}{k_2^{\frac{N}{4}}} = \|Q_{k_2}\|_{L^2} \geq \limsup_{n \to +\infty} \|Q_k(x_n(\tau_n))\|_{L^2} \geq \limsup_{n \to +\infty} \frac{\|Q\|_{L^2}}{\left( k(x_n(\tau_n)) \right)^{\frac{N}{4}}} \geq \frac{\|Q\|_{L^2}}{(k_2 - \varepsilon_0)^{\frac{N}{4}}},
\]

which is a contradiction. This concludes the proof of (6.8) and the fact that (6.7) implies (6.6).
Proof of (6.7). – We are now reduced to prove (6.7). Let us argue by contradiction: assume there is a sequence \( s_n \) such that

\[
|s_n| \leq c \quad \text{and} \quad \| \nabla u_n (s_n) \|_{L^2} + \| u_n (s_n) \|_{L^2} \leq c.
\]

There is then a \( \delta_0 > 0 \), by Sobolev imbedding such that

\[
\int_{|x-x_0| \leq \delta_0} |u_n (s_n, x)|^2 \leq \frac{1}{2} \| Q_{k_2} \|_{L^2}^2.
\]

The fact that \( x_0 \) is a strict local maximum implies that taking \( \delta_0 \) sufficiently small, there is a \( \varepsilon_0 > 0 \) such that

\[
k(x) \leq k_2 - \varepsilon_0 \quad \text{for } |x-x_0| = \delta_0.
\]

Consider now \( t_n \in [0, s_n] \) such that

\[
\int_{|x-x_0| \leq \delta_0} |u_n (t_n, x)|^2 \, dx = \frac{1}{2} \| Q_{k_2} \|_{L^2}^2,
\]

for \( t \in [0, t_n] \), \( \int_{|x-x_0| \leq \delta_0} |u_n (t, x)|^2 \, dx \geq \frac{1}{2} \| Q_{k_2} \|_{L^2}^2.
\]

We have then \( t_n \) such that for a \( c_0 > 0 \), \( \delta_0 > 0 \) and

\[
|t_n| \leq c_0,
\]

\[
\| \nabla u_n (t_n) \|_{L^2} \leq c_0,
\]

\[
\int_{|x-x_0| \leq \delta_0} |u_n (t_n, x)|^2 \, dx = \frac{1}{2} \| Q_{k_2} \|_{L^2}^2.
\]

We just have to check (6.19). We argue by contradiction: assume for a subsequence also denoted \( t_n \)

\[
\| \nabla u_n (t_n) \|_{L^2} \xrightarrow{n \to +\infty} + \infty.
\]

Then by Corollary 2.7 and Proposition 2.5 (see (6.13)), we have

\[
|u_n (t_n, x-x_n)|^2 \to \| Q_{k_2} \|_{L^2}^2 \| \delta_{x=x_0} \|_{L^2}^2.
\]
and

\[
(6.23) \quad \liminf_{n \to +\infty} \left\{ \frac{\| \phi_n \|_{L^2}}{\| Q_k(x_n) \|_{L^2}} \right\} \geq \liminf_{n \to +\infty} \left\{ \frac{\| u_n(t_n) \|_{L^2(B(x_n,1))}}{\| Q_k(x_n) \|_{L^2}} \right\} \geq 1.
\]

Since \( \| \phi_n \|_{L^2} \to \| Q_{k_2} \|_{L^2} \), we have from (6.17) and (6.22),

\[
(6.24) \quad x_n \to \hat{x} \quad |x_0 - \hat{x}| = \delta_0.
\]

(6.23) implies that

\[
(6.25) \quad \| Q_{k_2} \|_{L^2} \geq \| Q_{k}(\hat{x}) \|_{L^2},
\]

that is

\[
\frac{\| Q \|_{L^2}}{k_{2}^{\frac{3}{2}}} \geq \frac{\| Q \|_{L^2}}{[k(\hat{x})]^{\frac{3}{2}}} \quad \text{or} \quad k(\hat{x}) \geq k_2,
\]

which is a contradiction with (6.16) and (6.24). Thus (6.19) is proved. Let us now obtain a contradiction with \( u_n(t_n) \).

**Step 2. – Compactness of \( u_n(t_n) \) in \( L^2 \).**

**Lemma 6.2. – There is a \( \phi \in H^1 \) such that**

\[
(6.26) \quad u_n(t_n) \to \phi \quad \text{in} \quad L^2 \quad \text{as} \quad n \to +\infty
\]

(eventually subtracting a subsequence).

**Proof of Lemma 6.2. – From (6.19) and (6.20) and the fact that**

\[
(6.27) \quad \| u(t_n) \|_{L^2} = \| \phi_n \|_{L^2} \xrightarrow[n \to +\infty]{} \| Q_{k_2} \|_{L^2},
\]

we have, by standard compactness arguments, (eventually subtracting a subsequence) the existence of \( \phi \in H^1 \) such that

\[
(6.28) \quad u_n(t_n) \to \phi \quad \text{in} \quad L^2_{loc} \quad \text{as} \quad n \to +\infty.
\]

In addition,

\[
(6.29) \quad \| \nabla \phi \|_{H^1} \leq c,
\]

\[
(6.30) \quad \| \phi \|_{L^2} \leq \| Q_{k_2} \|_{L^2}.
\]
We claim that in fact

\[(6.31) \quad \| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} .\]

Then (6.31) together with (6.27)-(6.28) give that

\[(6.32) \quad u_n (t_n) \to \phi \text{ in } L^2 \text{ as } n \to +\infty.\]

We show (6.31) by contradiction. We have to avoid in some sense dichotomy. Assume that

\[(6.33) \quad \| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} - \delta \quad \text{where} \quad \delta > 0.\]

We can remark from (6.37), (6.33) and (6.28) that

\[(6.34) \quad \frac{1}{2} \| Q_{k_2} \|_{L^2} \leq \| \phi \|_{L^2} \text{ or } \delta < \frac{\| Q_{k_2} \|_{L^2}}{2}.\]

We then have the existence of \( R_0 \) and a sequence \( R_n \to +\infty \) such that for \( n \) large

\[(6.35) \quad \| u_n (t_n) \|_{L^2 (|x| > R_0)} \geq \| Q_{k_2} \|_{L^2} - \delta - \frac{\delta}{8};\]

and

\[(6.36) \quad \| u_n (t_n) \|_{L^2 (|x| > R_n)} = \delta - \frac{\delta}{8}.\]

We consider now \( \psi \) such that

\[\psi \in C^\infty, \quad |\psi| \leq 1, \quad \psi \equiv 0 \text{ for } |x| \leq \frac{1}{2}, \quad \psi \equiv 1 \text{ for } |x| \geq 1.\]

Let us consider \( t'_n \) such that

\[(6.37) \quad \text{for } t \in [t'_n, t_n], \quad \int \psi \left( \frac{x}{R_n} \right) |u_n (t, x)|^2 \, dx \geq \frac{\delta}{2},\]

\[(6.38) \quad \int \psi \left( \frac{x}{R_n} \right) |u_n (t'_n, x)|^2 \, dx = \frac{\delta}{2}.\]
We have from (6.15), (6.2)-(6.3) that

\begin{equation}
0 < t'_n < t_n \quad \text{and} \quad 0 \leq t_n - t'_n \leq c.
\end{equation}

In addition, we have, for \( c > 0 \),

\begin{equation}
\forall t \in [t'_n, t_n], \quad \|
\nabla u_n(t) \|_{L^2} \leq c.
\end{equation}

Indeed by contradiction Lemma 5.6 (ii) implies that for \( x_n \) and \( \tau_n \in [t'_n, t_n] \)

\begin{equation}
| u_n(\tau_n, x - x_n) |^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0}.
\end{equation}

We have in addition

\[ \| u_n \|_{L^2}^2 = \| \phi_n \|_{L^2} \to \| Q_{k_2} \|_{L^2}^2. \]

For \( n \) large \( \| u_n \|_{L^2}^2 - \| Q_{k_2} \|_{L^2}^2 \leq \frac{1}{8} \| Q_{k_2} \|_{L^2}^2 \) and from (6.17)

\[ \int_{|x-x_0| \leq \delta_0} | u_n(\tau_n, x) |^2 \, dx \geq \frac{1}{2} \| Q_{k_2} \|_{L^2}^2, \]

we obtain using (6.40) that

\begin{equation}
| x - x_0 | \leq 2 \delta_0.
\end{equation}

Then, from (6.41)-(6.42), we obtain for \( n \) large

\[ \| u_n(\tau_n) \|_{L^2 (|x-x_0| \geq 3\delta_0)} \leq \frac{\delta}{4} \]

or

\[ \| u_n(\tau_n) \|_{L^2 (|x| > R_n/2)} \leq \frac{\delta}{4}, \]

which is a contradiction with (6.37). Therefore (6.40) is proved.

Let \( y_n(s) = \int \psi(\frac{x}{R_n}) | u_n(t_n - s, x) |^2 \, dx \). We have

\begin{equation}
y_n(0) \geq \int_{|x| > R_n} | u_n(t_n, x) |^2 \, dx \geq \delta - \frac{\delta}{8},
\end{equation}

\begin{equation}
y_n(t_n - t'_n) = \int \psi(\frac{x}{R_n}) | u_n(t'_n, x) |^2 \, dx = \frac{\delta}{2},
\end{equation}

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for $s \in [0, t_n - t'_n]$,

$$\left| y'_n(s) \right| \leq \frac{c}{R_n} \left| \int \nabla \psi \left( \frac{x}{R_n} \right) \nabla u_n u_n \right| \leq \frac{c}{R_n}.$$ 

Integrating (6.45), we obtain from (6.39)

$$\left| y_n(t_n - t'_n) - y_n(0) \right| \leq \left| t_n - t'_n \right| \frac{c}{R_n} \to 0 \quad \text{as } n \to +\infty$$

which is a contradiction with (6.43)-(6.44). This concludes the proofs of (6.31) and of Lemma 6.2.

Remark. – In the case where

$$-\varepsilon_n = 0,$$

and the fact that there is no minimal blow-up solutions, the solution of Eq. (1.1) with initial data $\phi$, $u(t)$ is globally defined for all $t \in \mathbb{R}$ (using conjugation for $t > 0$ and for $t < 0$). Moreover, there is a $c > 0$ such that

$$\int |x|^2 |\phi_n(x)|^2 dx \leq c,$$

there is a simpler proof of (6.31).

Step 3. – Conclusion of the proof.

We have then the existence of $\phi \in H^1$ such that

$$u_n(t_n) \to \phi \quad \text{in } L^2 \quad \text{as } n \to +\infty.$$ 

Since $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ and the fact that there is no minimal blow-up solutions, the solution of Eq. (1.1) with initial data $\phi$, $u(t)$ is globally defined for all $t \in \mathbb{R}$ (using conjugation for $t > 0$ and for $t < 0$). Moreover, there is a $c > 0$ such that

$$\|\nabla u(t)\|_{L^2} \leq c,$$

(where $c_0$ is defined in (6.18)).

Continuity arguments with respect to the initial data in $L^2$ implies in fact that

$$u_n(t_n + t) \to u(t) \quad \text{in } C\left([-c_0, 0], L^2\right) \quad \text{as } n \to +\infty.$$ 

In the case $\varepsilon_n = 0$ it follows from a result of Cazenave and Weissler (Theorem 1.2 of [1′]). In the general case, we can see from Kato [6] that

$$u_n(t_n) \to \phi$$

in standard Cauchy space where continuity with respect initial data is true from (6.19) and $\phi \in H^1$. 

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Since \(|t_n| \leq c_0\), from (6.45) we have
\[
\int |u_n(t_n - t_n) - u(-t_n)|^2 \to 0 \quad \text{as } n \to +\infty
\]
or equivalently
\[
\int |\phi_n(x) - u(-t_n)|^2 \to 0 \quad \text{as } n \to +\infty.
\]
From (6.3), (6.48),
in the distribution sense which is a contradiction with the fact

\[
\|\nabla u(-t_n)\|_{L^2} \leq c.
\]
This concludes the proof of (6.7) and of Theorem 6.1.

REFERENCES


BLOW-UP FOR $iu_t = -\Delta u - k(x)|u|^{4/N} u$ IN $\mathbb{R}^N$


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